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LIE ALGEBRAS IN WHICH EVERY FINITELY GENERATED SUBALGEBRA IS A SUBIDEAL

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1. Introduction.

1.1. We prove that: *To every positive integer n there exist positive integers $\lambda_1(n)$ and $\lambda_2(n)$ such that every Lie algebra all of whose $\lambda_2(n)$-generator subalgebras are n-step subideals is nilpotent of class $\leqq \lambda_1(n)$.*

This result is the Lie theoretic analogue of that by Roseblade [4]. We leave unanswered the question of whether or not we can replace $\lambda_2(n)$ by n. However we give an example which shows that if $\lambda_2(n)$ is replaced by $n-2$, then the result is false.

1.2. Notation. All Lie algebras considered in this paper (unless otherwise specified) will have finite or infinite dimension over a fixed (but arbitrary) field $\boldsymbol{k}$.

We employ the notation of [3] and [5].

Let L be a Lie algebra and H a subspace of L. By $H \leqq L$, $H \lhd L$, H si L, $H \lhd^m L$ we shall mean (respectively) that H is a subalgebra, an ideal, *subideal* (in the sense of Hartley [3] p. 257), and m-step subideal of L.

Square brackets $[\,,\,]$ will denote Lie multiplication and triangular brackets $\langle\,,\,\rangle$ will denote the subalgebra generated by their contents. If A, B are subsets of L, then $[A, B]$ is the subspace spanned by all $[a, b]$ with $a \in A$, $b \in B$; and inductively, $[A, {}_0B] = A$ and $[A, {}_nB] = [[A, {}_{n-1}B], B] (n > 0)$. We let $\langle A^B \rangle$ be the smallest subalgebra of L containing A and invariant under Lie multiplication by the elements of B. If A, B are subspaces we define $A \circ B = \langle [A, B]^C \rangle$, where $C = \langle A, B \rangle$; and inductively $A \circ_1 B = A \circ B$, $A \circ_{n+1} B = (A \circ_n B) \circ B$; and $A + B$ is the vector space spanned by A and B.

$L^{(n)}$, L^n, $Z_n(L)$ denote respectively the n-th terms of the derived series, lower central series and upper central series of L. Inductively we define $L^{(0)} = L$, $L^{(n)} = [L^{(n-1)}, L^{(n-1)}]$, $L^1 = L$, $L^{n+1} = [L^n, L]$, $Z_0(L) = 0$, $Z_n(L)/Z_{n-1}(L) = Z(L/Z_{n-1}(L)) (n > 0)$ where $Z(L) =$ centre of $L = \{x \in L \mid [x, L] = 0\}$.

If $H \leqq L$, then the ideal closure series of H in L,

$$\cdots H_i \lhd H_{i-1} \lhd \cdots \lhd H_0 = L\ ,$$

is defined inductively by $H_0 = L$, $H_{i+1} = \langle H^{H_i} \rangle$. Evidently $H \lhd^n L$ if and only if $H = H_n$.

By a *class X of Lie algebras* (over k) we shall mean a class (in the usual sense) whose elements are Lie algebras and such that $(0) \in X$, and if $H \cong K$ with $K \in X$, then $H \in X$. (0) is the 0-dimensional Lie algebra. A *closure operation* A assigns to each class X another class AX in such a way that $A(0) = (0)$, $X \leqq AX$, $A(AX) = AX$, and if $X \leqq Y$ then $AX \leqq AY$ (here (0) is the class consisting only of the 0-dimensional Lie algebra; $\leqq$ denotes inclusion of classes). If X, Y are classes then XY is the class of all L with an X-ideal H such that $L/H \in Y$. We define the product of n classes by $X_1 \cdots X_n = (X_1 \cdots X_{n-1})X_n$; and if each $X_i = X$ we write X^n. If X is a class and A a closure operation we say X is *A-closed* if $X = AX$.

