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NORMALITY OF ALMOST CONTACT 3-STRUCTURE

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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0. Introduction. The almost contact 3-structure has been defined by Kuo [5, 6], Tachibana [6, 12], Yu [12] and studied by them and Eum [16], Kashiwada [4], Ki [16], Sasaki [10], Yano [16]. Some topics related to almost contact 3-structures have been considered by Ishihara, Konishi [1, 2, 3] and Tanno [13].

It is well known that the product of a manifold with almost contact 3-structure and a straight line admits an almost quaternion structure (cf. [5]). Recently, Ako and one of the present authors [14, 15] have proved that, if for an almost quaternion structure (F, G, H) the Nijenhuis tensors $[F, F]$ and $[G, G]$ vanish, then the other Nijenhuis tensors $[H, H]$, $[G, H]$, $[H, F]$ and $[F, G]$ vanish too (cf. Obata [7]), and that if the Nijenhuis tensor $[F, G]$ vanishes, then the other Nijenhuis tensors $[F, F]$, $[G, G]$, $[H, H]$, $[G, H]$, and $[H, F]$ vanish too. The main purpose of the present paper is to study almost contact 3-structures in the light of this work.

1. Almost contact 3-structure. Let M be an n-dimensional differentiable manifold[1] and let f, U and u be a tensor field of type (1, 1), a vector field and a 1-form in M, respectively. If f, U and u satisfy

$$f^2 = -I + u \otimes U\,, \quad fU = 0\,, \quad u \circ f = 0\,, \quad u(U) = 1\,,$$

the 1-form $u \circ f$ being defined by $(u \circ f)(x) = u(fx)$[2] and I being the identity tensor field of type (1, 1), then the set (f, U, u) is called an *almost contact structure* (cf. [8, 9, 11]).

Let f_1, f_2 be tensor fields of type (1, 1), U_1, U_2 vector fields and u_1, u_2 1-forms in M. If (f_1, U_1, u_1) and (f_2, U_2, u_2) are both almost contact structures and satisfy

$$f_1 f_2 + f_2 f_1 = u_1 \otimes U_2 + u_2 \otimes U_1\,, \quad f_1 U_2 + f_2 U_1 = 0\,,$$
$$u_1 \circ f_2 + u_2 \circ f_1 = 0\,, \quad u_1(U_2) = 0\,, \quad u_2(U_1) = 0\,,$$

1) Manifolds, vector fields, tensor fields and other geometric objects we discuss are assumed to be differentiable and of class C^∞.

2) Here and in the sequel, x, y and z denote arbitrary vector fields in the manifold M.

then the sets (f_1, U_1, u_1) and (f_2, U_2, u_2) are said to define an *almost contact 3-structure* in M.

If (f_1, U_1, u_1) and (f_2, U_2, u_2) define an almost contact 3-structure, putting

$$f_3 = f_1 f_2 - u_2 \otimes U_1 = -f_2 f_1 + u_1 \otimes U_2 ,$$
$$U_3 = f_1 U_2 = -f_2 U_1 , \qquad u_3 = u_1 \circ f_2 = -u_2 \circ f_1 ,$$

we can easily verify that (f_3, U_3, u_3) defines an almost contact structure. We can also verify

$$\begin{aligned} f_1 &= f_2 f_3 - u_3 \otimes U_2 & f_2 &= f_3 f_1 - u_1 \otimes U_3 \\ &= -f_3 f_2 + u_2 \otimes U_3 , & &= -f_1 f_3 + u_3 \otimes U_1 , \\ U_1 &= f_2 U_3 = -f_3 U_2 , & U_2 &= f_3 U_1 = -f_1 U_3 , \\ u_1 &= u_2 \circ f_3 = -u_3 \circ f_2 , & u_2 &= u_3 \circ f_1 = -u_1 \circ f_3 , \end{aligned}$$
$$u_2(U_3) = 0 , \quad u_3(U_2) = 0 , \quad u_3(U_1) = 0 , \quad u_1(U_3) = 0 .$$

Therefore any two of (f_1, U_1, u_1), (f_2, U_2, u_2) and (f_3, U_3, u_3) define essentially the same almost contact 3-structure. In this sense, we say that such almost contact structures $(f_\lambda, U_\lambda, u_\lambda)$ $(\lambda = 1, 2, 3)$ define in M an almost contact 3-structure $\{(f_\lambda, U_\lambda, u_\lambda);\ \lambda = 1, 2, 3\}$.

