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SOME EXAMPLES OF NON-REGULAR ALMOST CONTACT STRUCTURES ON EXOTIC SPHERES*

KINETSU ABE

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1. Introduction. Let M be a C^∞-manifold of dimension $2n+1$. An almost contact structure on M is a triple (ϕ, ξ, η) of C^∞-tensor fields of type (1, 1), (1, 0) and (0, 1), respectively; such that

(1) $\eta(\xi)=1$, i.e., the contraction of ξ and η equals 1.

(2) $\phi^2(X) = -X+\eta(X)\xi$ for all C^∞-vector fields X on M.

A manifold with an almost contact structure is called an almost contact manifold. See S. Sasaki [7], for more details on almost contact manifolds. An almost contact structure (ϕ, ξ, η) is called regular if the foliation on M given by the maximum integral curves of ξ is a regular foliation in the usual sense, and otherwise called non-regular. A simple but typical example of an almost contact structure is given by the Hopf fibration $S^{2n+1} \xrightarrow{\pi} CP^n$. This structure on S^{2n+1} is regular, because the leaves of the associated foliation are great circles. In fact, any principal circle bundles over almost complex manifolds can be made into regular almost contact manifolds in a natural way. Thus, there are many examples of regular almost contact manifolds. On the other hand, there seem to be not so many examples of non-regular almost contact structures on compact manifolds. It is our purpose in this paper to exhibit, in a unified fashion, some examples of non-regular almost contact structures on compact manifolds which are given as intersections of spheres and complex algebraic sets. Furthermore, these almost contact structures have closed curves as the leaves of the associated foliations. Many of these manifolds are exotic spheres. Incidentally, it will be shown that all the odd-dimensional standard spheres have non-regular almost contact structures whose associated leaves are closed curves. Finally, the author thanks S. K. Kim for the useful conversations with him.

2. Brieskorn and weighted homogeneous manifolds. Let $(a_0, \cdots, a_n)$ be an $(n+1)$-tuple of positive integers, and let $f(Z_0, \cdots, Z_n) = Z_0^{a_0} +$

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$\cdots + Z_n^{a_n}$ be a polynomial of complex $(n+1)$-variables $Z_0, \cdots, Z_n$. Then f can be regarded as a holomorphic mapping from $\boldsymbol{C}^{n+1}$ into $\boldsymbol{C}$. Denote by $V = V(a_0, \cdots, a_n)$ the locus of the zeros of f in $\boldsymbol{C}^{n+1}$, i.e., $V = \{(Z_0, \cdots, Z_n) \in \boldsymbol{C}^{n+1}: f(Z_0, \cdots, Z_n) = 0\}$, and denote by $\Sigma = \Sigma(a_0, \cdots, a_n)$ the intersection of V and the unit sphere S^{2n+1} in $\boldsymbol{C}^{n+1}$, i.e., $\Sigma = V \cap S^{2n+1}$, where $S^{2n+1} = \{(Z_0, \cdots, Z_n) \in \boldsymbol{C}^{n+1}: |Z_0|^2 + \cdots + |Z_n|^2 = 1\}$. Then it is easy to see that V has the origin of $\boldsymbol{C}^{n+1}$ as its only possible singular point. Thus, $X = X(a_0, \cdots, a_n) = V - \{0\}$ is a complex submanifold of $\boldsymbol{C}^{n+1}$; and therefore, it is a Kählerian submanifold of $\boldsymbol{C}^{n+1}$ with the induced Hermitian metric. It is well known that $\Sigma = V \cap S^{2n+1} = X \cap S^{2n+1}$ is a compact, smooth and $(n-2)$-connected manifold of dimension $2n-1$. For the details, see Milnor [6]. We call V and Σ the Brieskorn variety and the Brieskorn manifold, respectively. Next consider $\boldsymbol{C}$ as the natural additive Lie group. We introduce a $\boldsymbol{C}$-action on $\boldsymbol{C}^{n+1}$ by

$$t(Z_0, \cdots, Z_n) = (Z_0 \exp(td/a_0), \cdots, Z_n \exp(td/a_n)) ,$$

where d is the least common multiple of $a_0, \cdots, a_n$. Since

$$\begin{aligned}(Z_0 \exp(td/a_0))^{a_0} + \cdots + (Z_n \exp(td/a_n))^{a_n} \\ = (\exp td)(Z_0^{a_0} + \cdots + Z_n^{a_n}) = 0\end{aligned}$$

