Tôhoku Math. Journ.
27 (1975), 103-110.

CONFORMALLY FLAT RIEMANNIAN MANIFOLDS ADMITTING A TRANSITIVE GROUP OF ISOMETRIES

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(Received March 4, 1974)

1. **Introduction.** In the present paper, we shall classify conformally flat Riemannian manifolds admitting a transitive group of isometries. This class of manifolds contains the homogeneous Riemannian manifolds of constant curvature classified by J. A. Wolf ([4], [5]).

Theorem A in Section 2 imposes a restriction on the local Riemannian structure of the manifold in consideration. Using Theorem A, we get Theorem B in Section 3 and Theorem C in Section 4. They give the classification together with Theorem D in Section 4.

The author wishes to express his sincere thanks to Prof. S. Tanno and Prof. T. Sakai who gave him many valuable suggestions and guidances.

2. **Local structure.** Let M be a C^∞ Riemannian manifold of dimension n. On a neighborhood of a point of M, we take a field of orthonormal co-frame $\{\omega_1, \cdots, \omega_n\}$. Then we have the Cartan structural equations:

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j\ , \qquad \omega_{ij} + \omega_{ji} = 0 \tag{2.1}$$

and

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}\ , \tag{2.2}$$

where ω_{ij} and Ω_{ij} are the connection form and the curvature form respectively.

Now, we assume that M is conformally flat, that is, each point of M has a neighborhood where there exists a conformal diffeomorphism onto an open subset in a Euclidean space. Then each point of M has a coordinate neighborhood $\{U; x_1, \cdots, x_n\}$ where we can choose as a $\{\omega_1, \cdots, \omega_n\}$ one satisfying $\omega_i = (1/\rho)dx_i$ for each i, with certain positive C^∞ function ρ of $x_1, \cdots, x_n$. Then, by (2.1), we have

$$\omega_{ij} = \rho_i\omega_j - \rho_j\omega_i\ , \qquad \rho_i = \partial\rho/\partial x_i\ . \tag{2.3}$$

Then, by (2.2), we have

$$\Omega_{ij} = \phi_i \wedge \omega_j + \omega_i \wedge \phi_j \,, \tag{2.4}$$

where

$$\phi_i = \sum_j A_{ij}\omega_j,\ A_{ij} = \rho\rho_{ij} - (1/2)\delta_{ij}(\sum_k \rho_k^2),\ \rho_{ij} = \partial^2\rho/\partial x_i\partial x_j \,. \tag{2.5}$$

And moreover, by (2.1), (2.3) and (2.5), we have

$$d\phi_i = -\sum_j \omega_{ij} \wedge \phi_j \,. \tag{2.6}$$

Here we express $A = (A_{ij})$ in terms of the Ricci tensor $R = (R_{ij})$ and the scalar curvature S. That is, we have

$$A_{ij} = [1/(n-2)](R_{ij} - [1/2(n-1)]S\delta_{ij}) \,. \tag{2.7}$$

This is easily seen by the following definitions of the curvature tensors (R_{ijkl}), (R_{ij}) and S:

$$\Omega_{ij} = (1/2)\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l \,, \qquad R_{ijkl} = -R_{ijlk} \,,$$

$$R_{ij} = \sum_k R_{ikjk} \,, \qquad S = \sum_i R_{ii} \,.$$

Conversely, it is well known that every Riemannian manifold with 1-form $\phi_i = \sum_j A_{ij}\omega_j$ satisfying (2.4), (2.6) and (2.7) is conformally flat.

THEOREM A. *Let M be a connected conformally flat Riemannian manifold. If M is homogeneous, that is, M admits a transitive group of isometries, then M is isometric to certain one of the following manifolds*:

(1) *A space of constant curvature.*

(2) *A Riemannian manifold which is locally a product of a space of constant curvature $K(\neq 0)$ and a space of constant curvature $-K$.*

(3) *A Riemannian manifold which is locally a product of a space of constant curvature $K(\neq 0)$ and a 1-dimensional space.*