2. Almost quaternion structure. Let there be given, in a manifold $\bar{M}$, three tensor fields F_λ $(\lambda = 1, 2, 3)$[3] of type $(1, 1)$ satisfying

$$F_\lambda^2 = -I , \qquad F_\lambda F_\mu = -F_\mu F_\lambda = F_\nu ,$$

where (λ, μ, ν) is an even permutation of $(1, 2, 3)$. Then the set $\{F_\lambda; \lambda = 1, 2, 3\}$ is called an *almost quaternion structure* in $\bar{M}$, where $\bar{M}$ is necessarily $4m$-dimensional.

For two tensor fields P and Q of type $(1, 1)$ in $\bar{M}$, the Nijenhuis tensor $[P, Q]$ of P and Q is, by definition, a tensor field of type $(1, 2)$ such that

$$\begin{aligned} (2.1) \qquad 2[P, Q](X, Y) = {} & [PX, QY] - P[QX, Y] - Q[X, PY] \\ & + [QX, PY] - Q[PX, Y] - P[X, QY] + (PQ + QP)[X, Y] \end{aligned}$$

and hence the Nijenhuis tensor $[P, P]$ of P is given by

$$(2.2) \qquad [P, P](X, Y) = [PX, PY] - P[PX, Y] - P[X, PY] + P^2[X, Y] ,$$

where X and Y denote arbitrary vector fields in $\bar{M}$. Ako and one of the present authors [14] (cf. [7]) have proved

THEOREM A. *If, for an almost quaternion structure* $\{F_\lambda; \lambda = 1, 2, 3\}$,

[3] In the sequel, Greek indices λ, μ, ν run over the range $\{1, 2, 3\}$.

the Nijenhuis tensors $[F_1, F_1]$ *and* $[F_2, F_2]$ *vanish, then the other Nijenhuis tensors* $[F_3, F_3]$, $[F_2, F_3]$, $[F_3, F_1]$ *and* $[F_1, F_2]$ *vanish too.*

They have also proved in [15]

THEOREM B. *If, for an almost quaternion structure* $\{F_\lambda; \lambda = 1, 2, 3\}$, *the Nijenhuis tensor* $[F_1, F_2]$ *vanishes, then the other Nijenhuis tensors* $[F_1, F_1]$, $[F_2, F_2]$, $[F_3, F_3]$, $[F_2, F_3]$ *and* $[F_3, F_1]$ *vanish too.*

3. Almost contact 3-structure and almost quaternion structure. Let M be a manifold with almost contact 3-structure $\{(f_\lambda, U_\lambda, u_\lambda); \lambda = 1, 2, 3\}$. We now consider the product space $M \times R$, where R is a straight line. Let X be a vector field in $M \times R$, which is naturally represented by a pair of a vector field x in M and a function α in M, i.e.,[4]

$$X = \begin{pmatrix} x \\ \alpha \end{pmatrix}.$$

We define torsor fields F_λ ($\lambda = 1, 2, 3$) of type (1, 1) in $M \times R$ by

$$F_\lambda X = F_\lambda\begin{pmatrix} x \\ \alpha \end{pmatrix} = \begin{pmatrix} f_\lambda x - \alpha U_\lambda \\ u_\lambda(x) \end{pmatrix}. \tag{3.1}$$

Then, using (1.1) and (3.1), we see easily

$$F_\lambda^2 = -I, \qquad F_\lambda F_\mu = -F_\mu F_\lambda = F_\nu, \tag{3.2}$$

(λ, μ, ν) being an even permutation of (1, 2, 3), which shows that $\{F_\lambda; \lambda = 1, 2, 3\}$ defines an almost quaternion structure in $M \times R$. Thus we have (cf. [5, 12])

LEMMA 3.1. *If* M *is a manifold with an almost contact 3-structure* $\{(f_\lambda, U_\lambda, u_\lambda); \lambda = 1, 2, 3\}$, *then the product space* $M \times R$ *admits an almost quaternion structure* $\{F_\lambda; \lambda = 1, 2, 3\}$ *defined by* (3.2).

Since the almost quaternion manifold $M \times R$ is $4m$-dimensional, M with almost contact 3-structure is $(4m - 1)$-dimensional.

