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ALMOST SURE INVARIANCE PRINCIPLES FOR LACUNARY TRIGONOMETRIC SERIES

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1. Introduction. In this note let $\{n_m\}$ be a sequence of positive integers satisfying the gap condition

$$(1.1) \qquad n_{m+1}/n_m > 1 + cm^{-\alpha} \qquad (c > 0 \text{ and } 0 \leqq \alpha \leqq 1/2) ,$$

and $\{a_m\}$ be a sequence of positive numbers such that

$$(1.2) \qquad \begin{cases} A_k = \left(2^{-1} \sum_{m=1}^{k} a_m^2\right)^{1/2} \to +\infty , \\ a_k = O(A_k k^{-\alpha}(\log A_k)^{-\beta}) , \qquad \beta > 1/2 , \qquad \text{as} \quad k \to +\infty . \end{cases}$$

Further, we put

$$(1.3) \qquad \xi_m(\omega) = a_m \cos 2\pi(n_m\omega + \alpha_m) \quad \text{and} \quad T_k = \sum_{m=1}^{k} \xi_m ,$$

where $\{\alpha_m\}$ is a sequence of arbitrary real numbers, and consider ξ_m's as random variables on a probability space $([0, 1), \mathscr{F}, P)$ where $\mathscr{F}$ is the σ-field of all Borel sets on $[0, 1)$ and P is the Lebesgue measure on $\mathscr{F}$. Then we write, for $\omega \in [0, 1)$ and $t \geqq 0$,

$$(1.4) \qquad S(t) = S(t, \omega) = T_k(\omega) , \qquad \text{if} \quad A_k^2 \leqq t < A_{k+1}^2 ,$$

for $k \geqq 0$, where we put $A_0 = 0$ and $T_0 = 0$.

The purpose of the present paper is to prove the following.

THEOREM. *Without changing the distribution of $\{S(t), t \geqq 0\}$ we can redefine the process $\{S(t), t \geqq 0\}$ on a richer probability space together with standard Brownian motion $\{X(t), t \geqq 0\}$ such that*

$$S(t) = X(t) + o(t^{1/2}) \quad a.s. \qquad as \ t \to +\infty .$$

Using the almost sure limiting behavior of $\{X(t), t \geqq 0\}$ and the above theorem we can deduce the corresponding limiting properties of $\{S(t), t \geqq 0\}$ or $\{T_k(\omega)\}$. For example we can obtain the following

COROLLARY (cf. [3]). *Under the conditions* (1.1) *and* (1.2) *we have, for a.e. ω,*

$$(1.5)\qquad \limsup_{k\to+\infty}(2A_k^2\log\log A_k)^{-1/2}\sum_{m=1}^{k}a_m\cos 2\pi(n_m\omega+\alpha_m)=1\,.$$

For $\alpha=0$, that is, when the sequence $\{n_m\}$ satisfies the Hadamard gap condition, Weiss [4] proved that if $a_k=o(A_k(\log\log A_k)^{-1/2})$ as $k\to+\infty$, then (1.5) holds.

Recently, Philipp and Stout [1] have proved that if $\alpha=0$, $a_k=O(A_k^{1-\delta})$ for some $\delta>0$, and $\{n_k\}$ is a sequence of real numbers, then for any $\lambda<\delta/32$

$$S(t)=X(t)+O(t^{1/2-\lambda})\quad \text{a.s.}\qquad \text{as}\quad t\to+\infty\,.$$

For the proof of our theorem we approximate $\{T_k(\omega)\}$ by a martingale and then apply a martingale version of the Skorohod representation theorem due to Strassen ([2] Theorem 4.3 and also cf. [1]).

THEOREM OF STRASSEN. *Let $\{Y_k,\mathfrak{F}_k\}$ be a martingale difference sequence. Then without changing the distribution of $\{Y_k\}$ we can redefine the sequence $\{Y_k\}$ on a richer probability space together with a sequence $\{T_k\}$ of non-negative random variables and standard Brownian motion $\{X(t),t\geqq 0\}$ such that*

$$\sum_{m=1}^{k}Y_m=X\Big(\sum_{m=1}^{k}T_m\Big)\quad a.s.$$

Moreover, if $\mathfrak{G}_k$ is the σ-field generated by $\{X(t),0\leqq t\leqq\sum_{m=1}^{k}T_m\}$, then T_k is $\mathfrak{G}_k$-measurable and for some constant C

$$E(T_k|\mathfrak{G}_{k-1})=E(Y_k^2|\mathfrak{G}_{k-1})=E(Y_k^2|\mathfrak{H}_{k-1})\,,$$
$$E(T_k^2|\mathfrak{G}_{k-1})\leqq CE(Y_k^4|\mathfrak{H}_{k-1})\quad a.s.\,,$$

where $\mathfrak{H}_k$ is the σ-field generated by $\{Y_m,1\leqq m\leqq k\}$.

