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TAUBERIAN THEOREMS FOR $(\mathfrak{R}, p, \alpha)$ SUMMABILITY

Dedicated to Professor Gen-ichirô Sunouchi on his 60th birthday

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1. Introduction. Professor G. Sunouchi has introduced the summability $(\mathfrak{R}, \alpha)$ and $(\mathfrak{R}^*, \alpha)$ in his paper [4]. Later we [3] have introduced, as generalizations of these summability, the summability $(\mathfrak{R}, p, \alpha)$ defined as follows. Throughout this paper, p denotes a positive integer and α denotes a real number, not necessarily an integer, such that $0 < \alpha < p$. Let us put

$$C_{p,\alpha} = \int_0^\infty \frac{\sin^p x}{x^{\alpha+1}}\,dx\ ,$$

$$(1.1)\qquad \varphi(n, t) \equiv \varphi(nt) \equiv (C_{p,\alpha})^{-1}\int_{nt}^\infty \frac{\sin^p x}{x^{\alpha+1}}\,dx = (C_{p,\alpha})^{-1}\int_t^\infty \frac{\sin^p nu}{n^\alpha u^{\alpha+1}}\,du\ .$$

Then a series $\sum_{n=0}^\infty a_n$ is said to be summable $(\mathfrak{R}, p, \alpha)$ to s if the series in

$$f(p, \alpha, t) = a_0 + \sum_{n=1}^\infty a_n\varphi(nt)$$

converges for t positive and small and $f(p, \alpha, t) \to s$ as $t \to +0$. Under this definition, the summability $(\mathfrak{R}, \alpha)$ and the summability $(\mathfrak{R}^*, \alpha)$ are reduced to the summability $(\mathfrak{R}, 1, \alpha)$ and the summability $(\mathfrak{R}, 2, \alpha)$, respectively. On the other hand, for a series $\sum a_n$, let us write $\sigma_n^\beta = s_n^\beta/A_n^\beta$, where s_n^β and A_n^β are defined by the relations

$$(1.2)\qquad (1-x)^{-\beta-1} = \sum_{n=0}^\infty A_n^\beta x^n \quad\text{and}\quad (1-x)^{-\beta-1}\sum_{n=0}^\infty a_n x^n = \sum_{n=0}^\infty s_n^\beta x^n\ .$$

Then, if $\sigma_n^\beta \to s$ as $n \to \infty$, we say that the series $\sum_{n=0}^\infty a_n$ is summable (C, β), $\beta > -1$, to s.

Concerning $(\mathfrak{R}, p, \alpha)$ summability, we [3] have proved the following theorems.

THEOREM A. *Let $0 < \beta < \alpha < p$. Then, if a series $\sum_{n=0}^\infty a_n$ is summable (C, β) to s, the series $\sum_{n=0}^\infty a_n$ is summable $(\mathfrak{R}, p, \alpha)$ to s.*

THEOREM B. *Let $0 < \alpha < p$, $\lambda_n > 0$ $(n = 1, 2, 3, \cdots)$ and the series $\sum_{n=1}^\infty \lambda_n/n$ converge. Then, if*

$$s_n^\alpha - sA_n^\alpha = o(n^\alpha \lambda_n) ,$$

the series $\sum_{n=0}^\infty a_n$ *is summable* $(\mathfrak{R}, p, \alpha)$ *to* s.

In Theorem B, we may take $\lambda_n = 1/(\log(n+2))^{1+\delta}$, $\delta > 0$. Then we know that if

$$s_n^1 = o(n/(\log(n+2))^{1+\delta}) ,$$

the series $\sum_{n=0}^\infty a_n$ is summable $(\mathfrak{R}, p, 1)$, $p > 1$, to 0. However we have the following.

THEOREM 1. *There exists a series* $\sum_{n=0}^\infty a_n$ *which is not summable* $(\mathfrak{R}, p, 1)$, $p > 1$, *but satisfies the condition*

$$s_n^1 = o(n/\log(n+2)) . \tag{1.3}$$

This is proved in § 3. Since the condition (1.3) implies the $(C, 1)$ summability of the series $\sum_{n=0}^\infty a_n$, we see that the $(C, 1)$ summability does not necessarily imply the $(\mathfrak{R}, p, 1)$ summability when $p > 1$. One of the object of this paper is to study Tauberian condition for the $(\mathfrak{R}, p, 1)$ summability of the series which satisfies the condition (1.3). Concerning this problem we have the following.

