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A FIXED POINT THEOREM AND ITS APPLICATION IN ERGODIC THEORY

Dedicated to Professor Taro Yoshizawa on his sixtieth birthday

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The purpose of this paper is to prove a simple fixed point theorem in Banach spaces, and to show its application in ergodic theory. The theorem asserts the existence of a unique fixed point for affine transformations and the convergence of successive approximations to the fixed point. In the special case of linear operators in L^1 generated by point-to-point nonsingular transformations, this fixed point theorem demonstrates the existence and uniqueness of invariant measures and the exactness of corresponding measurable dynamical systems. The theorem thus gives a new tool for proving the exactness of some measurable endomorphisms.

The paper is divided into four parts. In Section 1 an abstract version of the fixed point theorem is proved. From the formal point of view it remembles some known results of Edelstein [1]. The proof, however, is based on ideas due to Pianigiani and Yorke [7]. Section 2 contains the specialization of the fixed point theorem to the space L^1. In Section 3 the general theory is examined in the case of expanding mappings of differentiable manifolds and a new simpler proof of the well known Krzyżewski-Szlenk theorem [5] is presented. In the proof once again the ideas of Pianigiani and Yorke are used. Finally, Section 4 is devoted to the study of a class of dynamical systems generated by piecewise convex transformations.

1. Fixed point theorem. Let E, $\|\ \|$ be a Banach space. A closed convex set $C \subset E$ is said to be imbedded in V ($V \subset E$) if for each two different points $x_1, x_2 \in C$ the closed interval $[0, 1]$ is contained in the interior of the set $\{\lambda \in R: \lambda x_1 + (1-\lambda)x_2 \in V\}$. The distance between a nonempty set $C \subset E$ and a point $x \in E$ is defined, as usual by

$$\rho(x, C) = \inf\{\|x - y\|: y \in C\}\ .$$

A sequence $\{x_n\} \subset E$ converges to C $(x_n \to C)$ if $\lim_n \rho(x_n, C) = 0$. In particular $x_n \to x_0$ always stands for $\|x_n - x_0\| \to 0$.

THEOREM 1. *Let C be a compact convex subset of a Banach space E, imbeded in a set $V \subset E$. Assume that an affine trasformation $U: E \to E$ satisfies the following two conditions*:

(1) *There exists a constant q such that $\|U^n x - U^n y\| \leqq q\|x - y\|$ for all $x, y \in E$ and all integers $n > 0$.*

(2) *The set $\{x: U^n x \to C\}$ is dense in V.*

Then U has in V a unique fixed point x_0. Further, $x_0 \in C$ and

$$\lim_n U^n x = x_0 \quad \textit{for each} \quad x \in V . \tag{3}$$

PROOF. From (1) it follows that $U^n x \to C$ for each $x \in V$. Since C is a compact set, $\{U^n x\}$ is relatively compact for $x \in V$. Thus according to the Kakutani Yosida ergodic theorem (see also Edelstein [1]) there exists a limit of the sequence $n^{-1}\sum_{k=0}^{n-1} U^k x$ which is a fixed point of U. The condition $U^n x \to C$ implies that any fixed point of U in V belongs to C. It remains to prove (3). Let $x_0 \in C$ be a fixed point and let $x \in V$. Suppose that (3) does not hold. Then there exists a subsequence $\{U^{\alpha_n} x\}$ such that $\lim_n U^{\alpha_n} x = x_1 \neq x_0$. Now let $\{\gamma_n\}$ be a subsequence of $\{\alpha_n\}$ such that $\beta_n = \gamma_n - \alpha_n \to \infty$. From (1) it follows that

$$\begin{aligned} \|U^{\beta_n} x_1 - x_1\| &\leqq \|U^{\beta_n} x_1 - U^{\alpha_n + \beta_n} x\| + \|U^{\gamma_n} x - x_1\| \\ &\leqq q\|x_1 - U^{\alpha_n} x\| + \|U^{\gamma_n} x - x_1\| . \end{aligned}$$