4. Nijenhuis tensors. For two vectors X and Y in $M \times R$ of the form $X = \begin{pmatrix} x \\ \alpha \end{pmatrix}$ and $Y = \begin{pmatrix} y \\ \beta \end{pmatrix}$, where x and y are arbitrary vector fields in M and α, β arbitrary functions in M, the bracket product of X and Y is a vector field of the form

$$[X, Y] = \begin{pmatrix} [x, y] \\ x\beta - y\alpha \end{pmatrix}. \tag{4.1}$$

[4] In the sequel, X, Y, and Z denote arbitrary vector fields of this type in $M \times R$, i.e., $X = \begin{pmatrix} x \\ \alpha \end{pmatrix}$, $Y = \begin{pmatrix} y \\ \beta \end{pmatrix}$, $Z = \begin{pmatrix} z \\ \gamma \end{pmatrix}$, x, y, z being vector fields and α, β, γ functions in M.

If we take account of (1.1), (2.2) and (4.1), we have, for the tensor field F defined by (3.1),

$$[F, F](X, Y) = \begin{pmatrix} [f, f](x, y) + (du)(x, y)U - \alpha(\mathfrak{L}_U f)y + \beta(\mathfrak{L}_U f)x \\ (du)(fx, y) + (du)(x, fy) - \alpha(\mathfrak{L}_U u)(y) + \beta(\mathfrak{L}_U u)(x) \end{pmatrix},$$

$\mathfrak{L}_U$ denoting the Lie derivation with respect to U and du being defined by $du(x, y) = xu(y) - yu(x) - u([x, y])$, where we have used the formulas

$$(\mathfrak{L}_U u)(x) = Uu(x) - u([U, x]) , \qquad (\mathfrak{L}_U f)x = [U, fx] - f[U, x] .$$

Thus we have $[F, F] = 0$ if and only if

$$\begin{cases} [f, f] + (du) \otimes U = 0 , & \mathfrak{L}_U f = 0 , \\ (du) \barwedge f = 0 , & \mathfrak{L}_U u = 0 , \end{cases} \tag{4.2}$$

where $du \barwedge f$ is a 2-form defined by

$$((du) \barwedge f)(x, y) = (du)(fx, y) + (du)(x, fy) .$$

On the other hand, it is well known that the first equation of (4.2) implies all the others (cf. [11]). Thus we have the following well known lemma:

LEMMA 4.1. *A necessary and sufficient condition that $[F, F] = 0$ in $M \times R$, that is, the almost complex structure F is integrable in $M \times R$ is that*

$$[f, f] + du \otimes U = 0 \tag{4.3}$$

holds in M.

If the condition (4.3) is satisfied, then the almost contact structure (f, U, u) is said to be *normal*. Thus, taking account of Theorem A and Lemma 4.1, we have

THEOREM 4.2. *If, for an almost contact 3-structure $\{(f_\lambda, U_\lambda, u_\lambda); \lambda = 1, 2, 3\}$, any two of almost contact structures $(f_\lambda, U_\lambda, u_\lambda)$ are normal, then the third is so* (cf. [5]).

Next, using (1.2), (1.3), (1.4), (2.1), and (4.1), we find for F_λ defined by (3.1)

$$2[F_1, F_2](X, Y)$$
$$= \begin{pmatrix} 2[f_1, f_2](x, y) + du_1(x, y)U_2 + du_2(x, y)U_1 - \alpha(\mathfrak{L}_{U_1} f_2 + \mathfrak{L}_{U_2} f_1)y \\ \qquad + \beta(\mathfrak{L}_{U_1} f_2 + \mathfrak{L}_{U_2} f_1)x \\ (du_1 \barwedge f_2 + du_2 \barwedge f_1)(x, y) - \alpha(\mathfrak{L}_{U_1} u_2 + \mathfrak{L}_{U_2} u_1)(y) + \beta(\mathfrak{L}_{U_1} u_2 + \mathfrak{L}_{U_2} u_1)(x) \end{pmatrix}.$$

Thus we have

LEMMA 4.2. *A necessary and sufficient condition that* $[F_1, F_2]$ *vanishes in* $M \times R$ *is that in* M

$$\left\{\begin{array}{ll} 2[f_1, f_2] + du_1 \otimes U_2 + du_2 \otimes U_1 = 0\ , & \mathfrak{L}_{U_1} f_2 + \mathfrak{L}_{U_2} f_1 = 0\ , \\ (du_1) \barwedge f_2 + (du_2) \barwedge f_1 = 0\ , & \mathfrak{L}_{U_1} u_2 + \mathfrak{L}_{U_2} u_1 = 0\ . \end{array}\right. \tag{4.4}$$

