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A DECOMPOSITION OF REDUCING SUBSPACE FOR SHIFT OPERATORS

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Abstract. Let $\mathscr{S}$ be a family of shift operators on $\mathfrak{H} = \sum_{n \in \boldsymbol{Z}} \oplus \mathfrak{K}_n$ such that the von Neumann algebra $M(\mathscr{S})$ generated by $\mathscr{S}$ is the crossed product of a factor M on $\mathfrak{K} = \mathfrak{K}_0$ by $\boldsymbol{Z}$ with respect to an action of spatial automorphisms $\{\mathrm{Ad}\, u^n\}_{n \in \boldsymbol{Z}}$ of M. It is proved that every reducing subspace for $\mathscr{S}$ is decomposed into two reducing subspaces: one is generated by a pure simply invariant subspace and the other contains no simply invariant subspace.

Introduction. This paper is a continuation of [7]. Let $\mathfrak{H}$ be the direct sum $\sum_{n \in \boldsymbol{Z}} \oplus \mathfrak{K}_n$ $(= \ell^2(\boldsymbol{Z}) \otimes \mathfrak{K})$ of a countable family $\{\mathfrak{K}_n\}_{n \in \boldsymbol{Z}}$ of copies of a separable Hilbert space $\mathfrak{K}$. A unitary operator U on $\mathfrak{H}$ is said to be a shift operator if

$$U: (\xi_n)_{n \in \boldsymbol{Z}} \to (w_n \xi_{n-1})_{n \in \boldsymbol{Z}} \quad (\xi_n \in \mathfrak{K},\ n \in \boldsymbol{Z}),$$

where w_n is a unitary operator on $\mathfrak{K}$. When $w_n = 1$ for all n in $\boldsymbol{Z}$, U becomes the usual shift operator and is denoted by $s \otimes 1$. Then each shift operator U is of the form $U = W \cdot (s \otimes 1)$, where $W = \sum_{n \in \boldsymbol{Z}} \oplus w_n$. Let $\mathscr{S}$ be a family of shift operators on $\mathfrak{H}$ and $W(\mathscr{S})$ denote the set of diagonal parts of shift operators in $\mathscr{S}$, that is, $W(\mathscr{S}) = \{W : U = W \cdot (s \otimes 1),\ U \in \mathscr{S}\}$.

Our purpose is to analyze the structure of reducing subspaces for $\mathscr{S}$. We here recall the case where $\mathscr{S} = \{s \otimes 1\}$. When $\dim \mathfrak{K} = 1$, $\mathfrak{H}$ is regarded as $L^2(\boldsymbol{T})$ and $s \otimes 1 = s$ is the multiplication operator by $f(z) = z$, where $\boldsymbol{T}$ is the unit circle in the complex plane. According to Beurling's theorem in [6], any non-trivial reducing subspace $M_{\chi_E} L^2(\boldsymbol{T})$ contains no simply invariant subspace, where M_{χ_E} is the multiplication operator by the characteristic function of a measurable subset E of $\boldsymbol{T}$. For an arbitrary $\mathfrak{K}$, $\mathfrak{H}$ is regarded as the Hilbert space $L^2(\boldsymbol{T}, \mathfrak{K})$ of measurable $\mathfrak{K}$-valued functions f such that the functions: $z \to \|f(z)\|$, are square integrable, and $s \otimes 1$ is the multiplication operator by $F = F(z) = z1$ $(z \in \boldsymbol{T})$. According to Halmos and Helson's theorem in [5] and

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[6], a reducing subspace $\mathfrak{N} = M_F L^2(T, \mathfrak{K})$ contains a simply invariant subspace if and only if $\dim F(z) \geqq 1$ for a.e. z in T, where F is a projection-valued measurable function on T.

In this paper, we seek a necessary and sufficient condition for reducing subspaces for $\mathscr{S}$ to contain a simply invariant subspace. In the above classical theory, the following two decomposition theorems for invariant subspaces play an important role.

(1) If $\mathfrak{M}$ is a simply invariant subspace, then we have

$$\mathfrak{M} = \mathfrak{M}_p \oplus \mathfrak{M}_r ,$$

where $\mathfrak{M}_p$ is a pure simply invariant subspace and $\mathfrak{M}_r$ is a reducing subspace.

(2) If $\mathfrak{M}$ is a pure simply invariant subspace, then $\mathfrak{M}$ is decomposed into the wandering subspaces; i.e.,

$$\mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots .$$

Although the above two theorems do not necessarily hold for an arbitrary family $\mathscr{S}$, the author [7] showed that they hold if $\mathscr{S}$ satisfies the following conditions:

$$(*) \qquad \begin{array}{ll} \text{(i)} & W(\mathscr{S}) \text{ is a group ,} \\ \text{(ii)} & SW(\mathscr{S})S^* = W(\mathscr{S}) . \end{array}$$

Throughout this paper, the condition $(*)$ is assumed.

For $\mathscr{S}$, we denote by $M(\mathscr{S})$ (resp. $D(\mathscr{S})$) the von Neumann algebra generated by $\mathscr{S}$ (resp. $W(\mathscr{S})$). Let J_i be the isometric operator of $\mathfrak{K}$ into $\mathfrak{H}$ defined by

$$J_i(\xi) = (\xi_n)_{n \in Z} \qquad (\xi_i = \xi,\ \xi_n = 0 \text{ for } n \neq i) .$$

Let M denote the von Neumann algebra $J_0^* D(\mathscr{S}) J_0$ and N denote the commutant of M on $\mathfrak{K}$. The main theorem in [7] says that every pure simply invariant subspace is of Beurling type if and only if there exists a unitary operator u on $\mathfrak{K}$ such that

(**)
(1) $[\mathrm{Ad}\, u](M) = M$, (hence $[\mathrm{Ad}\, u](N) = N$) ,
(2) $M(\mathscr{S})$ is the crossed product of M by Z with respect to the action $\{\mathrm{Ad}\, u^n\}_{n \in Z}$,
(3) $\mathrm{Ad}\, u$ leaves the finite central projections in N element-wise fixed ,

where $\mathrm{Ad}\, u$ means the *-automorphism of the full operator algebra $B(\mathfrak{K})$ defined by $[\mathrm{Ad}\, u](x) = uxu^*$ for x in $B(\mathfrak{K})$. In this paper, we assume the condition (**) in addition to (*). Moreover, in order to clarify our discussion, we restrict ourselves to the case where M is a factor.

In Section 2, for each case of $\mathscr{S}$ we give decomposition theorems for reducing subspaces for $\mathscr{S}$ (Theorems 2.5, 10, 11, 12, 13). To get the decomposition theorems in the case where $M(\mathscr{S})$ is semi-finite we use the center-valued trace, or a generalized center-valued trace of the commutant $N(\mathscr{S})$ of $M(\mathscr{S})$. For this reason, Section 1 is devoted to discussions about crossed products and center-valued traces as well as generalized center-valued traces. In the case where $N(\mathscr{S})$ is finite (resp. properly-infinite and semi-finite), we describe the center-valued trace (resp. a generalized center-valued trace) by producing a candidate directly, instead of constructing it along the proof of the existence theorem [10: V. Theorem 2.6, 2.34].

For standard results of crossed products we refer to the book of Strătilă [9: Section 22].