Since $\{\gamma_n\}$ is a subsequence of $\{\alpha_n\}$, this implies $\lim_n U^{\beta_n} x_1 = x_1$. Now consider the family of points $x_\lambda = (1 - \lambda)x_0 + \lambda x_1$. Since x_0 is a fixed point of U, one has $\lim_n U^{\beta_n} x_\lambda = (1 - \lambda)x_0 + \lambda x_1 = x_\lambda$. The limit belongs to C whenever $x_\lambda \in V$ and therefore the following implication is proved

$$x_\lambda \in V \Rightarrow x_\lambda \in C . \tag{4}$$

Define

$$\lambda_0 = \inf\{\lambda: x_\lambda \in C\} , \quad \lambda_1 = \sup\{\lambda: x_\lambda \in C\} , \quad y_0 = x_{\lambda_0} , \quad y_1 = x_{\lambda_1} .$$

Notice that $\lambda_0 \leqq 0$, $\lambda_1 \geqq 1$ and let $y_\lambda = (1 - \lambda)y_0 + \lambda y_1$. Now implication (4) may be rewritten in the form

$$y_\lambda \in V \Rightarrow y_\lambda \in C .$$

From the definition of y_0 and y_1 it follows that $y_\lambda \in C$ if and only if $\lambda \in [0, 1]$. Consequently, since C is imbedded in V there is an open $\Delta \supset [0, 1]$ such that $y_\lambda \in V$ for $\lambda \in \Delta$. This in turn implies that $y_\lambda \in C$ for $\lambda \in \Delta$ which is impossible according to the definition of y_0 and y_1.

2. Markov-Hopf processes. Let (X, Σ, m) be a measure space with σ-finite measure m. A linear operator $P: L^1 \to L^1$ $(L^1 = L^1(X, m))$ is called a Markov-Hopf process ([2], [3]) if it satisfies the following two conditions:

(a) $Pf \geqq 0$ for $f \geqq 0$, $f \in L^1$,

(b) $\|Pf\| = \|f\|$ for $f \geqq 0$, $f \in L^1$,

where $\| \quad \|$ stands for the norm in L^1.

Denote by $D = D(X, m)$ the set of all densities, that is, all $f \in L^1(X, m)$ such that $f \geqq 0$ and $\|f\| = 1$. From Theorem 1 follows immediately

COROLLARY 1. *Let $P: L^1 \to L^1$ be a Markov-Hopf process for which there exists a convex compact set C imbedded in D and such that the family $\{f: P^n f \to C\}$ is dense in D. Then there exists a unique $f_0 \in D$ which satisfies $Pf_0 = f_0$. Moreover $\lim_n P^n f = f_0$ for $f \in D$.*

The corollary is of special value for Markov-Hopf processes generated by point-to-point transformations of the space X into itself. Let $\varphi: X \to X$ be measurable and nonsingular. The last condition means that $m(\varphi^{-1}(A)) = 0$ whenever $m(A) = 0$ and $A \in \Sigma$. The operator P_φ, corresponding to φ, is defined by the formula

$$P_\varphi f = (d/dm)(\mu_f \circ \varphi^{-1}) , \quad d\mu_f = f dm .$$

P_φ is obviously a Markov-Hopf process. From the definition it follows that μ_f is invariant under φ if and only if $P_\varphi f = f$.

Corollary 1 gives, therefore, a sufficient condition for the existence and uniqueness of invariant measures for some nonsingular transformations. It will be shown below that it is also useful for proving the exactness of some dynamical systems.

Let (X, Σ, μ) be a measure space with normalized measure μ $(\mu(X) = 1)$ and let $\varphi: X \to X$ be a measure preserving transformation. The dynamical system $(X, \Sigma, \mu; \varphi)$ is called exact if the σ-algebra $\bigcap_{n=0}^{\infty} \varphi^{-n}(\Sigma)$ contains only sets of measure zero and their complements. Exactness is a strong property, implying ergodicity and mixing of all orders. It is equivalent [9] to the following condition: For each $A \in \Sigma$ such that $\varphi^n(A) \in \Sigma$ $(n = 1, 2, \cdots)$,

$$\mu(A) > 0 \Rightarrow \lim_n \mu(\varphi^n(A)) = 1 .$$

Using this definition it is easy to prove the following analog of M. Lin condition:

PROPOSITION 1. *Let (X, Σ, m) be a σ-finite measure space and let $\varphi: X \to X$ be a nonsingular tranformation. If there exists $f_0 \in D(X, m)$ such that $\lim_n P_\varphi^n f = f_0$ for each $f \in D$, then the system $(X, \Sigma, \mu_{f_0}; \varphi)$ is exact.*