We now prove that the first equation of (4.4) implies the last equation. If we put

$$S(x, y) = 2[f_1, f_2](x, y) + du_1(x, y) U_2 + du_2(x, y) U_1\ ,$$

then, computing $S(x, U_\lambda)$, we obtain

$$S(x, U_1) = (\mathfrak{L}_{U_3} f_1)x + f_1(\mathfrak{L}_{U_1} f_2)x + f_2(\mathfrak{L}_{U_1} f_1)x - (\mathfrak{L}_{U_1} u_1)(x) U_2 - (\mathfrak{L}_{U_1} u_2)(x) U_1\ , \tag{4.5}$$

$$S(x, U_2) = -(\mathfrak{L}_{U_3} f_2)x + f_1(\mathfrak{L}_{U_2} f_2)x + f_2(\mathfrak{L}_{U_2} f_1)x - (\mathfrak{L}_{U_2} u_1)(x) U_2 - (\mathfrak{L}_{U_2} u_2)(x) U_1\ , \tag{4.6}$$

$$\begin{aligned} S(x, U_3) = {} & f_2(\mathfrak{L}_{U_3} f_1)x + f_1(\mathfrak{L}_{U_3} f_2)x - (\mathfrak{L}_{U_1} f_1)x + (\mathfrak{L}_{U_2} f_2)x + du_1(x, U_3) U_2 \\ & + du_2(x, U_3) U_1\ . \end{aligned} \tag{4.7}$$

Thus, if $S(x, y) = 0$, using (4.5)-(4.7), we have

$$\begin{aligned} 0 &= f_2(S(x, U_1)) - f_1(S(x, U_2)) \\ &= S(x, U_3) - du_1(x, U_3) U_2 - du_2(x, U_3) U_1 + f_2 f_1\{(\mathfrak{L}_{U_1} f_2)x + (\mathfrak{L}_{U_2} f_1)x\} \\ &\quad - \{u_1((\mathfrak{L}_{U_2} f_2)x) + u_2((\mathfrak{L}_{U_2} f_1)x)\} U_1 - \{u_1((\mathfrak{L}_{U_2} f_1)x) - u_2((\mathfrak{L}_{U_1} f_1)x)\} U_2 \\ &\quad + \{(\mathfrak{L}_{U_1} u_2)(x) + (\mathfrak{L}_{U_2} u_1)(x)\} U_3 \\ &= f_2 f_1\{(\mathfrak{L}_{U_1} f_2)x + (\mathfrak{L}_{U_2} f_1)x\} + \{(\mathfrak{L}_{U_1} u_2)(x) + (\mathfrak{L}_{U_2} u_1)(x)\} U_3 \\ &\quad - \{u_1((\mathfrak{L}_{U_2} f_2)x) + u_2((\mathfrak{L}_{U_2} f_1)x) - (\mathfrak{L}_{U_3} u_2)(x)\} U_1 \\ &\quad - \{u_1((\mathfrak{L}_{U_2} f_1)x) - u_2((\mathfrak{L}_{U_1} f_1)x) - (\mathfrak{L}_{U_3} u_1)(x)\} U_2\ , \end{aligned}$$

from which, using $u_3 \circ (f_2 f_1) = -u_1 \circ f_1 = 0$, $u_3(U_1) = 0$, $u_3(U_2) = 0$, and $u_3(U_3) = 1$, we obtain

$$\mathfrak{L}_{U_1} u_2 + \mathfrak{L}_{U_2} u_1 = 0\ .$$

Thus we have

LEMMA 4.3. *The first equation of* (4.4) *implies the last equation.*

From Lemmas 4.2 and 4.3, we have

THEOREM 4.4. *Let* M *admit an almost contact* 3*-structure* $\{(f_\lambda, U_\lambda, u_\lambda);\ \lambda = 1, 2, 3\}$. *A necessary and sufficient condition that* $[F_1, F_2]$ *vanishes in* $M \times R$ *is that in* M

$$\begin{gathered} 2[f_1, f_2] + du_1 \otimes U_2 + du_2 \otimes U_1 = 0\ , \\ \mathfrak{L}_{U_1} f_2 + \mathfrak{L}_{U_2} f_1 = 0\ , \qquad du_1 \barwedge f_2 + du_2 \barwedge f_1 = 0\ . \end{gathered} \tag{4.8}$$

