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HARMONIC AND QUASIHARMONIC DEGENERACY OF RIEMANNIAN MANIFOLDS

LUNG OCK CHUNG AND LEO SARIO

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The harmonic and quasiharmonic classifications of Riemannian manifolds have been largely brought to completion (see Bibliography). In the present paper we shall discuss interrelations between harmonic null classes and quasiharmonic null classes.

Let H and Q be the classes of harmonic and quasiharmonic functions h and q, defined by $\Delta h = 0$ and $\Delta q = 1$, where $\Delta = d\delta + \delta d$ is the Laplace-Beltrami operator. Denote by P, B, D, C, L^p the classes of functions which are positive, bounded, Dirichlet finite, bounded Dirichlet finite, or of finite L^p norm, respectively. Here $1 \leqq p < \infty$, the value $p = \infty$ being excluded since for both harmonic and quasiharmonic functions, $L^\infty = B$. For $X = P, B, D, C, L^p$, set $HX = H \cap X$, $QX = Q \cap X$, and let O_F^N stand for the class of Riemannian N-manifolds, $N \geqq 2$, which do not carry nonconstant functions in a given class F. The complement of O_F^N with respect to the totality of Riemannian N-manifolds is designated by $\tilde{O}_F^N$.

We shall first show that $O_{HX}^N \cap O_{QY}^N \neq \emptyset$ for $X, Y = P, B, D, C, L^p$, $1 \leqq p < \infty$, $N \geqq 2$. In view of the Euclidean ball it is trivial that $\tilde{O}_{HX}^N \cap \tilde{O}_{QY}^N \neq \emptyset$, and we shall prove that $\tilde{O}_{HX}^N \cap O_{QY}^N \neq \emptyset$ for all X, Y, p, N.

The classes $O_{HX}^N \cap \tilde{O}_{QY}^N$ are intriguing. From the harmonic classification theory it is known that the class O_G^N of parabolic N-manifolds, characterized by the nonexistence of Green's functions, is related to other harmonic null classes by the strict inclusion relations $O_G^N < O_{HP}^N < O_{HB}^N < O_{HD}^N = O_{HC}^N$. The proof of the strictness, due mainly to Ahlfors, Royden, and Tôki, was one of the most challenging problems in the theory of harmonic functions. On the other hand, O_G^N is strictly contained also in all O_{QX}^N, $X = P, B, D, C$ [12, 20]. The problem of proving the nonemptiness of the classes $O_{HX}^N \cap \tilde{O}_{QY}^N$ thus amounts to finding manifolds which belong to the "narrow" spaces $\tilde{O}_G^N \cap O_{HX}^N$, yet carry QY-functions. For X, Y other than L^p we only have fragmentary results on this problem (see No. 11). On the other hand, the classes $O_{HX}^N \cap \tilde{O}_{QY}^N$ turn out to be

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nonvoid for $X = L^p$, all Y, p, N; $Y = L^p$, all X, p, N; and $X = L^p$, $Y = L^q$, all p, q, N.

1. Our first problem is to exclude both HX and QY functions:

THEOREM 1. *$O^N_{HX} \cap O^N_{QY} \neq \emptyset$ for X, $Y = P, B, D, C, L^p$; $1 \leqq p < \infty$; $N \geqq 2$.*

The proof will be given in Nos. 2–4.

2. By means of the Poisson integral and Harnack's inequality we see immediately that every $h \in HP$ in the Euclidean N-space E^N reduces to a constant. Therefore $E^N \in O^N_{HX}$ for $X = P, B, D, C$.

