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REFLECTION GROUPS AND THE EIGENVALUE PROBLEMS OF VIBRATING MEMBRANES WITH MIXED BOUNDARY CONDITIONS

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Introduction. Throughout this paper, (M, g) is an n-dimensional space form of constant curvature, that is, the Euclidean space $\boldsymbol{R}^n$, the standard sphere $\boldsymbol{S}^n$ or the hyperbolic space $\boldsymbol{H}^n$. Let Δ be the (non-negative) Laplacian of (M, g). Let Ω be a bounded domain in M with an appropriately regular boundary $\partial\Omega$. For an arbitrary fixed real number ρ, let us consider the following boundary value eigenvalue problem:

$$\begin{cases} \Delta f = \lambda f & \text{in } \Omega, \\ f = 0 & \text{on } \Gamma_1, \quad \text{and} \\ \partial f/\partial \boldsymbol{n} = \rho f & \text{a.e. } \Gamma_2, \text{ i.e., where the exterior normal } \boldsymbol{n} \text{ of } \Gamma_2 \text{ is defined.} \end{cases}$$

Here the boundary $\partial\Omega$ is a disjoint union of Γ_1 and Γ_2. It is called (cf. [B, p. 91]) to be

(D) the *fixed membrane* problem if $\Gamma_2 = \emptyset$,

(N) the *free membrane* problem if $\Gamma_1 = \emptyset$, or

(M_ρ) the membrane problem of *mixed boundary conditions* if $\Gamma_1 \neq \emptyset$ and $\Gamma_2 \neq \emptyset$.

It is well known that each problem has a discrete spectrum of the eigenvalues with finite multiplicity. We denote by $\mathrm{Spec}_D(\Omega)$, $\mathrm{Spec}_N(\Omega)$ and $\mathrm{Spec}_{M_\rho}(\Omega)$, the spectra of the problems (D), (N) and (M_ρ), respectively.

One of the important problems of the spectra is to research how the spectra $\mathrm{Spec}_D(\Omega)$, $\mathrm{Spec}_N(\Omega)$ or $\mathrm{Spec}_{M_\rho}(\Omega)$ reflect the shape of Ω. In his paper [K], M. Kac posed the following problem:

For two bounded domains $\Omega, \tilde{\Omega}$ in $\boldsymbol{R}^n$ $(n \geqq 2)$, assume that $\mathrm{Spec}_D(\Omega) = \mathrm{Spec}_D(\tilde{\Omega})$. *Are the domains $\Omega, \tilde{\Omega}$ congruent in $\boldsymbol{R}^n$?*

Here two domains $\Omega, \tilde{\Omega}$ are congruent in the space form (M, g) if there exists an isometry Φ of (M, g) such that $\Phi(\Omega) = \tilde{\Omega}$. Note that $\Omega, \tilde{\Omega}$ are isometric with respect to the induced metrics from (M, g) if and only if they are congruent in (M, g) because of simple connectedness of M (cf. [K.N., p. 252]).

In the paper [U], we gave the following answer:

THEOREM A (cf. [U, Theorem 4.4]). *There exist two domains $\Omega, \tilde{\Omega}$ in $\boldsymbol{R}^n$ $(n \geqq 4)$ such that*

$$\mathrm{Spec}_D(\Omega) = \mathrm{Spec}_D(\tilde{\Omega}) \quad \textit{and} \quad \mathrm{Spec}_N(\Omega) = \mathrm{Spec}_N(\tilde{\Omega}) ,$$

but Ω and $\tilde{\Omega}$ are not congruent in $\boldsymbol{R}^n$.

THEOREM B (cf. [U, Theorem 3.8] and Proposition 3.1, §3). *There exist two domains $\Omega, \tilde{\Omega}$ in $\boldsymbol{S}^{n-1}$ $(n \geqq 4)$ such that*

$$\mathrm{Spec}_D(\Omega) = \mathrm{Spec}_D(\tilde{\Omega}) \quad \textit{and} \quad \mathrm{Spec}_N(\Omega) = \mathrm{Spec}_N(\tilde{\Omega}) ,$$

but Ω and $\tilde{\Omega}$ are not congruent in $\boldsymbol{S}^{n-1}$.