Taking account of Theorems B and 4.4, we have

THEOREM 4.5. *A necessary and sufficient condition that, for an almost contact 3-structure* $\{(f_\lambda, U_\lambda, u_\lambda); \lambda = 1, 2, 3\}$, *the almost contact structures* $(f_\lambda, U_\lambda, u_\lambda)$ *are all normal is that the condition* (4.8) *is valid.*

5. A special case. In this section, we assume that the almost contact 3-structure $\{(f_\lambda, U_\lambda, u_\lambda); \lambda = 1, 2, 3\}$ satisfies the condition

$$2[f_1, f_2] + du_1 \otimes U_2 + du_2 \otimes U_1 = 0\,, \tag{5.1}$$

$$\mathfrak{L}_{U_1} f_2 = 2f_3\,, \quad \mathfrak{L}_{U_2} f_1 = -2f_3\,, \quad du_1 \barwedge f_2 + du_2 \barwedge f_1 = 0\,.$$

Then, by Theorem 4.4, all of the almost contact structures $(f_\lambda, U_\lambda, u_\lambda)$ are normal.

Forming $(\mathfrak{L}_{U_2} f_1) U_1 = -2f_3 U_1$, we find by means of (1.1) and (1.3)

$$-f_1 \mathfrak{L}_{U_2} U_1 = -2U_2\,, \quad \text{i.e.} \quad f_1[U_1, U_2] = -2U_2\,,$$

from which, applying f_1,

$$-[U_1, U_2] + u_1([U_1, U_2])U_1 = -2U_3\,.$$

On the other hand, using Lemma 4.3, we obtain

$$\begin{aligned} u_1([U_1, U_2]) &= -u_1(\mathfrak{L}_{U_2} U_1) = (\mathfrak{L}_{U_2} u_1)(U_1) \\ &= -(\mathfrak{L}_{U_1} u_2)(U_1) = \mathfrak{L}_{U_1} u_2(U_1) = 0 \end{aligned}$$

and consequently

$$[U_1, U_2] = 2U_3\,. \tag{5.2}$$

Forming next $(\mathfrak{L}_{U_2} f_1) U_2 = -2f_3 U_2$, we have

$$\mathfrak{L}_{U_2}(f_1 U_2) = 2U_1\,, \quad \text{i.e.,} \quad \mathfrak{L}_{U_2} U_3 = 2U_1$$

and hence

$$[U_2, U_3] = 2U_1\,. \tag{5.3}$$

Similarly, forming $(\mathfrak{L}_{U_1} f_2) U_1 = 2f_3 U_1$, we obtain

$$[U_3, U_1] = 2U_2\,. \tag{5.4}$$

Thus, summing up (5.2), (5.3), and (5.4), we have

THEOREM 5.1. *If, for an almost contact 3-structure* $\{(f_\lambda, U_\lambda, u_\lambda); \lambda = 1, 2, 3\}$, *the condition* (5.1) *is satisfied, then we have*

$$[U_\lambda, U_\mu] = 2U_\nu\,, \tag{5.5}$$

where (λ, μ, ν) *is an even permutation of* $(1, 2, 3)$.

Now, forming $u_1 \circ (\mathfrak{L}_{U_1} f_2) = 2u_1 \circ f_3$, we find

$$\mathfrak{L}_{U_1}(u_1 \circ f_2) = 2u_1 \circ f_3\,, \quad \text{i.e.,} \quad \mathfrak{L}_{U_1} u_3 = -2u_2 \tag{5.6}$$

and consequently, by means of Lemma 4.3,

$$\mathfrak{L}_{U_3} u_1 = 2u_2 . \tag{5.7}$$

Similarly, we find also

$$\mathfrak{L}_{U_1} u_2 = -\mathfrak{L}_{U_2} u_1 = 2u_3$$

and

$$\mathfrak{L}_{U_2} u_3 = -\mathfrak{L}_{U_3} u_2 = 2u_1 .$$

That is, we have

THEOREM 5.2. *If, for an almost contact* 3*-structure* $\{(f_\lambda, U_\lambda, u_\lambda); \lambda = 1, 2, 3\}$, *the condition* (5.1) *is satisfied, then we have*

$$\mathfrak{L}_{U_\lambda} u_\mu = -\mathfrak{L}_{U_\mu} u_\lambda = 2u_\nu \tag{5.8}$$

where (λ, μ, ν) *is an even permutation of* $(1, 2, 3)$.