To show that $E^N \in O^N_{HL^p}$ we first consider the case $p = 1$. Suppose there exists a nonconstant $h \in HL^1$. It has a unique representation $h = \sum_{n=0}^{\infty} r^n S_n$, with $S_n = S_n(\theta)$ spherical harmonics, characterized by $\Delta(r^n S_n) = 0$, and $(r, \theta) = (r, \theta_1, \cdots, \theta_{N-1})$ the polar coordinates. For some $n_0 > 0$, $S_{n_0} \not\equiv 0$. Take a function $\rho(r) \in C[0, \infty)$ with $\rho(r) = 1/r$ for $r \geqq 1$ and set $\varphi = \rho S_{n_0}$. Since $\varphi \in B$, we have $\| h\varphi \|_1 < \infty$. On the other hand,

$$\| h\varphi \|_1 \geqq |(h\rho, S_{n_0})| = c \int_0^\infty r^{n_0} \rho r^{N-1} dr$$
$$= a + c \int_1^\infty r^{n_0+N-2} dr = \infty \ ;$$

here and later c is a constant, not always the same. The contradiction shows that $E^N \in O^N_{HL^1}$.

Now let $p > 1$ and take p' with $1/p + 1/p' = 1$. Suppose there exists a nonconstant $h \in HL^p$. In the expansion $h = \sum_0^\infty r^n S_n$, $S_{n_0} \not\equiv 0$ for some $n_0 > 0$. Let $\rho(r) \in C[0, \infty)$ with

$$\rho(r) = r^{-(N+1)/p'} \quad \text{for} \quad r \geqq 1 \ ,$$

and set $\varphi = \rho S_{n_0}$. Since

$$\| \rho \|_{p'}^{p'} = a + c \int_1^\infty r^{-(N+1)} r^{N-1} dr < \infty \ ,$$

$\| \varphi \|_{p'} < \infty$, and $|(h, \varphi)| < \infty$. We again have a contradiction:

$$|(h, \varphi)| = \left| \mathrm{a} + \mathrm{c} \int_1^\infty r^{n_0} r^{-(N+1)/p'} r^{N-1} dr \right|$$
$$= \left| a + c \int_1^\infty r^{n_0+N/p-1/p'-1} dr \right| = \infty \ .$$

A fortiori $E^N \in O^N_{HL^p}$ for $p > 1$ as well.

3. To prove $E^N \in O^N_{QX}$ for $X = P, B, D, C$ we recall that $O^N_{QP} < O^N_{QB} \cap$

$O_{QD}^N < O_{QB}^N \cup O_{QD}^N = O_{QC}^N$ [12, 20]. Thus it suffices to establish $E^N \in O_{QP}^N$. Since

$$q_0 = -(2N)^{-1}r^2 \in Q ,$$

every $q \in Q$ can be written $q = q_0 + h$ with some $h \in H$. We are to show that $q \notin P$. Set

$$q = q_0 + h(0) + k , \quad k \in H , \quad k(0) = 0 ,$$

where $h(0)$, $k(0)$ are the values at the origin. By the mean value theorem there exists, for every r_n, an $\theta^n = (\theta_1^n, \cdots, \theta_{N-1}^n)$ such that $k(r_n, \theta^n) = 0$. If $\{r_n\}_0^\infty$ is an increasing sequence with $r_n \to \infty$, then

$$q(r_n, \theta^n) = -\frac{1}{2N}r_n^2 + h(0) \to -\infty ,$$

and therefore $q \notin P$.

4. It remains to show that $E^N \in O_{QL^p}^N$. Again we start with $p = 1$. For $q \in Q$, $q = q_0 + h(0) + k$, we have

$$k(0) = \int_{E^N} kdV = 0 ,$$

and therefore

$$\| q \|_1 \geqq \left| \int_{E^N} (q_0 + h(0))dV \right| .$$

The integrand (with respect to $drd\theta_1 \cdots d\theta_{N-1}$) is $\sim cr^2r^{N-1}$, hence $\| q \|_1 = \infty$, and $E^N \in O_{QL^1}^N$.

In the case $p > 1$, the choice $\varphi(r) \in C[0, \infty)$, $\varphi(r) = r^{-(N+1)/p'}$ for $r \geqq 1$ gives $\| \varphi \|_{p'} < \infty$. If there exists a $q \in QL^p$, $q = q_0 + h(0) + k$, then by $(k, \varphi) = 0$, the integrand in $(q, \varphi) = (q_0 + h(0), \varphi)$ is asymptotically

$$cr^2r^{-(N+1)/p'}r^{N-1} = cr^{N/p-1/p'+1} .$$

The exponent dominates N/p, hence $|(q, \varphi)| = \infty$, in violation of $\| \varphi \|_{p'} < \infty$.