In this paper, we give the following:

THEOREM C (cf. §2). *Let (M, g) be an n-dimensional simply connected space form of constant curvature. Assume that $n \geqq 4$. Then there exist two domains $\Omega, \tilde{\Omega}$ in (M, g) and disjoint subsets Γ_1, Γ_2* (resp. $\tilde{\Gamma}_1, \tilde{\Gamma}_2$) *of $\partial\Omega$* (resp. $\partial\tilde{\Omega}$) *such that*

$$\mathrm{Spec}_D(\Omega) = \mathrm{Spec}_D(\tilde{\Omega}),\ \mathrm{Spec}_N(\Omega) = \mathrm{Spec}_N(\tilde{\Omega}) \quad \textit{and}$$
$$\mathrm{Spec}_{M_\rho}(\Omega) = \mathrm{Spec}_{M_\rho}(\tilde{\Omega}) \quad \textit{for each real number } \rho ,$$

but Ω and $\tilde{\Omega}$ are not congruent in (M, g). Here $\mathrm{Spec}_{M_\rho}(\Omega)$ (resp. $\mathrm{Spec}_{M_\rho}(\tilde{\Omega})$) *are the spectra of the membrane problem (M_ρ) of the mixed boundary conditions for Ω, Γ_1 and Γ_2* (resp. $\tilde{\Omega}, \tilde{\Gamma}_1$ *and* $\tilde{\Gamma}_2$).

1. Preliminaries. Let (M, g) be an n-dimensional simply connected space form of constant curvature. Fix an origin o of M. Let exp: $T_oM \to M$ be the exponential mapping of (M, g) from the tangent space T_oM of M at o into M. Let $\boldsymbol{S}^{n-1} = \{\omega \in T_oM;\ \|\omega\| = 1\}$, where $\|\cdot\|$ is the norm of T_oM induced from the Riemannian metric g on M. We give the geodesic polar coordinate $(r, \omega) \in \boldsymbol{R}^+ \times \boldsymbol{S}^{n-1}$ around the origin o of M by

$$\omega = \omega(p) = \frac{1}{r(p)}\exp^{-1}(p) \in \boldsymbol{S}^{n-1} , \quad \text{and} \quad r = r(p) = d(o, p) ,$$

which is valid in $M - \{o\}$ in case of $M = \boldsymbol{R}^n$ or $\boldsymbol{H}^n$, or $M - \{o, \tilde{o}\}$, ($\tilde{o}$ the antipodal point of o in $\boldsymbol{S}^n$) in case of $M = \boldsymbol{S}^n$. Here $d(p, q)$, $p, q \in M$, is the geodesic distance between p and q in (M, g). Let g_0 be the Riemannian metric on $\boldsymbol{S}^{n-1} = \{\omega \in T_0M;\ \|\omega\| = 1\}$ of constant curvature 1 induced from the inner product g on T_0M. It is well known that the Riemannian metric g can be expressed using the geodesic polar coordinate (r, ω) as follows:

$$g = dr^2 + (\mathrm{Sn}\,(r))^2 g_0 \,. \tag{1.1}$$

Here the function $\mathrm{Sn}\,(r)$ of r is

$$\mathrm{Sn}\,(r) = \begin{cases} r & , \text{ if } M = \boldsymbol{R}^n \,, \\ \sin\,(r) & , \text{ if } M = \boldsymbol{S}^n \,, \quad \text{or} \\ \sinh\,(r) & , \text{ if } M = \boldsymbol{H}^n \,. \end{cases}$$