for all $t \in \boldsymbol{C}$, the $\boldsymbol{C}$-action leaves X invariant. Therefore, $\boldsymbol{C}$ acts on X. Now if we take t to be purely imaginary numbers, then $\exp(td/a_i)$'s are elements of the unit circle. Hence, the above $\boldsymbol{C}$-action restricted to the imaginary numbers induces a natural S^1-action on Σ. This is clear because this S^1-action leaves the unit sphere S^{2n+1} invariant. Next let t be real numbers. Then the action of t induces a diffeomorphism $F: \Sigma \times \boldsymbol{R}$ onto X defined by

$$F((Z_0, \cdots, Z_n), t) = (Z_0 \exp(td/a_0), \cdots, Z_n \exp(td/a_n)) ,$$

where $\sum_{i=0}^{n} |Z_i|^2 = 1$. As for the weighted case, let $(w_0, \cdots, w_n)$ be an $(n+1)$-tuple of positive rational numbers. Then a polynomial $f(Z_0, \cdots, Z_n)$ is said to be weighted homogeneous with weights $(w_0, \cdots, w_n)$ if f is a linear combination of monomials $Z_0^{i_0} Z_1^{i_1} \cdots Z_n^{i_n}$ for which $i_0/w_0 + \cdots + i_n/w_n = 1$. For more details and some examples, see Milnor [6]. As before, we define $V = V(w_0, \cdots, w_n)$ to be the locus of the zeros of $f(Z_0, \cdots, Z_n)$. If V has its only possible singular point, $X = X(w_0, \cdots, w_n) = V - \{0\}$ is a Kählerian submanifold of C^{n+1}. We denote by $\Sigma = \Sigma(w_0, \cdots, w_n)$ the intersection of X and S^{2n+1}, which is an $(n-2)$-connected smooth manifold of dimension $2n-1$. We call V and Σ the weighted homogeneous variety and the weighted homogeneous manifold of weight $(w_0, \cdots, w_n)$, respectively. Set $w_j = u_j/v_j$, $j = 0, \cdots, n$, where u_j and v_j are mutually prime positive integers. Let d be the least common multiple of $(u_0, \cdots, u_n)$. As in the Brieskorn manifold case, $\boldsymbol{C}$ acts on X by

$$t(Z_0, \cdots, Z_n) = (Z_0 \exp(td/w_0), \cdots, Z_n \exp(td/w_n)), \qquad \text{for all } t \in \boldsymbol{C} .$$

It also gives a S^1-action on Σ induced by its restriction to purely imaginary numbers.

3. Almost contact structures on Σ. From now on, we shall treat the Brieskorn manifolds alone, since the same argument works in the weighted homogeneous case. Let $(a_0, \cdots, a_n)$ be the given $(n+1)$ positive integers, and let $f(Z) = Z_0^{a_0} + \cdots + Z_n^{a_0}$ be the polynomial. It is well known that each element of the Lie algebra of a Lie transformation group generates a vector field in a natural way on the manifold on which it acts. In particular, 1 and $\sqrt{-1}$ considered to be elements of Lie algebra of $\boldsymbol{C}$ generate vector fields $\mathfrak{A}$ and $\mathfrak{B}$ on X given as follows:

$$\mathfrak{A} = ((d/a_0)Z_0, \cdots, (d/a_n)Z_n) \qquad \text{at } (Z_0, \cdots, Z_n) \in X$$

$$\mathfrak{B} = (\sqrt{-1}(d/a_0)Z_0, \cdots, \sqrt{-1}(d/a_n)Z_n) \qquad \text{at } (Z_0, \cdots, Z_n) \in X .$$