Since we have assumed (5.1), we have, from (4.5),

$$(\mathfrak{L}_{U_3} f_1) + f_1(\mathfrak{L}_{U_1} f_2)x + f_2(\mathfrak{L}_{U_1} f_1)x - (\mathfrak{L}_{U_1} u_1)(x) U_2 - (\mathfrak{L}_{U_1} u_2)(x) U_1 = 0 ,$$

$$(\mathfrak{L}_{U_3} f_1)x + 2f_1 f_3 x - 2u_3(x) U_1 = 0 ,$$

$$\mathfrak{L}_{U_3} f_1 = 2f_2 . \tag{5.9}$$

Hence, from Theorems 4.5 and B, we also have

$$\mathfrak{L}_{U_1} f_3 = -2f_2 \tag{5.10}$$

and

$$-\mathfrak{L}_{U_3} f_2 = \mathfrak{L}_{U_2} f_3 = 2f_1 . \tag{5.11}$$

Thus we have

THEOREM 5.3. *If, for an almost contact* 3*-structure* $\{(f_\lambda, U_\lambda, u_\lambda); \lambda = 1, 2, 3\}$, *the condition* (5.1) *is satisfied, then we have*

$$\mathfrak{L}_{U_\lambda} f_\mu = -\mathfrak{L}_{U_\mu} f_\lambda = 2f_\nu , \tag{5.12}$$

where (λ, μ, ν) *is an even permutation of* $(1, 2, 3)$.

6. Contact 3-structure. Let (M, γ) be a Riemannian manifold with metric tensor γ and let (f, U, u) be an almost contact structure in M. When the conditions

$$\gamma(x, y) = \gamma(fx, fy) + u(x)u(y) , \qquad u(x) = \gamma(U, x)$$

are satisfied, γ is said to be a metric associated with (f, U) and (f, U) is called an *almost contact metric structure* in (M, γ). If, for an almost contact metric structure (f, U) in (M, γ), we put

$$\Phi(x, y) = \gamma(fx, y) ,$$

then Φ is a skew-symmetric tensor field of type $(0, 2)$, i.e., a 2-form. When the condition $\Phi = (1/2)du$ is satisfied, (f, U) is called a *contact structure* in (M, γ). If, for a contact structure (f, U), U is a Killing vector, then it is called a *K-contact structure in* (M, γ). For a K-contact structure, we have $f = \nabla U$, where ∇ denotes the covariant differentiation with respect to the Riemannian connection of (M, γ) (cf. [9]).

When, for an almost contact 3-structure $\{(f_\lambda, U_\lambda, u_\lambda); \lambda = 1, 2, 3\}$ in (M, γ), each of $(f_\lambda, U_\lambda, u_\lambda)$ is a contact structure (resp. a K-contact structure), the set $\{(f_\lambda, U_\lambda); \lambda = 1, 2, 3\}$ is called a *contact 3-structure* (resp. a *K-contact 3-structure*) in (M, γ).

Let $\{(f_\lambda, U_\lambda); \lambda = 1, 2, 3\}$ be a contact 3-structure, then we have, from the definition,

$$du_\lambda(x, y) = 2\gamma(f_\lambda x, y) ,$$

from which, we obtain

$$du_\lambda \barwedge f + du_\mu \barwedge f_\lambda = 0 , \qquad (\lambda \neq \mu) .$$

Thus, for a contact 3-structure $\{(f_\lambda, U_\lambda); \lambda = 1, 2, 3\}$, the condition (4.8) is equivalent to

$$2[f_1, f_2] + du_1 \otimes U_2 + du_2 \otimes U_1 = 0 , \qquad \mathfrak{L}_{U_1} f_2 + \mathfrak{L}_{U_2} f_1 = 0 .$$

Therefore, from Theorem 4.6, we have

THEOREM 6.1. *A necessary and sufficient condition that, for contact 3-structure $\{(f_\lambda, U_\lambda); \lambda = 1, 2, 3\}$ in a Riemannian manifold, each of (f_λ, U_λ) is normal is that*

$$2[f_1, f_2] + du_1 \otimes U_2 + du_2 \otimes U_1 = 0 , \qquad \mathfrak{L}_{U_1} f_2 + \mathfrak{L}_{U_2} f_1 = 0 . \tag{6.1}$$

