Tôhoku Math. Journ.
27 (1975), 13-24.

ON THE CHARACTERISTIC FUNCTION OF HARMONIC KÄHLERIAN SPACES

YOSHIYUKI WATANABE

(Received July 25, 1973)

1. Introduction. An n-dimensional space of constant curvature ($k \neq 0$) is characterized as a harmonic Riemannian space with characteristic function

$$f(\Omega) = 1 + (n-1)\sqrt{2k\Omega}\cot\sqrt{2k\Omega}$$

where $\Omega = s^2/2$ and s means the geodesic distance. A. Lichnérowicz has obtained the following

THEOREM A. ([3], [10]) *In any harmonic Riemannian space H^n with positive definite metric, its characteristic function $f(\Omega)$ satisfies the inequality*

$$\dot{f}^2(0) + \frac{5}{2}(n-1)\ddot{f}(0) \leqq 0 . \tag{1.1}$$

The equality sign is valid if and only if H^n is of constant curvature.

Recently, S. Tachibana [7] has showed that a $2m$-dimensional space of constant holomorphic curvature ($k \neq 0$) is characterized as a harmonic Kählerian space with characteristic function given by

$$f(\Omega) = 1 + (2m-1)(ls)\cot(ls) - (ls)\tan(ls) , \tag{1.2}$$

or

$$f(\Omega) = 1 + (2m-1)(ls)\coth(ls) + (ls)\tanh(ls) \tag{1.2$'$}$$

according to $l = \sqrt{k}/2$, or $l = \sqrt{-k}/2$. He also has obtained

THEOREM B. *In any $n(=2m)$-dimensional harmonic Kählerian space H^n, its characteristic function $f(\Omega)$ satisfies the inequality*

$$\dot{f}^2(0) + \frac{5(m+1)^2}{m+7}\ddot{f}(0) \leqq 0 . \tag{1.3}$$

The equality sign is valid if and only if H^n is of constant holomorphic curvature.

In §2, we give some preliminaries. In §3, $\dddot{f}(0)$ is calculated in terms of curvatures of a harmonic Riemannian space (Proposition 3.3). In §4, we give an equation in a harmonic Kählerian space (Proposition 4.1), which plays an important role in this paper. By this we obtain the following

THEOREM 4.7. *Let H^6 be a 6-(real)dimensional compact harmonic Kählerian space. We denote the Euler characteristic of H^6 by $\chi(H^6)$. Then we have*

$$\chi(H^6) = -\frac{27}{64\pi^3}\dot{f}(0)\{\dot{f}^2(0) + 5\ddot{f}(0)\} \operatorname{Vol}(H^6)$$

where $\operatorname{Vol}(H^6)$ *is the volume of* H^6.

In §5, we prove the following main theorem and some related theorems as applications of Propositions 3.3 and 4.1.

THEOREM 5.2. *In any $n(=2m)$-dimensional harmonic Kählerian space H^n, its characteristic function $f(\Omega)$ satisfies the inequality*

$$2\dot{f}^3(0) + (13m + 28)\dot{f}(0)\ddot{f}(0) + 7(m + 1)(m + 2)\dddot{f}(0) \leqq 0 .$$

The equality sign is valid if and only if H^n is locally symmetric.

This theorem is related to Theorem B and a well known conjecture that any harmonic Riemannian space (with positive definite metric) is locally symmetric (cf. [5], [9]).

The last section will be devoted to examples of Theorems 4.7 and 5.2.

I wish to express my sincere thanks to Prof. S. Tachibana, who kindly has sent me his lecture note and gave me many valuable suggestions and guidances, and also thanks to Mr. M. Yasuda and Miss. N. Takeuchi, who have made sure a portion of my tedious calculations.

2. **Preliminaries.** We shall give some formulas which are used in the subsequent sections. Let (M^n, g) be a Riemannian space with Levi-Civita connection ∇. By $R = (R^i{}_{jkl})$*) we denote the Riemannian curvature tensor of ∇. Then $R_1 = (R^a{}_{ija}) = (R_{ij})$ and $S = (g^{ij}R_{ij})$ are Ricci tensor and scalar curvature respectively. Let (;) denote the covariant differentiation, and put $\nabla R = (R^i{}_{jkl;h})$. For a tensor field $T = (T_{ijk})$, for example, we denote $|T|^2 = T_{ijk}T^{ijk}$. We put

$$\alpha = |R|^2 ,$$
$$\beta = R^{abcd}R_{ab}{}^{uv}R_{cduv} ,$$
$$\gamma = R^{abcd}R_a{}^u{}_c{}^v R_{budv} .$$

Then they satisfy the following fundamental formulas.