PROOF. Assume that M is homogeneous. Then, by (2.7), the characteristic roots $\lambda_1, \cdots, \lambda_n$ of the tensor field A are constant on M. Then, on some neighborhood of each point of M we can take a field of orthonormal co-frame $\{\omega_1, \cdots, \omega_n\}$ satisfying $\phi_i = \lambda_i\omega_i$ for each i. Then, by (2.4), we have

$$\Omega_{ij} = (\lambda_i + \lambda_j)\omega_i \wedge \omega_j \,. \tag{2.8}$$

Here we put

$$(\lambda_i - \lambda_j)\omega_{ij} = \sum_k S_{ijk}\,\omega_k \,. \tag{2.9}$$

Then S_{ijk} is symmetric for all indices and hence, if $i \equiv j$, $j \equiv k$ or $k \equiv i$, then $S_{ijk} = 0$, where $i \equiv j$ means $\lambda_i = \lambda_j$. In fact, by (2.1) and (2.6), we have $\sum_j (\lambda_i - \lambda_j)\omega_{ij} \wedge \omega_j = 0$, from which we have $S_{ijk} = S_{ikj}$. By (2.2), (2.8) and (2.9), we have

$$\begin{aligned}(\lambda_i + \lambda_j)\omega_i \wedge \omega_j = {} & [1/(\lambda_i - \lambda_j)] \sum_k (dS_{ijk}) \wedge \omega_k \\ & + [1/(\lambda_i - \lambda_j)] \sum_k S_{ijk} d\omega_k \\ & + \sum_{\substack{k(\not\equiv i,j)\\ l,m}} [1/(\lambda_i - \lambda_k)(\lambda_k - \lambda_j)] S_{ikl} S_{kjm} \omega_l \wedge \omega_m \\ & + \sum_{k(\equiv i),l} [1/(\lambda_k - \lambda_j)] S_{kjl} \omega_{ik} \wedge \omega_l \\ & + \sum_{k(\equiv j),l} [1/(\lambda_i - \lambda_k)] S_{ikl} \omega_l \wedge \omega_{kj} \quad \text{for} \quad i \not\equiv j .\end{aligned}$$

Comparing the coefficients of $\omega_i \wedge \omega_j$ and taking account of the properties of S_{ijk}, we have

$$(\lambda_i + \lambda_j)/(\lambda_i - \lambda_j) = 2 \sum_{k(\not\equiv i,j)} (S_{ijk})^2/(\lambda_i - \lambda_j)(\lambda_k - \lambda_j)(\lambda_k - \lambda_i) \text{ for } i \not\equiv j ,$$

from which we have

$$\sum_{j(\not\equiv i)} (\lambda_i + \lambda_j)/(\lambda_i - \lambda_j) = 0 \quad \text{for each} \quad i .$$

From the last identity, we see that possible cases are only the following (a) and (b):

(a) $\lambda_1 = \cdots = \lambda_n$.

(b) $\lambda_1 = \cdots = \lambda_r = -\lambda_{r+1} = \cdots = -\lambda_n \neq 0$, $1 \leqq r \leqq n-1$, where we assume that $\lambda_1 \geqq \cdots \geqq \lambda_n$.

In fact, let us assume that the case (a) does not occur. Then $\lambda_i \neq 0$ for each i. Let $\lambda^2 (> 0)$ be the minimum of $\lambda_1^2, \cdots, \lambda_n^2$. Then we have

$$(\lambda + \lambda_j)/(\lambda - \lambda_j) = (\lambda^2 - \lambda_j^2)/(\lambda - \lambda_j)^2 \leqq 0 ,$$

which shows that $\lambda_j = \pm\lambda$ for each j.

The above fact shows that $S_{ijk} = 0$ for each i, j, k, which implies that, if $i \not\equiv j$, then $\omega_{ij} = 0$ by (2.9) and that, if $i \equiv j$, then $\Omega_{ij} = \pm 2\lambda\omega_i \wedge \omega_j$ by (2.8). This completes the proof (cf. [2], [3]).

3. Some lemmas and Theorem B. Let M be a conformally flat homogeneous Riemannian manifold and $\tilde{M}$ be the universal covering manifold of M with the metric induced from the projection $p: \tilde{M} \to M$. Since M is complete, so is $\tilde{M}$. Then, by Theorem A and the decomposition theorem of de Rham, $\tilde{M}$ is isometric to one of the following manifolds:

(1) $M^n(K)$,

(2) $M^r(K) \times M^{n-r}(-K)$, $K \neq 0$, $2 \leqq r \leqq n-2$,

(3) $M^{n-1}(K) \times E^1$, $K \neq 0$,

where $M^m(K)$ denotes an ordinary sphere S^m of radius $K^{-1/2}$, a Euclidean space E^m or a hyperbolic space H^m with sectional curvature K according as K is positive, zero or negative. And M is isometric to a quotient $\tilde{M}/\Gamma$, where Γ is a certain group of isometries of $\tilde{M}$ acting freely and properly discontinuously (cf. Wolf [6]). Thus the classification is reduced to analyze the structure of Γ. So, we prepare some lemmas.