2. Preliminaries. I. Let us put, for each k,

$$(2.1)\qquad \begin{cases}p(0)=0\,,\quad p(k)=\max\{m;\,n_m<2^k\}\,,\\ \Delta_k=\displaystyle\sum_{m=p(k)+1}^{p(k+1)}\xi_m\quad\text{and}\quad B_k=A_{p(k+1)}\,.\end{cases}$$

Then if $p(k)+1<p(k+1)$, we have, by (1.1),

$$\begin{aligned}2>n_{p(k+1)}/n_{p(k)+1}&>\prod_{m=p(k)+1}^{p(k+1)-1}(1+cm^{-\alpha})\\&>1+c\{p(k+1)-p(k)-1\}p^{-\alpha}(k+1)\,.\end{aligned}$$

Hence we have

$$(2.2)\qquad p(k+1)-p(k)=O(p^{\alpha}(k))\,,\qquad \text{as}\quad k\to+\infty\,,$$

and if $m_k=o(p^{1-\alpha}(k))$ as $k\to+\infty$, then

$$p(k+m_k)/p(k) \to 1\ , \qquad \text{as} \quad k \to +\infty\ . \tag{2.3}$$

Further, we obtain from (1.2) and (2.2)

$$\begin{cases} b_k = \max\limits_{p(k)<m\leqq p(k+1)} a_m = O(B_k p^{-\alpha}(k)(\log B_k)^{-\beta})\ , \\ \sum\limits_{m=p(k)+1}^{p(k+1)} a_m \leqq b_k\{p(k+1)-p(k)\} = O(B_k(\log B_k)^{-\beta})\ , \\ E\Delta_k^2 \leqq b_k^2\{p(k+1)-p(k)\} = O(B_k^2 p^{-\alpha}(k)(\log B_k)^{-2\beta})\ , \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{as} \quad k \to +\infty\ . \end{cases} \tag{2.4}$$

On the other hand, by (1.2) we have

$$\sum_{m=1}^{k} (\log A_m)^{2\beta} a_m^2/A_m^2 = O(k)\ , \qquad \text{as} \quad k \to +\infty\ .$$

Therefore, we have

$$\log\log B_k = O(\log p(k))\ , \qquad \text{as} \quad k \to +\infty\ . \tag{2.5}$$

II. LEMMA 1. *For any given integers k, j, q and h such that $p(j)+1 < h \leqq p(j+1) < p(k)+1 < q \leqq p(k+1)$, the number of solutions (n_r, n_i) of the equations*

$$n_q - n_r = n_h \pm n_i\ ,$$

where $p(j) < i < h$ and $p(k) < r < q$, is at most $C2^{j-k}p^{\alpha}(k)$ for some constant C which does not depend on k, j, q and h.

PROOF. If $k < j+5$, the lemma is evident by (2.2). We assume that $k \geqq j+5$. Let m denote the smallest index r of the solutions (n_r, n_i). Then the number of solutions is at most $q-m$. Since $(n_h \pm n_i) \leqq 2^{j+2}$ we have

$$n_m \geqq n_q - 2^{j+2} > n_q(1-2^{j+2-k}) \geqq n_q(1+2^{j-k}\cdot 5)^{-1}\ .$$

By (1.1) we have

$$1 + 2^{j-k}\cdot 5 > n_q/n_m > \prod_{s=m}^{q-1}(1+cs^{-\alpha}) > 1 + c(q-m)p^{-\alpha}(k+1)\ .$$

Therefore, by (2.3) we can prove the lemma.

LEMMA 2. *For any M and N $(M<N)$ we have*

$$E\left(\left|\sum_{m=M}^{N}\{\Delta_m^2 - E\Delta_m^2\}\right|^2\right) \leqq CB_N^2 \sum_{m=M}^{N} E\Delta_m^2(\log B_N)^{-2\beta}\ ,$$

where C is a positive constant which does not depend on M and N.

PROOF. For $k = 1, 2, \cdots$ let us put

$$U_k = \Delta_k^2 - E\Delta_k^2 - 2^{-1}\sum_{m=p(k)+1}^{p(k+1)} a_m^2 \cos 4\pi(n_m\omega + \alpha_m) .$$

Then by (1.2) and (2.4) we have

$$\left\{E\left|\sum_{m=M}^{N}(\Delta_m^2 - E\Delta_m^2)\right|^2\right\}^{1/2} \leqq \left\{E\left(\sum_{m=M}^{N} U_m\right)^2\right\}^{1/2} + 2^{-1}\left(\sum_{m=M}^{N}\sum_{j=p(m)+1}^{p(m+1)} a_j^4\right)^{1/2}$$
$$= \left|2\sum_{k=M+1}^{N}\sum_{j=M}^{k-1} EU_kU_j\right|^{1/2} + O\left(\left\{\sum_{m=M}^{N} E\Delta_m^2 B_N^2(\log B_N)^{-2\beta}\right\}^{1/2}\right), \quad \text{as } N \to +\infty .$$

Further, by Lemma 1 and (2.4) we have for $k > j$

$$|EU_kU_j| \leqq C2^{j-k}p^\alpha(k)\sum_{q=p(k)+1}^{p(k+1)} a_q b_k \sum_{h=p(j)+1}^{p(j+1)} a_h b_j$$
$$= O(2^{j-k}\{E\Delta_k^2 E\Delta_j^2 p^\alpha(k)p^{-\alpha}(j)\}^{1/2}B_N^2(\log B_N)^{-2\beta}) , \qquad \text{as} \quad N \to +\infty .$$

Since $p(j+1)/p(j) \to 1$ as $j \to +\infty$, we have for all k

$$\sum_{j=1}^{k-1} p^{-\alpha}(j)2^{j-k} \leqq Cp^{-\alpha}(k) , \qquad \text{for some} \quad C > 0 . \tag{2.6}$$