*) We follow the definition of the curvature tensor in [12].

(2.1) (Bianchi's identities)

(a) $$R_{ijkh} + R_{ikhj} + R_{ihjk} = 0\ ,$$

(b) $$R_{ijkh;l} + R_{ijhl;k} + R_{ijlk;h} = 0\ .$$

Now (2.2) and (2.3) are all easily derived from (2.1) (cf. [6]).

(2.2)

(a) $$R^{abcd;u}R_{adcb;u} = \frac{1}{2}|\nabla R|^2\ ,$$

(b) $$R^{abcd;u}R_{cbad;u} = \frac{1}{2}|\nabla R|^2\ ,$$

(c) $$R^{abcd;u}R_{abud;c} = \frac{1}{2}|\nabla R|^2\ ,$$

(d) $$R^{abcu;v}R_{abcv;u} = \frac{1}{2}|\nabla R|^2\ ,$$

(e) $$R^{abcd;u}R_{ubcd;a} = \frac{1}{2}|\nabla R|^2\ ,$$

(f) $$R^{abcd;u}R_{ubad;c} = \frac{1}{2}|\nabla R|^2\ .$$

(2.3)

(a) $$R^{abcd}R_{ab}{}^{uv}R_{cudv} = R^{abcd}R_a{}^u{}_b{}^v R_{cduv} = \frac{1}{2}\beta\ ,$$

(b) $$R^{abcd}R_a{}^u{}_b{}^v R_{cudv} = \frac{1}{4}\beta\ ,$$

(c) $$R^{abcd}R_a{}^u{}_b{}^v R_{cvdu} = -\frac{1}{4}\beta\ ,$$

(d) $$R^{abcd}R_{ac}{}^{uv}R_{bduv} = \frac{1}{2}\beta\ ,$$

(e) $$R^{abcd}R_a{}^u{}_c{}^v R_{bduv} = R^{abcd}R_{ac}{}^{uv}R_{budv} = \frac{1}{4}\beta\ ,$$

(f) $$R^{abcd}R_a{}^u{}_c{}^v R_{bvdu} = R^{abcd}R_a{}^v{}_c{}^u R_{budv} = \gamma - \frac{1}{4}\beta\ .$$

(2.4) (Lichnérowicz's formula ([4], [11]))

$$\frac{1}{2}\Delta\alpha = |\nabla R|^2 - 4R^{jihk}R_{ik;h;j} + 2R_{ij}R^{ihkl}R^j{}_{hkl} + \beta + 4\gamma$$

where Δ is the Laplace-Bertrami operator acting on differentiable functions on M^n.

(2.5) $$\alpha - \frac{2}{n-1}|R_1|^2 \geqq 0$$

where the equality sign is valid if and only if M^n is constant curvature (cf. [8]).

Let M^n be an n-dimensional analytic Riemannian space and x_0 a point of M^n. We denote by s the geodesic distance from x_0 to the point in a neighbourhood of x_0. If the Laplacian Δs is a function of s only, then M^n is called to be harmonic at x_0. When M^n is harmonic at any point, it is called harmonic and denoted by H^n. For a harmonic space H^n, if we put $\Omega = (1/2)s^2$, then it is well known that $\Delta\Omega = f(\Omega)$ is a function of Ω only and does not depend on the reference point x_0. $f(\Omega)$ is called the characteristic function of M^n.

