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QUASIHARMONIC L^p FUNCTIONS AND BIHARMONIC DEGENERACY

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In a recent study, we introduced a criterion for the existence of quasiharmonic L^p functions on the Poincaré N-ball B_α^N (Chung-Sario-Wang [3]). By definition, a function u is quasiharmonic or biharmonic on a Riemannian manifold if $\Delta u = 1$ or $\Delta^2 u = \Delta(\Delta u) = 0$, respectively, with $\Delta = d\delta + \delta d$ the Laplace-Beltrami operator. By the Poincaré N-ball B_α^N we mean the unit ball of N-space, $N \geqq 2$, endowed with the Poincaré-type metric $ds_\alpha = (1 - r^2)^\alpha ds_0$, where $\alpha \in \boldsymbol{R}$ and ds_0 is the Euclidean metric. In the present study, we will apply our criterion to explore relations between the B_α^N not carrying quasiharmonic L^p functions and the B_α^N not carrying bounded or Dirichlet finite nonharmonic biharmonic functions. The Poincaré N-ball, on which there is rather extensive literature (see Bibliography) is particularly suitable for studying such questions of dependence of the voidness of certain function classes on the metric.

First we give, in Propositions 1-4, the earlier results we shall make use of. We then consider bounded nonharmonic biharmonic functions and subsequently Dirichlet finite nonharmonic biharmonic functions. The new results are given in Lemmas 1-9 and Theorems 1-3.

1. Background. For any function class F, let O_F be the class of Riemannian manifolds R such that $F(R) \subset \boldsymbol{R}$. Denote by $\tilde{O}_F$ the complement of O_F. By Q, H^2, L^p, B, D we designate the classes of functions which are quasiharmonic, nonharmonic biharmonic, of finite L^p norm, bounded, or of finite Dirichlet norm $\int |\mathrm{grad}\cdot|^2 dV$. For any two function classes X, Y, set $XY = X \cap Y$. Thus, e.g., O_{QL^p} is the class of Riemannian manifolds which carry no quasiharmonic L^p functions.

The existence of bounded nonharmonic biharmonic functions on B_α^N is completely characterized by the following result:

PROPOSITION 1 (Hada-Sario-Wang [5]). *$B_\alpha^N \in \tilde{O}_{H^2B}$ if and only if $\alpha > -1$ for $N = 2, 3, 4$, and $\alpha \in (-1, 3/(N-4))$ for $N > 4$.*

The existence problem of Dirichlet finite nonharmonic biharmonic functions on B_α^N has also been completely solved:

PROPOSITION 2 (Hada-Sario-Wang [4]). *$B_\alpha^N \in \tilde{O}_{H^2D}$ if and only if $\alpha > -3/(N+2)$ for $N \leqq 6$, and $\alpha \in (-3/(N+2), 5/(N-6))$ for $N > 6$.*

In our criteria for the existence of quasiharmonic L^p functions on B_α^N, we separate the cases $N > 2$ and $N = 2$:

PROPOSITION 3 (Chung-Sario-Wang [3]). *Suppose $N > 2$.*

(a) *A necessary and sufficient condition for $B_\alpha^N \in O_{QL^p}$ is one of the following*: (i) $\alpha > 1/(N-2)$ *and* $\alpha[(N-2)p - N] \geqq p+1$; *or* (ii) $\alpha \in (-1, -1/N)$ *and* $2p(\alpha+1) + N\alpha + 1 \leqq 0$; *or* (iii) $\alpha \leqq -1$.

(b) *For* $\alpha \in [-1/N, 1/(N-2)]$, $B_\alpha^N \in \tilde{O}_{QL_p}$.

PROPOSITION 4 (Chung-Sario-Wang [3]). *Suppose $N = 2$.*

(a) *Then* $B_\alpha^N \in O_{QL^p}$ *if and only if* (i) $\alpha \in (-1, -1/2)$ *and* $2p(\alpha+1) + 2\alpha + 1 \leqq 0$; *or* (ii) $\alpha \leqq -1$.

(b) *For* $\alpha \geqq -1/2$, $B_\alpha^N \in \tilde{O}_{QL^p}$.

2. QL^p and bounded biharmonic functions. We will study the nonemptiness of the classes $O_{QL^p} \cap O_{H^2B}$, $O_{QL^p} \cap \tilde{O}_{H^2B}$, $\tilde{O}_{QL^p} \cap O_{H^2B}$, and $\tilde{O}_{QL^p} \cap \tilde{O}_{H^2B}$.