PROOF. First observe that for each $f \in D$ supported on a set A $(f =$

$1_A f$), the function $P_\varphi^n f$ vanishes outside of $\varphi^n(A)$ ($A, \cdots, \varphi^n(A)$ are assumed to be measurable). In fact, write $B_n = X \setminus \varphi^n(A)$. Form the definition of P_φ it follows that

$$\int_{B_n} P_\varphi^n f dm = \int_{\varphi^{-n}(B_n)} f dm = \int_{A \cap \varphi^{-n}(B_n)} f dm .$$

Since $A \cap \varphi^{-n}(B_n) = \emptyset$, the last integral is equal to zero. This proves that $P_\varphi^n f$ vanishes on B_n. Now assume that $\mu_{f_0}(A) > 0$ and define $f_A = 1_A f_0 / \mu_{f_0}(A)$. Of course $f_A \in D$ and consequently $P_\varphi^n f_A \to f_0$. From the condition $1_{\varphi^n(A)} P_\varphi^n f_A = P_\varphi^n f_A \to f_0$, it follows that $1_{\varphi^n(A)} f_0 \to f_0$, and finally

$$\mu_{f_0}(\varphi^n(A)) = \int_{\varphi^n(A)} f_0 dm \to \int_X f_0 dm = 1 .$$

The following result is a direct consequence of Corollary 1 and Proposition 1:

THEOREM 2. *Let φ: $X \to X$ be a nonsingular transformation of a σ-finite measure space (X, Σ, m). Assume that there exists a convex compact set C imbedded in $D(X, m)$ such that the family $\{f: P_\varphi^n f \to C\}$ is dense in D. Then there exists a unique normalized measure μ absolutely continuous with respect to m and invariant under φ. The system $(X, \Sigma, \mu; \varphi)$ is exact and* $\lim_n P_\varphi^n f = d\mu/dm$ *for each $f \in D$.*

3. Expanding mappings. In this section M will always denote a compact connected smooth (C^∞) manifold equipped with a Riemannian metric $\|\cdot\|$. The metric induces on M the natural (Borel) measure m and the distance ρ. A density $f \in D(M, m)$ will be called regular if there is a constant $c > 0$ such that $f(x) > 0$ and $|f(x) - f(y)| \leqq c\rho(x, y)$ for $x, y \in M$. The regularity of f (see [7]) is defined by

$$\operatorname{Reg} f = \sup_M (|f'|/f)$$

where $|f'|$ is the length of the gradient of f. An important property of regular densities is described by the following:

PROPOSITION 2. *If $f \in D(M, m)$ is regular and* $\operatorname{Reg} f \leqq \alpha$, *then*

$$ke^{-\alpha r} \leqq f(x) \leqq ke^{\alpha r} \quad \text{and} \quad |f'(x)| \leqq ke^{\alpha r} \quad \text{for} \quad x \in M , \tag{5}$$

where $r = \sup\{\rho(x, y): x, y \in M\}$ *and* $k = 1/m(M)$.

PROOF. Let $\gamma(t)$ $(0 \leqq t \leqq 1)$ be an arc joining the points $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$. The differentiation of $f(\gamma(t))$ gives

$$(d/dt)f(\gamma(t)) = \langle f'(\gamma(t)), \gamma'(t)\rangle \leqq \alpha \|\gamma'(t)\| f(\gamma(t))$$

and consequently

$$f(x_1) \leqq f(x_0) \exp\left(\alpha \int_0^1 \|\gamma'(t)\| dt\right) .$$

According to the definition of r this implies $f(x_1) \leqq f(x_0)e^{\alpha r}$ for $x_0, x_1 \in M$. Since f is a density, there is a point $\tilde{x} \in M$ such that $f(\tilde{x}) = k$. Substituting $x_0 = \tilde{x}$ and $x_1 = x$ give the first inequality (5). The second follows from the first one and the condition $\operatorname{Reg} f \leqq \alpha$.

A C^1-mapping $\varphi: M \to M$ is called expanding if there exists a constant $\lambda > 1$ such that at each point $x \in M$ the differential $d\varphi(x)$ satisfies

$$(6) \qquad \|d\varphi(x)\xi\| \geqq \lambda \|\xi\|$$

for each tangent vector ξ. The following theorem (proved in [5]) plays a crucial role in the ergodic theory of expanding mappings.