It is known (cf. [3], [5]) that any harmonic space satisfies the following curvature conditions.

$$f(0) = n \,, \tag{2.6}$$

$$R_{ij} = -\frac{3}{2}\dot{f}(0)g_{ij} \,, \quad S = -\frac{3n}{2}\dot{f}(0) \,, \tag{2.7}$$

where $(\cdot)$ means the operator taking the derivative with respect to Ω.

$$P(R^p{}_{ijq}R^q{}_{klp}) = -\frac{45}{8}\ddot{f}(0)P(g_{ij}g_{kl}) \,, \tag{2.8}$$

where P denotes the sum of terms obtained by permuting the given free indices, i.e.,

$$\begin{aligned} R^p{}_{ijq}(R^q{}_{klp} + R^q{}_{lkp}) + R^p{}_{ikq}(R^q{}_{ljp} + R^q{}_{jlp}) + R^p{}_{ilq}(R^q{}_{jkp} + R^q{}_{kjp}) \\ = -\frac{45}{4}\ddot{f}(0)(g_{ij}g_{kl} + g_{ik}g_{lj} + g_{il}g_{jk}) \,, \end{aligned} \tag{2.8$'$}$$

$$\alpha = -\frac{3n}{2}\left\{\dot{f}^2(0) + \frac{5(n+2)}{2}\ddot{f}(0)\right\} \,, \tag{2.9}$$

$$P(9R^p{}_{ijq;k}R^q{}_{lmp;n} - 32R^p{}_{ijq}R^q{}_{klr}R^r{}_{mnp}) = 315\dddot{f}(0)P(g_{ij}g_{kl}g_{mn}) \,. \tag{2.10}$$

Then taking account of (2.7) and (2.9), (2.4) and (2.5) take the following forms respectively.

$$|\nabla R|^2 + \frac{2}{n}S\alpha + \beta + 4\gamma = 0 \,, \tag{2.11}$$

and

$$\alpha - \frac{2}{n(n-1)}S^2 \geqq 0 \,. \tag{2.12}$$

3. Calculation of $\dddot{f}(0)$.

LEMMA 3.1[*]. *For a tensor field* $T=(T_{ijklmn})$, *we have*

$$\begin{aligned} g^{ij}g^{kl}g^{mn}P(T_{ijklmn}) &= 48(T^{i\ j\ k}_{\ i\ j\ k} + T^{i\ jk}_{\ i\ \ jk} + T^{i\ jk}_{\ i\ \ kj} + T^{ij\ \ k}_{\ \ ij\ k} + T^{ij\ k}_{\ \ i\ jk} \\ &+ T^{ij\ k}_{\ \ i\ kj} + T^{ij\ \ k}_{\ \ ji\ k} + T^{ij\ k}_{\ \ j\ ik} + T^{ij\ \ k}_{\ \ jk\ i} + T^{ijk}_{\ \ \ ijk} \\ &+ T^{ijk}_{\ \ \ ikj} + T^{ijk}_{\ \ \ jik} + T^{ijk}_{\ \ \ jki} + T^{ijk}_{\ \ \ kij} + T^{ijk}_{\ \ \ kji}) \,. \end{aligned}$$

PROOF. Each term of the left hand side is a scalar with three dummy indices such as $T^{k\ i\ \ j}_{\ i\ jk}$. We can rewrite these indices as follows; replace the first index by i, the next and the rest by j and k respectively, for example, $T^{k\ i\ \ j}_{\ i\ jk}$ by $T^{i\ j\ \ k}_{\ j\ ki}$ or $T^{ij\ k}_{\ \ j\ ik}$. Then they necessarily coincide with one of the fifteen terms arranged in alphabetical order in the right hand side. It is easily seen that each of the fifteen terms appears forty eight times in the left hand side.

Now we put

$$\begin{aligned} A_{ijklmn} &= R^p{}_{ijq;k}R^q{}_{lmp;n} \,, \\ B_{ijklmn} &= R^p{}_{ijq}R^q{}_{klr}R^r{}_{mnp} \,, \\ C_{ijklmn} &= g_{ij}g_{kl}g_{mn} \,. \end{aligned}$$

Then we have the following

LEMMA 3.2. *In any harmonic Riemannian space, we have*

(a) $$g^{ij}g^{kl}g^{mn}P(A_{ijklmn}) = 48\cdot 3|\nabla R|^2 \,,$$

(b) $$g^{ij}g^{kl}g^{mn}P(B_{ijklmn}) = 48\Big(\frac{1}{n^2}S^3 + \frac{9}{2n}S\alpha - \frac{7}{2}\beta + \gamma\Big) \,,$$

(c) $$g^{ij}g^{kl}g^{mn}P(C_{ijklmn}) = 48(n^3+6n^2+8n) \,.$$

PROOF. Each straightforward calculation using Lemma 3.1, (2.1), (2.2), (2.3) and (2.7) yields (a), (b) and (c). We write calculations of (a) and (b):

I. Calculation of (a).

$$A^{ijk}{}_{ijk} = \frac{1}{2}|\nabla R|^2 \,, \quad A^{ijk}{}_{ikj} = \frac{1}{4}|\nabla R|^2 \,,$$

$$A^{ijk}{}_{jik} = |\nabla R|^2 \,, \quad A^{ijk}{}_{jki} = \frac{1}{2}|\nabla R|^2 \,,$$

$$A^{ijk}{}_{kij} = \frac{1}{2}|\nabla R|^2 \,, \quad A^{ijk}{}_{kji} = \frac{1}{4}|\nabla R|^2 \,,$$

all the other $=0$.