LEMMA 1. *A necessary and sufficient condition for $B_\alpha^N \in O_{QL^p} \cap O_{H^2B}$ is*

(a) *for $N \leqq 4$, that $\alpha \leqq -1$;*

(b) *for $N > 4$, that either* (i) $\alpha \leqq -1$, *or* (ii) $3/(N-4) \leqq \alpha$ *and* $p + 1 \leqq \alpha[(N-2)p - N]$.

PROOF. For the necessity of (b), suppose $B_\alpha^N \in O_{QL^p} \cap O_{H^2B}$. By Proposition 1, either $\alpha \leqq -1$ or $3/(N-4) \leqq \alpha$. Since for $\alpha \leqq -1$, we have (i), we may assume $3/(N-4) \leqq \alpha$. By Proposition 3 (i), $p + 1 \leqq \alpha[(N-2)p - N]$. For the sufficiency of (b), we reverse the argument. The proof of (a) is contained in the proof of (b).

LEMMA 2. *A necessary and sufficient condition for $B_\alpha^N \in O_{QL^p} \cap \tilde{O}_{H^2B}$ is*

(a) *for $N = 2$, that $-1 < \alpha < -1/2$ and $2p(\alpha+1) + 2\alpha + 1 \leqq 0$;*

(b) *for $N = 3, 4$, that either* (i) $-1 < \alpha < -1/N$ *and* $2p(\alpha+1) + N\alpha + 1 \leqq 0$; *or* (ii) $1/(N-2) < \alpha$ *and* $\alpha[(N-2)p - N] \geqq p + 1$;

(c) *for $N > 4$, that either* (i) *holds*; *or* (ii) *together with* $1/(N-2) < \alpha < 3/(N-4)$ *holds.*

PROOF. To show the necessity of (b), suppose $B_\alpha^N \in O_{QL^p} \cap \tilde{O}_{H^2B}$. By Proposition 1, we have $-1 < \alpha$. By Proposition 3, either $1/(N-2) < \alpha$ and $\alpha[(N-2)p - N] \geqq p + 1$, or $-1 < \alpha < -1/N$ and $2p(\alpha+1) + N\alpha + 1 \leqq 0$. These are the conditions of (b). The sufficiency is proved by reversing the argument. The proofs of (a) and (c) are basically the same.

LEMMA 3. (a) *A necessary and sufficient condition for $B_\alpha^N \in \tilde{O}_{QL^p} \cap$*

O_{H^2B} *is that* $N > 4$, $\alpha \geqq 3/(N-4)$, *and* $\alpha[(N-2)p - N] < p+1$.

(b) *For* $N > 4$, *a necessary and sufficient condition for the existence of a* B_α^N *in* $\tilde{O}_{QL^p} \cap O_{H^2B}$ *is that* $p < 2$.

PROOF. To show the necessity of (a), suppose $B_\alpha^N \in \tilde{O}_{QL^p} \cap O_{H^2B}$. If $N \leqq 4$, Proposition 1 gives $\alpha \leqq -1$. This conclusion and Proposition 3 imply $B_\alpha^N \in O_{QL^p}$, in violation of the assumption. Thus, $N > 4$.

By Proposition 3, $\alpha > -1$. If $\alpha < 3/(N-4)$, then by Proposition 1, $B_\alpha^N \in \tilde{O}_{H^2B}$, contrary to the assumption. Thus $\alpha \geqq 3/(N-4)$. If $\alpha[(N-2)p - N] \geqq p+1$, then, by Proposition 3 (i), $B_\alpha^N \in O_{QL^p}$. This is again a contradiction. Thus $\alpha[(N-2)p - N] < p+1$. We leave the proof of the sufficiency to the reader.

We proceed to part (b) of the lemma. Let $N > 4$ and $B_\alpha^N \in \tilde{O}_{QL^p} \cap O_{H^2B}$. We will show that $p < 2$. By (a), $\alpha > 3/(N-4)$ and $\alpha[(N-2)p - N] < p+1$. Since $p+1 > 0$ and $\alpha > 0$, if $\alpha[(N-2)p - N] < p+1$ holds for any $\alpha > 3/(N-4)$, it also holds for $\alpha = 3/(N-4)$. Thus, we have successively the inequalities

$$\alpha[(N-2)p - N] < p+1\,,$$
$$3[Np - 2p - N] < (p+1)(N-4)\,,$$
$$2p(N-1) - 4(N-1) < 0\,,$$

so that $p < 2$, We have proved the necessity of (b). To show the sufficiency, let $p < 2$ be given. Then the above inequalities show that for $\alpha = 3/(N-4)$, the other condition is also satisfied. Since we assume $N > 4$, (a) applies and we have $B_\alpha^N \in \tilde{O}_{QL^p} \cap O_{H^2B}$.