We shall need the following classes.

$$\boldsymbol{F},\ \boldsymbol{F}_m,\ \boldsymbol{G},\ \boldsymbol{G}_r,\ \boldsymbol{A},\ \boldsymbol{N},\ \boldsymbol{N}_c$$

will denote the classes of finite dimensional, finite dimensional of dimension $\leqq m$, finitely generated, finitely generated by $\leqq r$ elements, abelian, nilpotent, nilpotent of class $\leqq c$, Lie algebras respectively. We also let

$$\begin{aligned} \boldsymbol{D} &= \{L \mid \text{every subalgebra of } L \text{ is a subideal}\}\,, \\ \boldsymbol{D}_n &= \{L \mid H \leqq L \Longrightarrow H \lhd {}^n L\}\,, \\ \boldsymbol{D}_{n,r} &= \{L \mid H \leqq L \ \text{ and } \ H \in \boldsymbol{G}_r \Longrightarrow H \lhd^n L\}\,, \\ \boldsymbol{X}_n{}^* &= \{L \mid H \leqq L \ \text{ and } \ H \in \boldsymbol{G}_n \Longrightarrow \langle H^L \rangle^n \leqq H\}\,, \\ \boldsymbol{B} &= \{L \mid x \in L \Longrightarrow \langle x \rangle \text{ si } L\}\,. \end{aligned}$$

Evidently

$$\begin{aligned} &\boldsymbol{D}_n \leqq \boldsymbol{D}_{n,r} \leqq \boldsymbol{D}_{n,1} \leqq \boldsymbol{B}\,, \\ (1) \qquad &\boldsymbol{D}_n \leqq \boldsymbol{D} \leqq \boldsymbol{B}\,, \\ &\boldsymbol{X}_n^* \leqq \boldsymbol{B}\,. \end{aligned}$$

The closure operations we need are S, I, Q, E, L defined as follows; $L \in SX \Leftrightarrow L$ is isomorphic to a subalgebra of an X-algebra; $L \in IX \Leftrightarrow L$ is isomorphic to an ideal of an X-algebra; $L \in QX \Leftrightarrow L$ is isomorphic to a quotient of an X-algebra; $L \in EX \Leftrightarrow L \in X^n$ for some n; $L \in LX \Leftrightarrow$ every finite subset of L is contained in an X-subalgebra of L. We call LX is the class of locally X-algebras.

Thus EA is the class of soluble Lie algebras, A^d is the class of Lie algebras soluble of derived length $\leqq d$, and LN is the class of locally nilpotent Lie algebras.

(2) Clearly every class in (1) is Q-closed and S-closed.

We call $\boldsymbol{B}$ the class of *Baer algebras* (this extends the definition of Hartley [3]); we will show that

$$\boldsymbol{B} \leqq \boldsymbol{LN}.$$

2. The class $\boldsymbol{B}$.

The main result of this section is

THEOREM 2.1. $\boldsymbol{B} \leqq \boldsymbol{LN}$.

COROLLARY 2.11. *The classes $\boldsymbol{D}$, $\boldsymbol{D}_n$, $\boldsymbol{D}_{n,r}$ and $\boldsymbol{X}_n^*$ are all contained in the class $\boldsymbol{LN}$ of locally nilpotent Lie algebras.*

The inequality in 2.1 is strict as is well known. The special case of the result in characteristic zero follows from the fact that $\boldsymbol{B}$ is the class of Baer algebras in the sense of Hartley [3]. We remark also that in characteristic $p > 0$, a Lie algebra generated by 1-dimensional subideals need not be locally nilpotent (see Amayo [2]); but by Hartley [3], in characteristic zero, it is locally nilpotent.

What makes 2.1 possible is the following result.

THEOREM 2.2. (The Derived Join Theorem) *In any Lie algebra, the join of finitely many soluble subideals is soluble.*

PROOF. See Amayo [1].

We need two more results.

LEMMA 2.3. *Suppose that $L \in \boldsymbol{A}^2$, $x \in L$ and $[L^2, {}_n x] = 0$. Then*
$$\langle x^L \rangle \in \boldsymbol{N}_n.$$

PROOF. Let $B = L^2$, so that $B^2 = 0$ and $B \lhd L$. Evidently $\langle x^L \rangle \leqq \langle x \rangle + B$. A simple induction on r yields

$$(\langle x \rangle + B)^{r+1} = [B, {}_r x] .$$

In particular

$$(\langle x \rangle + B)^{n+1} = [B, {}_n x] = 0 .$$

If in 2.3, $[L, {}_n x] = 0$, then it is easy to show that $\langle x^L \rangle \in \boldsymbol{N}_m$, where $m = \max\{1, n - 1\}$. Clearly if $X = \langle x \rangle \lhd^n L$, then $[L, {}_n x] = 0$ or else $x \in L^2$. Thus we have

COROLLARY 2.31. *If $L \in \boldsymbol{A}^2$, $x \in L$ and $\langle x \rangle \lhd^n L$, then*

$$\langle x^L \rangle \in \boldsymbol{N}_n .$$

THEOREM 2.4. (Stewart [5]) *Let L be a Lie algebra and $H \lhd L$ such that $H \in \boldsymbol{N}_c$ and $L/H^2 \in \boldsymbol{N}_d$. Then*

$$L \in \boldsymbol{N}_{\mu_1(c,d)} ,$$

where $\mu_1(c, d) = cd + (c-1)(d-1)$.