---

[*] This lemma is essentially due to M. Yasuda.

II. Calculation of (b).

$$B^{i\ i\ k}_{\ i\ j\ k}=\frac{1}{n^2}S^3\ ,\quad B^{i\ jk}_{\ i\ \ jk}=\frac{1}{2n}S\alpha\ ,\quad B^{i\ jk}_{\ i\ \ kj}=\frac{1}{n}S\alpha\ ,$$

$$B^{ij\ k}_{\ \ ij\ k}=\frac{1}{2n}S\alpha\ ,\quad B^{ij\ k}_{\ \ i\ jk}=-\frac{1}{4}\beta\ ,\quad B^{ij\ k}_{\ \ i\ kj}=-\frac{1}{2}\beta\ ,$$

$$B^{ij\ k}_{\ \ ji\ k}=\frac{1}{n}S\alpha\ ,\quad B^{ij\ k}_{\ \ j\ ik}=-\frac{1}{2}\beta\ ,\quad B^{ij\ k}_{\ \ j\ ki}=-\beta\ ,$$

$$B^{ijk}_{\ \ \ ijk}=\gamma-\frac{1}{4}\beta\ ,\quad B^{ijk}_{\ \ \ ikj}=-\frac{1}{4}\beta\ ,\quad B^{ijk}_{\ \ \ jik}=-\frac{1}{4}\beta\ ,$$

$$B^{ijk}_{\ \ \ jki}=-\frac{1}{2}\beta\ ,\quad B^{ijk}_{\ \ \ kij}=\frac{1}{2n}S\alpha\ ,\quad B^{ijk}_{\ \ \ kji}=\frac{1}{n}S\alpha\ .$$

Transvecting (2.10) with $g^{ij}g^{kl}g^{mn}$, by Lemma 3.2 we obtain the following

PROPOSITION 3.3.*) *In any harmonic Riemannian space, we have*

$$27|\nabla R|^2-32\Big(\frac{1}{n^2}S^3+\frac{9}{2n}S\alpha-\frac{7}{2}\beta+\gamma\Big)=315n(n+2)(n+4)\ddot{f}(0)\ . \tag{3.1}$$

4. **Harmonic Kählerian spaces.** Let us consider an $n(=2m)$ real dimensional Kählerian space M^n with real coordinate $\{x^i\}$. Then the (positive definite) Kählerian metric $g=(g_{ij})$ and the almost complex structure $F=(F_j^{\ i})$ satisfy the following equations (cf. [11]).

$$\begin{aligned} &g_{kh}F_i^{\ k}F_j^{\ h}=g_{ij}\ ,\quad F_i^{\ h}F_h^{\ j}=-\delta_i^j\ ,\quad F_{ij}=-F_{ji}\ ,\\ &F_j^{\ h}{}_{;i}=0\ ,\quad R_{ijkh}=R_{ijab}F_k^{\ a}F_h^{\ b}\ ,\\ &F^r{}_jR^h{}_{rkl}=F^h{}_rR^r{}_{jkl}\ ,\quad F^r{}_hR_{rjkl}=-F^r{}_jR_{hrkl}\ ,\\ &F^{ij}R_{ijkh}=-2F_k^{\ j}R_{jh}\ ,\quad F^{jk}R_{ijkh}=F_i^{\ j}R_{jh}\ . \end{aligned} \tag{4.1}$$