The proof of Theorem 1 is herewith complete.

5. In view of the Euclidean N-ball it is trivial that $\tilde{O}_{HX}^N \cap \tilde{O}_{QY}^N \neq \varnothing$ for X, $Y = P, B, D, C, L^p$; $1 \leqq p < \infty$; $N \geqq 2$. Our next problem is to find an N-manifold which carries HX functions but no QY functions.

THEOREM 2. *$\tilde{O}_{HX}^N \cap O_{QY}^N \neq \varnothing$ for X, $Y = P, B, C, D, L^p$; $1 \leqq p < \infty$; $N \geqq 2$.*

The proof will be given in Nos. 6–10.

6. For X, $Y = P, B, D, C$ it suffices to show that $\tilde{O}_{HD}^N \cap O_{QP}^N \neq \varnothing$.

Consider the "beam"

$$T = \{(x, y_1, \cdots, y_{N-1}) \mid |x| < \infty, |y_i| \leqq 1, i = 1, \cdots, N-1\} ,$$

with each face $y_i = 1$ identified with the opposite face $y_i = -1$ by a parallel translation perpendicular to the x-axis. Endow T with a metric with volume element $\sqrt{g}\, dx dy_1 \cdots dy_{N-1}$, where g is the determinant of the metric tensor (g_{ij}) with inverse (g^{ij}). Let each g_{ij} depend on x only and set

$$\rho(x) = \sqrt{g(x)} , \qquad \sigma(x) = g^{11}(x) .$$

Then $h_0(x)$ is harmonic if and only if

$$\Delta h_0 = -\rho^{-1}(\rho\sigma h_0')' = 0 ,$$

that is,

$$h_0(x) = c \int_a^x \rho^{-1}\sigma^{-1} dt .$$

The Dirichlet integral over the subspace from $-x$ to x is

$$D_x(h_0) = c \int_{-x}^x h_0'^2 \sigma\rho dt = c \int_{-x}^x \rho^{-1}\sigma^{-1} dt .$$

We recall (loc. cit.) that a manifold belongs to $\tilde{O}_{QP}^N$ if and only if the potential $G1$ of the constant function 1,

$$G1(\xi) = \int_{\eta \in T} g(\xi, \eta) dV(\eta)$$

exists at some, and hence every, $\xi \in T$. In the present case this potential depends on the x-coordinate $x(\xi)$ of ξ only, and we can use the notation $G1(x)$. Suppose $G1(x)$ exists. Since $\Delta G1(x) \equiv 1$,

$$G1(x) = q_0(x) + h_0(x) ,$$

where

$$q_0(x) = -\int_0^x \rho^{-1}\sigma^{-1} \int_0^t \rho ds dt$$

is quasiharmonic by $\Delta q_0 = -\rho^{-1}(\rho\sigma q_0')' = 1$. Our problem thus reduces to finding functions $\rho(x)$, $\sigma(x)$ such that

$$\int_{-x}^x \rho^{-1}\sigma^{-1} dt \in B ,$$

$$\int_0^x \rho^{-1}\sigma^{-1} \int_0^t \rho(s) ds dt \longrightarrow \infty \quad \text{as} \quad x \longrightarrow \infty .$$

The choice $\rho = 1$, $\sigma = 1 + x^2$ gives

$$\int_0^x \rho^{-1}\sigma^{-1}dt = \int_0^x (1+t^2)^{-1}dt = \arctan x \in HD ,$$

hence $T \in \tilde{O}^N_{HD}$. An arbitrary $h_0(x) \in H$ is

$$h_0(x) = a \arctan x + b ,$$

and

$$q_0(x) = -\int_0^x (1+t^2)^{-1}t\,dt = -\frac{1}{2}\log(1+x^2) ,$$

which tends to $-\infty$ as $|x| \to \infty$. Therefore $G1 = q_0 + h_0 \to -\infty$, $G1 \notin P$, and consequently $T \in O^N_{QP}$.