It is clear that these two nowhere vanishing vector fields $\mathfrak{A}$ and $\mathfrak{B}$ are tangent to the orbit of $(Z_0, \cdots, Z_n)$ of $\boldsymbol{C}$-action in X. Since $\mathfrak{B} = \sqrt{-1}$ or $=5$ and since the complex structure J on X is induced from that of $\boldsymbol{C}^{n+1}$, we see that the tangent space to the orbit of $(Z_0, \cdots, Z_n)$ is J-invariant. In fact, each $\boldsymbol{C}$-orbit in X is a complex curve. It is easy to show that $\mathfrak{A}$ and $\mathfrak{B}$ are orthogonal to each other with respect to the induced metric from that of $\boldsymbol{C}^{n+1}$. Let TX be the tangent bundle of X. Then $TX = A \oplus B \oplus (A \oplus B)^{\perp}$, where A, B and $(A \oplus B)^{\perp}$ are the line bundle over X generated by $\mathfrak{A}$, the line bundle over X generated by $\mathfrak{B}$ and the orthogonal complement of $A \oplus B$ in TX, respectively, and $\oplus$ denotes an orthogonal direct sum. Since X is a Kählerian manifold and since $A \oplus B$ is a J-invariant subbundle of TX, $(A \oplus B)^{\perp}$ is also a J-invariant subbundle of TX. Next, let Σ have the Riemannian metric induced from that of $\boldsymbol{C}^{n+1}$, which is the same metric as the induced metric from that of S^{2n+1} or X; and let $\boldsymbol{R}$ have the natural metric. We consider that $\Sigma \times \boldsymbol{R}$ has the product Riemannian structure. Thus, the tangent bundle $T(\Sigma \times \boldsymbol{R})$ has the orthogonal decomposition $T(\Sigma \times \boldsymbol{R}) = (T\Sigma)' \oplus (T\boldsymbol{R})'$ with respect to the product metric, where $(T\Sigma)' = (T\Sigma) \times \boldsymbol{R}$ and $(T\boldsymbol{R})' = \Sigma \times (T\boldsymbol{R})$. Since $F: \Sigma \times \boldsymbol{R} \to X$ is a diffeomorphism, we have a C^{∞}-vector bundle isomorphism $F_*: T(\Sigma \times \boldsymbol{R}) \to TX$ such that the following diagram commutes:

$$\begin{array}{ccc} T(\Sigma \times \boldsymbol{R}) & \xrightarrow{F_*} & TX \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \Sigma \times \boldsymbol{R} & \xrightarrow{F} & X \end{array}$$

Here π_1 and π_2 are the bundle projections. Denote by F_*^{-1} the inverse isomorphism which covers $F^{-1}\colon X \to \Sigma \times \boldsymbol{R}$. By the definition of F, F_*^{-1} maps the A-factor of TX onto the $(T\boldsymbol{R})'$ factor of $T(\Sigma \times \boldsymbol{R})$ and the B-factor of TX into $(T\Sigma)'$ factor of $T(\Sigma \times \boldsymbol{R})$, respectively. Now if we denote by B' the line bundle over $\Sigma \times \boldsymbol{R}$ generated by $F_*^{-1}(\mathfrak{B})$, we have the following orthogonal decomposition of $T(\Sigma \times \boldsymbol{R})$ with respect to the product metric:

$$T(\Sigma \times \boldsymbol{R}) = (T\boldsymbol{R})' \oplus B' \oplus ((T\boldsymbol{R})' \oplus B')^{\perp} .$$

Note here that $((T\boldsymbol{R})' \oplus B')^{\perp}$ is actually the orthogonal complement of B' in $(T\Sigma)'$. We next make a C^∞-bundle isomorphism from $(A \oplus B)^{\perp}$ onto $((T\boldsymbol{R})' \oplus B')^{\perp}$ by making use of F_*^{-1}. Let x be a point of X, and let v be a vector of $(A \oplus B)^{\perp}_x$, i.e., the fibre of $(A \oplus B)^{\perp}$ at x. Then $F_{*x}^{-1}(v)$ has the unique decomposition $\alpha \oplus \beta \oplus \gamma$, where $\alpha \in (T\boldsymbol{R})'_{F^{-1}(x)}$, $\beta \in B'_{F^{-1}(x)}$ and $\gamma \in ((T\boldsymbol{R})' \oplus B')^{\perp}_{F^{-1}(x)}$. Define a mapping g_x by

$$g_x\colon (A \oplus B)^{\perp}_x \to ((T\boldsymbol{R})' \oplus B')^{\perp}_{F^{-1}(x)} \text{ by } g_x(v) = \gamma .$$