By (4.1), we have

$$F^{ij}F^{kh}R_{ijkh}=-2S\ ,\quad F^{ih}F^{jk}R_{ijkh}=S\ . \tag{4.2}$$

Henceforward let M^n be an $n(=2m)$-dimensional harmonic Kählerian space and denoted by H^n.
Transvecting (2.8)′ with $F_a^{\ i}F_b^{\ j}R^{kabl}$, we have from (4.2)

$$\begin{aligned} &\{R^p{}_{ijq}(R^q{}_{klp}+R^q{}_{lkp})+R^p{}_{ikq}(R^q{}_{ljp}+R^q{}_{jlp})\\ &\quad+R^p{}_{ilq}(R^q{}_{jkp}+R^q{}_{kjp})\}F_a^{\ i}F_b^{\ j}R^{kabl}\\ &=-45S\ddot{f}(0)\ . \end{aligned} \tag{4.3}$$

*) We received an information from S. Tachibana such that S. Yamaguchi has obtained independently this equation.

Now using (4.1) we calculate the left hand side of (4.3). By (2.3), we have

$$\begin{aligned} &F_a{}^i F_b{}^j R_{pijq} R^q{}_{kl}{}^p R^{kabl} \\ &\quad = -F_q{}^s R_{pibs} R^q{}_{kl}{}^p (-F_r{}^k R^{ribl}) \\ &\quad = F_q{}^s R_{pibs} F_t{}^q R^t{}_{rl}{}^p R^{ribl} \\ &\quad = -R_{pibs} R^s{}_{rl}{}^p R^{ribl} \\ &\quad = R^{abcd} R_a{}^u{}_c{}^v R_{bvdu} \\ &\quad = \gamma - \frac{1}{4}\beta \,. \end{aligned}$$

By (2.1) and (2.3), we have

$$\begin{aligned} &F_a{}^i F_b{}^j R_{pijq} R^q{}_{lk}{}^p R^{kabl} \\ &\quad = F_a{}^i F_b{}^j R_{pijq} R^q{}_{lk}{}^p (-R^{kbla} - R^{klab}) \\ &\quad = R_{pibs} R^s{}_{ak}{}^p R^{kbai} - R_{pijq} R^q{}_{lk}{}^p R^{klij} \\ &\quad = \gamma - \frac{1}{4}\beta - \frac{1}{4}\beta \,. \end{aligned}$$

Similary, we have

$$\begin{aligned} F_a{}^i F_b{}^j R_{pikq} R^q{}_{lj}{}^p R^{kabl} &= -\frac{1}{4}\beta \,, \\ F_a{}^i F_b{}^j R^p{}_{ik}{}^q R_{qjlp} R^{kabl} &= -\frac{1}{4}\beta \,, \\ F_a{}^i F_b{}^j R^p{}_{il}{}^q R_{qjkp} R^{kabl} &= \gamma - \frac{1}{2}\beta \,, \\ F_a{}^i F_b{}^j R^p{}_{il}{}^q R_{qkjp} R^{kabl} &= \gamma - \frac{1}{4}\beta \,. \end{aligned}$$

Substituting these into (4.3), we obtain

PROPOSITION 4.1. *In any harmonic Kählerian space, we have*

$$4\gamma - 2\beta = -45S\ddot{f}(0) \,. \tag{4.4}$$

Substituting (4.4) into (2.11), we have

$$|\nabla R|^2 + \frac{2}{n}S\alpha + 3\beta - 45S\ddot{f}(0) = 0 \,. \tag{4.5}$$

By (2.9), we have

$$\ddot{f}(0) = -\frac{4}{15n(n+2)}\left(\alpha + \frac{2}{3n}S^2\right) \leqq 0 \,, \tag{4.6}$$

from which (4.4) and (4.5) take the following forms respectively.

$$2\gamma - \beta = \frac{6S}{n(n+2)}\left(\alpha + \frac{2}{3n}S^2\right), \tag{4.7}$$

and

$$|\nabla R|^2 + 3\beta + \frac{2(n+8)}{n(n+2)}S\alpha + \frac{8}{n^2(n+2)}S^3 = 0. \tag{4.8}$$

By (4.7) and (4.8), we get

$$|\nabla R|^2 + 6\gamma + \frac{2(n-1)}{n(n+2)}S\alpha - \frac{4}{n^2(n+2)}S^3 = 0. \tag{4.9}$$

Thus we obtain

THEOREM 4.2. *Any n-(real)dimensional harmonic Kählerian space H^n satisfies the following two inequalities.*

$$\beta \leqq \frac{-2S}{3n(n+2)}\left\{(n+8)\alpha + \frac{4}{n}S^2\right\}, \tag{4.10}$$

$$\gamma \leqq \frac{-(n-1)S}{3n(n+2)}\left\{\alpha - \frac{2}{n(n-1)}S^2\right\}. \tag{4.11}$$

Each equality sign is valid if and only if H^n is locally symmetric.