1. Crossed products and their centers. Let M be a factor on a separable Hilbert space $\mathfrak{K}$ and u a unitary operator on $\mathfrak{K}$ such that $\mathrm{Ad}\,u$ is a spatial automorphism of M. Throughout this paper N denotes the commutant of M. Then $\mathrm{Ad}\,u$ is a spatial automorphism of N, too. The crossed product $\mathscr{R}(M, \mathrm{Ad}\,u)$ of M by $\boldsymbol{Z}$ with respect to the action $\{\mathrm{Ad}\,u^n\}_{n\in\boldsymbol{Z}}$ is a von Neumann algebra on the direct sum $\mathfrak{H} = \sum_{n\in\boldsymbol{Z}}\bigoplus \mathfrak{K}_n$ $(=\ell^2(\boldsymbol{Z})\otimes\mathfrak{K})$. For x in M, we put $I(x) = \sum_{n\in\boldsymbol{Z}}\bigoplus [\mathrm{Ad}\,u^{-n}](x)$. Let s denote the usual shift operator on $\ell^2(\boldsymbol{Z})$: $(\xi_n)_{n\in\boldsymbol{Z}} \to (\xi_{n-1})_{n\in\boldsymbol{Z}}$. Then $\mathscr{R}(M, \mathrm{Ad}\,u)$ is the von Neumann algebra generated by $\{I(x): x\in M\}$ and $\{s^n\otimes 1: n\in\boldsymbol{Z}\}$. It is well-known the commutant of $\mathscr{R}(M, \mathrm{Ad}\,u)$ is the von Neumann algebra generated by $\{1\otimes x: x\in N\}$ and $\{s^n\otimes u^{-n}: n\in\boldsymbol{Z}\}$, and it is denoted by $\mathscr{L}(M, \mathrm{Ad}\,u^{-1})$. Hereafter we use letters $\mathscr{R}$ and $\mathscr{L}$ instead of $\mathscr{R}(M, \mathrm{Ad}\,u)$ and $\mathscr{L}(N, \mathrm{Ad}\,u^{-1})$ for short when there is no confusion.

We here consider the matrix representations of elements of $\mathscr{R}$ and $\mathscr{L}$. For A in $B(\mathfrak{H})$ and i, j in $\boldsymbol{Z}$, we put $A_{i,j} = J_i^*AJ_j$, where $B(\mathfrak{H})$ is the full operator algebra on H. Moreover we put $P_i = J_iJ_i^*$. Then $\{P_i\}_{i\in\boldsymbol{Z}}$ is a family of orthogonal projections on $\mathfrak{H}$ such that $\sum_{i\in\boldsymbol{Z}} P_i = I$ and we have $P_iAP_j = J_iA_{i,j}J_j^*$. We need the following two well-known representations of elements of $\mathscr{R}$ and $\mathscr{L}$ for our discussions.

(1-1) A $(\in B(\mathfrak{H}))$ belongs to $\mathscr{R}$ if and only if there exists a mapping $f: \boldsymbol{Z}\to M$ such that $A_{i,j} = u^{-i}f(i-j)u^i$.

(1-2) A $(\in B(\mathfrak{H}))$ belongs to $\mathscr{L}$ if and only if there exists a mapping $g: \boldsymbol{Z}\to N$ such that $A_{i,j} = g(i-j)u^{j-i}$.

To get the matrix representation of the center $\mathscr{Z} = \mathscr{R} \cap \mathscr{L}$, we examine whether the automorphism $\mathrm{Ad}\, u^n$ is inner or not for each n in $\boldsymbol{Z}$. For a von Neumann algebra $\mathscr{A}$ such that $[\mathrm{Ad}\, u](\mathscr{A}) = \mathscr{A}$, we put

$$I(\mathrm{Ad}\, u, \mathscr{A}) = \{n \in \boldsymbol{Z}\colon \mathrm{Ad}\, u^n \text{ is an inner automorphism of } \mathscr{A}\}\,.$$

Then $I(\mathrm{Ad}\, u, M)$ is a subgroup of $\boldsymbol{Z}$ and $I(\mathrm{Ad}\, u, M) = I(\mathrm{Ad}\, u, N)$. When $I(\mathrm{Ad}\, u, M) = \{0\}$, $\mathscr{R}$ and $\mathscr{L}$ are factors (cf. [9: 22.6, Corollary 1]). Thus, of course, the center $\mathscr{Z}$ is the trivial algebra. Next we assume that $I(\mathrm{Ad}\, u, M) \neq \{0\}$. Let p be the smallest number in $\{n \in I(\mathrm{Ad}\, u, M)\colon n \geqq 0\}$. Then $\mathrm{Ad}\, u^p = \mathrm{Ad}\, v$ on M for some unitary operator v in M. We can easily find that the unitary operator v in M is uniquely determined up to constant multiple. Put $w = v^* u^p$. Then w is a unitary operator in N and $u^p = vw$. Let γ be a complex number with absolute value 1 such that $[\mathrm{Ad}\, u](v) = \gamma v$. Then γ is uniquely determined and $[\mathrm{Ad}\, u](w) = \bar{\gamma} w$. According to Connes [2], p and $m = pq$ (q = order of γ) are said to be the outer period and the minimal period of $\mathrm{Ad}\, u$, respectively and these numbers are fixed in this paper. We can see examples of such automorphisms in the preface of [3] added by Lance, in addition to [2].

The following is the matrix representations of operators in the center $\mathscr{Z}$ and this is easily obtained by (1–1) and (1–2) (cf. [9: 22.6, Theorem]).

(1–3) A ($\in B(\mathfrak{H})$) belongs to $\mathscr{Z}$ if and only if $A_{i,j} = f(i-j)$ for some mapping $f\colon \boldsymbol{Z} \to M$ such that $f(mk+i) = \delta_{i,0} \lambda_{mk} v^{qk}$ ($i = 0, \cdots, m-1$), where $\delta_{i,j}$ are Kronecker symbols and λ_{mk} complex numbers.

Since A is regarded as a bounded linear operator on $\ell^2(\boldsymbol{Z}) \otimes \mathfrak{K}$ the following representations are useful in what follows.

(1–4) $A \sim \sum_{n \in \boldsymbol{Z}} s^n \otimes x_n u^{-n}$ means that A is the operator in $\mathscr{L}$ defined by (1–2), where $x_n = g(n)$.

(1–5) $A \sim \sum_{k \in \boldsymbol{Z}} s^{mk} \otimes \lambda_{mk} v^{-qk}$ means that A is the operator in $\mathscr{Z}$ defined by (1–3).

We remark that symbol $\sum$ in (1–4) and (1–5) does not mean σ-weak convergence in $\mathscr{L}$ or $\mathscr{Z}$ but merely formal sum.

Let $M(s)$ denote the von Neumann algebra generated by the shift s on $\ell^2(\boldsymbol{Z})$. Then $M(s)$ is *-isomorphic to the W^*-algebra $L^\infty(\boldsymbol{T})$ of all essentially bounded measurable functions on the unit circle $\boldsymbol{T}$, in which two functions are identified if they coincide almost everywhere. Furthermore the tensor product $M(s) \mathbin{\overline{\otimes}} B(\mathfrak{K})$ is *-isomorphic to the W^*-algebra of $B(\mathfrak{K})$-valued essentially bounded measurable functions on $\boldsymbol{T}$, which is denoted by $L^\infty(\boldsymbol{T}, B(\mathfrak{R}))$ ([10: IV. Theorem 7.17]). An operator-valued measurable function $F\colon \boldsymbol{T} \to B(\mathfrak{R})$, means that for ξ and η in $\mathfrak{R}$, the scalar-valued function: $z \to \langle F(z)\xi, \eta\rangle$ is measurable on $\boldsymbol{T}$, where $\langle\cdot,\cdot\rangle$ is the inner product of $\mathfrak{R}$. As in the scalar case, we define the Fourier coefficients of F in $L^\infty(\boldsymbol{T}, B(\mathfrak{R}))$ as follows: For n in $\boldsymbol{Z}$, let $\hat{F}(n)$ be the operator in $B(\mathfrak{R})$ satisfying the following equalities:

$$\langle \hat{F}(n)\xi, \eta\rangle = (1/2\pi)\int_0^{2\pi} \langle F(e^{it})\xi, \eta\rangle e^{-int}dt \qquad (\xi, \eta \in \mathfrak{R}) .$$

Let π be the canonical *-isomorphism of $M(s) \bar{\otimes} B(\mathfrak{R})$ onto $L^\infty(\boldsymbol{T}, B(\mathfrak{R}))$. Then, for each x in $B(\mathfrak{R})$, $\pi(s \otimes x)(z) = zx$ for all z in $\boldsymbol{T}$ and immediately we get the following:

(1-6) $\pi(\mathscr{L}) = \{F \in L^\infty(\boldsymbol{T}, B(\mathfrak{R}))\colon u^n\hat{F}(n) \in N\}$,

(1-7) $\pi(\mathscr{Z}) = \{F \in L^\infty(\boldsymbol{T}, B(\mathfrak{R}))\colon \hat{F}(mk + i) = 0$ for $i = 0, 1, \cdots, mk - 1$, $\hat{F}(mk) = \lambda_{mk}v^{-mk}\}$.