THEOREM 3 (Krzyżewski, Szlenk). *Assume that $\varphi: M \to M$ is an expanding mapping of class C^2. Then there exists a unique normalized measure μ absolutely continuous with respect to m and invariant under φ. The system (M, μ, φ) is exact and the density $f_0 = d\mu/dm$ is regular. Moreover*

$$(7) \qquad \lim_n P_\varphi^n f = f_0 \qquad \text{for} \quad f \in D(M, m) .$$

PROOF. Since $d\varphi$ is nonsingular, for each point $x \in M$ there is a neighbourhood W_x of x such that $\varphi^{-1}(W_x)$ can be written as the union of disjoint sets $V_1, \cdots, V_N$ and φ restricted to V_i $(i = 1, \cdots, N)$ is a homeomorphism (from V_i onto W_x). Thus on W_x the operator P has an explicit formula

$$P_\varphi f(x) = \sum_i |\operatorname{Det} d\psi_i(x)| (f \circ \psi_i(x)) ,$$

where ψ_i denotes the inverse function to $\varphi|_{V_i}$. Differentiation of $P_\varphi f$ gives

$$\begin{aligned}\frac{|(P_\varphi f)'|}{P_\varphi f} &\leqq \frac{|\sum_i J_i'(f \circ \psi_i)|}{\sum_i J_i(f \circ \psi_i)} + \frac{|\sum_i J_i(d\psi_i)(f' \circ \psi_i)|}{\sum_i J_i(f \circ \psi_i)} \\ &\leqq \max_i \frac{|J_i'|}{J_i} + \max_i \|d\psi_i\| \frac{|f' \circ \psi_i|}{(f \circ \psi_i)} ,\end{aligned}$$

where $J_i(x) = |\operatorname{Det} d\psi_i(x)|$. From (6) it follows that $\|d\psi_i\| \leqq 1/\lambda$. Therefore $\operatorname{Reg} P_\varphi f \leqq \lambda^{-1} \operatorname{Reg} f + K$, where $K = \sup_{i,x} |J_i'(x)|/J_i(x)$ and consequently by induction $\operatorname{Reg} P_\varphi^n f \leqq \lambda^{-n} \operatorname{Reg} f + K(\lambda - 1)^{-1}$. Choose a real $\alpha > K/(\lambda - 1)$. Then $\operatorname{Reg} P_\varphi^n f \leqq \alpha$ for sufficiently large n and, according to Proposition 2, the sequence $P_\varphi^n f$ belongs to the set $C =$

$\{g \in D: ke^{-\alpha r} \leqq g \leqq ke^{\alpha r}, |g'| \leqq ke^{\alpha r}\}$. Since C is convex compact and imbedded in D, this, in virtue of Theorem 2, finishes the proof.

REMARK 1. From the proof it follows that for each regular f the sequence $\{P_\varphi^n f\}$ is relatively compact in the space of continuous functions on M. Thus for such f the convergence in (7) is not only strong in L^1 but also uniform.

4. Piecewise convex transformations. A real valued function g defined on an interval Δ is convex if

$$g(\alpha x + (1 - \alpha)y) \leqq \alpha g(x) + (1 - \alpha)g(y) \quad \text{for} \quad x, y \in \Delta;\ 0 \leqq \alpha \leqq 1 .$$

In general, the density of a measure invariant with respect to a piecewise convex mapping is not differentiable (not even continuous) and the notion of regularity is rather useless now. A somewhat analogous role will be played by "positive variation":

$$\bigvee_a^b {}^+ f = \sup \sum_{i=0}^{n-1} (f(x_{i+1}) - f(x_i))^+ \qquad (f\colon [a, b] \to R) ,$$

where $z^+ = \max(0, z)$ and the supremum is taken over all possible partitions $a = x_0 < x_1 < \cdots < x_n = b$. A simple but useful property of densities (on the unit interval $[0, 1]$) with finite positive variation is described by the following

PROPOSITION 3. *Assume that* $f \in D([0, 1])$ *and*

$$\bigvee_0^1 {}^+ f \leqq \alpha ,$$

then

$$f(x) \leqq (1 + \alpha)/x \qquad \textit{for} \quad x \in (0, 1] .$$

PROOF. According to the definition of positive variation $f(s) \geqq f(x) - \alpha$ for $x \geqq s$. Hence $1 \geqq \int_0^x f(s)ds \geqq \int_0^x (f(x) - \alpha)ds \geqq xf(x) - \alpha$.