Therefore, we have

$$\sum_{k=M+1}^{N}\sum_{j=M}^{k-1} 2^{j-k}\{E\Delta_k^2 E\Delta_j^2 p^\alpha(k)p^{-\alpha}(j)\}^{1/2} \leqq C\left\{\sum_{k=M+1}^{N} E\Delta_k^2\right\}^{1/2}\left\{\sum_{k=M+1}^{N}\sum_{j=M}^{k-1} E\Delta_j^2 2^{j-k}\right\}^{1/2}$$
$$\leqq C\left\{\sum_{k=M+1}^{N} E\Delta_k^2\right\}^{1/2}\left\{\sum_{j=M}^{N-1} E\Delta_j^2 \sum_{k=j+1}^{N} 2^{j-k}\right\}^{1/2} \leqq C\sum_{k=M}^{N} E\Delta_k^2 .$$

Also we need the following

LEMMA 3. *For any M and N $(M < N)$ we have*

$$E\left(\max_{M \leqq r \leqq N}\left|\sum_{k=M}^{r} \Delta_k\right|^4\right) \leqq C\sum_{k=M}^{N} E\Delta_k^2\left\{B_N^2(\log B_N)^{-2\beta} + \sum_{k=M}^{N} E\Delta_k^2\right\} ,$$

where C is a positive constant independent of M and N.

PROOF. From the definition of Δ_m we obtain
(i) $E(\max_{M \leqq r \leqq N}|\sum_{k=M}^{r}\Delta_k|^4) \leqq CE|\sum_{k=M}^{N}\Delta_k|^4$,
(ii) $E|\sum_{k=M}^{N}\Delta_k|^4 \leqq CE(\sum_{k=M}^{N}\Delta_k^2)^2$,
which are (4.4) and (2.7), respectively, of Chapter XV in [5]. Hence for our proof it is sufficient to show that

$$E\left(\sum_{k=M}^{N} \Delta_k^2\right)^2 \leqq C\sum_{k=M}^{N} E\Delta_k^2\left\{B_N^2(\log B_N)^{-2\beta} + \sum_{k=M}^{N} E\Delta_k^2\right\} .$$

By Lemma 2 we have

$$E\left|\sum_{k=M}^{N} \Delta_k^2\right|^2 \leqq 2\sum_{k=M}^{N} E|\Delta_k^2 - E\Delta_k^2|^2 + 2\left(\sum_{k=M}^{N} E\Delta_k^2\right)^2$$
$$\leqq C\sum_{k=M}^{N} E\Delta_k^2\left\{B_N^2(\log B_N)^{-2\beta} + \sum_{k=M}^{N} E\Delta_k^2\right\} .$$

3. Division into blocks. I. Let us put $q(0) = 1$ and for every $k \geqq 1$

$$q(k) = \min\{m;\, B_m^2 - B_{q(k-1)}^2 \geqq B_{q(k-1)}^2(\log B_{q(k-1)})^{-1-5\varepsilon}\} , \tag{3.1}$$

where ε is a positive number such that $2\beta = 1 + 10\varepsilon$.

Then by (2.4) and (3.1) we have

$$\begin{cases} B_{q(k)}/B_{q(k-1)} \to 1 , & \text{as} \quad k \to +\infty , \\ q(k) - q(k-1) > Cp^\alpha(q(k-1))(\log B_{q(k-1)})^{5\varepsilon} , & \text{for some} \quad C > 0 . \end{cases} \tag{3.2}$$

Putting $\psi(k) = [\{\alpha \log p(q(k-1)) + 2\beta \log\log B_{q(k-1)}\}/\log 2]$, (2.5) implies that

$$\psi(k) = \begin{cases} O(\log p(q(k-1))) , & \text{if} \quad \alpha > 0 , \\ O(\log\log B_{q(k-1)}) , & \text{if} \quad \alpha = 0 , \quad \text{as} \quad k \to +\infty . \end{cases} \tag{3.3}$$

Since $\psi(k) = o(q(k) - q(k-1))$ as $k \to +\infty$, if we put

$$q'(k) = q(k-1) + \psi(k) + 1 ,$$

then $q'(k) < q(k)$ for all $k > k_0$. Without loss of generality we may assume that $q'(k) < q(k)$ for all k. We write

$$\begin{cases} V_k = \sum_{m=q'(k)}^{q(k)-1} \Delta_m , \qquad W_k = \sum_{m=q(k-1)}^{q'(k)-1} \Delta_m , \\ C_k^2 = \sum_{m=q(k-1)}^{q(k)-1} E\Delta_m^2 \quad \text{and} \quad D_N^2 = \sum_{k=1}^{N} C_k^2 . \end{cases} \tag{3.4}$$

Then from (3.1), (3.2), (3.3) and (2.4) we obtain

$$C_k^2 = D_k^2(\log D_k)^{-1-5\varepsilon}(1 + o(1)) \tag{3.5}$$

and

$$\begin{aligned} EW_k^2 &= O(D_k^2\psi(k)/(\log D_k)^{2\beta}p^\alpha(q(k-1))) \\ &= o(C_k^2/(\log D_k)^{4\varepsilon}) , \qquad \text{as} \quad k \to +\infty . \end{aligned} \tag{3.6}$$

LEMMA 4. *Let μ_k and μ_k' denote respectively the maximum and minimum frequencies of a trigonometric polynomial $\sum_{m=q(k-1)}^{q'(k)-1} (\Delta_m^2 - E\Delta_m^2)$. Then we have*

$$\mu_k'/\mu_{k-1} \to +\infty \quad \textit{and} \quad \mu_k/\mu_k' \to +\infty ,$$

as $k \to +\infty$. The same conclusion holds for $V_k^2 - EV_k^2$.