Since $|\nabla R|^2 \geqq 0$ and $\alpha \geqq 0$, (2.12), (4.7), (4.10) and (4.11) give the following

THEOREM 4.3. *Let H^n be a harmonic Kählerian space.*
(1) *If $S \geqq 0$, then $0 \geqq 2\gamma \geqq \beta$.*
(2) *If $S < 0$, then $2\gamma < \beta$.*

Here we shall consider about the Euler characteristic of a 6-dimensional compact harmonic Kählerian space.

To prove Theorem 4.5, we need the following

LEMMA 4.4.*) ([2], [6]) *Let M be a 6-dimensional compact orientable Riemannian space. Then the Euler characteristic $\chi(M)$ is given as follows:*

$$\chi(M) = \frac{1}{384\pi^3}\int_M \{S^3 - 12S|R_1|^2 + 3S\alpha + 16R^{ab}R_a{}^cR_{bc} - 24R^{ab}R^{cd}R_{acbd} - 24R^{uv}R_u{}^{abc}R_{vabc} + 8\gamma - 4\beta\}dV \tag{4.12}$$

where dV is the volume element of M.

If M is Einsteinian, (4.12) *takes the following form;*

$$\chi(M) = \frac{1}{384\pi^3}\int_M \left\{\frac{1}{9}S^3 - S\alpha + 8\gamma - 4\beta\right\}dV. \tag{4.13}$$

*) Our curvature tensor is different from Sakai's in sign.

Let H^6 be a 6-(real)dimensional compact harmonic Kählerian space. Since H^6 is Einsteinian, we get from (4.4) and (4.13)

$$\chi(H^6) = \frac{1}{384\pi^3}\int_M S\left\{\frac{1}{9}S^2 - \alpha - 90\ddot{f}(0)\right\}dV. \tag{4.14}$$

Since S and α are constant (cf. [3], [5]), (4.6) and (4.14) give the following

THEOREM 4.5. *Let H^6 be a 6-(real)dimensional compact harmonic Kählerian space. We denote the Euler characteristic of H^6 by $\chi(H^6)$. Then we have*

$$\chi(H^6) = \frac{S}{768\pi^3}\left\{\frac{1}{3}S^2 - \alpha\right\}\operatorname{Vol}(H^6) \tag{4.15}$$

where $\operatorname{Vol}(H^6)$ *is the volume of* H^6.

COROLLARY 4.6. *Let H^6 be a 6-(real)dimensional compact harmonic Kählerian space. We denote the Euler characteristic of H^6 by $\chi(H^6)$.*

(1) *If $S > 0$ and $S^2/3 \geqq \alpha$ ($S^2/3 \leqq \alpha$, resp.) then $\chi(H^6) \geqq 0$ ($\chi(H^6) \leqq 0$, resp.).*

(2) *If $S = 0$, then $\chi(H^6) = 0$.*

(3) *If $S < 0$ and $S^2/3 \leqq \alpha$ ($S^2/3 \geqq \alpha$, resp.) then $\chi(H^6) \geqq 0$ ($\chi(H^6) \leqq 0$, resp.).*

Representing the right hand side of (4.15) in terms of $\dot{f}(0)$ and $\ddot{f}(0)$, we get from (2.7) and (2.9),

THEOREM 4.7. *Let H^6 be a 6-dimensional compact harmonic Kählerian space. We denote the Euler characteristic of H^6 by $\chi(H^6)$. Then we have*

$$\chi(H^6) = -\frac{27}{64\pi^3}\dot{f}(0)\left\{\dot{f}^2(0) + 5\ddot{f}(0)\right\}\operatorname{Vol}(H^6) \tag{4.16}$$

where $\operatorname{Vol}(H^6)$ *is the volume of* H^6.