Let $\{(f_\lambda, U_\lambda); \lambda = 1, 2, 3\}$ be a K-contact 3-structure in (M, γ). Then we have $f_\lambda = \nabla U_\lambda$ and hence, from (1.2),

$$[U_\lambda, U_\mu] = 2U_\nu , \tag{6.2}$$

(λ, μ, ν) being an even permutation of $(1, 2, 3)$. On the other hand, since U_λ are all Killing vectors, we have

$$\mathfrak{L}_{U_\lambda} \nabla = \nabla \mathfrak{L}_{U_\lambda} . \tag{6.3}$$

Thus, taking account of (6.2) and (6.3), we have

$$2f_3 = 2\nabla U_3 = \nabla \mathfrak{L}_{U_1} U_2 = \mathfrak{L}_{U_1} \nabla U_2 = \mathfrak{L}_{U_1} f_2 ,$$

and similarly

$$2f_3 = -\mathfrak{L}_{U_2} f_1 .$$

Consequently, for a K-contact 3-structure $\{(f_\lambda, U_\lambda); \lambda = 1, 2, 3\}$ in (M, γ), (5.1) is equivalent to

$$2[f_1, f_2] + du_1 \otimes U_2 + du_2 \otimes U_1 = 0 .$$

Therefore we have

THEOREM 6.2. *A necessary and sufficient condition that, for a K-contact 3-structure* $\{(f_\lambda, U_\lambda); \lambda = 1, 2, 3\}$ *in a Riemannian manifold, each of* (f_λ, U_λ) *is normal is that*

$$2[f_1, f_2] + du_1 \otimes U_2 + du_2 \otimes U_1 = 0 .$$

BIBLIOGRAPHY

[1] S. ISHIHARA AND M. KONISHI, Fibred Riemannian spaces with Sasakian 3-structure, Differential Geometry, in honor of K. Yano, Kinokuniya, Tokyo, 1972, 179-194.
[2] S. ISHIHARA AND M. KONISHI, On f-three-structure, Hokkaido Math. J., 1 (1972), 127-135.
[3] S. ISHIHARA AND M. KONISHI, Fibred Riemannian space with triple of Killing vectors, Kōdai Math. Sem. Rep. 25 (1973).
[4] T. KASHIWADA, On three framed f-structures with some relations, to appear in Nat. Sci. Rep. Ochanomizu Univ., 22 (1972). 91-99.
[5] Y. Y. KUO, On almost contact 3-structure, Tôhoku Math. J., 22 (1970), 235-332.
[6] Y. Y. KUO AND S. TACHIBANA, On the distribution appeared in contact 3-structure, Taita J. of Math., 2 (1970), 17-24.
[7] M. OBATA, Affine connections of manifolds with almost complex, quaternion or Hermitian structure, Japan. J. Math., 26 (1956), 43-77.
[8] S. SASAKI, On differentiable manifolds with certain structure which are closely related to contact structure I, Tôhoku Math. J., 12 (1960), 459-476.
[9] S. SASAKI, Almost contact manifolds, Lecture Note I, Tôhoku Univ.
[10] S. SASAKI, Spherical space forms with normal contact metric 3-structure, J. Diff. Geom., 6 (1972), 307-315.
[11] S. SASAKI AND Y. HATAKEYAMA, On differentiable manifolds with certain structures which are closely related to contact structure II, Tôhoku Math. J., 13 (1961), 281-294.
[12] S. TACHIBAMA AND W. N. YU, On a Riemannian space admitting more than one Sasakian structure, Tôhoku Math. J., 22 (1970), 536-540.
[13] S. TANNO, Killing vectors on contact Riemannian manifolds and fibering related to the Hopf fibrations, Tôhoku Math. J., 23 (1971), 313-334.
[14] K. YANO AND M. AKO, Integrability conditions for almost quaternion structures, Hokkaido Math. J., 1 (1972), 63-86.
[15] K. YANO AND M. AKO, An affine connection in an almost quaternion manifold, to appear in J. Diff. Geom.
[16] K. YANO, SANG-SEUP EUM AND U-HANG KI, On almost contact affine 3-structure, Kōdai Math. Sem. Rep., 25 (1973).

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