THEOREM 2. *Let* $0 < \alpha < p$ *and let* $0 < \delta < \alpha$. *Suppose that*

$$s_n^\alpha - sA_n^\alpha = o(n^\alpha/\log(n+2)) \tag{1.4}$$

and

$$\sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) = O(n^{\alpha-\delta}) . \tag{1.5}$$

Then the series $\sum_{n=0}^\infty a_n$ *is summable* $(\mathfrak{R}, p, \alpha)$ *to* s.

On the other hand we have the following theorems.

THEOREM 3. *Let* $0 < \alpha < p$ *and let* $0 < \gamma < \beta \leqq p - 1$. *Suppose that*

$$s_n^\beta - sA_n^\beta = o(n^\gamma) \tag{1.6}$$

and

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu} = O(n^{-(1-\delta)}) , \tag{1.7}$$

where $0 < \delta < 1$ *and* $\delta = \alpha(\beta-\gamma)/(\beta+1-\alpha)$. *Then the series* $\sum_{n=0}^\infty a_n$ *is summable* $(\mathfrak{R}, p, \alpha)$ *to* s.

THEOREM 4. *Let* p *be an odd integer. Then, in Theorem* 3, *we may* *replace* $\beta \leqq p-1$ *and* $\delta = \alpha(\beta-\gamma)/(\beta+1-\alpha)$ *by* $\beta \leqq p$ *and* $\delta = (\alpha+1)(\beta-\gamma)/(\beta-\alpha)$, *respectively.*

Since, for fixed α, β, γ,

$$\alpha(\beta-\gamma)/(\beta+1-\alpha) < (\alpha+1)(\beta-\gamma)/(\beta-\alpha),$$

we see that Theorem 4 is better than Theorem 3. If $\gamma < \alpha$, the condition (1.6) implies $s_n^\gamma - s A_n^\gamma = o(n^\gamma)$, that is, the series $\sum_{n=0}^\infty a_n$ is summable (C, γ) to s. Then, by Theorem A, the series $\sum_{n=0}^\infty a_n$ is summable $(\mathfrak{R}, p, \alpha)$ to s. Thus we see that Theorems 3 and 4 are significant for $\alpha \leqq \gamma$. In § 6, Theorems 2, 3 and 4 are proved by means of the following theorems.

THEOREM 5. *Let* ω *be a positive integer and let* $0 < \tau < \omega$. *Let* $\chi(t)$ *be a function defined for* $t \geqq 0$ *such that*

$$\chi(0) = \chi(+0) = 1, \qquad \chi(t) = O(t^{-\tau}), \tag{1.8}$$

$$\Delta^m \chi(nt) = O(t^{m-\tau-1} n^{-\tau-1}), \qquad 0 < m \leqq \omega + 1, \tag{1.9}$$

and, in addition, when τ *is an integer*

$$\Delta^{\tau+1} \chi(nt) = O(tn^{-\tau}), \tag{1.10}$$

where $\Delta^m \chi(nt)$ *denotes the* m-th *difference of* $\chi(nt)$ *with respect to* n *and* m *denotes an integer. Let* $0 < \delta < \tau$. *Suppose that*

$$s_n^\tau - sA_n^\tau = o(n^\tau/\log(n+2)) \tag{1.11}$$

and

$$\sum_{\nu=n}^{2n} (|a_\nu| - a_\nu) = O(n^{\tau-\delta}). \tag{1.12}$$

Then the series $\sum_{n=0}^\infty a_n \chi(nt)$ *converges for* t, *positive and small, and its sum tends to* s *as* $t \to +0$.

THEOREM 6. *Let* ω *be a positive integer and let* $0 < \tau < \omega$. *Let* $\chi(t)$ *be a function defined for* $t \geqq 0$ *such that*

$$\chi(0) = \chi(+0) = 1, \qquad \chi(t) = O(t^{-\tau}) \tag{1.13}$$

and

$$\Delta^m \chi(nt) = O(t^{m-\tau} n^{-\tau}), \qquad 0 < m \leqq \omega, \tag{1.14}$$

where m *denotes an integer. Let* $\tau - 1 < \gamma < \beta \leqq \omega - 1$. *Suppose that a series* $\sum_{n=0}^\infty a_n$ *satisfies the conditions* (1.6) *and* (1.7) *in which*

$$0 < \delta < 1 \quad \text{and} \quad \delta = \tau(\beta-\gamma)/(\beta+1-\tau).$$

Then the series $\sum_{n=0}^\infty a_n \chi(nt)$ *converges for* t, *positive and small, and its sum tends to* s *as* $t \to +0$.