Our next lemma would give a necessary and sufficient condition for $B_\alpha^N \in \tilde{O}_{QL^p} \cap \tilde{O}_{H^2B}$. However, in view of the above pattern, we may and do omit the lengthy statement of the necessity. The sufficiency follows from Propositions 1 and 3.

LEMMA 4. *If* $-1/N \leqq \alpha \leqq 1/(N-2)$, *then* $B_\alpha^N \in \tilde{O}_{QL^p} \cap \tilde{O}_{H^2B}$.

We are ready to draw the conclusion from the above lemmas:

THEOREM 1. (a) *For* $N \leqq 4$; *or* $N > 4$ *and* $p \geqq 2$, *the family* $\mathscr{B}$ *of the Poincaré N-balls decomposes into the following three disjoint nonempty sets*:

$$\mathscr{B} = O_{QL^p} \cap O_{H^2B} \oplus O_{QL^p} \cap \tilde{O}_{H^2B} \oplus \tilde{O}_{QL^p} \cap \tilde{O}_{H^2B}\,.$$

For $N > 4$ *and* $p < 2$, *the decomposition is*

$$\mathscr{B} = O_{QL^p} \cap O_{H^2B} \oplus O_{QL^p} \cap \tilde{O}_{H^2B} \oplus \tilde{O}_{QL^p} \cap O_{H^2B} \oplus \tilde{O}_{QL^p} \cap \tilde{O}_{H^2B}\,.$$

(b) *For* $p < 2$, *the class* $\mathscr{R}$ *of all Riemannian manifolds of dimension*

$N > 4$ decomposes into the following four disjoint nonempty sets:

$$\mathscr{R} = O_{QL^p} \cap O_{H^2B} \oplus O_{QL^p} \cap \widetilde{O}_{H^2B} \oplus \widetilde{O}_{QL^p} \cap O_{H^2B} \oplus \widetilde{O}_{QL^p} \cap \widetilde{O}_{H^2B} .$$

PROOF. (b) follows from (a), so that it suffices to prove (a). Since all classes involved in the decomposition are already disjoint, it suffices to show that they are nonempty and that their union is $\mathscr{R}$. But these properties are consequences of Lemmas 1-4.

3. QL^p and Dirichlet finite biharmonic functions. We turn to relations of quasiharmonic L^p functions to Dirichlet finite biharmonic functions. This time, instead of giving necessary and sufficient conditions for B_α^N to belong to the four possible sets, we will only give sufficient conditions for the two less interesting sets.

LEMMA 5. (a) *If $\alpha \leqq -1$, then $B_\alpha^N \in O_{QL^p} \cap O_{H^2D}$.*
(b) *If $-1/N \leqq \alpha \leqq 1/(N-2)$, then $B_\alpha^N \in \widetilde{O}_{QL^p} \cap \widetilde{O}_{H^2D}$.*

The proof is by Propositions 3 and 2.

LEMMA 6. (a) *For $N = 2$, there exist no B_α^N in $O_{QL^p} \cap \widetilde{O}_{H^2D}$.*
(b) *For $N > 2$, a necessary and sufficient condition for $B_\alpha^N \in O_{QL^p} \cap \widetilde{O}_{H^2D}$ is* (i) *for $N \leqq 6$, that $\alpha > 1/(N-2)$ and $\alpha[(N-2)p - N] \geqq p + 1$;* (ii) *for $N > 6$, that $1/(N-2) < \alpha < 5/(N-6)$ and $\alpha[(N-2)p - N] \geqq p + 1$.*

PROOF. (a) If $N = 2$ and $B_\alpha^N \in O_{QL^p} \cap \widetilde{O}_{H^2D}$, then, by Propositions 2 and 3, $-3/(N+2) < \alpha < -1/N$ and $2p(\alpha+1) + N\alpha + 1 \leqq 0$. The latter inequality implies $\alpha \leqq -(1+2p)/(N+2p)$. But $-(1+2p)/(N+2p) \leqq -3/(N+2)$ for $p \geqq 1$ and $N > 0$. Thus no α in $(-3/(N+2), -1/N)$ would satisfy $2p(\alpha+1) + N\alpha + 1 \leqq 0$, and we conclude that there exists no B_α^N in $O_{QL^p} \cap \widetilde{O}_{H^2D}$.