5. Main results. In this section, we shall give some results by combining Proposition 3.3 and Proposition 4.1.

Now substituting (4.4) into (3.1), we get

$$\begin{aligned} 27|\nabla R|^2 - \frac{32}{n^2}S^3 - \frac{144}{n}S\alpha + 96\beta + 360S\ddot{f}(0) \\ = 315n(n+2)(n+4)\ddddot{f}(0). \end{aligned} \tag{5.1}$$

By (4.5) and (5.1), we get

$$\begin{aligned} 5|\nabla R|^2 + \frac{32}{n^2}S^3 + \frac{208}{n}S\alpha \\ = 1800S\ddot{f}(0) - 315n(n+2)(n+4)\ddddot{f}(0). \end{aligned} \tag{5.2}$$

Since $|\nabla R|^2 \geqq 0$ and $\alpha \geqq 0$, we get from (4.6) and (5.2)

THEOREM 5.1. *An n-(real)dimensional harmonic Kählerian space with positive scalar curvature satisfies* $\dddot{f}(0) < 0$.

Taking account of (2.7) and (2.9), (5.2) takes the following form.

$$|\nabla R|^2 + 9n\{8\dot{f}^3(0) + 2(13n + 56)\dot{f}(0)\ddot{f}(0) + 7(n + 2)(n + 4)\dddot{f}(0)\} = 0 . \tag{5.3}$$

Thus we have

THEOREM 5.2. *In any* $n(=2m)$*-dimensional harmonic Kählerian space* H^n*, its characteristic function* $f(\Omega)$ *satisfies the inequality*

$$8\dot{f}^3(0) + 2(13n + 56)\dot{f}(0)\ddot{f}(0) + 7(n + 2)(n + 4)\dddot{f}(0) \leqq 0 , \tag{5.4}$$

or

$$2\dot{f}^3(0) + (13m + 28)\dot{f}(0)\ddot{f}(0) + 7(m + 1)(m + 2)\dddot{f}(0) \leqq 0 . \tag{5.4$'$}$$

The equality sign is valid if and only if H^n *is locally symmetric.*

On the other hand, we know ([3], [5]) that $\alpha, \dot{f}(0), \ddot{f}(0)$ and $\dddot{f}(0)$ are independent of the reference point. Therefore (5.3), (4.8) and (4.9) give the following

THEOREM 5.3. *In any* $n(=2m)$*-dimensional harmonic Kählerian space* H^n, $|\nabla R|^2$, β *and* γ *are all constant.*

We transform (5.4)′ as follows;

$$\begin{aligned} 2(m + 7)\left\{\dot{f}(0) + \frac{5(m + 1)^2}{m + 7}\ddot{f}(0)\right\}\dot{f}(0) & \\ + (m + 2)\{3(m + 31)\dot{f}(0)\ddot{f}(0) + 7(m + 1)(m + 7)\dddot{f}(0)\} &\leqq 0 . \end{aligned} \tag{5.5}$$

If $S > 0$, then $\dot{f}(0) < 0$. Hence by (1.3) we have

$$\left\{\dot{f}^2(0) + \frac{5(m + 1)^2}{m + 7}\ddot{f}(0)\right\}\dot{f}(0) \geqq 0 . \tag{5.6}$$

Thus we have

THEOREM 5.4. *An* $n(=2m)$*-dimensional harmonic Kählerian space* H^n *with positive scalar curvature satisfies the inequality*

$$3(m + 31)\dot{f}(0)\ddot{f}(0) + 7(m + 1)(m + 7)\dddot{f}(0) \leqq 0 . \tag{5.7}$$

The equality sign is valid if and only if H^n *is of constant holomorphic curvature.*

6. Examples.

Let H^n be an $n(=2m)$-dimensional space of constant holomorphic curvature ($k > 0$) characterized as a harmonic Kählerian space with the characteristic function given by (1.2). It is known that

$$
\begin{aligned}
x \cot x &= 1 - \sum_{n=1}^{\infty} \frac{2^{2n} B_n}{(2n)!} x^{2n} \\
&= 1 - \frac{1}{3}x^2 - \frac{1}{45}x^4 - \frac{2}{945}x^6 - \cdots
\end{aligned} \tag{6.1}
$$

where B_n is the Bernoulli number. Therefore we have

$$
\begin{aligned}
x \tan x &= x \cot x - 2x \cot 2x \\
&= x^2 + \frac{1}{3}x^4 + \frac{126}{945}x^6 + \cdots .
\end{aligned} \tag{6.2}
$$