PROOF OF THEOREM 2.1. Let $L \in \boldsymbol{B}$ and $X = \langle x_1, \cdots, x_n \rangle \leqq L$. Then by (2), $X \in \boldsymbol{B}$. Each $\langle x_i \rangle$ si L and so X is the join of n abelian subideals. By the Derived Join Theorem (2.2) $X \in \boldsymbol{A}^d$ for some d. So

$$X \in \boldsymbol{G} \cap \boldsymbol{A}^d \cap \boldsymbol{B} .$$

We use induction on d to show that $X \in \boldsymbol{N}$. If $d \leqq 2$, then by 2.31 $\langle x_i^X \rangle \in \boldsymbol{N}$ for each i. Thus $X = \langle x_1^X \rangle + \cdots + \langle x_n^X \rangle$, a sum of finitely many nilpotent ideals and so (by Hartley [3] p. 261) X is nilpotent.

Let $d > 2$ and assume inductively that

$$\boldsymbol{G} \cap \boldsymbol{A}^{d-1} \cap \boldsymbol{B} \leqq \boldsymbol{N} .$$

Since by (2) $\boldsymbol{B}$ is Q-closed, we have $X/X^{(d-1)} \in \boldsymbol{G} \cap \boldsymbol{A}^{d-1} \cap \boldsymbol{B} \leqq \boldsymbol{G} \cap \boldsymbol{N}$, by induction. Now by Lemma 3.3.5 of Stewart [5], we know that $\boldsymbol{G} \cap \boldsymbol{N} \leqq \boldsymbol{F} \cap \boldsymbol{N}$. Thus if $B = X^{(d-2)}$ and $A = X^{(d-1)}$ then $B^2 = A$, $A^2 = 0$ and $X/A \in \boldsymbol{G} \cap \boldsymbol{N} \leqq \boldsymbol{F} \cap \boldsymbol{N}$. Hence $B/A \in \boldsymbol{F}$. So we can find $y_1, \cdots, y_r \in B$ such that

$$B = \langle y_1, \cdots, y_r \rangle + A .$$

But each $\langle y_i \rangle$ si B (for $X \in \boldsymbol{B}$ implies that $B \in \boldsymbol{B}$) and $B \in \boldsymbol{A}^2$ and so by 2.31, $\langle y_i^B \rangle \in \boldsymbol{N}$. Thus

$$B = \langle y_1^B \rangle + \cdots + \langle y_r^B \rangle + A ,$$

a sum of finitely many nilpotent ideals, so B is nilpotent. But X/A is nilpotent and $B^2 = A$ and $B \triangleleft X$ and so by 2.4, X is nilpotent.

This completes our induction on d and with it the proof of Theorem 2.1.

From Stewart [5] we have

LEMMA 2.5. $\boldsymbol{G}_r \cap \boldsymbol{N}_c \leqq \boldsymbol{F}_{\mu_2(c,r)}$,

where for $r > 1$, $\mu_2(c, r) = (r^{c+1} - 1)/(r - 1)$.

And from Hartley [3] and Stewart [5],

LEMMA 2.6. *If* $L \in L\boldsymbol{N}$ *and* M *is a minimal ideal of* L *then* $M \leqq Z_1(L)$, *the centre of* L. *In particular if* $L \in L\boldsymbol{N}$ *and* Y *is an* $\boldsymbol{F}_h$*-ideal of* L *then* $Y \leqq Z_h(L)$.

3. The main theorem.

We will prove

THEOREM 3.1. *To every positive integer* n *there correspond positive integers* $\lambda_1(n)$ *and* $\lambda_2(n)$, *depending only on* n, *such that*

$$\boldsymbol{D}_{n,\lambda_2(n)} \leqq N_{\lambda_1(n)} .$$

It is not very hard to show that if $A, B \leqq L$, then

$$A \circ B = \bigcup \{H \circ K \mid H, K \in \mathfrak{G} \quad \text{and} \quad H \leqq A, K \leqq B\} .$$

Inductively it follows that

$$A \circ_n B = \bigcup \{H \circ_n K \mid H, K \in \mathfrak{G} \quad \text{and} \quad H \leqq A, K \leqq B\} .$$

From this and (1) we deduce that

$$\mathfrak{D}_n = \bigcap_{r=1}^{\infty} \mathfrak{D}_{n,r} .$$

So we have

COROLLARY 3.11. $\mathfrak{D}_n \leqq \mathfrak{N}_{\lambda_1(n)}$.