Let φ be a given transformation of the unit interval $[0, 1]$ into itself. We shall assume that it satisfies the following conditions:

(i) There exists a partition $0 = a_0 < \cdots < a_N = 1$ such that for each integer i $(i = 1, \cdots, N)$ the restriction φ_i of φ to the interval $[a_{i-1}, a_i)$ is continuous and convex.

(ii) $\varphi_i(a_{i-1}) = 0$, $\varphi_i'(a_{i-1}) > 1$ for $i = 1, \cdots, N$.

(iii) $\varphi_1([a_0, a_1)) = [0, 1)$, $\sup \varphi_1' < \infty$.

From (ii) and the convexity of φ_i it follows that $\varphi_i'(x) \geqq \varphi_i'(a_{i-1}) > 1$ for all $x \in [a_{i-1}, a_i]$.

The foregoing conditions are satisfied in particular for the r-adic transformations $\varphi(x) = rx \pmod 1$ if $r > 1$. The existence of an absolutely continuous invariant measure for these transformations was proved by Rényi [8] and the exactness of the corresponding dynamical systems by Rochlin [9].

The main result of this section is the following

THEOREM 4. *If φ: $[0, 1] \to [0, 1]$ satisfies* (i)-(iii), *then there exists a unique normalized absolutely continuous measure μ invariant under φ. The system* $([0, 1], \mu, \varphi)$ *is exact and the density $f_0 = d\mu/dx$ is positive* $(\inf f_0 > 0)$, *bounded and increasing. Moreover*

$$\lim_n P_\varphi^n f = f_0 \qquad \textit{for} \quad f \in D([0, 1]) .$$

PROOF. A simple computation shows that the operator P_φ can be written in the form

$$P_\varphi f(x) = \sum_{i=1}^{N} \psi_i'(x) f(\psi_i(x)) , \tag{8}$$

where

$$\psi_i(x) = \begin{cases} \varphi_i^{-1}(x) , & x \in \varphi_i([a_{i-1}, a_i)) \\ a_i , & x \in [0, 1] \setminus \varphi_i([a_{i-1}, a_i)) . \end{cases}$$

From (i) and (ii) it follows that the functions ψ_i are increasing, continuous and differentiable except on a set of at most countable number of points. At these points ψ_i' is defined as the right hand derivative. The functions ψ_i' are decreasing and $0 \leqq \psi_i'(x) \leqq \lambda^{-1}$ with $\lambda = \min_i \varphi_i'(a_{i-1}) > 1$. Now consider the set $C = \{g \in D: \delta \leqq g(x) \leqq K,\ g \text{ decreasing}\}$ where the numbers $K \geqq \delta \geqq 0$ will be defined later. It is obvious that C is a convex compact subset of L^1 imbedded in D. Thus in order to finish the proof it is sufficient to show that $P_\varphi^n f \to C$ for each $f \in D$ of bounded variation. The proof of this convergence depends upon the fact that the operator P_φ has the property of shrinking the positive variation. From (8) it follows that

$$\bigvee_0^1{}^+ P_\varphi f \leqq \sum_{i=1}^{N} \bigvee_0^1{}^+ \psi_i'(f \circ \psi_i) .$$

Since ψ_i' are decreasing, one has

$$\bigvee_0^1{}^+ \psi_i'(f \circ \psi_i) \leqq (\sup \psi_i') \bigvee_0^1{}^+ f \circ \psi_i \leqq \lambda^{-1} \bigvee_{a_{i-1}}^{a_i}{}^+ f$$

and consequently

$$\bigvee_0^1{}^+ P_\varphi f \leqq \lambda^{-1} \sum_{i=1}^{N} \bigvee_{a_{i-1}}^{a_i}{}^+ f = \lambda^{-1} \bigvee_0^1{}^+ f .$$