The relation $\tilde{O}^N_{HD} \cap O^N_{QP}$ we have thus established can be generalized to polyharmonic functions [3].

7, The relation

$$\tilde{O}^N_{HL^p} \cap O^N_{QY} \neq \emptyset$$

was proved in [1] for $Y = P, B, D, C$; $1 \leqq p < \infty$; $N \geqq 2$. We proceed to show that

$$\tilde{O}^N_{HX} \cap O^N_{QL^p} \neq \emptyset$$

for $X = P, B, D, C$; $1 \leqq p < \infty$; $N \geqq 2$. Consider the manifold

$$T: \{(x, y_1, \cdots, y_{N-1}) \mid x > 1, |y_i| \leqq \pi, i = 1, \cdots, N-1\} .$$

It shall be henceforth tacitly understood that the opposite faces $y_i = \pi$ and $y_i = -\pi$ of such "beams" are identified by pairs as in No. 6. For the metric we choose

$$ds^2 = dx^2 + x^{2\alpha/(N-1)} \sum_{i=1}^{N-1} dy_i^2 ,$$

with α a constant > 1.

To show that $T \in \tilde{O}^N_{HX}$ it suffices to take $X = C$. The function $x^{-\alpha+1}$ is in HB, by virtue of

$$\Delta x^{-\alpha+1} = -x^{-\alpha}(x^{\alpha}(x^{-\alpha+1})')' = 0 .$$

Moreover,

$$D(x^{-\alpha+1}) = c\int_1^\infty (x^{-\alpha})^2 x^{\alpha} dx < \infty .$$

Therefore $T \in \tilde{O}^N_{HC}$.

8. To prove $T \in O^N_{QL^p}$ we first take $p = 1$. Since

$$q_0(x) = -\frac{1}{2}(\alpha+1)^{-1}x^2 \in Q ,$$

every $q \in Q$ can be written

$$q = q_0 + ax^{-\alpha+1} + b + \Sigma' f_n(x)G_n(y) ,$$

where $f_n G_n \in H$ and G_n ranges over all products of the form

$$G_n(y) = \prod_{i=1}^{N-1} \begin{matrix}\cos\\ \sin\end{matrix}\, n_i y_i$$

with $n = (n_1, \cdots, n_{N-1}) \neq (0, \cdots, 0)$. Since $\int_T f_n G_n dV = 0$,

$$\| q \|_1 \geqq \left| \int_T (q_0 + ax^{-\alpha+1} + b) dV \right| .$$

The integrand is $\sim cx^{2+\alpha}$, hence $\| q \|_1 = \infty$ and $T \in O^N_{QL^1}$.

Now let $p > 1$, and p' as before. The function $\varphi(x) = x^{-(\alpha+p')/p'}$ belongs to $L^{p'}$ by virtue of

$$\int_T |\varphi|^{p'} dV = c \int_1^\infty x^{-(\alpha+p')} x^\alpha dx < \infty .$$

If there exists a $q \in L^p$, then $|(q, \varphi)| < \infty$. On the other hand,

$$|(q, \varphi)| = \left| \int_T (q_0 + ax^{-\alpha+1} + b) x^{-(\alpha+p')/p'} dV \right| ,$$

where the integrand with respect to $dx dy_1 \cdots dy_{N-1}$ is asymptotically $x^{\alpha/p+1}$. A fortiori $|(q, \varphi)| = \infty$, and the contradiction gives $T \in O^N_{QL^p}$.

9. It remains to show that

$$\tilde{O}^N_{HL^s} \cap O^N_{QL^t} \neq \varnothing$$

for $1 \leqq s < \infty, 1 \leqq t < \infty$. First fix t and consider the manifold

$$T = \{(x, y_1, \cdots, y_{N-1}) \mid x > 1, |y_i| \leqq \pi, i = 1, \cdots, N-1\}$$

with the metric

$$ds^2 = e^{-x/t} dx^2 + e^{(2x+x/t)/(N-1)} \sum_{i=1}^{N-1} dy_i^2 .$$

The function $h_0(x) = e^{-x-x/t}$ belongs to HL^s, and $T \in \tilde{O}^N_{HL^s}$.