For convenience, we seek a unitary operator V on $\mathfrak{H}$ such that $\pi([\mathrm{Ad}\ V](\mathscr{Z})) = L^\infty(\boldsymbol{T}, m, \boldsymbol{C}(\mathfrak{R}))$, where the right hand side means the set of periodic $\boldsymbol{C}(\mathfrak{R})$-valued essentially bounded measurable functions on $\boldsymbol{T}$ with period $2\pi/m$, i.e., $F(z) = F(ze^{2\pi i/m})$ for a.e. z in $\boldsymbol{T}$ and $\boldsymbol{C}(\mathfrak{R})$ is the algebra of all scalar multiples. We put $e_n = (x_m)$ $(\in \ell^2(\boldsymbol{Z}))$, where $x_n = 1$ and $x_m = 0$ for $m \neq n$. Then the required operator is the unitary operator V defined by $V(e_{pn+i} \otimes \xi) = e_{pn+i} \otimes v^n\xi$ for $0 \leqq i \leqq p - 1$ and n in $\boldsymbol{Z}$. In the remainder of this section, we construct the mapping of $\mathscr{L}$ onto $L^\infty(\boldsymbol{T}, m, \boldsymbol{C}(\mathfrak{R}))$ which corresponds to the center-valued trace or a generalized center-valued trace of $\mathscr{L}$. Let α represent the *-automorphism of $L^\infty(\boldsymbol{T}, B(\mathfrak{R}))$ defined by $\alpha(F)(z) = F(ze^{-2\pi i/m})$ $(z \in \boldsymbol{T})$ for F in $L^\infty(\boldsymbol{T}, B(\mathfrak{R}))$. We put

$$\Psi_1(F) = (1/m)(F + \alpha(F) + \cdots + \alpha^{m-1}(F)) \qquad (F \in L^\infty(\boldsymbol{T}, B(\mathfrak{R}))$$

and

$$\Phi_1(A) = (\pi^{-1}\cdot\Psi_1\cdot\pi)(A) \qquad (A \in \mathscr{L}) .$$

LEMMA 1.1. *Let A be an operator in $\mathscr{L}$. If $A \sim \sum_{n\in\boldsymbol{Z}} s^n \otimes x_n u^{-n}$, then $\Phi_1(A) \sim \sum_{k\in\boldsymbol{Z}} s^{mk} \otimes x_{mk}u^{-mk}$.*

PROOF. For F in $L^\infty(\boldsymbol{T}, B(\mathfrak{R}))$, we have

$$\begin{aligned}(\Psi_1(F))\hat{}\,(n) &= (1/2m\pi)\sum_{k=0}^{m-1}\int_0^{2\pi} F(e^{i(t+2k\pi/m)})e^{-int}dt \\ &= (1/2m\pi)\sum_{k=0}^{m-1}\left(\int_0^{2\pi} F(e^{it})e^{-int}dt\right)e^{2nk\pi i/m}\end{aligned}$$

$$= (1/m)\hat{F}(n)\sum_{k=0}^{m-1} e^{2nk\pi i/m} .$$

Thus we have $(\Psi_1(F))^\wedge(n) = \hat{F}(n)$ if $n = mk$ $(k \in \boldsymbol{Z})$ and $=0$ otherwise.
q.e.d.

Now we consider the matrix representations of operators in $\Phi_1(\mathscr{L})$. Since $u^p = vw$ $(v \in M, w \in N)$, we have $s^{mk} \otimes x_{mk}u^{-mk} = s^{mk} \otimes x_{mk}v^{-qk}w^{-qk} = s^{mk} \otimes y_{mk}v^{-qk}$, $(y_{mk} = x_{mk}w^{qk} \in N)$. Hence it follows that

$$\Phi_1(\mathscr{L}) = \{A \in \mathscr{L}: A \sim \sum_{k \in \boldsymbol{Z}} s^{mk} \otimes x_{mk}v^{-qk},\ x_{mk} \in N\} .$$

Thus we have

$$\pi(\Phi_1(\mathscr{L})) = \{F \in L^\infty(\boldsymbol{T}, B(\mathfrak{H})): \hat{F}(mk+i) = \delta_{i,0}x_{mk}v^{-qk},\ x_{mk} \in N\} .$$

For the unitary operator V defined above, we have $[\mathrm{Ad}\ V](s^{mk} \otimes x_{mk}v^{-qk}) = s^{mk} \otimes x_{mk}$, so that

$$\mathscr{L}_1 = [\mathrm{Ad}\ V](\Phi_1(\mathscr{L})) = M(s^m)\mathbin{\overline{\otimes}} N \quad \text{and} \quad \pi(\mathscr{L}_1) = L^\infty(\boldsymbol{T}, m, N) .$$

For a moment, N is assumed to be a finite factor with the unique trace τ. For F in $L^\infty(\boldsymbol{T}, m, N)$, we put $\Psi_2(F)(z) = \tau(F(z))1$. Then Ψ_2 coincides with the canonical center-valued trace of $L^\infty(\boldsymbol{T}, m, N)$ onto $L^\infty(\boldsymbol{T}, m, \boldsymbol{C}(\mathfrak{K}))$, and we put

$$\Phi_2(A) = (\pi^{-1}\cdot\Psi_2\cdot\pi)(A) \qquad (A \in \mathscr{L}_1) .$$

Then we have the following lemma.

LEMMA 1.2. *Let A be an operator in $\mathscr{L}_1$. If $A \sim \sum_{k \in \boldsymbol{Z}} s^{mk} \otimes x_{mk}$, then $\Phi_2(A) \sim \sum_{k \in \boldsymbol{Z}} s^{mk} \otimes \tau(x_{mk})1$.*

PROOF. For each operator A of the form $A = s^{mk} \otimes x (x \in N)$, we have

$$\begin{aligned}\int_0^{2\pi} \Psi_2(\pi(A))(e^{it})e^{-int}dt &= \int_0^{2\pi} \tau(\pi(A)(e^{it}))1e^{-int}dt \\ &= \int_0^{2\pi} \tau(x)1e^{mkt}e^{-int}dt = \tau(x)1\int_0^{2\pi} e^{i(mk-n)t}dt \\ &= \begin{cases} 2\pi\cdot\tau(x)1 & \text{if} \quad n = mk \ (k \in \boldsymbol{Z}) , \\ 0 & \text{otherwise} . \end{cases}\end{aligned}$$

Thus $(\pi^{-1}\cdot\Psi_2\cdot\pi)(A) = s^{mk} \otimes \tau(x)1$. Hence the statement holds for all operators A of the form $A = \sum_{k=-K}^{K} s^{mk} \otimes x_{mk}$ $(K \geqq 0)$. We shall prove the statement in the general case. As in the case of scalar-valued functions [4: p. 20, Theorem], the Cesàro means $\sigma_n(F)$ of F in $L^\infty(\boldsymbol{T}, m, N)$ converge σ-weakly to F (cf. [8: Lemma 2.5]). Since Ψ_2 is σ-weakly continuous, $\Psi_2(\sigma_n(F))$ converge σ-weakly to $\Psi_2(F)$ and thus $(\Psi_2(\sigma_n(F)))^\wedge(i)$ converge to $(\Psi_2(F))^\wedge(i)$ for each i in $\boldsymbol{Z}$. For $A \sim \sum_{k\in\boldsymbol{Z}} s^{mk} \otimes x_{mk}$ in $\mathscr{L}_1$, we have

$$(\Psi_2(\sigma_n(\pi(A))))^\wedge(i) = \begin{cases}(1 - |i|/n)\tau(x_i)1 & \text{if } \quad -n < i = mk < n \,, \\ 0 & \text{otherwise} \,. \end{cases}$$

Hence we have $(\Psi_2(\pi(A)))^\wedge(i) = \tau(x_i)1$ if $i = mk$ $(k \in \boldsymbol{Z})$ and $= 0$ otherwise. q.e.d.

REMARK. In [8], the Cesàro means are associated with a periodic flow acting on a von Neumann algebra. In the present paper, it is the periodic *-automorphism group $\{\beta_t\}_{t\in\boldsymbol{R}}$ of $L^\infty(\boldsymbol{T}, m, N)$ defined by $\beta_t(F)(z) = F(ze^{-it})$ for F in $L^\infty(\boldsymbol{T}, m, N)$.