Theorems 5 and 6 are proved in § 4 and § 5, respectively.

2. Some Lemmas.

LEMMA 1. *Let* $0 < \delta < \tau$ *and let* $s_n^\beta = o(n^\beta)$, $\beta > 0$. *Then the condition* (1.12) *implies*

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu^\tau} = O(n^{-\delta}) \quad \text{and} \quad s_n \equiv s_n^0 = O(n^{\tau-\delta}) . \tag{2.1}$$

Proof is similar to the proof of Lemma 3 in [2], so we omit it.

LEMMA 2. *Let* $0 < \delta < \tau$. *Then the condition* (1.7) *implies*

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu^\tau} = O(n^{\delta-\tau}) \quad \text{and} \quad s_n \equiv s_n^0 = O(n^\delta) . \tag{2.2}$$

PROOF. Let $r_n = \sum_{\nu=n}^{\infty} |a_\nu|/\nu$. Then

$$\sum_{\nu=n}^{2n} |a_\nu| = \sum_{\nu=n}^{2n} \nu(r_\nu - r_{\nu+1}) = \sum_{\nu=n+1}^{2n} r_\nu + nr_n - 2nr_{2n+1} = O(n^\delta) .$$

Hence we have

$$\sum_{\nu=n}^{\infty} \frac{|a_\nu|}{\nu^\tau} = \sum_{\mu=0}^{\infty} \sum_{\nu=2^\mu n}^{2^{\mu+1}n-1} \frac{|a_\nu|}{\nu^\tau} \leqq n^{-\tau} \sum_{\mu=0}^{\infty} 2^{-\mu\tau} \sum_{\nu=2^\mu n}^{2^{\mu+1}n-1} |a_\nu|$$
$$= O\left(n^{-\tau} \sum_{\mu=0}^{\infty} 2^{-(\tau-\delta)\mu} n^\delta\right) = O(n^{\delta-\tau}) ,$$

which proves the required result. $s_n = O(n^\delta)$ is similarly proved.

LEMMA 3. *Let* $0 < \alpha < p$ *and let* m *be an integer. Then, for* $\varphi(t)$ *in* (1.1),

$$\Delta^m \varphi(nt) = O(t^{m-\alpha} n^{-\alpha}) \qquad \text{when } 0 \leqq m \leqq p , \tag{2.3}$$

$$\Delta^m \varphi(nt) = O(t^{m-\alpha-1} n^{-\alpha-1}) \qquad \text{when } 0 < m \leqq p+1 \tag{2.4}$$

and, for an odd integer p,

$$\varphi(nt) = O(t^{-\alpha-1} n^{-\alpha-1}) . \tag{2.5}$$

PROOF. This lemma for $m \geqq 1$ is Lemma 1 in [3] and (2.3) for $m = 0$ is trivial. (2.5) is proved by means of the identity

$$(-1)^{(p-1)/2} 2^{p-1} (\sin t)^p$$
$$= \sin pt - \binom{p}{1} \sin(p-2)t + \cdots + (-1)^{(p-1)/2} \binom{p}{(p-1)/2} \sin t$$

and, for a constant $k \neq 0$,

$$\int_u^\infty \frac{\sin kx}{x^{\alpha+1}}\,dx = \frac{\cos ku}{ku^{\alpha+1}} - \frac{\alpha+1}{k}\int_u^\infty \frac{\cos kx}{x^{\alpha+2}}\,dx = O(u^{-\alpha-1}) .$$

3. Proof of Theorem 1. Omitting the constant factor in (1.1), let

$$\varphi_0(n, t) \equiv \varphi_0(nt) = \int_{nt}^\infty \frac{\sin^p x}{x^2}\,dx .$$

By the Abel transformation two times, we have, by (1.3) and (2.3),

$$\sum_{n=1}^\infty a_n\varphi_0(nt) = \sum_{n=1}^\infty s_n^1 \Delta^2\varphi_0(nt) = \sum_{n=1}^\infty \varepsilon_n c_n(t) ,$$

where $\varepsilon_n = s_n^1 \log(n+2)/n$ and $c_n(t) = n\Delta^2\varphi_0(nt)/\log(n+2)$ when $n \geqq 1$. For the proof of Theorem, it is sufficient to prove that the sequence-to-function transformation $\sum \varepsilon_n c_n(t)$ is not convergence-preserving. In order that the transformation is convergence-preserving, $\sum_{n=1}^\infty |c_n(t)|$ must be bounded in $0 < t < t_0$. But this series is divergent at some point in an arbitrary neighbourhood of origin. The proof of this is as follows.