(b) We will show the necessity of (i). Assume $B_\alpha^N \in O_{QL^p} \cap \widetilde{O}_{H^2D}$. Again by Proposition 2, $\alpha > -3/(N+2)$. By Proposition 3, either both $1/(N-2) < \alpha$ and $\alpha[(N-2)p - N] \geqq p + 1$ or both $-3(N+2) < \alpha < -1/N$ and $2p(\alpha+1) + N\alpha + 1 \leqq 0$. The latter case was proved to be impossible in (a), where the assumption $N = 2$ was not invoked in proving the contradiction. Thus, we only have one possibility which is exactly our condition. The sufficiency of (b) follows by Propositions 3 and 2.

(ii) has the same proof as (i), except that the extra condition $\alpha < 5/(N-6)$ was imposed.

Examination of the conditions in Lemma 6 shows that for $p = 1$, there is no B_α^N in $O_{QL^p} \cap \widetilde{O}_{H^2D}$. The exact range for p is given below.

LEMMA 7. *For each $N > 2$, a necessary and sufficient condition for* *the existence of a* B_α^N *in* $O_{QL^p} \cap \tilde{O}_{H^2D}$ *is*

(a) *for* $N \leqq 6$, *that* $p > N/(N-2)$,

(b) *for* $N > 6$, *that* $p > 3/2$.

PROOF. (a) Suppose that there exists a B_α^N in $O_{QL^p} \cap \tilde{O}_{H^2D}$. By Lemma 6, $\alpha[(N-2)p - N] \geqq p + 1$ for some $\alpha > 1/(N-2)$. This implies that $(N-2)p - N > 0$. Thus the condition is necessary.

To show that the condition is sufficient, assume $(N-2)p - N > 0$. Then we simply choose $\alpha > 1/(N-2)$ so large that $\alpha[(N-2)p - N] \geqq p + 1$. Again by Lemma 6, $B_\alpha^N \in O_{QL^p} \cap \tilde{O}_{H^2D}$.

(b) Suppose that there exists a B_α^N in $O_{QL^p} \cap \tilde{O}_{H^2D}$. By Lemma 6, $1/(N-2) < \alpha < 5/(N-6)$ and $\alpha[(N-2)p - N] \geqq p + 1$. The latter inequality implies $(N-2)p - N > 0$ and $\alpha \geqq p + 1/[(N-2)p - N]$. By the former inequality, $p + 1/[(N-2)p - N] < 5/(N-6)$, which yields $p > 3/2$. The sufficiency follows by Lemma 6.

LEMMA 8. *A necessary and sufficient condition for* $B^N \in \tilde{O}_{QL^p} \cap O_{H^2D}$ *is*:

(a) *for* $N \leqq 6$, *that* $-1 < \alpha \leqq -3/(N+2)$ *and* $2p(\alpha+1) + N\alpha + 1 > 0$;

(b) *for* $N > 6$, *that either* (i) $-1 < \alpha \leqq -3/(N+2)$ *and* $2p(\alpha+1) + N\alpha + 1 > 0$, *or* (ii) $5/(N-6) \leqq \alpha$ *and* $\alpha[(N-2)p - N] < p + 1$.

This follows immediately from Propositions 2 and 3.

LEMMA 9. (a) *For* $N \leqq 6$, *a necessary and sufficient condition for the existence of a* B_α^N *in* $\tilde{O}_{QL^p} \cap O_{H^2D}$ *is that* $p > 1$.

(b) *If* $N > 6$, *then for each* $p \geqq 1$, *there exists a* B_α^N *in* $\tilde{O}_{QL^p} \cap O_{H^2D}$.

PROOF. (a) To show the necessity, take $p = 1$ and suppose there exists a B_α^N in the above class. By Lemma 8, $2p(\alpha+1) + N\alpha + 1 > 0$. Thus $2 \cdot 1 \cdot (\alpha+1) + N\alpha + 1 = (N+2)\alpha + 3 > 0$. As a consequence, we have $\alpha > -3/(N+2)$, which, by Proposition 2, gives $B_\alpha^N \in \tilde{O}_{H^2D}$, in violation of our assumption $B_\alpha^N \in O_{H^2D}$.

To prove the sufficiency, take a $p > 1$ and choose $\alpha = -3/(N+2)$. Then

$$\begin{aligned} 2p(\alpha+1) + N\alpha + 1 &= 2p(-3(N+2)^{-1} + 1) + N\cdot(-3(N+2)^{-1}) + 1 \\ &= (N+2)^{-1}(N-1)[2p-2] > 0 . \end{aligned}$$

Thus for such α and p, $B_\alpha^N \in \tilde{O}_{QL^p} \cap O_{H^2D}$.