It is obvious that g_x is linear. Next we show that g_x is 1-1; hence onto, because $\dim (A \oplus B)^{\perp}_x = \dim ((T\boldsymbol{R})' \oplus B')^{\perp}_{F^{-1}(x)} = 2(n-1)$. Now let $g_x(v) = 0$. Then $F_{*x}^{-1}(v) = \alpha \oplus \beta \oplus 0$. Therefore, $v = F_{*F^{-1}(x)} \circ F_{*x}^{-1}(v) = F_{*F^{-1}(x)}(\alpha \oplus \beta) = F_{*F^{-1}(x)}(\alpha) \oplus F_{*F^{-1}(x)}(\beta)$. Since the definition of F tells us that F_* maps $(T\boldsymbol{R})'$ and B' onto A and B, respectively, v belongs to $(A \oplus B)^{\perp} \cap (A \oplus B) = \{0\}$, i.e., $v = 0$. This shows that g_x is 1-1; hence, an isomorphism. It is easy to prove that the vector bundle map $g\colon (A \oplus B)^{\perp} \to ((T\boldsymbol{R})' \oplus B')^{\perp}$ which is pointwisely given by g_x as above is a C^∞-isomorphism which covers F^{-1}. Now we have come to the definition of the triple (ϕ, ξ, η). First of all, we denote by P the natural projection from $B' \oplus ((T\boldsymbol{R})' \oplus B')^{\perp} = (T\Sigma)'$ onto the second factor $((T\boldsymbol{R})' \oplus B')^{\perp}$. Then define a bundle map ϕ' from $(T\Sigma)'$ into itself by $\phi' = g \circ J \circ g^{-1} \circ P$, where J is the complex structure restricted to $(A \oplus B)^{\perp}$. Define $\xi' = F_*^{-1}(\mathfrak{B})$. Finally, define η' to be the C^∞-section of $\mathrm{Hom}((T\Sigma)', \boldsymbol{R})$ given by $\eta'(\xi') = 1$ and $\eta'(T(T\boldsymbol{R}' \oplus B')^{\perp}) = 0$. It is easy to show that $(\phi')^2(X) = -X + \eta'(X)\xi'$ for any C^∞-section X of the bundle $(T\Sigma)'$. As before, we identify $(T\Sigma)'$ with $(T\Sigma) \times \boldsymbol{R}$. Then, for any $t \in \boldsymbol{R}$, (ϕ', ξ', η') restricted to $((T\Sigma), t)$ gives a triple (ϕ_t, ξ_t, η_t) on Σ which satisfies the definition of the almost contact structure. We next show that most of these structures are non-regular with closed curves as their leaves. As is mentioned before, the C-action on X restricted to the imaginary numbers induces the circle action on X given by

$$t(Z_0, \cdots, Z_n) = (Z_0 \exp(\sqrt{-1}td/a_0), \cdots, Z_n \exp(\sqrt{-1}td/a_n)) ,$$

where $t \in S^1 = [0, 2\pi)$. Since the leaves of the associated foliation of (ϕ_t, ξ_t, η_t) are precisely the F^{-1}-image of the orbits of the above S^1-action on X, ξ_t's are the same vector fields for all $t \in \boldsymbol{R}$. Thus, it is sufficient to show that $\xi_0 = \xi$ is a non-regular vector field. Since we defined ξ' to be $F_*^{-1}(\mathfrak{B})$ and since F is the identity mapping at $t = 0$ (or on Σ), the leaves of the associated foliation of (ϕ_0, ξ_0, η_0) are precisely the S^1-orbits on Σ. This S^1-action gives a non-regular foliation if the slice diagram of the S^1-action contains at least two different slice types. Thus, for most of $(n + 1)$-tuples of positive integers, $\Sigma(a_0, \cdots, a_n) = \Sigma$ has a one-parameter family of non-regular almost contact structures (ϕ_t, ξ_t, η_t). For example, if $a_0 = 3$, $a_1 = 2$, $a_2 = 2$ and $a_3 = 2$, the orbit of a point $(0, Z_1, Z_2, Z_3)$ such that $Z_1^2 + Z_2^2 + Z_3^2 = 0$ and $|Z_1|^2 + |Z_2|^2 + |Z_3|^2 = 1$ is a closed curve, and it is an exceptional orbit with the isotropy group $\boldsymbol{Z}_3$. Thus, any cubical Frobenius neighborhood meets some of principal orbits precisely in three distinct slices of the neighborhood.