$$\begin{aligned}\Delta^2\varphi_0(nt) &= \Delta\left(\int_{nt}^{(n+1)t} \frac{\sin^p x}{x^2}dx\right)\\ &= \int_0^t\left\{\frac{\sin^p(nt+x)}{(nt+x)^2} - \frac{\sin^p((n+1)t+x)}{((n+1)t+x)^2}\right\}dx .\end{aligned}$$

We now take $t = 2\pi/k$, $k = 8, 9, 10, \cdots$, and $n = km$, $m = 1, 2, 3, \cdots$. Then

$$\begin{aligned}\Delta^2\varphi_0\left(km, \frac{2\pi}{k}\right) &= \int_0^t\left\{\frac{\sin^p x}{(2m\pi+x)^2} - \frac{\sin^p(x+t)}{(2m\pi+x+t)^2}\right\}dx\\ &= \int_0^t \frac{(t^2+2xt+4m\pi t)\sin^p x}{(2m\pi+x)^2(2m\pi+x+t)^2}dx - \int_0^t \frac{\sin^p(x+t) - \sin^p x}{(2m\pi+x+t)^2}dx\\ &= b_1(m) - b_2(m), \quad \text{say} .\end{aligned}$$

Hence

$$|\Delta^2\varphi_0(km, 2\pi/k)| \geqq b_2(m) - b_1(m) .$$

On the other hand, when $t = 2\pi/k$,

$$\begin{aligned}b_2(m) &= \int_0^t \frac{\sin^p(x+t) - \sin^p x}{(2m\pi+x+t)^2}dx \geqq \frac{\sin^{p-1} t}{4(m\pi+t)^2}\int_0^t(\sin(x+t)-\sin x)dx\\ &= \frac{\sin^{p-1}t\cdot\cos t\cdot(1-\cos t)}{2(m\pi+t)^2} \geqq \frac{\sin^{p-1} t\cdot\cos t\cdot(1-\cos t)}{8\pi^2}\cdot\frac{1}{m^2}\end{aligned}$$

and

$$b_1(m) = \int_0^t \frac{(t^2+2xt+4m\pi t)\sin^p x}{(2m\pi+x)^2(2m\pi+x+t)^2} dx < \frac{1}{m^3} .$$

Thus we have, for $t = 2\pi/k$, $k = 8, 9, 10, \cdots$,

$$\begin{aligned}\sum_{n=1}^{\infty} | c_n(t) | &= \sum_{n=1}^{\infty} \frac{n}{\log (n+2)} | \Delta^2\varphi_0(n, 2\pi/k) | \\ &\geqq \sum_{m=1}^{\infty} \frac{km}{\log (km+2)} | \Delta^2\varphi_0(km, 2\pi/k) | \\ &\geqq \sum_{m=1}^{\infty} \left(\frac{\sin^{p-1} t \cdot \cos t \cdot (1 - \cos t)}{8\pi^2} \cdot \frac{k}{m \log(km + 2)} - \frac{km}{\log(km)} \cdot \frac{1}{m^3} \right) \\ &= \infty ,\end{aligned}$$

and the proof is completed.

4. **Proof of Theorem 5.** We shall first prove Theorem when τ is not an integer. For the proof, we may assume, without loss of generality, that $s = 0$ and $a_0 = 0$. We now take r such that $r\delta - \tau > 0$ and $(2\tau + 1 - [\tau])r > 2\tau+1$*) and we put $\xi = [t^{-r}]$, $0 < t < 1$. Let us write

$$\sum_{n=1}^{\infty} a_n \chi(nt) = \left(\sum_{n=1}^{\xi+1} + \sum_{n=\xi+2}^{\infty} \right) = U(t) + V(t) ,$$

where, by (1.8) and (2.1),

$$V(t) = \sum_{n=\xi+2}^{\infty} a_n\chi(nt) = O\left(t^{-\tau} \sum_{n=\xi}^{\infty} | a_n |/n^{\tau}\right) = O(t^{-\tau}\xi^{-\delta}) = O(t^{r\delta-\tau}) = o(1) ,$$

which proves the convergence of the series $\sum_{n=0}^{\infty} a_n\chi(nt)$ for t positive.