10. To see that $T \in O^N_{QL^t}$, note that $q_0(x) = te^{-x/t} \in Q$. Every $q \in Q$ has the form

$$q = q_0 + ah_0 + b + \Sigma' f_n G_n .$$

For $t = 1$,

$$\| q \|_1 \geqq c \left| \int_1^\infty (q_0 + ah_0 + b) e^x dx \right| .$$

If $b \neq 0$, then the integrand is $\sim be^x$, and $\|q\|_1 = \infty$. If $b = 0$, the dominating term in the integrand is $q_0 e^x$, so that again $\|q\|_1 = \infty$, and $T \in O^N_{QL^1}$.

In the case $t > 1$, take t' such that $1/t + 1/t' = 1$. The function $\varphi(x) = e^{-x/t'}x^{-1}$ belongs to $L^{t'}$. If there exists a $q \in QL^t$, then $|(q, \varphi)| < \infty$. But

$$|(q, \varphi)| = |(q_0 + ah_0 + b + \Sigma' f_n G_n, \varphi)| = |(q_0 + ah_0 + b, \varphi)| .$$

If $b \neq 0$, the integrand is $\sim be^{-x/t'}x^{-1}e^x = be^{x/t}x^{-1}$, and we have the contradiction $|(q, \varphi)| = \infty$. If $b = 0$, since q_0 dominates h_0, the integrand is asymptotically

$$ce^{-x/t-x/t'}x^{-1}e^x = cx^{-1} ,$$

and again we have divergence. Therefore $T \in O^N_{QL^t}$.

The proof of Theorem 2 is herewith complete.

11. Our final problem is to find N-manifolds which carry QY functions but no HX functions. For X, Y other than L^p we only have two fragmentary results, both quite immediate. First, $O^N_{HD} \cap \tilde{O}^N_{QC} \neq \emptyset$ for $N \geqq 5$ is a consequence of what is known of the Poincaré N-ball B^N_α, that is, the ball $\{r < 1\}$ with the metric $ds = (1 - r^2)^\alpha |dx|$, α a real constant. It was shown in [5] that $B^N_\alpha \in O^N_{HD}$ if and only if $|\alpha| \geqq (N-2)^{-1}$, $N \geqq 3$, and in [26] that

$$B^N_\alpha \in \tilde{O}^N_{QB} \cap \tilde{O}^N_{QD} = \tilde{O}^N_{QC}$$

if and only if $-3/(N+2) < \alpha < 1/(N-2)$, $N \geqq 3$. A fortiori,

$$B^N_\alpha \in O^N_{HD} \cap \tilde{O}^N_{QC} , \qquad N \geqq 5 ,$$

if and only if $-3/(N+2) < \alpha \leqq -1/(N-2)$.

Another result, $O^N_{HD} \cap \tilde{O}^N_{QB} \neq \emptyset$ and $O^N_{HD} \cap \tilde{O}^N_{QP} \neq \emptyset$, both for $N \geqq 4$, is also offered by B^N_α. We know [26] that $B^N_\alpha \in \tilde{O}^N_{QB}$ if and only if $-1 < \alpha < 1/(N-2)$, and $B^N_\alpha \in \tilde{O}^N_{QP}$ is characterized by the same range of α. As a consequence,

$$B^N_\alpha \in O^N_{HD} \cap \tilde{O}^N_{QB}(\tilde{O}^N_{QP}) , \qquad N \geqq 4 ,$$

if and only if $-1 < \alpha \leqq -1/(N-2)$.

12. In the case of L^p functions we have a complete result:

THEOREM 3. *$O^N_{HX} \cap \tilde{O}^N_{QY} \neq \emptyset$ for $X = L^p$, $Y = P, B, D, C$, $1 \leqq p < \infty$, $N \geqq 2$; $X = P, B, D, C$, $Y = L^p$, $1 \leqq p < \infty$, $N \geqq 2$; and $X = L^s$, $Y = L^t$, $1 \leqq s < \infty$, $1 \leqq t < \infty$, $N \geqq 2$.*

The proof will be given in Nos. 13–14.