THEOREM 1.3. *Suppose that N is a finite factor and $I(\mathrm{Ad}\, u, N) \neq \{0\}$. Let $\Phi = \mathrm{Ad}\, V^* \cdot \Phi_2 \cdot \mathrm{Ad}\, V \cdot \Phi_1$. Then Φ is the unique center-valued trace of $\mathscr{L}$ onto $\mathscr{Z}$.*

Before proving the theorem, let us recall the properties of center-valued trace. The center-valued trace is the unique mapping Φ satisfying the following conditions: (1) Φ is a linear mapping of $\mathscr{L}$ onto $\mathscr{Z}$, (2) $\Phi(I) = I$, (3) $\Phi(A^*A) = \Phi(AA^*) \geqq 0$, (4) $\Phi(A^*A) \neq 0$ if $A \neq 0$, (5) $\Phi(XA) = X\Phi(A)$ for X in $\mathscr{Z}$ and A in $\mathscr{L}$. In our proof, instead of Φ, we consider the mapping $\Psi_2 \cdot \pi \cdot \mathrm{Ad}\, V \cdot \Phi_1$ of $\mathscr{L}$ onto $L^\infty(\boldsymbol{T}, m, \boldsymbol{C}(\mathfrak{K}))$, which is denoted by T, and show that T satisfies the following conditions; (i) T is linear, (ii) $T(I) = I$, (iii) $T(A^*A) = T(AA^*) \geqq 0$, (iv) $T(A^*A) \neq 0$ if $A \neq 0$, (v) $T(XA) = \pi([\mathrm{Ad}\, V](X))T(A)$. If this is done, the statement in the theorem mentioned above trivially holds.

PROOF OF THEOREM 1.3. We first see the positivity of Ψ_1 and Ψ_2. Let F be a positive function in $L^\infty(\boldsymbol{T}, B(\mathfrak{K}))$. Then $F(z)$ and $\alpha^n(F)(z)$ are positive operators in $B(\mathfrak{K})$ for a.e. z in $\boldsymbol{T}$ and $n = 1, \cdots, m-1$. Hence $\Psi_1(F)(z) = (1/m)\sum_{n=0}^{m-1} \alpha^n(F)(z) \geqq 0$ for a.e. z in $\boldsymbol{T}$. Next, let G be a positive operator in $L^\infty(\boldsymbol{T}, m, N)$. Then we have $\Psi_2(G)(z) = \tau(G(z))1 \geqq 0$ for a.e. z in $\boldsymbol{T}$. Since π and $\mathrm{Ad}\, V$ are *-isomorphisms, T satisfies (i) and (iv). Condition (ii) holds trivially.

To prove (iii) and (v), let $A \sim \sum_{n\in\boldsymbol{Z}} s^n \otimes x_n u^{-n}$ and $X \sim \sum_{k\in\boldsymbol{Z}} s^{mk} \otimes \lambda_{mk} v^{-qk}$. Since $\pi(X)$ is periodic of period $2\pi/m$, we have $\alpha(\pi(XA)) = \pi(X)\alpha(\pi(A))$. From this, we have $\Phi_1(XA) = X\Phi_1(A)$. Since $[\mathrm{Ad}\, V](X) \sim \sum_{k\in\boldsymbol{Z}} s^{mk} \otimes \lambda_{mk}$, $\pi([\mathrm{Ad}\, V](X))$ is a $\boldsymbol{C}(\mathfrak{K})$-valued function, so that

$$\begin{aligned}(\Psi_2 \cdot \pi)([\mathrm{Ad}\, V](X\Phi_1(A)) &= \Psi_2(\pi([\mathrm{Ad}\, V](X))\pi([\mathrm{Ad}\, V](\Phi_1(A))) \\ &= \pi([\mathrm{Ad}\, V](X))\Psi_2(\pi([\mathrm{Ad}\, V](\Phi_1(A)))) = \pi([\mathrm{Ad}\, V](X))T(A) \,.\end{aligned}$$

Hence Condition (v) holds. We now show $T(A^*A) = T(AA^*)$. Since $(AA^*)_{i,j} = \sum_{k\in Z} A_{i,k}A^*_{k,j}$, we have

$$AA^* \sim \sum_{n\in Z} s^n \otimes (\sum_{i\in Z} x_i u^{-n} x^*_{i-n} u^n) u^{-n}$$

and

$$A^*A \sim \sum_{n\in Z} s^n \otimes (\sum_{i\in Z} u^{i-n} x^*_{i-n} x_i u^{n-i}) u^{-n}$$

where $\sum_{i\in Z} x_i u^{-n} x^*_{i-n} u^n$ and $\sum_{i\in Z} u^{i-n} x^*_{i-n} x_i u^{n-i}$ converges in N with respect to the σ-weak topology.

Thus we have

$$\Phi_1(AA^*) \sim \sum_{k\in Z} s^{mk} \otimes (\sum_{i\in Z} x_i u^{-mk} x^*_{i-mk} u^{mk}) u^{-mk}$$

and

$$\Phi_1(A^*A) \sim \sum_{k\in Z} s^{mk} \otimes (\sum_{i\in Z} u^{i-mk} x^*_{i-mk} x_i u^{mk-i}) u^{-mk} .$$

Hence we have

$$\begin{aligned}(\Phi_2 \cdot \mathrm{Ad}\, V \cdot \Phi_1)(AA^*) &\sim \sum_{k\in Z} s^{mk} \otimes \tau((\sum_{i\in Z} x_i u^{-mk} x^*_{i-mk} u^{mk}) w^{-qk})1 \\ &\sim \sum_{k\in Z} s^{mk} \otimes \sum_{i\in Z} \tau(x_i w^{-qk} x^*_{i-mk})1\end{aligned}$$

and

$$\begin{aligned}(\Phi_2 \cdot \mathrm{Ad}\, V \cdot \Phi_1)(A^*A) &\sim \sum_{k\in Z} s^{mk} \otimes \tau((\sum_{i\in Z} u^{i-mk} x^*_{i-mk} x_i u^{mk-i}) w^{-qk})1 \\ &\sim \sum_{k\in Z} s^{mk} \otimes \sum_{i\in Z} \tau(u^{i-mk} x^*_{i-mk} x_i u^{mk-i} w^{-qk})1 .\end{aligned}$$

Since $[\mathrm{Ad}\, u](w^q) = w^q$ and $\tau([\mathrm{Ad}\, u](x)) = \tau(x)$ $(x \in N)$, we have

$$\begin{aligned}\tau(u^{i-mk} x^*_{i-mk} x_i u^{mk-i} w^{qk}) &= \tau(u^{i-mk} x^*_{i-mk} x_i w^{-qk} u^{mk-i}) \\ &= \tau(x^*_{i-mk} x_i w^{-qk}) = \tau(x_i w^{-qk} x^*_{i-mk}) .\end{aligned}$$

Therefore it follows that $(\Phi_2 \cdot \mathrm{Ad}\, V \cdot \Phi_1)(AA^*) = (\Phi_2 \cdot \mathrm{Ad}\, V \cdot \Phi_1)(A^*A)$, so that $T(AA^*) = T(A^*A)$. q.e.d.