PROOF. Since (2.3) and (3.3) imply that $p(q(k-1)-\psi(k))/p(q(k-1))\to 1$, as $k\to+\infty$, we have, by (2.4) and (3.3),

$$B^2_{q(k-1)} - B^2_{q(k-1)-\psi(k)} \geqq \psi(k)D^2_{k-1}/p^{\alpha}(q(k-1)-\psi(k))(\log D_{k-1})^{2\beta}$$
$$= o(D^2_{k-2}/(\log D_{k-2})^{1+9\varepsilon}) , \quad \text{as} \quad k\to+\infty .$$

Therefore, by (3.1) it is seen that $q'(k-1) < q(k-1) - \psi(k)$, if $k > k_0$. On the other hand from the definition of Δ_m we can see that the frequencies of terms of $\sum_{m=q(k-1)}^{q'(k)-1} (\Delta_m^2 - E\Delta_m^2)$ lie in the interval

$$[c2^{q(k-1)}/p^{\alpha}(q(k-1)),\ 2^{q'(k)+1}] .$$

Hence we have

$$\mu_k'/\mu_k > c2^{q(k-1)-q'(k-1)-1}p^{-\alpha}(q(k-1)) > c2^{\psi(k)}p^{-\alpha}(q(k-1))$$
$$> c(\log D_{k-1})^{2\beta} \to +\infty , \quad \text{as} \quad k\to+\infty .$$

In the same way we can prove the remaining part of the lemma.

LEMMA 5. *We have*
(i) $\sum_{k=1}^{N} \sum_{m=q(k-1)}^{q'(k)-1} \Delta_m^2 = o(D_N^2(\log D_N)^{-2\varepsilon})$ *a.s.*,
(ii) $\sum_{k=1}^{N} V_k^2 = D_N^2 + o(D_N^2(\log D_N)^{-2\varepsilon})$ *a.s., as* $N\to+\infty$.

PROOF. (i) By Lemma 2 and Lemma 4 we have, for some $C>0$,

$$E\left|\sum_{k=1}^{\infty} D_k^{-2}(\log D_k)^{2\varepsilon} \sum_{m=q(k-1)}^{q'(k)-1} (\Delta_m^2 - E\Delta_m^2)\right|^2$$
$$\leqq C\sum_{k=1}^{\infty} D_k^{-4}(\log D_k)^{4\varepsilon} E\left|\sum_{m=q(k-1)}^{q'(k)-1} (\Delta_m^2 - E\Delta_m^2)\right|^2$$
$$\leqq C\left(\sum_{k=1}^{\infty} D_k^{-2}(\log D_k)^{4\varepsilon-2\beta} EW_k^2\right) < +\infty .$$

This shows that the series $\sum D_k^{-2}(\log D_k)^{2\varepsilon} \sum_{m=q(k-1)}^{q'(k)-1} (\Delta_m^2 - E\Delta_m^2)$ is the Fourier series of some square integrable function and by Lemma 4 this series converges a.s. Hence by Kronecker's lemma we have

$$\lim_{N\to\infty} D_N^{-2}(\log D_N)^{2\varepsilon} \sum_{k=1}^{N} \sum_{m=q(k-1)}^{q'(k)-1} (\Delta_m^2 - E\Delta_m^2) = 0 , \quad \text{a.s.}$$

(ii) In the same way as in the proof of (i) we have

$$E\left|\sum_{k=1}^{\infty} D_k^{-2}(\log D_k)^{2\varepsilon}(V_k^2 - EV_k^2)\right|^2 \leqq C\sum_{k=1}^{\infty} D_k^{-4}(\log D_k)^{4\varepsilon}E(V_k^2 - EV_k^2)^2$$
$$\leqq C\sum_{k=1}^{\infty} D_k^{-4}(\log D_k)^{4\varepsilon}\{EV_k^4 - (EV_k^2)^2\} .$$

On the other hand, by Lemma 3, (3.4) and (3.5) we have

$$\begin{aligned} EV_k^4 &= O(D_k^2 EV_k^2(\log D_k)^{-2\beta} + (EV_k^2)^2) \\ &= O(D_k^2 C_k^2(\log D_k)^{-2\beta} + C_k^2 D_k^2(\log D_k)^{-1-5\varepsilon}) \\ &= O(D_k^2 C_k^2(\log D_k)^{-1-5\varepsilon}) , \qquad \text{as} \quad k \to +\infty . \end{aligned}$$

Hence we have

$$E\left|\sum_{k=1}^{\infty} D_k^{-2}(\log D_k)^{2\varepsilon}(V_k^2 - EV_k^2)\right|^2 < +\infty ,$$

and in the same way as in (i) we can see that

$$\lim_{N\to\infty} D_N^{-2}(\log D_N)^{2\varepsilon} \sum_{k=1}^{N} (V_k^2 - EV_k^2) = 0 \quad \text{a.s.}$$

Since (3.6) implies that

$$D_N^2 - \sum_{k=1}^{N} EV_k^2 = \sum_{k=1}^{N-1} EW_k^2 = o(D_N^2(\log D_N)^{-2\varepsilon}) , \qquad \text{as} \quad N \to +\infty ,$$

we can prove the second part of the lemma.