4. Concluding remarks.

REMARK 1. Brieskorn [3] showed that for $n \neq 2$ and every $(2n - 1)$-dimensional homotopy sphere bounding a parallelizable manifold, there are infinitely many $(n + 1)$-tuples of positive integers $(a_0, \cdots a_n)$ such that $\Sigma(a_0, \cdots, a_n)$ is diffeomorphic to the homotopy sphere. Thus such homotopy spheres admit non-regular almost contact structures. In particular, every 7-dimensional homotopy spheres admits a non-regular almost contact structure. Now let $\Sigma(3, 2, \cdots, 2) = V(3, 2, \cdots, 2) \cap S^{2n+1}$ be a Brieskorn manifold. It is known that $\Sigma(3, 2, \cdots, 2)$ is homeomorphic to S^{2n-1} for every odd value of n. If $n = 5$, $\Sigma(3, 2, 2, 2, 2, 2)$ is an exotic 9-sphere, and it admits a non-regular almost contact structure. If $n = 3$, $\Sigma(3, 2, 2, 2)$ is diffeomorphic to the standard 5-sphere, since there is no exotic sphere of dimension 5. Therefore, our almost contact structure on $\Sigma(3, 2, 2, 2)$ is a non-regular almost contact structure on S^5. If we take a Brieskorn polynomial of the form $f(Z_0, \cdots, Z_n) = Z_0^{a_0} + \cdots + Z_n^{a_n}$ such that one of a_i's is equal to 1, then $\Sigma(a_0, \cdots, a_n)$ is diffeomorphic to the standard $(2n - 1)$-sphere [6]. Thus, our almost contact structure on $\Sigma(a_0, \cdots, a_n)$ is a non-regular structure on S^{2n-1}.

REMARK 2. $\Sigma(1, \cdots, 1)$ is the total space of the Hopf fibration over CP^{n-1}; therefore, the standard S^{2n-1}. $\Sigma(2, \cdots, 2)$ is a circle bundle over the complex quadric CQ^{n-1}. In general, $\Sigma(a, \cdots, a)$ gives the total space of a circle bundle over a complex submanifold of CP^n, i.e., $(n-1)$-drics. They, of course, admit a regular almost contact structure. It would be interesting to ask whether or not an exotic sphere can admit a regular almost contact structure. To this end, it might be of some help to observe that some exotic 7-spheres are circle bundle over 6-dimensional manifolds which have the same homotopy type as CP^3. Also notice that some exotic 7-spheres can be considered as the total spaces of fiber bundles over S^4 with S^3 as its typical fiber.

REMARK 3. If we consider a finite number of generalized Brieskorn polynomials or weighted homogeneous polynomials under certain conditions, we can obtain more examples of compact manifold admitting almost contact structures whose associated foliations have closed curves as their leaves. Some of them are regular and some are non-regular. In fact, in the forthcoming paper [1], we study these cases. We show that these manifolds admit many 1-parameter families of (almost) contact structures. In particular, one of them is normal; therefore, we are able to introduce complex structures on products of these manifolds. These complex structures are closely related to those of Calabi-Eckmann [5] and Brieskorn-Van de Ven [4].

REMARK 4. It is well known that any orientable submanifold of codimension 1 of a Kählerian manifold admits an almost contact structure which is induced from the complex structure of the Kählerian manifold in a natural way. Since our Σ is a codimension one submanifold of X, Σ certainly admits an almost contact structure. Neither is known to the author under what conditions our examples and these natural structures coincide, nor whether the latter is regular (or non-regular).

REMARK 5. Having known that these manifolds admit almost contact structures, it would be natural to ask whether or not they admit contact structures. Recently, Erbacher and the author [1] and [2] have shown that a family of C^∞-manifolds which are given as intersections of spheres and complex algebraic sets admit contact structures (also see [8]). In particular, all the generalized Brieskorn manifolds belong to this family.

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THE UNIVERSITY OF CONNECTICUT
STORRS, CONNECTICUT 06268