LEMMA 1. *Let M_i $(i = 1, 2)$ be a connected Einstein Riemannian manifold with the metric tensor g_i, that is, the Ricci tensor R_i is written as $R_i = c_i g_i$ over M_i with constant c_i. If $c_1 \neq c_2$, then $I(M_1 \times M_2) = I(M_1) \times I(M_2)$, where $I(M_1 \times M_2)$, $I(M_1)$ and $I(M_2)$ denote the groups of all isometries of $M_1 \times M_2$, M_1 and M_2 respectively.*

PROOF. If we put $\tilde{M} = M_1 \times M_2$, then the tangent space $T_z(\tilde{M})$ at a point $z = (x, y) \in \tilde{M}$ is identified with direct sum $T_x(M_1) + T_y(M_2)$. Let R^1 be a field of symmetric endomorphism which corresponds to the Ricci tensor R of $\tilde{M}$, that is, $R(X, Y) = g(R^1X, Y)$ for any tangent vectors X and Y, where g is the metric tensor of the direct product $\tilde{M}$. Then, $X \in T_z(\tilde{M})$ is contained in $T_x(M_1)$ (resp. $T_y(M_2)$) if and only if $R^1X = c_1X$ (resp. $R^1X = c_2X$).

Now, let $f \in I(\tilde{M})$. Then we have

$$df_z \circ R^1_z = R^1_{f(z)} \circ df_z \quad \text{for each} \quad z \in \tilde{M}\,,$$

which shows that df_z maps $T_x(M_1)$ (resp. $T_y(M_2)$) into $T_{f_1(x,y)}(M_1)$ (resp. $T_{f_2(x,y)}(M_2)$), where we put $f(z) = f(x, y) = (f_1(x, y), f_2(x, y))$. Making use of this fact, we shall show that $f_1(x, y)$ (resp. $f_2(x, y)$) does not depend on y (resp. x) which completes the proof. Let $\{x_1, \cdots, x_r, y_1, \cdots, y_s\}$ be a coordinate system on a neighborhood $U_x \times V_y$ of $\tilde{M}$ at $z = (x, y)$ such that $\{x_1, \cdots, x_r\}$ and $\{y_1, \cdots, y_s\}$ are coordinate systems on U_x and V_y respectively. Let $\{u_1, \cdots, u_r, v_1, \cdots, v_s\}$ be a coordinate system on a neighborhood $U_{f_1(x,y)} \times V_{f_2(x,y)}$ of $\tilde{M}$ at $f(z) = (f_1(x, y), f_2(x, y))$ such that $\{u_1, \cdots, u_r\}$ and $\{v_1, \cdots, v_s\}$ are coordinate systems on $U_{f_1(x,y)}$ and $V_{f_2(x,y)}$ respectively. Let f be represented locally by the functions,

$$\begin{cases} u_i = f_{1i}(x_1, \cdots, x_r, y_1, \cdots, y_s) & (i = 1, \cdots, r) \\ v_i = f_{2i}(x_1, \cdots, x_r, y_1, \cdots, y_s) & (i = 1, \cdots, s)\,. \end{cases}$$

Since $df(\partial/\partial x_i) \in T(M_1)$ and $df(\partial/\partial y_i) \in T(M_2)$, we have

$$\partial f_{1i}/\partial y_j = 0 \quad \text{for} \quad i = 1, \cdots, r;\ j = 1, \cdots, s$$

and

$$\partial f_{2i}/\partial x_j = 0 \quad \text{for} \quad i = 1, \cdots, s;\ j = 1, \cdots, r\,,$$

that is,

$$\begin{cases} u_i = f_{1i}(x_1, \cdots, x_r) & \text{for} \quad i = 1, \cdots, r \\ v_i = f_{2i}(y_1, \cdots, y_s) & \text{for} \quad i = 1, \cdots, s \, . \end{cases}$$

This means that $f_1(x, y) = f_1(x, y')$ (resp. $f_2(x, y) = f_2(x', y)$), if y is sufficiently near to y' (resp. x is sufficiently near to x'). Since M_2 (resp. M_1) is connected, $f_1(x, y) = f_1(x, y')$ (resp. $f_2(x, y) = f_2(x', y)$) for any pair y and y' of M_2 (resp. for any pair x and x' of M_1). q.e.d.