This corollary has been obtained by Stewart [5].

THEOREM 3.2. *For every positive integer $n > 0$,*

i) *If $L \in \mathfrak{X}_n^*$ and $x \in L$, then $\langle x^L \rangle \in \mathfrak{N}_n$ and,*

ii) *$\mathfrak{X}_n^* \leqq \mathfrak{N}_{\mu_3(n)}$, where $\mu_3(n) = \mu_2(n^2, n) + n - 1$.*

PROOF. i) By the definition of $\mathfrak{X}_n^*$, $\langle x^L \rangle^n \leqq \langle x \rangle$. Since $\langle x^L \rangle^n \lhd L$, we must have $\langle x \rangle \lhd L$ or else $\langle x^L \rangle^n = 0$. Hence $\langle x^L \rangle \in \mathfrak{N}_m$, $m = \max\{1, n-1\}$.

(ii). Let $L \in \mathfrak{X}_n^*$. Then by (i) or 2.11, $L \in L\mathfrak{N}$. Clearly

$$L^n = \langle \langle [x_1, \cdots, x_n]^L \rangle \mid \quad \text{for all} \quad x_1, \cdots, x_n \in L \rangle .$$

Let $x_1, \cdots, x_n$ be fixed but arbitrary elements of L and put $X = \langle x_1, \cdots, x_n \rangle$ and $T = \langle X^L \rangle$. By (i) each $\langle x_i^L \rangle \in \mathfrak{N}_n$ and since $T = \sum_{i=1}^n \langle x_i^L \rangle$, we have $T \in \mathfrak{N}_{n^2}$. By the definition of $\mathfrak{X}_n^*$, $T^n \leqq X \in \mathfrak{N}_{n^2} \cap \mathfrak{G}_n$. Thus if $Y = \langle [x_1, \cdots, x_n]^L \rangle$, then $Y \leqq T^n$ and so by 2.5, $Y \in \mathfrak{F}_h$, where $h = \mu_2(n^2, n)$. But $Y \lhd L$ and $L \in L\mathfrak{N}$ and so by 2.6, $Y \leqq Z_h(L)$. Since the x_i's were arbitrarily chosen and $L^n = \langle \langle [x_1, \cdots, x_n]^L \rangle \mid \text{all } x_i\text{'s in } L \rangle$ we have $L^n \leqq Z_h(L)$ and so $L = Z_{h+n-1}(L)$.

LEMMA 3.3. *For any positive integer $n > 0$,*

i) *$\mathfrak{X}_n^* \leqq \mathfrak{D}_{n,n}$ and*

ii) *$\mathfrak{D}_{n,n} \cap \mathfrak{A}^2 \leqq \mathfrak{X}_n^*$.*

PROOF. (i). Trivial.

(ii) Let $L \in \mathfrak{D}_{n,n} \cap \mathfrak{A}^2$ and H be a $\mathfrak{G}_n$-subalgebra of L. Let $H_1 = \langle H^L \rangle$ and $A = H_1 \cap L^2$. Then $A^2 = 0$, $A \lhd L$ and $H_1 = H + A$. A simple induction on r gives

$$(H + A)^r = H^r + [A, {}_{r-1}H]$$

and in particular $H_1^n = (H + A)^n = H^n + [A, {}_{n-1}H] \leqq H$, since $H \lhd^n L$, implies that $[H_1, {}_{n-1}H] \leqq H$ and so $[A, {}_{n-1}H] \leqq H$.

THEOREM 3.4. *To every pair n, m of positive integers there corresponds an integer $\mu_4(n, m)$ such that*

$$D_{n,n} \cap A^m \leqq N_{\mu_4}(n, m)$$

where for $m > 1$, $\mu_4(n, m) = \mu_1(\mu_4(n, m-1), \mu_3(n))$.