Finally, by induction

$$\bigvee_0^1{}^+ P_\varphi^n f \leqq \lambda^{-n} \bigvee_0^1{}^+ f .$$

Choose a function $f \in D$ of bounded variation. For each $\varepsilon > 0$ there is an integer $n_0(\varepsilon)$ such that the sequence $f_n = P_\varphi^n f$ satisfies

$$(9) \qquad \bigvee_0^1{}^+ f_n \leqq \varepsilon \qquad \text{for} \quad n \geqq n_0(\varepsilon) .$$

Thus, according to Proposition 3, $f_n(x) \leqq 2/x$ for $n \geqq n_1 = n_0(1)$. This inequality allows one to evaluate $f_n(0)$. In fact

$$\begin{aligned} f_{n+1}(0) = P_\varphi f_n(0) &= \psi_1'(0) f_n(0) + \sum_{i=2}^{N} \psi_i'(0) f_n(a_{i-1}) \\ &\leqq \lambda^{-1} f_n(0) + \lambda^{-1} \sum_{i=2}^{N} 2/a_{i-1} \end{aligned}$$

and by induction

$$f_n(0) \leqq f_{n_1}(0)\lambda^{-n+n_1} + K_0 \quad \text{for} \quad n \geqq n_1 , \quad \text{where} \quad K_0 = 2(\lambda - 1)^{-1} \sum_{i=2}^{n} (a_{i-1})^{-1} .$$

From this and (9) it follows that $f_n(x) \leqq f_n(0) + \bigvee_0^{1+} f_n \leqq f_{n_1}(0)\lambda^{-n+n_1} + K_0 + 1$ for $n \geqq n_1$. Let $K = K_0 + 2$. Then there is $n_2 \geqq n_1$ such that

$$(10) \qquad f_n(x) \leqq K \qquad \text{for} \quad n \geqq n_2 .$$

Now it is easy to evaluate f_n from below. In fact $f_{n+1}(x) = P_\varphi f_n(x) \geqq \psi_1'(x) f_n(\psi_1(x))$. By induction this implies

$$(11) \qquad f_{n+r}(x) \geqq \alpha^r f_n(\psi_1^r(x)) \qquad \text{for} \quad n \geqq n_2 , \quad r > 0 ,$$

where, according to (iii), $\alpha = \inf \psi_1' = 1/\sup \varphi_1' > 0$. From (ii) it follows that for sufficiently large r ($r \geqq r_0$) we have $\psi_1^r(x) \leqq (4K)^{-1}$. It is easy to see that $f_n(y) \geqq 1/2$ for $y \leqq (4K)^{-1}$ and large n, namely $n \geqq n_3 = n_2 + n_0(1/4)$. In fact suppose not, then

$$\begin{aligned} 1 = \int_0^1 f_n dx = \int_0^y f_n dx + \int_y^1 f_n dx &\leqq Ky + \sup_{[y,1]} f_n \\ &\leqq \frac{1}{4} + \left(f_n(y) + \bigvee_0^1{}^+ f_n \right) < \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1 \end{aligned}$$

which is impossible. Thus, for $r = r_0$, inequality (11) implies

$$(12) \qquad f_n(x) \geqq \delta \qquad \text{for} \quad n \geqq n_4 = n_3 + r_0$$

where $\delta = \alpha^{r_0}/2$. Now write $g_n(x) = (1 - \theta) \sup \{f_n(s) \colon x \leqq s \leqq 1\} + \theta \inf \{f_n(s) \colon 0 \leqq s \leqq x\}$ where $\theta \in [0, 1]$ and is chosen such that $\|g_n\| = 1$. From the definition it follows that g_n is decreasing. According to (10) and (12), $\delta \leqq g_n \leqq K$ for $n \geqq n_4$. Thus $g_n \in C$ for $n \geqq n_4$. On the other hand, from (9) it follows that $\sup |f_n - g_n| \leqq \varepsilon$ for $n \geqq n_0(\varepsilon)$. This implies $\rho(f_n, C) \leqq \varepsilon$ for $n \geqq n_0(\varepsilon) + n_4$ and finishes the proof of the convergence $f_n = P_\varphi^n f \to C$.

REMARK 2. The existence of absolutely continuous invariant measures for piecewise convex transformations has been proved in [4] and [6] under weaker assumptions than (i)-(iii). In particular, Condition (iii) can be fully omitted. In this case, however, little can be said about the ergodic properties of corresponding dynamical systems.

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