II. LEMMA 6. *We have* $\lim_{N\to\infty} D_N^{-1}\sum_{k=1}^{N} W_k = 0$ *a.s.*

PROOF. For every positive integer N let us put $I_N = \{m; q(k-1) \leqq m < q'(k), k = 1, 2, \cdots, N\}$, $I'_N = \{m; m \in I_N$ and m is even$\}$ and $I''_N = \{m; m \in I_N$ and m is odd$\}$. If $m \in I_N$ and $\lambda_N = (\log D_N)^{2\varepsilon}/D_N$, then $|\lambda_N \Delta_m| < 1/4$ for all large N. Since $|x| < 1/2$ implies that $\exp(x) \leqq (1+x)\exp(x^2)$, we have

$$\begin{aligned} \exp\Big(\lambda_N \sum_{m\in I_N} \Delta_m\Big) &= \Big\{\exp\Big(2\lambda_N \sum_{m\in I'_N} \Delta_m\Big)\exp\Big(2\lambda_N \sum_{m\in I''_N} \Delta_m\Big)\Big\}^{1/2} \\ &\leqq \Big\{\prod_{m\in I'_N} (1 + 2\lambda_N\Delta_m) \prod_{m\in I''_N} (1 + 2\lambda_N\Delta_m)\Big\}^{1/2} \exp\Big(2\lambda_N^2 \sum_{m\in I_N} \Delta_m^2\Big) . \end{aligned}$$

Hence we have

$$\begin{aligned} &E\Big\{\exp\Big(\lambda_N \sum_{m\in I_N} \Delta_m - 2\lambda_N^2 \sum_{m\in I_N} \Delta_m^2\Big)\Big\} \\ &\qquad \leqq E\Big\{\prod_{m\in I'_N} (1 + 2\lambda_N\Delta_m) \prod_{m\in I''_N} (1 + 2\lambda_N\Delta_m)\Big\}^{1/2} \\ &\qquad \leqq \Big\{E\prod_{m\in I'_N} (1 + 2\lambda_N\Delta_m) E\prod_{m\in I''_N} (1 + 2\lambda_N\Delta_m)\Big\}^{1/2} . \end{aligned}$$

Estimating the frequencies of terms of Δ_m for $m \in I'_N$, we have

$$E\Delta_m \prod_{j\in I'_N}^{j<m} (1 + 2\lambda_N\Delta_j) = 0 .$$

Therefore, we have

$$E \prod_{m \in I'_N} (1 + 2\lambda_N \Delta_m) = E \prod_{m \in I''_N} (1 + 2\lambda_N \Delta_m) = 1 ,$$

and we obtain

$$E\left\{\exp\left(\lambda_N \sum_{m \in I_N} \Delta_m - 2\lambda_N^2 \sum_{m \in I_N} \Delta_m^2\right)\right\} \leqq 1 .$$

If we take $x_N = D_N/(\log D_N)^\varepsilon$, then we have

$$P\left\{\sum_{m \in I_N} \Delta_m > 2\lambda_N \sum_{m \in I_N} \Delta_m^2 + x_N\right\} \leqq \exp\{-(\log D_N)^\varepsilon\} . \tag{3.7}$$

Next we take $m_k = \min\{m; D_m^2 \geqq \exp(k^\gamma)\}$, where γ is a positive number such that $1/(2+5\varepsilon) < \gamma < 1/2$. Since

$$\{\exp(k^\gamma)\}k^{-\gamma(1+5\varepsilon)} = o(\exp(k+1)^\gamma - \exp(k^\gamma)) , \qquad \text{as} \quad k \to +\infty ,$$

(3.5) implies that there exists an integer k_0 such that if $k > k_0$, then

$$\exp\{(k+1)^\gamma\} > D_{m_k}^2 \geqq \exp(k^\gamma) . \tag{3.8}$$

By (3.7) we have

$$\sum_k P\left\{\sum_{m \in I_{m_k}} \Delta_m > 2D_{m_k}^{-1}(\log D_{m_k})^{2\varepsilon} \sum_{m \in I_{m_k}} \Delta_m^2 + D_{m_k}(\log D_{m_k})^{-\varepsilon}\right\} < +\infty .$$

Therefore, by Lemma 5 (i) we have

$$\limsup_{k\to+\infty} D_{m_k}^{-1} \sum_{m \in I_{m_k}} \Delta_m \leqq 0 \quad \text{a.s.} \tag{3.9}$$

Putting $Z_k = \max\{|\sum_{m=q(m_{k-1})}^r \Delta_m|; m \in I_{m_k}, q(m_{k-1}) \leqq r < q'(m_k)\}$, we have, by Lemma 3 and (3.8),

$$\begin{aligned} E|Z_k|^4 &= O(D_{m_k}^2(D_{m_k}^2 - D_{m_{k-1}}^2)(\log D_{m_k})^{-2\beta} + (D_{m_k}^2 - D_{m_{k-1}}^2)^2) \\ &= O(D_{m_k}^4 k^{\gamma-1-2\gamma\beta} + D_{m_k}^4 k^{-2+2\gamma}) , \qquad \text{as} \quad k \to +\infty . \end{aligned}$$