We shall next consider $U(t)$. Using Abel's transformation

$$U(t) = \sum_{n=1}^{\xi+1} a_n\chi(nt) = \sum_{n=1}^{\xi} s_n\Delta\chi(nt) + s_{\xi+1}\chi((\xi+1)t) ,$$

where, by (1.8) and (2.1),

$$s_{\xi+1}\chi((\xi+1)t) = O(\xi^{\tau-\delta}\cdot\xi^{-\tau}t^{-\tau}) = O(t^{r\delta-\tau}) = o(1) .$$

Now, by the well-known formula

$$s_n = \sum_{\nu=1}^{n} A_{n-\nu}^{-\tau-1} s_{\nu}^{\tau} ,$$

$$\begin{aligned}\sum_{n=1}^{\xi} s_n\Delta\chi(nt) &= \sum_{\nu=1}^{\xi} s_{\nu}^{\tau} \sum_{n=\nu}^{\xi} A_{n-\nu}^{-\tau-1} \Delta\chi(nt) = \sum_{\nu=1}^{\xi} s_{\nu}^{\tau} G(\nu, \xi, t) \\ &= \left(\sum_{\nu=1}^{\eta} + \sum_{\nu=\eta+1}^{\xi} \right) = U_1(t) + U_2(t), \quad \text{say} ,\end{aligned}$$

*) Throughout this paper, $[x]$ denotes the greatest integer less than x.

where $G(\nu, \xi, t) = \sum_{n=\nu}^{\xi} A_{n-\nu}^{-\tau-1} \Delta\chi(nt)$ and $\eta = [1/t]$. Then we have, by the method similar to that of the proof of Lemma 2 in [3], for ν, ξ and t,

$$G(\nu, \xi, t) = O(\nu^{-\tau-1})$$

and

$$G(\nu, \xi, t) = O(\nu^{[\tau]-2\tau} t^{[\tau]-\tau+1}) + O(\xi^{[\tau]-2\tau-1} t^{-\tau}) \,. \tag{4.1}$$

Hence

$$\begin{aligned} U_1(t) &= o\Big(\sum_{\nu=1}^{\eta} \frac{\nu^{\tau}}{\log(\nu+2)} \cdot \nu^{[\tau]-2\tau} t^{[\tau]-\tau+1}\Big) + o\Big(\sum_{\nu=1}^{\eta} \frac{\nu^{\tau}}{\log(\nu+2)} \cdot \xi^{[\tau]-2\tau-1} t^{-\tau}\Big) \\ &= o\Big(t^{[\tau]-\tau+1} \sum_{\nu=1}^{\eta} \nu^{[\tau]-\tau}\Big) + o\Big(\xi^{[\tau]-2\tau-1} t^{-\tau} \sum_{\nu=1}^{\eta} \nu^{\tau}\Big) \\ &= o(1) \,, \end{aligned}$$

because $(2\tau-[\tau]+1)\, r > 2\tau + 1$, by the our assumption, and

$$\begin{aligned} U_2(t) &= o\Big(\sum_{\nu=\eta+1}^{\xi} \frac{\nu^{\tau}}{\log \nu} \cdot \frac{1}{\nu^{\tau+1}}\Big) = o(\log\log \xi - \log\log \eta) \\ &= o(\log r) = o(1) \,. \end{aligned}$$

Summing up the above estimates, we obtain

$$\sum_{n=1}^{\infty} a_n \chi(nt) \to 0 \quad \text{as} \quad t \to +0 \,,$$

which is the required result.

Next we shall prove Theorem when τ is an integr. The method of the proof runs similarly to that of the proof when τ is not an integer. So we shall remark some of different points and omit the proof in detail. In this case we take r such that $r\delta - \tau > 0$ and $r > \tau$ and use (1.10) in place of (4.1) in the estimation of $U_1(t)$ above. Then the remaining part is similarly proved.