PROOF OF 3.4. If $m \leqq 2$, this follows by 3.3(ii) and 3.2(ii). Let $m > 1$, and assume that the result is true for $m-1$ in place of m. Let $L \in D_{n,n} \cap A^m$. Then as $D_{n,n}$ is Q-closed and S-closed,

$$L^2 \in D_{n,n} \cap A^{m-1} \leqq N_{\mu_4(n,m-1)}$$

and

$$L/(L^2)^2 \in D_{n,n} \cap A^2 \leqq N_{\mu_3(n)}$$

by 3.2(ii) and 3.3(ii). Therefore by 2.4 the result follows.

We need one more purely technical result before proving Theorem 3.1.

LEMMA 3.5. *Let s, t be positive integers with $1 \leqq t < s$. Suppose that $L \in D_{n,s}$ and H is a G_t-subalgebra of L. If H_j, $0 \leqq j \leqq n$, denotes the j-th term of the ideal closure series of H in L, then for each j, $0 < j < n$,*

$$H_j/H_{j+1} \in D_{(n-j),(s-t)} .$$

PROOF. Suppose that $0 < j < n$, and Y/H_{j+1} is a $G_{(s-t)}$-subalgebra of H_j/H_{j+1}. Then it is sufficient to show that $Y \lhd^{(n-j)} H_j$.

Evidently there exists a $G_{(s-t)}$ subalgebra X of H_j such that $Y = X + H_{j+1}$. Let $K = \langle X, H\rangle$ so that $K \in G_s$ and so $K \lhd^n L$. If K_i is the i-th term of the ideal closure series of K in L, then by simple induction we have

$$H_i = H + L\circ_i H \leqq K + L\circ_i K = K_i \leqq H_j + L\circ_i H_j .$$

Thus we have $K_j = H_j$, since $H_j \lhd^j L$. But

$$K \lhd^{n-j} K_j = H_j$$

and $Y = K + H_{j+1}$, and so the result follows.

PROOF OF THEOREM 3.1. We want to prove that: to every posivive integer n there correspond positive integers $\lambda_1(n)$ and $\lambda_2(n)$ depending only on n, such that

$$D_{n,\lambda_2(n)} \leqq N_{\lambda_1(n)} .$$

We use induction on n. For $n = 1$, take $\lambda_1 = \lambda_2 = 1$; for $D_{1,1} = A = N_1$. Let $n > 1$ and assume that for each r, $1 \leqq r \leqq n-1$, we have determined $\lambda_1(r)$ and $\lambda_2(r)$ such that

$$D_{r,\lambda_2(r)} \leqq N_{\lambda_1(r)} . \tag{3}$$

Define

$$\mu_5(n) = 1 + \mu_4(n, (n-1)\cdot\lambda_1(n-1)) , \tag{4}$$

and

$$\lambda_2(n) = \lambda_2(n-1) + \mu_5(n) . \tag{5}$$

We will show that

$$D_{n,\lambda_2(n)} \leqq X^*_{\mu_5(n)} ,$$

so by 3.2(ii) we may take

$$\lambda_1(n) = \mu_3(\mu_5(n)) . \tag{6}$$

Let $L \in D_{n,\lambda_2(n)}$ and H be a $G_{\mu_5(n)}$-subalgebra of L. From (4) and (5), $1 \leqq \mu_5(n) < \lambda_2(n)$ and so $H \lhd^n L$. If H_j is the j-th term of the ideal closure series of H in L, then by 3.5 for each j, $0 < j < n$,

$$H_j/H_{j+1} \in D_{(n-j),\lambda_2(n-1)} \leqq D_{n-1,\lambda_2(n-1)}$$

(for any positive r, $D_{r,k} \leqq D_{r+1,k}$ for all k). Thus by the inductive hypothesis (3),

$$H_j/H_{j+1} \in N_{\lambda_1(n-1)} .$$

Hence

$$H_j^{(\lambda_1(n-1))} \leqq H_j^{\lambda_1(n-1)} \leqq H_{j+1} .$$

Let $k = (n-1)\lambda_1(n-1)$. Then we have from above,

$$H_1^{(k)} \leqq H_n = H . \tag{7}$$

Now by definition, $n \leqq \lambda_2(n)$ and so $D_{n,\lambda_2(n)} \leqq D_{n,n}$. Since also $D_{n,n}$ is Q-closed and S-closed we have

$$H_1/H_1^{(k)} \in D_{n,n} \cap A^k \leqq N_{\mu_4(n,k)} ,$$

by 3.4. From this and (7) we have

$$H_1^{\mu_4(n,k)} \leqq H_1^{(k)} \leqq H .$$

But $\mu_5(n) = 1 + \mu_4(n, k)$, $H_1 = \langle H^L \rangle$, and H was an arbitrary $G_{\mu_5(n)}$-subalgebra of L. Thus

$$L \in X^*_{\mu_5}(n) ,$$

and the proof is complete.