Hence, we have $\sum D_{m_k}^{-4} E|Z_k|^4 < +\infty$ and this implies that

$$\lim_{k\to+\infty} D_{m_k}^{-1} Z_k = 0 \qquad \text{a.s.} \tag{3.10}$$

Since $D_{m_k}/D_{m_{k-1}} \to 1$ as $k \to +\infty$, (3.9) and (3.10) show that

$$\limsup_{k\to+\infty} D_N^{-1} \sum_{k=1}^N W_k \leqq 0 , \qquad \text{a.s.} ,$$

and replacing $\{W_k\}$ by $\{-W_k\}$, we have

$$\liminf_{k\to+\infty} D_N^{-1} \sum_{k=1}^N W_k \geqq 0 , \qquad \text{a.s.}$$

III. LEMMA 7. *For any $t \geqq 0$ let $N(t)$ and $M(t)$ denote the integers such that $D_{M(t)}^2 \leqq A_{N(t)}^2 \leqq t < A_{N(t)+1}^2 \leqq D_{M(t)+1}^2$. Then we have*

$$S(t) = \sum_{k=1}^{M(t)} V_k + o(t^{1/2}) \;, \quad a.s., \qquad as \quad t \to +\infty \;.$$

PROOF. By (1.4) and (2.4) we have

$$S(t) = T_{N(t)} = \sum_{k=1}^{M(t)} V_k + \sum_{k=1}^{M(t)} W_k + T_{N(t)} - T_{p(q(M(t)))}$$

and $|T_{N(t)} - T_{p(q(M(t)))}| \leqq Z_{M(t)} + o(D_{M(t)}(\log D_{M(t)})^{-\beta})$ as $t \to +\infty$, where $Z_k = \max_r \{|\sum_{m=q(k)}^{r} \Delta_m|; q(k) \leqq r < q(k+1)\}$. By Lemma 3 and (3.5) we have

$$\begin{aligned}\sum_{k=1}^{\infty} D_k^{-4} E Z_k^4 &= O\Big(\sum_{k=1}^{\infty} D_k^{-2} C_k^2 (\log D_k)^{-2\beta} + \sum_{k=1}^{\infty} D_k^{-4} C_k^4\Big) \\ &= O\Big(\sum_{k=1}^{\infty} D_k^{-2} C_k^2 (\log D_k)^{-1-5\varepsilon}\Big) < +\infty \;.\end{aligned}$$

Hence by Lemma 6 we can prove Lemma 7.

4. **Martingale representation.** For each positive integer k let $r(k) = q(k) + [(2^{-1}\alpha \log p(q(k)) + \beta \log\log D_k)/\log 2]$ and $\mathfrak{F}_k$ be the σ-field generated by the intervals $\{[\nu 2^{-r(k)}, (\nu+1)2^{-r(k)}); 0 \leqq \nu < 2^{r(k)}\}$. Then we put

$$X_k = V_k - E(V_k|\mathfrak{F}_k) \quad \text{and} \quad Y_k = E(V_k|\mathfrak{F}_k) - E(V_k|\mathfrak{F}_{k-1}) \;.$$

Clearly $\{Y_k, \mathfrak{F}_k\}$ is a martingale difference sequence.

LEMMA 8. *We have*

(i) $|X_k| = o(C_k^2 D_k^{-1}(\log D_k)^{-2\varepsilon})$ *a.s.*,

(ii) $E(V_k|\mathfrak{F}_{k-1}) = o(C_k^2 D_k^{-1}(\log D_k)^{-2\varepsilon})$ *a.s.* *as* $k \to +\infty$.

PROOF. (i) Since $|\xi_j - E(\xi_j|\mathfrak{F}_k)| \leqq a_j n_j 2^{-r(k)}$ a.s., we have by (2.2)

$$\begin{aligned}|\Delta_m - E(\Delta_m|\mathfrak{F}_k)| &\leqq \sum_{j=p(m)+1}^{p(m+1)} a_j n_j 2^{-r(k)} \\ &= O(\{E\Delta_m^2 p^\alpha(m)\}^{1/2} 2^{m-r(k)}) \quad \text{a.s.} \quad \text{as } k \to +\infty \;.\end{aligned}$$

On the other hand we have, by (2.6), (2.3) and (3.5),

$$\begin{aligned}\sum_{m=q'(k)}^{q(k)-1} \Big\{ E\Delta_m^2 p^\alpha(m)\}^{1/2} 2^{m-r(k)} &= O\Big\{C_k^2 \sum_{m=q'(k)}^{q(k)-1} p^\alpha(m) 2^{2m-2r(k)}\Big\}^{1/2} \\ &= O(C_k p^{\alpha/2}(q(k)-1) 2^{q(k)-r(k)}) = O(C_k p^{\alpha/2}(q(k)-1) p^{-\alpha/2}(q(k))(\log D_k)^{-\beta}) \\ &= o(C_k^2 D_k^{-1}(\log D_k)^{-2\varepsilon}) \;, \qquad \text{as } k \to +\infty \;.\end{aligned}$$

By the above two relations we can complete the proof of (i).