Then the volume element dv is

$$dv = (\mathrm{Sn}\,(r))^{n-1} dr\, d\omega \,, \tag{1.2}$$

where $d\omega$ is the volume element of $(\boldsymbol{S}^{n-1}, g_0)$. The (non-negative) Laplacian $\Delta = -\sum_{i,j} g^{ij}(\partial^2/\partial x_i \partial x_j - \sum_k \Gamma^k_{ij}\partial/\partial x_k)$, can be expressed by

$$\Delta = -\partial^2/\partial r^2 - (n-1)\,\mathrm{Ct}\,(r)\partial/\partial r + (\mathrm{Sn}\,(r))^{-2}\Delta_S \,, \tag{1.3}$$

where (g^{ij}) is the inverse of the matrix (g_{ij}), $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$, $(x_1, \cdots, x_n)$ is a local coordinate, Γ^k_{ij} are the Christoffel symbols, the function $\mathrm{Ct}\,(r)$ of r is

$$\begin{cases} 1/r & , \text{ if } M = \boldsymbol{R}^n \,, \\ \cot\,(r) & , \text{ if } M = \boldsymbol{S}^n \,, \quad \text{or} \\ \coth\,(r) & , \text{ if } M = \boldsymbol{H}^n \,, \end{cases}$$

and Δ_S is the (non-negative) Laplacian of $(\boldsymbol{S}^{n-1}, g_0)$.

2. Reduction of Theorem C to Theorem B. Throughout this paper, we consider the truncated cone D_ε in (M, g) as follows: For $0 < \varepsilon < \varepsilon_1$ and a domain C_1 in the unit sphere $\boldsymbol{S}^{n-1}$ of the tangent space $T_o M$, let $D_\varepsilon = \{\exp\,(r\omega);\ \varepsilon < r < \varepsilon_1,\ \omega \in C_1\}$, where the number ε_1 is 1 if $M = \boldsymbol{R}^n$, $\boldsymbol{H}^n$ or $\pi/2$ if $M = \boldsymbol{S}^n$. Then the boundary ∂D_ε of D_ε in M is given by

$$\partial D_\varepsilon = \exp\,(\varepsilon C_1) \cup \exp\,(\varepsilon_1 C_1) \cup \{\exp\,(r\omega);\ \varepsilon \leqq r \leqq \varepsilon_1,\ \omega \in \partial C_1\} \,,$$

where ∂C_1 is the boundary of C_1 in $\boldsymbol{S}^{n-1}$. Put

$$\Gamma_1 = \{\exp\,(r\omega);\ \varepsilon \leqq r \leqq \varepsilon_1,\ \omega \in \partial C_1\} \,, \quad \text{and}$$
$$\Gamma_2 = \exp\,(\varepsilon C_1) \cup \exp\,(\varepsilon_1 C_1) \quad \text{(cf. Figure 1).}$$

Let us consider the following problems for the truncated cones D_ε:

$$(D)\begin{cases} \Delta f = \lambda f & \text{in } \ D_\varepsilon \,, \\ f = 0 & \text{on } \ \partial D_\varepsilon \,, \end{cases}$$
$$(N)\begin{cases} \Delta f = \lambda f & \text{in } \ D_\varepsilon \,, \\ \partial f/\partial \boldsymbol{n} = 0 & \text{a.e. } \ \partial D_\varepsilon \,, \end{cases}$$

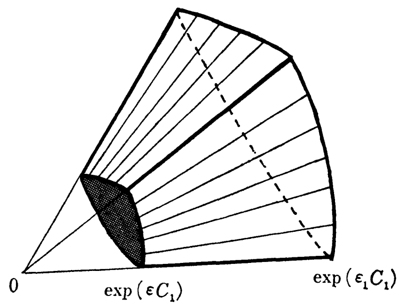


FIGURE 1. The domain D_ε and the boundary ∂D_ε.

$$(M_\rho)\begin{cases} \Delta f = \lambda f & \text{in } D_\varepsilon\,, \\ f = 0 & \text{on } \Gamma_1\,, \\ \partial f/\partial \boldsymbol{n} = \rho f & \text{a.e. } \Gamma_2\,, \end{cases}$$

where $\partial/\partial \boldsymbol{n}$ is the derivative with respect to the exterior normal unit vector of ∂D_ε. Then we have the following:

THEOREM 2.1. *For $0 < \varepsilon < \varepsilon_1$ and two domains C_1, $\tilde{C}_1$ in S^{n-1}, define the truncated cones D_ε, $\tilde{D}_\varepsilon$ by*

$$D_\varepsilon = \{\exp(r\omega);\, \varepsilon < r < \varepsilon_1,\, \omega \in C_1\}\,,$$
$$\tilde{D}_\varepsilon = \{\exp(r\omega);\, \varepsilon < r < \varepsilon_1,\, \omega \in \tilde{C}_1\}\,, \quad \textit{respectively}.$$

(i) *If* $\mathrm{Spec}_D(C_1) = \mathrm{Spec}_D(\tilde{C}_1)$, *then we have*

$$\mathrm{Spec}_D(D_\varepsilon) = \mathrm{Spec}_D(\tilde{D}_\varepsilon) \quad \textit{and} \quad \mathrm{Spec}_{M_\rho}(D_\varepsilon) = \mathrm{Spec}_{M_\rho}(\tilde{D}_\varepsilon)$$

for each real number ρ.

(ii) *If* $\mathrm{Spec}_N(C_1) = \mathrm{Spec}_N(\tilde{C}_1)$, *then we have* $\mathrm{Spec}_N(D_\varepsilon) = \mathrm{Spec}_N(\tilde{D}_\varepsilon)$. *Here* $\mathrm{Spec}_D(C_1)$ (*resp.* $\mathrm{Spec}_N(C_1)$) *stands for the spectrum of the fixed* (*resp. free*) *membrane problem of the Laplacian Δ_S for a domain C_1 in S^{n-1}.*

Theorem C follows from Theorem 2.1 because of Theorem B. In fact, two truncated cones D_ε, $\tilde{D}_\varepsilon$ are congruent in (M, g) if and only if C_1, $\tilde{C}_1$ are congruent in (S^{n-1}, g_0). Theorem 2.1 follows from Proposition 2.2, which is proved in §4.

PROPOSITION 2.2. *For $0<\varepsilon<\varepsilon_1$ and a domain C_1 in S^{n-1}, let D_ε be the truncated cone as in Theorem 2.1. Then we have the following:*

(i) *The spectra* $\mathrm{Spec}_D(D_\varepsilon)$ *and* $\mathrm{Spec}_{M_\rho}(D_\varepsilon)$ *depend only upon ε and* $\mathrm{Spec}_D(C_1)$.

(ii) *The spectrum* $\mathrm{Spec}_N(D_\varepsilon)$ *depends only upon ε and* $\mathrm{Spec}_N(C_1)$.

3. Case of spherical domains.

3.1. In this section, we generalize Theorem B, which is proved in

[U]. We preserve the notations as in [U].