Let γ be an isometry of a metric space N with distance function d. We say that γ is a Clifford translation if $d(x, \gamma(x)) = d(y, \gamma(y))$ for each pair of points x and y of N.

LEMMA 2. *Let M_1 and M_2 be complete Riemannian manifolds. Let α and β be isometries of M_1 and M_2 respectively. Then $\gamma = (\alpha, \beta)$ is a Clifford translation on $M_1 \times M_2$ if and only if α and β are Clifford translations on M_1 and M_2 respectively.*

PROOF. Let (x, y) and (u, v) be points of $M_1 \times M_2$. Then, as easily seen, the following identity of Pythagoras is valid:

$$[d((x, y), (u, v))]^2 = [d(x, u)]^2 + [d(y, v)]^2 \, ,$$

where d denotes the distance functions of M_1, M_2 and $M_1 \times M_2$. Now the lemma is evident.

LEMMA 3. (cf. Wolf [6]) *Let M and $\tilde{M}$ be Riemannian manifolds and let $M = \tilde{M}/\Gamma$, where Γ is a group of isometries of $\tilde{M}$ acting freely and properly discontinuously. Let G be the centralizer of Γ in the group $I(\tilde{M})$ of all isometries of $\tilde{M}$. Then M is homogeneous if and only if G is transitive on $\tilde{M}$. And if M is homogeneous, then every element of Γ is a Clifford translation of $\tilde{M}$.*

THEOREM B. *Let M be a connected conformally flat homogeneous Riemannian manifold. Then M is isometric to one of the following manifolds:*

(I) *A homogeneous space of constant curvature.*

(II) *A direct product of a homogeneous space of constant curvature $K(> 0)$ and a homogeneous space of constant curvature $-K$.*

(III) *A direct product of a homogeneous space of constant curvature $-K(< 0)$ and a 1-dimensional homogeneous space.*

(IV) *A homogeneous Riemannian manifold which is locally a product of a space of constant curvature $K(> 0)$ and a 1-dimensional space.*

PROOF. Since the only Clifford translation of H^m is the identity transformation 1 (cf. Wolf [6]), every Clifford translation of $H^{n-r} \times S^r$ or $H^{n-1} \times E^1$ must be of the form $(1, \beta)$, by Lemma 1 and Lemma 2. Now this proves the theorem by Lemma 3. q.e.d.

Thus the only problem left to us is to check up the space of the form $(S^{n-1} \times E^1)/\Gamma$.

4. $(S^{n-1} \times E^1)/\Gamma$. S^{n-1} is considered as the set of vectors of norm $K^{-1/2}$ in a Euclidean vector space $\boldsymbol{R}^n$. Then $I(S^{n-1})$ is the orthogonal group $O(n)$. The group of all Clifford translations of E^1 is identified with the additive group $\boldsymbol{R}$. The natural projections $I(S^{n-1}) \times I(E^1) \to I(S^{n-1})$ and $I(S^{n-1}) \times I(E^1) \to I(E^1)$ are denoted by p and q respectively.

LEMMA 4. *Let $(S^{n-1} \times E^1)/\Gamma$ be homogeneous. Then $p(\Gamma)$ is contained in one of the following closed subgroups* (1), (2) *and* (3) *of $O(n)$, according as $n = 2m + 1$, $n = 2m$ (m: odd) or $n = 4m$:*

(1) $\{\pm E\}$,

(2) $\{aE + bI;\ a^2 + b^2 = 1,\ a, b \in \boldsymbol{R}\}$,

(3) $\{aE + bI + cJ + dK;\ a^2 + b^2 + c^2 + d^2 = 1,\ a, b, c, d \in \boldsymbol{R}\}$,

where E is the identity transformation of $\boldsymbol{R}^n$ and I, J, K are elements of $O(n)$ satisfying the conditions, $I^2 = J^2 = K^2 = -E$, $IJ = -JI = K$, $JK = -KJ = I$, $KI = -IK = J$.