4. A counterexample. We remarked earlier on that the question of whether $D_{n,n} \leqq N_{\lambda(n)}$ for a suitable $\lambda(n)$ is still unsettled. The next result seems to point to an answer in the negative.

THEOREM 4.1. $(\bigcap_{n=1}^{\infty} D_{n+2,n}) \cap A^2 \nleqq N$.

PROOF. Let k be a field of characteristic 2, and B an abelian Lie algebra over k with basis $\{b_1, b_2, \cdots\}$. Define $U = U(B)$ to be the universal algebra of B; then U has $\{b_{i_1} \cdots b_{i_m} \mid 0 \leqq m, i_1 \leqq \cdots \leqq i_m\}$ and is a polynomial ring in the b_i's. Now let V be the subspace of U with basis $\{b_{i_1} \cdots b_{i_m} \mid m > 1$, and for some j, $b_{i_j} = b_{i_{j+1}}\}$.

Now U is a B-module under the usual action and evidently so is V. Let $A = U/V$, so that A is a B-module. Consider A as an abelian Lie algebra and form the split extension,

$$L = A + B,\ A^2 = 0,\ A \lhd L \quad \text{and} \quad A \cap B = 0\ .$$

Clearly $L^{(2)} = 0$ and $L \notin \boldsymbol{N}$.

For any $x, y \in B$ and $a \in A$ we have

$$axy = ayx\ . \tag{8}$$

Suppose that $x = \sum l_i b_i$ $(l_i \in k)$. Then considering x as an element of U and since U is commutative and k has characteristic 2, we have

$$x^2 = \sum l_i^2 b_i^2 \in V\ ,$$

and so

$$ax^2 = axx = 0\ . \tag{9}$$

Thus if $x_1, \cdots, x_n \in B$ and $X = A + \langle x_1, \cdots, x_n \rangle$, then $X \lhd L$ and it follows easily by induction that

$$X^{r+1} = \sum_{m_1 + \cdots + m_n = r} A x_1^{m_1} \cdots x_n^{m_n}\ .$$

If $r = n + 1$, then in any particular term $Ax_1^{m_1} \cdots x_n^{m_n}$ some $m_i > 1$, and so by (8) and (9), each such term is zero. Hence $X \in \boldsymbol{N}_{n+1}$. Now pick any $a_1, \cdots, a_n \in A$ and let $H = \langle a_1 + x_1, \cdots, a_n + x_n \rangle$. Then $H \leqq X$ and so $H \lhd^{n+1} X \lhd L$. Clearly any $\boldsymbol{G}_n$-subalgebra of L is of the same form as H and hence is a $(n+2)$-step subideal of L. So $L \in \boldsymbol{D}_{n+2,n}$ for each $n > 0$, and the proof is complete.

The example above can be extended to give: in any field of characteristic $p > 0$, $(\bigcap_{n=1}^{\infty} \boldsymbol{D}_{n(p-1)+2,n}) \cap \boldsymbol{A}^2 \nleqq \boldsymbol{N}$. However it can be proved that in any field of characteristic zero, $\boldsymbol{D}_{n,1} \cap \boldsymbol{A}^d \leqq \boldsymbol{N}_{\mu(n,d)}$ for some $\mu(n, d)$ depending only on n and d; the result will hold for fields of characteristic p provided $n \leqq p$.

Let $\boldsymbol{B}_c^* = \{L \mid x \in L \Rightarrow \langle x^L \rangle \in \boldsymbol{N}_c\}$ then by 4.1 $\boldsymbol{B}_2^* \nleqq \boldsymbol{N}$. But it can be proved that $\boldsymbol{D}_{n,n} \leqq \boldsymbol{N}$, provided $\boldsymbol{D}_{n,n} \cap \boldsymbol{B}_c^* \leqq \boldsymbol{N}$ for all c.

REMARK: It will be shown in a forthcoming paper that there exists $\lambda(n)$ for which $\boldsymbol{D}_{n+1,n} \leqq \boldsymbol{N}_{\lambda(n)}$ (over any field)

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