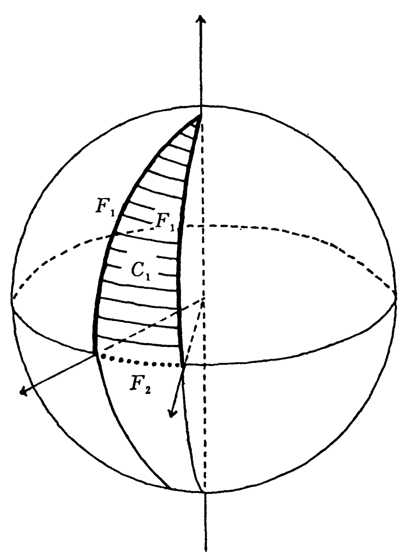
Let $(E, (\ , \ ))$ be the n-dimensional Euclidean space. Let (W, E) be a finite reflection group acting essentially on E (cf. [U] or [B.N]). We assume that (W, E) is a direct product of two reflection groups $(W_{(i)}, E_{(i)})$, $i = 1, 2$, that is, $W = W_{(1)} \times W_{(2)}$, $E = E_{(1)} \times E_{(2)}$, (direct product). Put $n_{(i)} = \dim E_{(i)}$, $i = 1, 2$. We choose and fix a chamber C of (W, E). Then it is given by $C = C_{(1)} \times C_{(2)}$, where $C_{(i)}$ is a chamber of $(W_{(i)}, E_{(i)})$. Let $\mathscr{M}_{(i)} = \{H^j_{(i)};\ j = 1, \cdots, n_{(i)}\}$ be the set of all walls $H^j_{(i)}$ of the chamber $C_{(i)}$, $i = 1, 2$. We consider the spherical domain $C_1 = C \cap S^{n-1}$, where $S^{n-1} = \{\omega \in E;\ \|\omega\| = 1\}$, $\|\omega\| = \sqrt{(\omega, \omega)}$. The boundary ∂C_1 of C_1 in S^{n-1} is $\partial C_1 = F_1 \cup F_2$. Here

$$F_1 = \overline{(\partial C_{(1)} \times C_{(2)}) \cap S^{n-1}} \quad \text{(the closure in } S^{n-1}\text{)} \quad \text{and}$$
$$F_2 = (C_{(1)} \times \partial C_{(2)}) \cap S^{n-1} ,$$

where $\partial C_{(i)}$ is the boundary of the chamber $C_{(i)}$ in $E_{(i)}$, $i = 1, 2$.

Let us consider the following membrane problem of mixed boundary conditions.

$$(3.1) \quad \begin{cases} \Delta_S \Psi = \lambda \Psi & \text{in } C_1 , \\ \Psi = 0 & \text{on } F_1 , \text{ and} \\ \partial\Psi/\partial \boldsymbol{n} = 0 & \text{a.e. } F_2, \text{ i.e., where the exterior normal } \boldsymbol{n} \text{ of } F_2 \\ & \qquad \text{in } S^{n-1} \text{ is defined.} \end{cases}$$



FIGUER 2. Membrane problem with free condition for the dotted set and fixed condition for the dark lined set.

As an example, let $W = I_2(p) \times A_1$ (cf. [U]). We can choose a chamber C of W as the domain in Figure 2, F_1 is the dark lined set and F_2 is the dotted set.

3.2. The method of Bérard-Besson [B.B] is valid for the problem (3.1). We sketch briefly how to determine the spectrum $\mathrm{Spec}_{M_0}(C_1)$ of the membrane problem (3.1) of mixed boundary conditions.

Consider a C^∞ function f on S^{n-1} satisfying the conditions

$$\Delta_S f = \lambda f \quad \text{in} \quad S^{n-1}, \quad \text{and} \tag{3.2}$$

$$w \cdot f = \varepsilon(w) f, \quad w \in W, \tag{3.3}$$

where $(w \cdot f)(x) = f(w^{-1}(x))$, $x \in S^{n-1}$, $w \in W$, and $\varepsilon(w)$, $w \in W$, is given by

$$\varepsilon(w) = \det w_1, \quad w = (w_1, w_2) \in W = W_{(1)} \times W_{(2)}. \tag{3.4}$$

Then the restriction to C_1 of f satisfies (3.1). Furthermore the set of all restrictions to C_1 of C^∞ eigenfunctions of Δ_S on S^{n-1} with the condition (3.3) is dense in the space $L^2(C_1)$ of all square integrable functions on C_1 with respect to the volume element $d\omega$ of (S^{n-1}, g_0). Thus to determine the spectrum $\mathrm{Spec}_{M_0}(C_1)$ of (3.1), we have only to consider the set of all C^∞ eigenfunctions of Δ_S on S^{n-1} with (3.3).