PROOF. Let $(S^{n-1} \times E^1)/\Gamma$ be homogeneous. Then by Lemma 3, the centralizer G of Γ in $I(S^{n-1} \times E^1) = I(S^{n-1}) \times I(E^1)$ is transitive on $S^{n-1} \times E^1$. Then the group $p(G)$ is the centralizer of $p(\Gamma)$ in $O(n)$ and it is transitive on S^{n-1}. In particular, $\boldsymbol{R}^n$ has no $p(G)$-invariant linear subspace. Then, every element A of the centralizer $\boldsymbol{F}$ of $p(G)$ in the algebra of all linear transformations of $\boldsymbol{R}^n$ is written as $A = aE$ or $A = aE + bI$, where $b \neq 0$ and I is a linear transformations of $\boldsymbol{R}^n$ satisfying $I^2 = -E$ (cf. p. 277 [1]). It should be noted that, if an element of the form $aE + bI$ $(b \neq 0)$ is contained in $O(n)$, then $a^2 + b^2 = 1$ and $I \in O(n)$. In fact, let $aE + bI \in O(n)$. Then we have $aE - bI = (a^2 + b^2)(aE + b{}^tI)$ and $aE - b{}^tI = (a^2 + b^2)(aE + bI)$, which implies that $a^2 + b^2 = 1$ and ${}^tI = -I$.

By definition of $\boldsymbol{F}$, $p(\Gamma) \subset \boldsymbol{F}$. Thus, if n is odd, then $\boldsymbol{F} = \{aE;\ a \in \boldsymbol{R}\}$ and hence $p(\Gamma) \subset \{\pm E\}$.

Now, let n be even. If $p(\Gamma)$ contains an element of the form $aE + bI$ $(b \neq 0)$, then $I \in O(n)$ and $\boldsymbol{F} \supset \boldsymbol{C}$, where $\boldsymbol{C} = \{aE + bI;\ a, b \in \boldsymbol{R}\}$. $\boldsymbol{C}$ is written as follows; $\boldsymbol{C} = \{A \in \boldsymbol{F};\ AI = IA\}$. In fact, let A be an element of $\boldsymbol{F}$ satisfying $AI = IA$. Since $A \in \boldsymbol{F}$, A is of the form $aE + bL$, where $L^2 = -E$. Hence, if $b \neq 0$, then $IL = LI$. It is sufficient to show that $L = I$ or $L = -I$. So we show that $\boldsymbol{R}^n = W_1 + W_2$ (direct sum), where $W_1 = \{v \in \boldsymbol{R}^n; Iv = Lv\}$ and $W_2 = \{v \in \boldsymbol{R}^n; Iv = -Lv\}$. Clearly, $W_1 \cap W_2 = \{0\}$. Every $v \in \boldsymbol{R}^n$ is of the form $w_1 + w_2$ with $w_1 \in W_1$ and $w_2 \in W_2$ by setting $w_1 = (1/2)(v - ILv)$ and $w_2 = (1/2)(v + ILv)$. Since $I, L \in \boldsymbol{F}$, W_1 and W_2 are invariant by $p(\Gamma)$, which implies that $L = I$ or $-I$.

Next we assume that $p(\Gamma) \not\subset \boldsymbol{C}$. And we take an element A of $p(\Gamma)$ such that $A \notin \boldsymbol{C}$. If we put $B = A + IAI$, then $B \notin \boldsymbol{C}$ and $B^2 \in \boldsymbol{C}$, because $BI = -IB$, $B^2I = IB^2$ and $AI \neq IA$. On the other hand, B may be written as $B = cE + dJ$, where $J^2 = -E$, because $B \in \boldsymbol{F}$. Since $B \notin \boldsymbol{C}$ and $B^2 \in \boldsymbol{C}$, then $c = 0$ and $d \neq 0$. We show that $J \in O(n)$. Since $A \in \boldsymbol{F} \cap O(n)$, A is of the form $aE + bL$, where $L^2 = -E$ and ${}^tL = -L$. Then $B = b(L + ILI)$, which implies that ${}^tB = -B$ and hence ${}^tJ = -J$. Here, we put $\boldsymbol{D} = \{(aE + bI)J; a, b \in \boldsymbol{R}\}$. Then $\boldsymbol{D} = \{A \in \boldsymbol{F}; AI = -IA\}$. For, if $A \in \boldsymbol{F}$ satisfies $AI = -IA$, then $AJ \in \boldsymbol{C}$, that is, AJ is written as $AJ = -aE - bI$ and hence $A = (aE + bI)J$. Now we show that $\boldsymbol{F} = \boldsymbol{C} + \boldsymbol{D}$ (direct sum). Clearly, $\boldsymbol{C} \cap \boldsymbol{D} = \{0\}$. Every $A \in \boldsymbol{F}$ is of the form $C + D$ with $C \in \boldsymbol{C}$ and $D \in \boldsymbol{D}$ by setting $C = (1/2)(A - IAI)$ and $D = (1/2)(A + IAI)$.