5. Proof of Theorem 6. We shall prove Theorem when β is not an integer, the case in which β is an integer being easily proved by the method analogous to the following argument. For the proof we may assume, without loss of generality, that $s = 0$. We first remark that

$$\tau - \delta = \tau(\gamma+1-\tau)/(\beta+1-\tau) > 0 \,.$$

Hence, by Lemma 2, we have (2.2). Let $k = [\beta] + 1$. Then, by $\beta < \omega - 1$, we get $k + 1 \leqq w$. Let us now write

$$\sum_{n=0}^{\infty} a_n \chi(nt) = \Big(\sum_{n=0}^{\xi+k+1} + \sum_{n=\xi+k+2}^{\infty}\Big) = U(t) + V(t) \,,$$

where $\xi = [(\varepsilon t)^{-\rho}]$, ε being an arbitrary fixed positive number, and

$$\rho = \frac{\tau}{\tau - \delta} = \frac{\beta + 1 - \tau}{\gamma + 1 - \tau} .$$

Then, by (1.13) and (2.2),

$$V(t) = \sum_{n=\xi+k+2}^{\infty} a_n \chi(nt) = O\Big(t^{-\tau} \sum_{n=\xi}^{\infty} \frac{|a_n|}{n^\tau}\Big) = O(\xi^{\delta-\tau} t^{-\tau}) = O(\varepsilon^\tau) .$$

We shall next prove $U(t) = o(1) + O(\varepsilon^\tau)$. By the Abel transformation $(k+1)$ times, we have

$$U(t) = \sum_{n=0}^{\xi+k+1} a_n \chi(nt) = \sum_{n=0}^{\xi} s_n^k \Delta^{k+1} \chi(nt) + \sum_{\nu=0}^{k} s_{\xi+k-\nu+1}^{\nu} \Delta^{\nu} \chi((\xi+k-\nu+1)t)$$

$$= U_0(t) + \sum_{\nu=0}^{k} W_\nu(t), \quad \text{say} .$$

Using the Dixon and Ferrar convexity theorem [1], we have, by (1.6) and (2.2),

$$s_n^\nu = o(n^{(\delta(\beta-\nu)+\gamma\nu)/\beta}) , \qquad 0 < \nu < \beta .$$

Hence, by (1.13), (1.14) and (2.2),

$$\begin{aligned} W_0(t) &= O(\xi^\delta \cdot \xi^{-\tau} t^{-\tau}) = O(\xi^{\delta-\tau} t^{-\tau}) = O(\varepsilon^\tau) , \\ W_\nu(t) &= O(\xi^{\{\delta(\beta-\nu)+\gamma\nu\}/\beta} \cdot \xi^{-\tau} t^{\nu-\tau}) \\ &= o(t^{\nu-\tau-\tau\{\delta(\beta-\nu)+\gamma\nu-\beta\tau\}/\beta(\tau-\delta)}) \\ &= o(t^{(\tau\nu/\beta(\tau-\delta))\cdot(\beta-\gamma)/(\beta+1-\tau)}) = o(1) , \qquad \text{for } \nu = 1, 2, \cdots, k-1 , \end{aligned}$$

and, since $s_n^k = o(n^{k-\beta+\gamma})$,

$$\begin{aligned} W_k(t) &= o(\xi^{k-\beta+\gamma} \cdot \xi^{-\tau} t^{k-\tau}) = o(\xi^{k+\gamma-\beta-\tau} t^{k-\tau}) \\ &= o(t^{k-\tau-(k+\gamma-\beta-\tau)\tau/(\tau-\delta)}) = o(1) . \end{aligned}$$

It remains to prove that $U_0(t) = o(1)$. But this is proved by the method analogous to that of the proof of $U_0(t) = o(1)$ in the proof of Theorem 1 in [2]. Thus, summing up the above estimates, we obtain

$$\limsup_{t \to +0} \Big| \sum_{n=0}^{\infty} a_n \chi(nt) \Big| = 0(\varepsilon^\tau) .$$

Since ε is an arbitrary positive number, we have

$$\lim_{t \to +0} \sum_{n=0}^{\infty} a_n \chi(nt) = 0 ,$$

and Theorem is completely proved.

6. Proofs of Theorems 2, 3 and 4. Under the assumptions of

Theorem 2, by Lemma 3, we can take $\chi(t) = \varphi(t)$, $\tau = \alpha$ and $w = p$, in Theorem 5. Then Theorem 2 is proved by means of Theorem 5. On the other hand, if we take $\chi(t) = \varphi(t)$, $\tau = \alpha$ and $w = p$, in Theorem 6, then, combining the remark to Theorem 3 given in §1, we have Theorem 3. Similarly, by (2.4) and (2.5), if we take $\chi(t) = \varphi(t)$, $\tau = \alpha + 1$ and $\omega = p + 1$, in Theorem 6, then we have Theorem 4.

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