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A NOTE ON THE VON NEUMANN ALGEBRA WITH A CYCLIC AND SEPARATING VECTOR

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Let M be a von Neumann algebra on a Hilbert space H with a cyclic and separating vector. If, for some cyclic and separating vector ξ_0 in H for M,

$$M\xi_0 = M'\xi_0 , \tag{$*$}$$

then we shall call that M has the property (J).

Note that M satisfying the equality $(*)$ for ξ_0 does not always imply for the other cyclic and separating vector.

In [Theorem 1] we show that M has the property (J) if and only if M is a finite von Neumann algebra. In terms of the Hilbert algebra, we can consider $M\xi_0$ as an achieved left Hilbert algebra $\mathfrak{A}$ with the product: $(x\xi_0)(y\xi_0) = xy\xi_0$, and the involution: $S(x\xi_0) = x^*\xi_0$, $x, y \in M$, and $M'\xi_0$ as the right Hilbert algebra $\mathfrak{A}'$ of $\mathfrak{A}$. (see [2], [3]) The analysis in this paper may be the special case of the characterization of the type of the left von Neumann algebra $\mathfrak{L}(\mathfrak{A})$ associated to the achieved left Hilbert algebra $\mathfrak{A}$ under the condition $\mathfrak{A} = \mathfrak{A}'$ as a set. Without difficulty, we can prove that an achieved left Hilbert algebra $\mathfrak{A}$ is equal to $\mathfrak{A}'$ as a set if and only if $\mathfrak{A}$ is a Tomita algebra.

In [Theorem 2] we shall give a characterization of a finite von Neumann algebra via the Radon-Nikodym theorem for the state. We mainly refer [1] and [2].

Now, we state here that if M is finite, then M has the property (J).

In fact, let ξ_0 be a cyclic and separating trace vector in H for M. Then we have

$$||S(x\xi_0)||^2 = ||x^*\xi_0||^2 = (xx^*\xi_0|\xi_0) = (x^*x\xi_0|\xi_0) = ||x\xi_0||^2 ,$$

for all x in M. Therefore $M\xi_0$ is a uni-modular Hilbert algebra. From [2] Cor. 10.1, we have $M\xi_0 = M'\xi_0$.

Now we need the following lemma to prove [Theorem 1].

LEMMA (cf. [1] Chap. I §1 ex. 5). *Suppose that M is a von Neumann algebra on a Hilbert space H such that $M\xi_0 = M'\xi_0$ for a cyclic and separating vector ξ_0 in H, that is, for any element x in M, there exists* *a unique element x' in M' such that $x\xi_0 = x'\xi_0$. Then the mapping $\Phi: x \mapsto x'$ is a norm bi-continuous anti-isomorphism of M onto M'.*

PROOF. It is clear that Φ is anti-isomorphic. Let $\{x_n\}$ be a sequence in M such that $x_n \to x$ and $\Phi(x_n) \to y'$, $x \in M$, $y' \in M'$. Then $x_n\xi_0 \to x\xi_0$ and $\Phi(x_n)\xi_0 \to y'\xi_0$. Thus we have $x\xi_0 = y'\xi_0$, i.e., $\Phi(x) = y'$. Therefore Φ is norm continuous by the closed graph theorem. We see the continuity of Φ^{-1} from the symmetrical argument. q.e.d.

THEOREM 1. *Let M be a von Neumann algebra on a Hilbert space H with a cyclic and separating vector. Then M is finite if and only if M has the property (J).*

PROOF. We must prove that if M is not finite, then M does not have the property (J). As any von Neumann algebra is uniquely decomposed into direct sum of a finite and a properly infinite algebra, we may assume that M is properly infinite. Then M is spatially isomorphic to $M \otimes \mathscr{B}(K)$ where $\mathscr{B}(K)$ is the algebra of all bounded operators on an infinite dimensional separable Hilbert space K. If M has a cyclic and separating vector, then $M \otimes \mathscr{B}(K)$ has also a cyclic and separating vector. We see that M has not the property (J) if and only if $M \otimes \mathscr{B}(K)$ has not the property (J).