If we put $K = IJ$, then $K \in O(n)$ and the I, J and K satisfy the conditions stated in this lemma. And $p(\Gamma) \subset (\boldsymbol{C} + \boldsymbol{D}) \cap O(n)$. Of course, if $p(\Gamma) \not\subset \boldsymbol{C}$, then $n = 4m$. q.e.d.

The groups (1), (2) and (3) in Lemma 4 are isomorphic to $\{\pm 1\}$, S^1 and $Spin(3)$ respectively. Hereafter, we mean these groups as closed subgroups of $O(n)$ acting S^{n-1} by the above fashion.

THEOREM C. *$(S^{n-1} \times E^1)/\Gamma$ is homogeneous if and only if Γ is a discrete subgroup of $\{\pm 1\} \times \boldsymbol{R}$, $S^1 \times \boldsymbol{R}$ or $Spin(3) \times \boldsymbol{R}$ according as $n = 2m + 1$, $n = 2m$ (m: odd) or $n = 4m$.*

PROOF. Let $(S^{n-1} \times E^1)/\Gamma$ be homogeneous. Then $p(\Gamma)$ is contained in $\{\pm 1\}$, S^1 or $Spin(3)$ by Lemma 4 and $q(\Gamma) \subset \boldsymbol{R}$ by Lemma 2 and Lemma 3. And moreover Γ is discrete in $O(n) \times I(E^1)$, since the action of Γ is free and discontinuous (cf. [1]). Conversely, let Γ be a discrete subgroup of $\{\pm 1\} \times \boldsymbol{R}$, $S^1 \times \boldsymbol{R}$ or $Spin(3) \times \boldsymbol{R}$. Then the centralizer G of Γ in $O(n) \times I(E^1)$ actually contains $O(n) \times \boldsymbol{R}$, $U(m) \times \boldsymbol{R}$ or $Sp(m) \times \boldsymbol{R}$. In particular G is transitive on $S^{n-1} \times E^1$. By the way of action, Γ acts freely. Since $S^{n-1} \times E^1$ is a homogeneous space and the isotropy group is compact, the discrete subgroup Γ of $I(S^{n-1} \times E^1)$ acts on $S^{n-1} \times E^1$ properly discontinuously (cf. [1]). q.e.d.

THEOREM D. *Let H be one of the groups $\{\pm 1\}$, S^1 and $Spin(3)$. Then a discrete subgroup Γ of $H \times \boldsymbol{R}$ is one of the following forms:*

(1) $\Gamma_1 \times \{0\}$,

(2) *A group which is semi-direct product of the infinite cyclic group* $\langle(\alpha, \beta)\rangle$ *generated by* (α, β) *and* $\Gamma_1 \times \{0\}$,

where Γ_1 *is a finite subgroup of* H, α *an element of the normalizer of* Γ_1 *in* H *and* $\beta(\neq 0) \in \boldsymbol{R}$.

PROOF. First, the projection $q(\Gamma)$ is a discrete subgroup of $\boldsymbol{R}$ and hence it is the infinite cyclic group $\langle\beta\rangle$ generated by an element $\beta(\neq 0) \in \boldsymbol{R}$ or $\{0\}$. If $q(\Gamma) = \{0\}$, then Γ is a discrete subgroup of compact group $H \times \{0\}$ and hence Γ is of the form (1). If $q(\Gamma) = \langle\beta\rangle$, then $q^{-1}(k\beta) \cap \Gamma$ $(k \in \boldsymbol{Z})$ is a finite set and the number of elements of the set does not depend on k. In fact, we have a one to one correspondence between the finite group $q^{-1}(0) \cap \Gamma$ and the set $q^{-1}(k\beta) \cap \Gamma$, that is, $(\gamma, k\beta)$ maps $(\alpha_i, 0) \in q^{-1}(0) \cap \Gamma$ to $(\gamma\alpha_i, k\beta) \in q^{-1}(k\beta) \cap \Gamma$, where $(\gamma, k\beta)$ is an arbitrary fixed element of $q^{-1}(k\beta) \cap \Gamma$. Thus, taking an element $(\alpha, \beta) \in q^{-1}(\beta) \cap \Gamma$, Γ may be written as $\Gamma = \bigcup_{k \in Z} (\alpha^k, k\beta)(\Gamma_1 \times \{0\})$. For Γ to be closed with respect to the compositions of $H \times \boldsymbol{R}$, α must be contained in the normalizer of Γ_1 in H. q.e.d.

REMARK 1. The finite subgroups of H are completely classified by Wolf ([4], [5]).

REMARK 2. The classification for the case that the dimension of M is equal to 1 or 2 is well known.

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