(b) Assume $N > 6$. If $p > 1$, the same argument as in (a) gives $B_\alpha^N \in \tilde{O}_{QL^p} \cap O_{H^2D}$. If $p = 1$, we have for $\alpha \geqq 5/(N-6)$, $\alpha[(N-2)p - N] = \alpha[(N-2)\cdot 1 - N] < 1 + 1 = p + 1$. Thus, by Lemma 8, $B_\alpha^N \in \tilde{O}_{QL^p} \cap O_{H^2D}$.

We are ready to draw the conclusion:

THEOREM 2. *The family $\mathscr{B}$ of the Poincaré N-balls decomposes into the following disjoint nonempty sets*:

(a) *for* $N=2$, $p=1$, $\mathscr{B}=O_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{H^2D}$;
for $N=2$, $p>1$, $\mathscr{B}=O_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap\tilde{Q}_{H^2D}$;

(b) *for* $N=3, 4, 5, 6$, *and* $p=1$, $\mathscr{B}=O_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{H^2D}$;
for $N=3, 4, 5, 6$, *and* $1<p\leqq N/(N-2)$,

$$\mathscr{B}=O_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{H^2D}\ ;$$

for $N=3, 4, 5, 6$, *and* $p>N/(N-2)$,

$$\mathscr{B}=O_{QL^p}\cap O_{H^2D}\oplus O_{QL^p}\cap\tilde{O}_{H^2D}\oplus\tilde{O}_{H^2D}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{H^2D}\ ;$$

(c) *for* $N>6$, $p\leqq 3/2$, $\mathscr{B}=O_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{H^2D}$;
for $N>6$, $p>3/2$,

$$\mathscr{B}=O_{QL^p}\cap O_{H^2D}\oplus O_{QL^p}\cap\tilde{O}_{H^2D}\oplus\tilde{O}_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{H^2D}\ .$$

The proof is by examination of Lemmas 5-9.

As an immediate consequence of Theorem 2, we have the following result in the general classification theory for biharmonic functions.

THEOREM 3. *The class $\mathscr{R}$ of Riemannian N-manifolds decomposes into the following disjoint nonempty sets*:
for $N=3, 4, 5, 6$, *and* $p>N/(N-2)$,

$$\mathscr{R}=O_{QL^p}\cap O_{H^2D}\otimes O_{QL^p}\cap\tilde{O}_{H^2D}\oplus\tilde{O}_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{H^2D}\ ;$$

for $N>6$, $p>3/2$,

$$\mathscr{R}=O_{QL^p}\cap O_{H^2D}\oplus O_{QL^p}\cap\tilde{O}_{H^2D}\oplus\tilde{O}_{QL^p}\cap O_{H^2D}\oplus\tilde{O}_{QL^p}\cap\tilde{O}_{H^2D}\ .$$

BIBLIOGRAPHY

[1] L. O. CHUNG, L. SARIO AND C. WANG, Quasiharmonic L^p-functions on Riemannian manifolds, Ann. Scuola Norm. Sup. Pisa, Ser. IV, 2 (1975), 469-478.

[2] L. O. CHUNG, L. SARIO AND C. WANG, Harmonic L^p functions on the Poincaré N-ball, Bull. Inst. Math. Acad. Sinica (to appear).

[3] L. O. CHUNG, L. SARIO AND C. WANG, Quasiharmonic L^p functions on the Poincaré ball, Kōdai Math. J. (to appear).

[4] D. HADA, L. SARIO AND C. WANG, Dirichlet finite biharmonic functions on the Poincaré N-ball, J. Reine Angew. Math. 272 (1975), 92-101.

[5] D. HADA, L. SARIO AND C. WANG, Bounded biharmonic functions on the Poincaré N-ball, Kōdai Math. Sem. Rep. 26 (1975), 327-342.

[6] L. SARIO, M. NAKAI, C. WANG AND L. CHUNG, Classification Theory of Riemannian Manifolds, Lecture Notes in Math. 605, Springer-Verlag, 1977.

[7] L. SARIO AND C. WANG, Quasiharmonic functions on the Poincaré N-ball, Rend. Mat. 6 (1973), 1-14.

[8] L. SARIO AND C. WANG, Existence of Dirichlet finite biharmonic functions on the Poincaré 3-ball, Pacific J. Math. 48 (1973), 267–274.

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