If we develop $f(\Omega)$ in the power series of Ω and s respectively, then taking account of (6.2) it holds that

$$
\begin{aligned}
f(\Omega) &= 2m + \dot{f}(0)\Omega + \frac{1}{2}\ddot{f}(0)\Omega^2 + \frac{1}{3!}\dddot{f}(0)\Omega^3 + \cdots \\
&= 2m - \frac{2(m+1)}{3}(ls)^2 - \frac{2(m+7)}{45}(ls)^4 - \frac{4(m+31)}{945}(ls)^6 - \cdots .
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\dot{f}(0) = -\frac{4(m+1)}{3}l^2 , \quad \ddot{f}(0) = -\frac{16(m+7)}{45}l^4 , \\
&\dddot{f}(0) = -\frac{64(m+31)}{315}l^6 .
\end{aligned} \tag{6.3}
$$

It is easily seen that (6.3) satisfies (5.4)′ and (5.7), i.e.,

$$
2\dot{f}(0) + (13m + 28)\dot{f}(0)\ddot{f}(0) + 7(m+1)(m+2)\dddot{f}(0) = 0 , \tag{6.4}
$$

$$
3(m + 31)\dot{f}(0)\ddot{f}(0) + 7(m+1)(m+7)\dddot{f}(0) = 0 . \tag{6.5}
$$

On the other hand, it is well known (cf. [11]) that a Kählerian space of constant holomorphic curvature is locally symmetric.

REMARK. Similarly, it can be seen that (6.4) is valid for the $f(\Omega)$ of (1.2)′ in the case of $l = \sqrt{-k}/2$.

Now let S^{2m+1} be the unit sphere in C^{m+1} and S^1 the multiplicative group of complex numbers of absolute value 1. Then S^{2m+1} is a principal fibre bundle over the complex projective space CP^m with group S^1, called Hopf fibering. It is well known that CP^m carries the canonical Kählerian metric of constant holomorphic curvature 4. We know (cf. [1]).

$$
\mathrm{Vol}\,(CP^m) = \frac{\pi^m}{m!} . \tag{6.6}
$$

Especially for CP^3, we have from (6.3) and (6.6)

$$\dot{f}(0) = -\frac{16}{3},\ \ddot{f}(0) = -\frac{32}{9},\ \mathrm{Vol}\,(CP^3) = \frac{\pi^3}{6}\,. \tag{6.7}$$

Substituting (6.7) into (4.16), we get the well known result; $\chi(CP^3) = 4$.

REFERENCES

[1] M. BERGER, P. GAUDUCHON et E. Mazet, Le spectre d'une variété riemannienne, Springer-Lecture Notes in Mathematics, no 194, 1971.

[2] J. C. DESSERTINE, Expressions nouvelles de la formule de Gauss-Bonnet en dimensions 4 et 6, C. R. Acad. Sci. Paris, t. 273 (1971).

[3] A. LICHNEROWICZ, Sur les espaces riemanniens complètement harmoniques, Bull. Soc. Math., France 72 (1944) 146–168.

[4] A. LICHNEROWICZ, Geometrie des groupes de transformations, Dund, Paris, 1958.

[5] H. S. RUSE, A. G. WALKER AND T. J. WILLMORE, Harmonic Spaces, Edizioni Cremonese, Roma, 1961.

[6] T. SAKAI, On eigenvalues of Laplacian and curvature of Riemannian manifold, Tôhoku Math. J., 23 (1971) 589–603.

[7] S. TACHIBANA, On the characteristic function of spaces of constant holomorphic curvature, Colloq. Math. 26 (1972) 149–155.

[8] S. TANNO, Euler-Poincare characteristic and curvature tensors, Tôhoku Math. J., 25 (1973), 33–52.

[9] A. G. WALKER, On Lichnerowicz's conjecture for harmonic 4-spaces, J. Lond. Math. Soc., 24 (1948–1949) 21–28.

[10] T. J. WILLMORE, Some properties of harmonic Riemannian manifold, Convegno di Geometria Differenzial, Venice, (1953) 141–147.

[11] K. YANO, Differential geometry on complex and almost complex spaces, Pergamon Press, 1965.

[12] K. YANO AND S. BOCHNER, Curvature and Betti numbers, Ann. of Math. studies, 32, Princeton, 1953.

FACULTY OF SCIENCE,
TOYAMA UNIVERSITY
TOYAMA, JAPAN