(ii) Since $|E(\xi_j|\mathfrak{F}_{k-1})| \leqq 2(2\pi n_j)^{-1} a_j 2^{r(k-1)}$ a.s., we have

$$|E(\Delta_m|\mathfrak{F}_{k-1})| = O(\{E\Delta_m^2 p^\alpha(m)\}^{1/2} 2^{r(k-1)-m}) \quad \text{a.s.} \qquad \text{as } k \to +\infty \;.$$

On the other hand by (2.6), (2.3), (3.5) and the definitions of $\{r(k)\}$ and $\{q'(k)\}$ we have

$$\begin{aligned}\sum_{m=q'(k)}^{q(k)-1} \{E\Delta_m^2 p^\alpha(m)\}^{1/2} 2^{r(k-1)-m} &= O\Big(C_k\Big\{\sum_{m=q'(k)}^{q(k)-1} p^\alpha(m)2^{-2m}\Big\}^{1/2} 2^{r(k-1)}\Big)\\ &= O(C_k p^{\alpha/2}(q'(k))2^{r(k-1)-q'(k)}) = O(C_k p^{\alpha/2}(q'(k))p^{-\alpha/2}(q(k-1))(\log D_k)^{-\beta})\\ &= o(C_k^2 D_k^{-1}(\log D_k)^{-2\varepsilon}), \qquad \text{as } k\to+\infty .\end{aligned}$$

Hence we can prove (ii).

LEMMA 9. *We have*
(i) $\sum_{k=1}^N |Y_k - V_k| = o(D_N(\log D_N)^{-2\varepsilon})$ *a.s.*
(ii) $\sum_{k=1}^N Y_k^2 = D_N^2 + o(D_N^2(\log D_N)^{-2\varepsilon})$ *a.s. as* $N\to+\infty$.

PROOF. (i) follows trivially from Lemma 8.
(ii) By Lemma 5 (ii) it is sufficient to show that

$$\Big|\sum_{k=1}^N Y_k^2 - \sum_{k=1}^N V_k^2\Big| = o(D_N^2(\log D_N)^{-2\varepsilon}) \quad \text{a.s.} \qquad \text{as } N\to+\infty .$$

Since $\max_{1\le k\le N}|Y_k + V_k| \leqq \max_{1\le k\le N}(2|V_k| + |X_k| + |E(V_k|\mathfrak{F}_{k-1})|) = O(D_N)$ a.s. as $N\to+\infty$, (i) implies (ii). Therefore, by Lemma 7 and Lemma 9 (i) we have

$$S(t) = \sum_{k=1}^{M(t)} Y_k + o(t^{1/2}), \quad \text{a.s.} \qquad \text{as } t\to+\infty . \tag{4.1}$$

5. **Embedding procedure.** We apply the theorem of Strassen stated in §1. Let $\{X(t), t\geqq 0\}$ be standard Brownian motion. Then there exist non-negative random variables T_k such that

$$\Big\{X\Big(\sum_{m=1}^k T_m\Big), k\geqq 1\Big\} \quad \text{and} \quad \Big\{\sum_{m=1}^k Y_m, k\geqq 1\Big\}$$

have the same distribution. Hence without loss of generality we can redefine $\{Y_k\}$ by

$$Y_k = X\Big(\sum_{m=1}^k T_m\Big) - X\Big(\sum_{m=1}^{k-1} T_m\Big) \tag{5.1}$$

and can keep the same notation. Thus $\mathfrak{H}_k$ becomes the σ-field generated by $\{X(\sum_{j=1}^m T_j), m\leqq k\}$ and $\mathfrak{G}_k$ is the σ-field generated by $\{X(t), 0\leqq t\leqq \sum_{m=1}^k T_m\}$. Note that $\mathfrak{H}_k\subset\mathfrak{G}_k$, $k\geqq 1$ and each T_k is $\mathfrak{G}_k$-measurable. Moreover, for some constant C we have

$$\begin{cases} E(T_k|\mathfrak{G}_{k-1}) = E(Y_k^2|\mathfrak{H}_{k-1}) \quad \text{a.s.,}\\ E(T_k^2|\mathfrak{G}_{k-1}) \leqq CE(Y_k^4|\mathfrak{H}_{k-1}) \quad \text{a.s.}\end{cases} \tag{5.2}$$

LEMMA 10. *We have*

$$\sum_{k=1}^{N} T_k = D_N^2 + o(D_N^2(\log D_N)^{-2\varepsilon}) \quad a.s., \qquad as\ N \to +\infty\ .$$

PROOF. By (5.2) we have

$$\sum_{k=1}^{N} T_k - D_N^2 = \sum_{k=1}^{N} \{T_k - E(T_k | \mathfrak{G}_{k-1})\}$$
$$- \sum_{k=1}^{N} \{Y_k^2 - E(Y_k^2 | \mathfrak{H}_{k-1})\} + \sum_{k=1}^{N} Y_k^2 - D_N^2\ , \quad \text{a.s.}$$

Since $EY_k^4 \leqq 16EV_k^4$ we have, by Lemma 3 and (3.5),

$$EY_k^4 = O(D_k^2 EV_k^2(\log D_k)^{-2\beta} + (EV_k^2)^2)$$
$$= O(D_k^2 C_k^2(\log D_k)^{-1-5\varepsilon})\ , \qquad \text{as } k \to +\infty\ .$$