In the remainder of this section, N is assumed to be a semi-finite properly-infinite factor with an $\mathrm{Ad}\, u$-invariant faithful normal trace τ. The condition $I(\mathrm{Ad}\, u, N) \neq \{0\}$ is still assumed. Then we can get a generalized center-valued trace of $\mathscr{L}$ as in the finite case above. In this case, instead of $L^\infty(\boldsymbol{T}, m, C(\mathfrak{R}))$ (resp. $\mathscr{Z}$), we consider the set $\hat{L}^\infty(\boldsymbol{T}, m, C(\mathfrak{R}))_+$ (resp. $\hat{\mathscr{Z}}_+$) of all the supremums of increasing net of positive operators in $L^\infty(\boldsymbol{T}, m, C(\mathfrak{R}))$ (resp. $\mathscr{Z}$). Since π is an order preserving mapping of $[\mathrm{Ad}\, V](\mathscr{Z}_+)$ onto $L^\infty(\boldsymbol{T}, m, C(\mathfrak{R}))_+$, it can be canonically extended to an order isomorphism of $[\mathrm{Ad}\, V](\hat{\mathscr{Z}}_+)$ onto

$\hat{L}^\infty(\boldsymbol{T}, m, C(\mathfrak{K}))_+$, which is again denoted by π, where $[\mathrm{Ad}\, V](B) = \sup [\mathrm{Ad}\, V](A_i)$ for $B = \sup A_i$ in $\hat{\mathscr{Z}}_+$. In this situation, we put $\Psi_2(F)(z) = \tau(F(z))1$ $(z \in \boldsymbol{T})$ for a positive function F in $L^\infty(\boldsymbol{T}, m, N)$. Then $\Psi_2(F)$ is a $[0, \infty]$-valued measurable function on $\boldsymbol{T}$. Further we put $\Phi_2 \cdot \pi^{-1} \cdot \Psi_2 \cdot \pi$ and $\Phi = \mathrm{Ad}\, V^* \cdot \Phi_2 \cdot \mathrm{Ad}\, V \cdot \Phi_1$. Let N_τ denote the definition ideal of τ (cf. [10: V. Definition 2.17]). Then τ can be extended to a linear functional $\dot{\tau}$ on N_τ. Then, for B in $\mathscr{L}_1 = [\mathrm{Ad}\, V](\Phi_1(\mathscr{L}))$ of the form $B = \sum_{k=-K}^{K} s^{mk} \otimes x_{mk}$ $(x_{mk} \in N_\tau)$, we have $\Phi_2(B) = \sum_{k=-K}^{K} s^{mk} \otimes \dot{\tau}(x_{mk})$ (cf. Lemma 1.4). Therefore, for A in $\mathscr{L}$ of the form $A = \sum_{n=-K}^{K} s^n \otimes x_n u^{-n}$ $(x \in N_\tau)$, we have $\Phi(A^*A) = \Phi(AA^*)$. Since τ is normal, so is the positive linear mapping Φ on the set $\mathscr{L}_+$ of positive operators in $\mathscr{L}$. Hence $\Phi(A^*A) = \Phi(AA^*)$ for every A in $\mathscr{L}$ because the set $\{A \in \mathscr{L}: A = \sum_{n=-K}^{K} s^n \otimes x_n u^{-n}, x_n \in N_\tau, K \geqq 0\}$ is σ-weakly dense in $\mathscr{L}$. Therefore Φ enjoys the properties of generalized semi-finite trace on $\mathscr{L}_+$ onto $\hat{\mathscr{Z}}_+$, that is, (1) $\Phi(A + B) = \Phi(A) + \Phi(B)$ for A and B in $\mathscr{L}_+$, (2) $\Phi(XA) = X\Phi(A)$ for A in $\mathscr{L}_+$ and X in $\mathscr{Z}_+$, (3) $\Phi(A^*A) = \Phi(AA^*)$ for A in $\mathscr{L}$, (4) $\Phi(A^*A) \neq 0$ if $A \neq 0$, (5) $\Phi(\sup A_i) = \sup \Phi(A_i)$ for any bounded increasing net $\{A_i\}$ in $\mathscr{L}_+$ (cf. [10: p. 330]). Thus we get the following:

THEOREM 1.4. *Suppose that N is a semi-finite and properly-infinite factor with an* $\mathrm{Ad}\, u$*-invariant faithful normal trace τ, and* $I(\mathrm{Ad}\, u, N) \neq \{0\}$. *Let $\Phi = \mathrm{Ad}\, V^* \cdot \Phi_2 \cdot \mathrm{Ad}\, V \cdot \Phi_1$ on $\mathscr{L}_+$. Then Φ is a generalized centervalued trace of $\mathscr{L}_+$ onto $\hat{\mathscr{Z}}_+$.*

In Section 2, instead of Φ in the above theorem, we use the mapping $\Psi_2 \cdot \pi \cdot \mathrm{Ad}\, V \cdot \Phi_1$ of $\mathscr{L}_+$ onto $\hat{L}^\infty(\boldsymbol{T}, m, C(\mathfrak{K}))$, which is denoted by $\hat{T}$.

2. A decomposition of reducing subspaces.

Let $\mathscr{S}$ be a family of shift operators on $\mathfrak{H} = \sum_{n \in \boldsymbol{Z}} \bigoplus \mathfrak{K}_n$ $(= \ell^2(\boldsymbol{Z}) \otimes \mathfrak{K})$. In the present paper we are assuming that $\mathscr{S}$ satisfies the conditions $(*)$ and $(**)$ in the introduction. In additon, we assume that $M(\mathscr{S})$ is the crossed product $\mathscr{R}(M, \mathrm{Ad}\, u)$ of a factor M on $\mathfrak{K}$ by $\boldsymbol{Z}$ with respect to a spatial automorphism $\mathrm{Ad}\, u$. For convenience, we denote by $N(\mathscr{S})$ the commutant of $M(\mathscr{S})$ on $\mathfrak{H}$. We here note that $N(\mathscr{S})$ is *-isomorphic to $\mathscr{R}(N, \mathrm{Ad}\, u^{-1})$ because it is the commutant of the crossed product $M(\mathscr{S}) = \mathscr{R}(M, \mathrm{Ad}\, u)$ (cf. [10: the end of V. 7]).

Let $\mathfrak{M}$ be a pure simply invariant subspace for $\mathscr{S}$. Then, by [7: Proposition 1.7], $\mathfrak{M}$ is decomposed as

$$\mathfrak{M} = \mathfrak{M}_0 \oplus \mathfrak{M}_1 \oplus \mathfrak{M}_2 \oplus \cdots,$$

where $\mathfrak{M}_n = (s^n \otimes 1)[W(\mathscr{S})\mathfrak{M}_0]$ for $n \geqq 1$. We denote by $R(\mathfrak{M})$ the reducing subspace generated by $\mathfrak{M}$, i.e., the smallest reducing subspace containing $\mathfrak{M}$. Then we have that

$$R(\mathfrak{M}) = \cdots \oplus \mathfrak{M}_{-1} \oplus [W(\mathscr{S})\mathfrak{M}_0] \oplus \mathfrak{M}_1 \oplus \cdots ,$$

where $\mathfrak{M}_n = (s^n \otimes 1)[W(\mathscr{S})\mathfrak{M}_0]$ for all integer $n \neq 0$. Moreover by [7: Theorem 2.12], it follows that

$$\mathfrak{M} = U\mathfrak{N} \quad (\mathfrak{N} = \mathfrak{N}_0 \oplus \mathfrak{N}_1 \oplus \mathfrak{N}_2 \oplus \cdots) ,$$

where $\mathfrak{N}_n = (s^n \otimes 1)[W(\mathscr{S})\mathfrak{N}_0] = e\mathfrak{K}_n$ for some projection e in N and U is a partial isometry in $N(\mathscr{S})$ such that $UU^*\mathfrak{H} = R(\mathfrak{M})$ and $U^*U\mathfrak{H} = \sum_{n\in Z} \oplus e\mathfrak{H}_n$. Namely the projections of $\mathfrak{H}$ onto $R(\mathfrak{M})$ and $\sum_{n\in Z} \oplus e\mathfrak{H}_n$ are equivalent in the von Neumann algebra $N(\mathscr{S})$. Two projections P and Q in a von Neumann algebra $\mathscr{A}$ are said to be equivalent of $P = V^*V$ and $Q = VV^*$ for some partial isometry in $\mathscr{A}$, and this relation is denoted by $P \sim Q$. A projection P is said to be dominated by a projection Q if P is equivalent to a sub-projection of Q, and this relation is denoted by $P \prec Q$. For a subspace $\mathfrak{M}$ of $\mathfrak{H}$, let $P_{\mathfrak{M}}$ represent the projection of $\mathfrak{H}$ onto $\mathfrak{M}$. Then, from the above discussion, we get the following:

LEMMA 2.1. *Let $\mathfrak{R}$ be a reducing subspace. Then the following three statements are equivalent.*

(1) *$\mathfrak{R}$ contains a simply invariant subspace.*

(2) *$\mathfrak{R}$ contains a pure simply invariant subspace.*

(3) *$P_{\mathfrak{R}} \succ 1 \otimes e$ for some projection e in N.*

To see the equivalence between the projections of the form $1 \otimes e$ and the other projections, we use the center-valued trace. We thus need the following, whose proof by easy calculation is omitted:

LEMMA 2.2. *Let e be a projection in N. Then we have the following*:

(1) $(\mathrm{Ad}\, V \cdot \Phi_1)(1 \otimes e) = 1 \otimes e$.