13. The first case,

$$O_{HL^p}^N \cap \tilde{O}_{QY}^N \neq \varnothing$$

for $Y = P, B, D, C,\ 1 \leqq p < \infty,\ N \geqq 2$, was established in [1], and we proceed to the case $O_{HX}^N \cap \tilde{O}_{QL^p}^N \neq \varnothing$ for $X = P, B, D, C, 1 \leqq p < \infty, N \geqq 2$. It suffices to prove

$$O_{HP}^N \cap \tilde{O}_{QL^p}^N \neq \varnothing \ .$$

Take the manifold

$$T = \{(x, y_1, \cdots, y_{N-1}) \mid |x| < \infty, |y_i| \leqq 1, i = 1, \cdots, N-1\}$$

with the metric

$$ds^2 = e^{-x^2}dx^2 + e^{-x^2/(N-1)} \sum_{i=1}^{N-1} dy_i^2 \ .$$

Since $x \in H$, T is parabolic, hence in O_{HP}^N. The function

$$q_0(x) = -\int_0^x \int_0^t e^{-s^2} ds dt$$

is quasiharmonic and for $1 \leqq p < \infty$

$$\|q_0\|_p^p = c\int_{-\infty}^{\infty} |q_0|^p e^{-x^2} dx < \infty \ .$$

In fact, $|q_0|^p \leqq \left|\int_0^x a dx\right|^p \leqq a^p |x|^p$, where $a = \int_0^\infty e^{-t^2}dt$. Therefore $T \in \tilde{O}_{QL^p}^N$.

14. The remaining case of Theorem 3 is

$$O_{HL^s}^N \cap \tilde{O}_{QL^t}^N \neq \varnothing$$

for $1 \leqq s < \infty$, $1 \leqq t < \infty$, $N \geqq 2$. Take the N-space M with the metric

$$ds^2 = \varphi(r)dr^2 + \psi(r)^{1/(N-1)} \sum_{i=1}^{N-1} \lambda_i(\theta) d\theta_i^2 \ ,$$

where φ, $\psi \in C[0, \infty)$ and the λ_i are trigonometric functions of $\theta = (\theta_1, \cdots, \theta_{N-1})$ such that the metric is Euclidean for $r \leqq 1/2$. For $r \geqq 1$ we choose

$$\varphi(r) = e^{-2r} \ , \qquad \psi(r) = e^{2r} \ .$$

The volume element of M for $r \geqq 1$ is simply $dr d\theta_1 \cdots d\theta_{N-1}$.

To prove $M \in O_{HL^s}^N$ take a nonconstant

$$h \in H \ , \qquad h = \Sigma f_n(r) S_n(\theta) \ .$$

In each term, if $f_n \not\equiv 0$, then by the maximum principle, $f_n(r) \neq 0$ for every $r > 0$, and $\lim_{r\to\infty} f_n(r) \neq 0$, so that $|f_n(r)| > c_n > 0$ for $r > 1$, say.

For some n_0, $f_{n_0} \not\equiv 0$. Take a function $\rho(r) \in C[0, \infty)$ with $\rho(r) = 1/r$ for $r \geqq 1$, and set $\varphi = \rho S_{n_0}$. Since $\|\rho\|_{s'} < \infty$, $\|\varphi\|_{s'} < \infty$. If $h \in L^s$, then $|(h, \varphi)| < \infty$. But

$$|(h, \varphi)| = \left| a + b \int_1^\infty f_{n_0} \rho dr \right| ,$$

where $|f_{n_0}\rho| \geqq c_{n_0}/r$ and f_{n_0} is of constant sign, hence $|(h, \varphi)| = \infty$. This contradiction gives $M \in O^N_{HL^s}$.

To see that $M \in \tilde{O}^N_{QL^t}$, $1 \leqq t < \infty$, we note that the function

$$q_0(r) = \int_r^\infty \varphi^{1/2}\psi^{-1/2} \int_0^t (\varphi\psi)^{1/2} ds dt$$

is quasiharmonic and $\|q_0\|_t < \infty$.

This completes the proof of Theorem 3.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
LOS ANGELES 90024, CALIFORNIA
U.S.A.