Therefore, we have

$$\sum_{k=1}^{\infty} D_k^{-4}(\log D_k)^{4\varepsilon} EY_k^4 < +\infty\ .$$

Hence by (5.2) we have

$$\begin{cases} \sum_{k=1}^{N} \{T_k - E(T_k | \mathfrak{G}_{k-1})\} = o(D_N^2(\log D_N)^{-2\varepsilon}) \quad \text{a.s.}\ , \\ \sum_{k=1}^{N} \{Y_k^2 - E(Y_k^2 | \mathfrak{H}_{k-1})\} = o(D_N^2(\log D_N)^{-2\varepsilon}) \quad \text{a.s.} \qquad \text{as } N \to +\infty\ , \end{cases}$$

for two martingales. Therefore, by Lemma 9 (ii) we can prove the lemma.

Next let us define a random process $\{S^*(t), t \geqq 0\}$ by

$$S^*(t) = \sum_{k=1}^{N} Y_k\ , \qquad \text{if} \quad D_N^2 \leqq t < D_{N+1}^2\ . \tag{5.3}$$

Observe that $\{S(t)\}$ and $\{S^*(t)\}$ are not necessarily defined on the same probability space, since we redefined $\{Y_k\}$ by (5.1). But we can redefine $\{S(t)\}$, $\{S^*(t)\}$ and $\{X(t)\}$ on still another probability space so that the joint distribution of $\{S^*(t)\}$ and $\{X(t)\}$ as well as that of $\{S(t)\}$ and the old version of $\{S^*(t)\}$ remains unchanged. Hence without loss of generality we can assume that $\{S(t)\}$, $\{S^*(t)\}$ and $\{X(t)\}$ are defined on the same probability space and that the lemmas proved so far continue to hold in this new setup.

Therefore, by (4.1) and (5.3) it is enough for the proof of our theorem to show the following.

LEMMA 11. *We have* $S^*(t) = X(t) + o(t^{1/2})$ *a.s.*, *as* $t \to +\infty$.

PROOF. Let $\varepsilon_n = (\log \log D_n)^{-\varepsilon}$ and define the sets as follows:

$$E_n = \left\{\max\left(\left|X\left(\sum_{k=1}^{n} T_k\right) - X(t)\right|; D_n^2 \leqq t < D_{n+1}^2\right) > 4\varepsilon_n D_n\right\} ,$$

$$F_n = \left\{\left|\left(\sum_{k=1}^{n} T_k\right) - D_n^2\right| > D_n^2(\log D_n)^{-2\varepsilon}\right\}$$

$$G_n = \left\{\left|X\left(\sum_{k=1}^{n} T_k\right) - X(D_n^2)\right| > 2\varepsilon_n D_n\right\}$$

$$H_n(r, s) = \{\max(|X(D_n^2 + h) - X(D_n^2)|; 0 < |h| \leqq rD_n^2(\log D_n)^{-2\varepsilon}) > s\varepsilon_n D_n\} , \quad \text{for} \quad 0 < r, s < \infty .$$

For the proof it is sufficient to show that $P(\limsup_{n\to+\infty} E_n) = 0$. Since (3.5) implies

$$E_n \subset G_n \cup H_n(1, 2) \subset F_n \cup \{(F_n^c \cap H_n(1, 2))\} \cup H_n(1, 2) \subset F_n \cup H_n(1, 2)$$

for $n \geqq n_0$ and since Lemma 10 implies $P(\limsup_{n\to+\infty} F_n) = 0$, it is sufficient to show that

$$P\left\{\limsup_{n\to+\infty} H_n(1, 2)\right\} = 0 . \tag{5.4}$$

Let $m_k = \min\{m; D_m^2 \geqq \exp(\sqrt{k})\}$. Then by (3.5) there exists an integer k_0 such that $k > k_0$ implies

$$\begin{cases} \exp(\sqrt{k}) \leqq D_{m_k}^2 < \exp(\sqrt{k+1}) , \\ D_{m_{k+1}}^2\{1 + (\log D_{m_{k+1}})^{-2\varepsilon}\} < D_{m_k}^2\{1 + 2(\log D_{m_k})^{-2\varepsilon}\} . \end{cases} \tag{5.5}$$

For $n \geqq m_{k_0}$ and $m_k \leqq n < m_{k+1}$ (5.5) implies that $H_{m_k}^c(2, 1) \subset H_n^c(1, 2)$. Therefore, by (5.4) it is sufficient to show that

$$P\{\limsup_{k\to+\infty} H_{m_k}(2, 1)\} = 0 . \tag{5.6}$$

Using Lévy's maximal inequality we have

$$\begin{aligned} P\{H_{m_k}(2, 1)\} &\leqq 2P\{|X(4D_{m_k}^2(\log D_{m_k})^{-2\varepsilon})| > \varepsilon_{m_k} D_{m_k}\} \\ &\leqq 2P\{|X(1)| > \varepsilon_{m_k}(\log D_{m_k})^{\varepsilon}/2\} < \exp\{-(\sqrt{k})^{\varepsilon}/8\} , \qquad \text{for } k > k_0 . \end{aligned}$$

Hence by the Borel-Cantelli lemma we can prove (5.6).

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