(2) $T(1 \otimes e)(z) = \tau(e)$ (resp. $\hat{T}(1 \otimes e)(z) = \tau(e)$) *provided N and $I(\mathrm{Ad}\, u, N)$ satisfy the hypothesis in Theorem* 1.3 (resp. *Theorem* 1.4).

DEFINITION 2.3. Let $\mathfrak{R}$ be a reducing subspace for $\mathscr{S}$. Then we say that

(1) $\mathfrak{R}$ is fat if $\mathfrak{R}$ is of the form $\mathfrak{R} = R(\mathfrak{M})$ for a pure simply invariant subspace $\mathfrak{M}$ for $\mathscr{S}$,

(2) $\mathfrak{R}$ is thin if $\mathfrak{R}$ contains no simply invariant subspace.

From the definition of fat reducing subspaces, we get the following lemma immediately.

LEMMA 2.4. *$\mathfrak{R}$ is a fat reducing subspace if and only if $P_{\mathfrak{R}} \sim 1 \otimes e$*

in $M(\mathscr{S})$ *for some projection* e *in* N.

From now on, we break down the remainder of this section into two parts [I] and [II]. In the first part, we assume that N is a semi-finite factor with an $\mathrm{Ad}\,u$-invariant trace τ and $I(\mathrm{Ad}\,u, N) \neq \{0\}$. In the second part, we consider the other cases.

[I] Let p, γ, q and m be as in Section 1. Then the center $\mathscr{Z} = M(\mathscr{S}) \cap N(\mathscr{S})$ is non-trivial and $\pi([\mathrm{Ad}\,V](\mathscr{Z})) = L^\infty(\boldsymbol{T}, m, \boldsymbol{C}(\mathfrak{K}))$. For a projection P in $N(\mathscr{S})(=\mathscr{L} = \mathscr{L}(N, \mathrm{Ad}\,u^{-1}))$, we put

$$\varepsilon(P) = \text{ess. inf}\,\{T(P)(z) : z \in \boldsymbol{T}\} \quad (N \text{ is finite}) ,$$

$$\hat{\varepsilon}(P) = \text{ess. inf}\,\{\hat{T}(P)(z) : z \in \boldsymbol{T}\} \quad (N \text{ is properly-infinite}) ,$$

where ess. inf E is the number $-$ess. sup $(-E)$ for a subset E of $[0, \infty]$.

In the case where $N(\mathscr{S})$ $(=\mathscr{L})$ is finite (resp. semi-finite and properly-infinite), Condition (3): $P_{\mathfrak{R}} \succ 1 \otimes e$ $(e \in N)$, in Lemma 2.1 is equivalent to $\Phi(P_{\mathscr{R}}) \geqq \Phi(1 \otimes e)$ because Φ is the center-valued trace (resp. a generalized center-valued trace) of $\mathscr{L}$ [10: V. Corollary 2.8] (resp. [9: p. 175]). The latter is equivalent to $T(P_{\mathfrak{R}}) \geqq T(1 \otimes e)$ (resp. $\hat{T}(P_{\mathfrak{R}}) \geqq \hat{T}(1 \otimes e)$), since T (resp. $\hat{T}$) is the product of the order preserving mappings $\mathrm{Ad}\,V$ and Φ. In our main theorem, looking at the values $T(P_{\mathfrak{R}})(z)$ (resp. $\hat{T}(P_{\mathfrak{R}})(z)$) $(z \in \boldsymbol{T})$, we get a condition under which $\mathfrak{R}$ contains a simply invariant subspace.

THEOREM 2.5. *Suppose that* N *is a semi-finite factor and* $I(\mathrm{Ad}\,u, N) \neq \{0\}$. *Then every reducing subspace* $\mathfrak{R}$ *for* $\mathscr{S}$ *has a decomposition*:

$$\mathfrak{R} = \mathfrak{R}_f \oplus \mathfrak{R}_t ,$$

where $\mathfrak{R}_f$ *is a fat reducing subspace and* $\mathfrak{R}_t$ *is a thin reducing subspace.*

PROOF. Let $\mathfrak{R}$ be a reducing subspace such that $\varepsilon(P_{\mathfrak{R}}) = 0$ (when N is finite) or $\hat{\varepsilon}(P_{\mathfrak{R}}) = 0$ (when N is infinite). Then $\mathfrak{R}$ itself is thin. In fact, if $\mathfrak{R}$ contains a pure simply invariant subspace, then $P_{\mathfrak{R}} \succ 1 \otimes e$ for some projection e in N by Lemma 2.1. Hence $T(P_{\mathfrak{R}})(z)$ (resp. $\hat{T}(P_{\mathfrak{R}}(z))) \geqq T(1 \otimes e)$ (resp. $\hat{T}(1 \otimes e)) = \tau(e)1$ for a.e. z in $\boldsymbol{T}$. This contradicts the assumption. When $\varepsilon(P_{\mathfrak{R}})$ (resp. $\hat{\varepsilon}(P_{\mathfrak{R}})) > 0$, we consider the following four cases.

CASE 1. N is of type I and $\varepsilon = \varepsilon(P_{\mathfrak{R}})$ (resp. $= \hat{\varepsilon}(P_{\mathfrak{R}})) > 0$. Since every *-automorphism of a factor of type I is inner (cf. [10: V. 1. Exercise 4]), we have $I(\mathrm{Ad}\,u, N) = \boldsymbol{Z}$, that is, $p = q = m = 1$. Thus Φ_1 becomes the identity mapping. Hence $\pi \cdot \mathrm{Ad}\,u \cdot \Phi_1$ is a *-isomorphism of $\mathscr{L}$ onto $L^\infty(\boldsymbol{T}, N)$. Thus an operator A in $\mathscr{L}$ is a projection if and only if $((\pi \cdot \mathrm{Ad}\,u \cdot \Phi_1)(A))(z)$ are projections in N for a.e. z in $\boldsymbol{T}$. Since N

is discrete, it follows that $\{\tau(e)\colon e$ is a projection in $N\} = \{n\tau(e_0)\}_{n=1}^{k}$ for some k $(1 \leqq k \leqq \infty)$, where e_0 is a minimal projection in N. Hence $\varepsilon = n\tau(e_0)$ for some integer n. We take n-mutually orthogonal projections $\{e_i\}_{i=0}^{n-1}$ which are equivalent to e_0 and put $e = e_0 + e_1 + \cdots + e_{n-1}$. Then we have $P \succ 1 \otimes e$. Hence there exists a sub-projection Q of $P_{\mathfrak{R}}$ which is equivalent to $1 \otimes e$. We put $\mathfrak{R}_f = Q\mathfrak{H}$ and $\mathfrak{R}_t = \mathfrak{R} \ominus \mathfrak{R}_f$. Then $T(Q)(z)$ (resp. $\hat{T}(Q)(z)) = n\tau(e_0)$ for a.e. z in $\boldsymbol{T}$ and $\varepsilon(P_{\mathfrak{R}} - Q)$ (resp. $\hat{\varepsilon}(P_{\mathfrak{R}} - Q)) = 0$, so that they are the reducing subspaces desired.

CASE 2. N is of type I_∞ and $\hat{\varepsilon}(P_{\mathfrak{R}}) = \infty$. Since $\hat{T}(P_{\mathfrak{R}})(z) = \infty$ for a.e. z in $\boldsymbol{T}$, we have that $P_{\mathfrak{R}} \sim 1 \otimes e$ for any infinite projection e in N. Hence $\mathfrak{R}$ itself is fat.