In fact, we assume that $M\xi \neq M'\xi$ for any cyclic and separating vector ξ in H for M. For any cyclic and separating vector η in $H \otimes K$ for $M \otimes \mathscr{B}(K)$, there exists a cyclic and separating vector ξ in H for M such that $\eta = U\xi$ where U is an isometry of H onto $H \otimes K$ with $UMU^{-1} = M \otimes \mathscr{B}(K)$. Then,

$$\begin{aligned}(M \otimes \mathscr{B}(K))\eta &= (UMU^{-1})U\xi = UM\xi \neq UM'\xi \\ &= UM'U^{-1}U\xi = (UMU^{-1})'U\xi = (M \otimes \mathscr{B}(K))'\eta\,.\end{aligned}$$

Now, we will prove that $M \otimes \mathscr{B}(K)$ has not the property (J). Suppose that

$$(M \otimes \mathscr{B}(K))\eta = (M \otimes \mathscr{B}(K))'\eta\,,$$

for some cyclic and separating vector η in $H \otimes K$. Let η be a form $\sum_{i=1}^{\infty} \xi_i \otimes \varepsilon_i$ where $\xi_i \in H$, and $\{\varepsilon_i\}$ is a completely orthonormal system in K. Let v_j, $j = 1, 2, \cdots$, be partial isometries in $\mathscr{B}(K)$ such that $v_j\varepsilon_1 = \varepsilon_j$, $v_j\varepsilon_i = 0$ $(i = 2, 3, \cdots)$. Then there exists an element y_j in M' for each j such that

$$(1 \otimes v_j)\eta = (y_j \otimes 1)\eta$$

because of $(M \otimes \mathscr{B}(K))' = M' \otimes I_K$. We have, for each j,

$$(1 \otimes v_j)\eta = (1 \otimes v_j)(\sum_{i=1}^{\infty} \xi_i \otimes \varepsilon_i) = \xi_1 \otimes \varepsilon_j ,$$

and,

$$(y_j \otimes 1)\eta = (y_j \otimes 1)(\sum_{i=1}^{\infty} \xi_i \otimes \varepsilon_i) = \sum_{i=1}^{\infty} y_j \xi_i \otimes \varepsilon_i .$$

Hence we have

$$\xi_1 \otimes \varepsilon_j = (y_j \otimes 1)(\xi_j \otimes \varepsilon_j) ,$$

and,

$$||\xi_1|| = ||\xi_1 \otimes \varepsilon_j|| \leq ||y_j \otimes 1|| \, ||\xi_j \otimes \varepsilon_j|| = ||y_j \otimes 1|| \, ||\xi_j|| ,$$

for each $j = 1, 2, \cdots$. Then we have $\xi_1 = 0$, because the sequence $\{\xi_j\}$ is convergent to 0 and $\{y_j \otimes 1\}$ is bounded from the lemma.

Applying this argument to the other elements ξ_i, $i = 2, 3, \cdots$, we obtain $\xi_i = 0$ for each i. Thus we have $\eta = 0$. Therefore,

$$(M \otimes \mathscr{B}(K))\eta \neq (M \otimes \mathscr{B}(K))'\eta ,$$

for any cyclic and separating vector η in $H \otimes K$. This completes the proof. q.e.d.

Next, we state the following theorem.

THEOREM 2. *Let M be a von Neumann algebra on a Hilbert space H with a cyclic and separating vector. Then the following statements are equivalent;*

i) *M is finite.*

ii) *We can find a cyclic and separating vector ξ_0 in H satisfying the following condition:*

For any element a in M, there exists a positive number γ such that

$$a\omega_{\xi_0}a^* \leq \gamma\omega_{\xi_0} ,$$

where ω_{ξ_0} is a vector state on M for ξ_0.

PROOF. We immediately see that i) implies ii). In fact, if M is finite, then there exists a cyclic and separating vector ξ_0 such that $M\xi_0 = M'\xi_0$ from [Theorem 1]. Then, if $a\xi_0 = a'\xi_0$, $a \in M$, $a' \in M'$, then we have

$$(a^*x^*xa\xi_0|\xi_0) = ||xa\xi_0||^2 = ||xa'\xi_0||^2 = ||a'x\xi_0||^2$$
$$\leq ||a'||^2 ||x\xi_0||^2 = ||a'||^2(x^*x\xi_0|\xi_0) ,$$

for all x in M.

Conversely, suppose that we choose the element ξ_0 satisfying the condition in ii). Then, for each element a in M, there exists a positive element h' in M' such that

$$\omega_{h'\xi_0} = \omega_{a\xi_0} .$$

Then we have $||xh'\xi_0|| = ||xa\xi_0||$ for all x in M. Put $u_0(xh'\xi_0) = xa\xi_0$, $x \in M$, then u_0 can be extended to a partial isometry u' in M'. Therefore, $a\xi_0 = u'h'\xi_0$, that is, an element $a\xi_0$ falls in $M'\xi_0$, i.e., $M\xi_0 \subset M'\xi_0$. Then we have

$$M'\xi_0 = JM\xi_0 \subset JM'\xi_0 = M\xi_0 ,$$

where J is a modular conjugation operator of an achieved left Hilbert algebra $M\xi_0$. (see [2] Cor. 10.1) Hence we have $M\xi_0 = M'\xi_0$, and then M is finite from [Theorem 1]. q.e.d.

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