CASE 3. N is of type II and $0 < \varepsilon = \varepsilon(P_{\mathfrak{R}})$ (resp. $= \hat{\varepsilon}(P_{\mathfrak{R}})) < \infty$. Since N is continuous, there exists a projection e in N such that $\tau(e) = \varepsilon$. This implies that $P_{\mathfrak{R}} \succ 1 \otimes e$. Namely there exists a sub-projection Q of $P_{\mathfrak{R}}$ which is equivalent to $1 \otimes e$. The reducing subspaces $\mathfrak{R}_f = Q\mathfrak{H}$ and $\mathfrak{R}_t = \mathfrak{R} \ominus \mathfrak{R}_f$ are fat and thin respectively.

CASE 4. N is of type II_∞ and $\hat{\varepsilon}(P_{\mathfrak{R}}) = \infty$. As in Case (2), $\mathfrak{R}$ itself is fat. q.e.d.

COROLLARY 2.6. *Let N and $I(\mathrm{Ad}\,u, N)$ be as in Theorem 2.5. Then a reducing subspace $\mathfrak{R}$ contains a simply invariant subspace if and only if $\varepsilon(P_{\mathfrak{R}}) > 0$ (when N is finite) or $\hat{\varepsilon}(P_{\mathfrak{R}}) > 0$ (when N is infinite).*

COROLLARY 2.7. *Suppose that N is of type I. Then a reducing subspace $\mathfrak{R}$ contains a simply invariant subspace if and only if $T(P_{\mathfrak{R}})(z) > 0$ (when N is finite) or $\hat{T}(P_{\mathfrak{R}})(z) > 0$ (when N is infinite) for* a.e. *z in $\boldsymbol{T}$.*

By Corollary 2.7, when $N = B(\mathfrak{K})$ and $\mathrm{Ad}\,u =$ the identity on N, a reducing subspace $\mathfrak{R}$ contains a simply invariant subspace if and only if $\tau(\pi(P_{\mathfrak{R}})(z))1 \geqq 1$ for a.e. z in $\boldsymbol{T}$, where τ is the usual trace of positive operators in $B(\mathfrak{K})$. This is the case treated in [5]. In the continuous cases, the condition $T(P_{\mathfrak{R}}(z)) > 0$ for a.e. z in $\boldsymbol{T}$, is not a sufficient condition for $\mathfrak{R}$ to contain a simply invariant subspace as is shown in the following:

EXAMPLE 2.8. Let N be a factor of type II_1 with the normalized trace τ. Let $\mathscr{S}$ be a family of shift operators satisfying the condition $(*)$ in the introduction and such that $D(\mathscr{S}) = 1 \bar{\otimes} N$ on $\ell^2(\boldsymbol{Z}) \otimes \mathfrak{K}$. Then $M(\mathscr{S})$ is *-isomorphic to $L^\infty(\boldsymbol{T}, N)$. Let F be a measurable function in $L^\infty(\boldsymbol{T}, N)$ such that $\tau(F(e^{it})) = t/2\pi$ $(0 \leqq t < 2\pi)$. For a projection e in N we put $G(z) = e$ for all z in $\boldsymbol{T}$. If G is dominated by F in $L^\infty(\boldsymbol{T}, N)$, so is $G(z)$ by $F(z)$ in N for a.e. z in $\boldsymbol{T}$. Hence $\tau(G(z)) \leqq \tau(F(z))$ for a.e. z in T. But this is impossible. Hence the corresponding reducing subspace for $\mathscr{S}$ contains no simply invariant subspace.

REMARK. If N is finite, the decomposition in Theorem 2.5 is unique in the following sense. For another decomposition $\mathfrak{N} = \mathfrak{N}'_f \oplus \mathfrak{N}'_t$, it follows that $P_{\mathfrak{N}_f} \sim P_{\mathfrak{N}'_f}$ (resp. $P_{\mathfrak{N}_t} \sim P_{\mathfrak{N}'_t}$). Indeed, $P_{\mathfrak{N}'_f}$ is equivalent to a projection in $\mathscr{L}$ of the from $1 \otimes e'$, where e' is a projection in N. Hence, as in the proof of Theorem 2.5, we get $\tau(e') = \varepsilon = \tau(e)$ because $\mathfrak{N}'_t$ is thin. Thus we have $P_{\mathfrak{N}'_f} \sim 1 \otimes e' \sim 1 \otimes e \sim P_{\mathfrak{N}_f}$. Since $\mathscr{L}$ is finite, $P_{\mathfrak{N}'_t} = P_{\mathfrak{N}'} - P_{\mathfrak{N}'_f}$ is equivalent to $P_{\mathfrak{N}_t} = P_{\mathfrak{N}} - P_{\mathfrak{N}_f}$.

In the case where N is semi-finite and properly-infinite the decomposition is not unique in the sense mentioned above, as we see in the following:

EXAMPLE 2.9. Suppose that the dimension of $\mathfrak{K}$ is infinite and $\mathscr{S} = \{s \otimes 1\}$ on $L^2(\boldsymbol{T}) \otimes \mathfrak{K}$. Then $M(\mathscr{S}) = M_{L^\infty(\boldsymbol{T})} \mathbin{\bar{\otimes}} \boldsymbol{C}(\mathfrak{K})$ and $\mathfrak{L} = M(\mathscr{S})' = M_{L^\infty(\boldsymbol{T})} \mathbin{\bar{\otimes}} B(\mathfrak{K})$, which is *-isomorphic to $L^\infty(\boldsymbol{T}, B(\mathfrak{K}))$. Let τ be the usual trace on $B(\mathfrak{K})_+$. We take a one-dimensional projection e_0 in $B(\mathfrak{K})$ and put $P(e^{it}) = 1 - e_0$ $(0 \leqq t < 2\pi)$ and $Q(e^{it}) = 1$ if $0 \leqq t < \pi$, $=1 - e_0$ if $\pi < t < 2\pi$. Then P and Q are properly-infinite projections in $L^\infty(\boldsymbol{T}, B(\mathfrak{K}))$ whose central support are equal to the identity. Thus we have $P \sim Q \sim I$ in $\mathscr{L}$. Hence $\mathfrak{N} = Q\mathfrak{H}$ itself is a reducing subspace for $\mathscr{S}$. But $\mathfrak{N}$ has a decomposition $\mathfrak{N} = P\mathfrak{H} + (Q - P)\mathfrak{H}$ such that $P\mathfrak{H}$ is fat and $(Q - P)\mathfrak{H}$ is thin. Obviously $Q - P$ is not equivalent to 0.

[II] In this part, we consider all the cases except the ones examined in part [I]. We recall that every *-automorphism of a factor of type I is inner and the unique trace of a II_1-factor is *-automorphism-invariant. Furthermore, when N is of type II_∞ and $I(\mathrm{Ad}\, u, N) \neq \{0\}$, every trace of N is $\mathrm{Ad}\, u$-invariant. Hence the following cases remain:

(II-1) N is of type II_1 and $I(\mathrm{Ad}\, u, N) = \{0\}$.
(II-2) N is of type II_∞ and $I(\mathrm{Ad}\, u, N) = \{0\}$.
(II-3) N is of type III and $I(\mathrm{Ad}\, u, N) = \{0\}$.
(II-4) N is of type III and $I(\mathrm{Ad}\, u, N) \neq \{0\}$.

In each proof of the following theorems, we use the fact that the von Neumann algebra $\mathscr{L} = \mathscr{L}(N, \mathrm{Ad}\, u^{-1})$ is spatially isomorphic to the crossed product $\mathscr{R}(N, \mathrm{Ad}\, u^{-1})$ of N by $\boldsymbol{Z}$ with respect to the action $\{\mathrm{Ad}\, u^{-n}\}_{n \in \boldsymbol{Z}}$.

THEOREM 2.10. *Let N and $I(\mathrm{Ad}\, u, N)$ be as in* (II-1). *Then every non-zero reducing subspace for $\mathscr{S}$ is fat.*

PROOF. The von Neumann algebra $N(\mathscr{S}) = \mathscr{L}$ is a factor of type II_1, because $\mathscr{R}(N, \mathrm{Ad}\, u^{-1})$ becomes such a factor by [9: 22.6, Corollary

1 and 22.7, Theorem 1]. Let Tr be the unique normalized trace of $\mathscr{L}$. Then, considering the conditional expectation of $\mathscr{L}$ onto $1 \bar{\otimes} N$, we find that the set $\{Tr(1 \otimes e)\colon e \text{ is a projection in } N\}$ is just the closed interval $[0, 1]$. Hence, for each projection P in $\mathscr{L}$, there exists a projection e in N such that $Tr(P) = Tr(1 \otimes e)$. This implies that P is equivalent to $1 \otimes e$. q.e.d.

THEOREM 2.11. *Let N and $I(\mathrm{Ad}\, u, N)$ be as in* (II-2). *Then every non-zero reducing subspace for $\mathscr{S}$ is fat.*

PROOF. Let τ be a faithful normal semi-finite trace of N_+. Then there exists a number λ $(0 < \lambda \leqq 1)$ such that $\tau([\mathrm{Ad}\, u](x)) = \lambda\tau(x)$ or $\tau([\mathrm{Ad}\, u^{-1}](x)) = \lambda\tau(x)$ $(x \in N)$. When $\lambda = 1$, $\mathscr{R}(N, \mathrm{Ad}\, u^{-1})$ is a factor of type II_∞ [9: 22.7, Theorem 2]. If $\lambda \neq 1$, $\mathscr{R}(N, \mathrm{Ad}\, u^{-1})$ becomes a factor of type III_λ [9: 29.1, Proposition]. Hence $N(\mathscr{S}) = \mathscr{L}$ is factor of type II_∞ or III_λ. In the former case, we can show as in Case (II-1) that every projection in $\mathscr{L}$ is equivalent to $1 \otimes e$ for some projection e in N. In the latter case, every projection in $\mathscr{L}$ is equivalent to $1 \otimes e$ for any projection e in N. q.e.d.

THEOREM 2.12. *Let N and $I(\mathrm{Ad}\, u, N)$ be as in* (II-3). *Then every non-zero reducing subspace for $\mathscr{S}$ is fat.*

PROOF. It is sufficient to note that $\mathscr{R}(N, \mathrm{Ad}\, u^{-1})$ is a factor of type III [9: 10.21, Proposition]. q.e.d.

THEOREM 2.13. *Let N and $I(\mathrm{Ad}\, u, N)$ be as in* (II-4). *Then a reducing subspace for $\mathscr{S}$ is either fat or thin.*

PROOF. We first point out that $N(\mathscr{S}) = \mathscr{L}$ is of type III. In fact, the crossed product of a factor of type III by a discrete group is always of type III (cf. [9: 10.21, Proof of (1) in Proposition]). Next we see that the central support of $1 \otimes e$ $(e \in N)$ in $\mathscr{L}$ is the identity. Let X be a central projection in $\mathscr{L}$ such that $(1 \otimes e)X = 1 \otimes e$. Then we have $(1 \otimes e)([\mathrm{Ad}\, V](X)) = [\mathrm{Ad}\, V]((1 \otimes e)(X)) = [\mathrm{Ad}\, V](1 \otimes e) = 1 \otimes e$. Since $\pi([\mathrm{Ad}\, V](\mathscr{X})) = L^\infty(\boldsymbol{T}, m, \boldsymbol{C}(\mathfrak{K}))$, it follows that $\pi((1 \otimes e)([\mathrm{Ad}\, V](X)))(z) = e \cdot \pi([\mathrm{Ad}\, V](X))(z) = e$ for a.e. z in $\boldsymbol{T}$. Since, for each z in $\boldsymbol{T}$, $[\mathrm{Ad}\, V](X)(z)$ is a projection in $\boldsymbol{C}(K)$, we have $\pi([\mathrm{Ad}\, V](X)) = 1$, thus $[\mathrm{Ad}\, V](X) = X = I$.

Therefore a projection P in $\mathscr{L}$ is equivalent to $1 \otimes e$ if and only if the central support of P is the identity. q.e.d.

For a projection P in $\mathscr{L}$, let $\tilde{\pi}(P)$ denote the function defined by $\tilde{\pi}(P)(z) = 1$ if $\pi(P)(z) \neq 0$ and $=0$ if $\pi(P)(z) = 0$. Then we get the following.

PROPOSITION 2.14. *Suppose that N is of type* III *and $v=1$* (i.e., $u^p = w \in N$). *Then the following statements hold.*

(1) *For any projection P in $\mathscr{L}$, $\tilde{\pi}(P)$ is periodic of period $2\pi/p$.*

(2) *A reducing subspace $\mathfrak{N}$ is fat if and only if $\tilde{\pi}(P_{\mathfrak{N}}) = 1$ for* a.e. *z in $\boldsymbol{T}$.*

(3) *A reducing subspace $\mathfrak{N}$ is thin if and only if $\tilde{\pi}(P_{\mathfrak{N}}) = 0$ for z in a measurable subset of $\boldsymbol{T}$ with positive measure.*

PROOF. (1) From the assumption, we have $\pi(\mathscr{Z}) = L^\infty(\boldsymbol{T}, p, C(\mathfrak{K}))$. Let $C(P)$ denote the central support of P. Then $P \sim C(P)$, thus $\pi(P) \sim \pi(C(P))$. Hence $\pi(P)(z) \sim \pi(C(P))(z)$ for a.e. z in $\boldsymbol{T}$. Since $\tilde{\pi}(C(P))$ is periodic of period $2\pi/p$, so is $\tilde{\pi}(P)$.

(2) Since $C(1 \otimes e) = I$, $P_{\mathfrak{N}}$ is equivalent to $1 \otimes e$ if and only if $C(P_{\mathfrak{N}}) = I$. The latter is equivalent to $\tilde{\pi}(P_{\mathfrak{N}}) = I$.

(3) This is derived from (2). q.e.d.

Now we notice that the statements (2) and (3) in the above proposition closely resemble the original theorem of Beurling, though the structure of $M(\mathscr{S})$ is mose complex.

REMARK. From the preceding theorems, it seems that the structure of reducing subspace for $\mathscr{S}$ is easily analyzed in the case where the von Neumann algebra $M(\mathscr{S})$ is factor, even if $\mathscr{S}$ does not satisfy the condition $(**)$. For instance, for the unitary operator u on $L^2(\boldsymbol{T})$ defined by $uf(z) = f(ze^{2\pi i\theta})$, where θ is an irrational number in (0, 1), we consider the crossed product $\mathscr{R} = \mathscr{R}(M_{L^\infty(T)}, \operatorname{Ad} u)$ on $\ell^2(\boldsymbol{Z}) \otimes L^2(\boldsymbol{T})$. It is well-known that $\mathscr{R}$ is a factor of type II_1. Let $\mathscr{S}$ be a family of shift operators on $\ell^2(\boldsymbol{Z}) \otimes L^2(\boldsymbol{T})$ such that $W(\mathscr{S})$ satisfies the condition $(*)$ and $D(\mathscr{S})$ be the diagonal part of $\mathscr{R}$. Then $M(\mathscr{S}) = \mathscr{R}$. Since $\mathscr{S}$ does not satisfy the condition $(**)$, there is at least one simply invariant subspace $\mathfrak{M}$ which cannot be of the form $\mathfrak{M} = U\sum_{n=0}^{\infty} \bigoplus \mathfrak{N}_n$, where $(s^n \otimes 1)[W(\mathscr{S})\mathfrak{N}_0] = \mathfrak{N}_n = e\mathfrak{K}$ $(e \in M_{L^\infty(T)})$. However every projection of $\mathfrak{H}$ onto a reducing subspace is equivalent to a projection of the form $1 \otimes e$, hence so is that of $\mathfrak{H}$ onto the reducing subspace generated by $\mathfrak{M}$. This means that every projection onto the reducing subspace generated by a simply invariant subspace is equivalent to $1 \otimes e$ for some projection e in $M_{L^\infty(T)}$, though a simply invariant subspace is not necessarily of Beurling type. Hence it seems to be useful for the study of invariant subspaces for $\mathscr{S}$ to analyze the reducing subspaces generated by simply invariant subspaces.

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Addendum. Recently the author received a preprint [11] from Professor B. Solel. In that paper he solved the invariant subspace problem mentioned in the above remark. Namely he expressed all pure simply invariant subspaces for $\mathscr{S}$ such that $M(\mathscr{S}) = \mathscr{R}$ as the ranges of canonical models by the partial isometries in the commutant of $M(\mathscr{S})$.

The author however believes that it is still very interesting to study the relation between the reducing subspaces and the simply invariant subspaces for the above family $\mathscr{S}$.

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