The set of the eigenvalues of Δ_S on S^{n-1} is $\{k(k+n-2); k = 0, 1, 2, \cdots\}$ and the corresponding eigenfunctions are given by the restrictions to S^{n-1} of all harmonic polynomials in E. That is, for $k = 0, 1, 2, \cdots$, let $P_k(E)$ be the set of all homogeneous polynomials in E of degree k, $H_k(E) = \{P \in P_k(E); \Delta P = 0\}$, where Δ is the Laplacian of the standard Euclidean space (E, g). Set

$$H_{k,W}(E) = \{P \in H_k(E); w \cdot P = \varepsilon(w) P \quad \text{for all} \quad w \in W\},$$

where $w \cdot P(x) = P(w^{-1}(x))$, $w \in W$, $x \in E$. Put $h_{k,W} = \dim H_{k,W}(E)$, $k = 0, 1, 2, \cdots$. Then the number $k(k+n-2)$ is really an eigenvalue of (3.1) with multiplicity $h_{k,W}$ if and only if $h_{k,W} \neq 0$.

To determine all $h_{k,W}$, $k = 0, 1, 2, \cdots$, consider the Poincaré series

$$F_W(T) = \sum_{k=0}^{\infty} h_{k,W} T^k,$$

where T is an indeterminate. Using the invariant theory of finite reflection group (cf. [B.N]), the series $F_W(T)$ can be determined as

$$F_W(T) = (1 - T^2) T^{d_1} / \prod_{j=1}^{n} (1 - T^{m_j+1}), \tag{3.5}$$

where $\{m_j\}_{j=1}^n$ is the set of all the exponents of the reflection group W and d_1 is the sum of all the exponents of the reflection group $W_{(1)}$.

Thus we have:

PROPOSITION 3.1. *Let $W, \tilde{W}$ be two finite reflection groups acting essentially on the same n-dimensional Euclidean space $(E, (\ ,\ ))$. Assume that (W, E), $(\tilde{W}, E)$ are decomposed as*

$$W = W_{(1)} \times W_{(2)},\ E = E_{(1)} \times E_{(2)};\ \tilde{W} = \tilde{W}_{(1)} \times \tilde{W}_{(2)}\ , \quad \text{and} \quad E = \tilde{E}_{(1)} \times \tilde{E}_{(2)}\ .$$

Let $C=C_{(1)}\times C_{(2)}$, $\tilde{C}=\tilde{C}_{(1)}\times\tilde{C}_{(2)}$ be the chambers of $W, \tilde{W}$, respectively. Put $C_1=C\cap S^{n-1}$, $\tilde{C}_1=\tilde{C}\cap S^{n-1}$. Set $F_1=\overline{(\partial C_{(1)}\times C_{(2)})\cap S^{n-1}}$, $F_2=(C_{(1)}\times\partial C_{(2)})\cap S^{n-1}$; $\tilde{F}_1 = \overline{(\partial\tilde{C}_{(1)} \times \tilde{C}_{(2)})\cap S^{n-1}}$ and $\tilde{F}_2 = (\tilde{C}_{(1)} \times \partial\tilde{C}_{(2)})\cap S^{n-1}$. Let $\mathrm{Spec}_{M_0}(C_1)$ *(resp.* $\mathrm{Spec}_{M_0}(\tilde{C}_1)$*) be the spectrum of the membrane problem* (3.1) *of mixed boundary conditions for* (C_1, F_1, F_2) (resp. $(\tilde{C}_1, \tilde{F}_1, \tilde{F}_2)$).

(i) *If the sets of all the exponents of $W, \tilde{W}$ and the sums of all the exponents of $W_{(1)}, \tilde{W}_{(1)}$ coincide each other, then* $\mathrm{Spec}_{M_0}(C_1) = \mathrm{Spec}_{M_0}(\tilde{C}_1)$.

(ii) *The domains $C_1, \tilde{C}_1$ are congruent in S^{n-1} if and only if the Coxeter graphs of $W, \tilde{W}$ coincide.*

EXAMPLE 1. Let $W_{(1)} = A_3, W_{(2)} = A_1 \times G_2;\ \tilde{W}_{(1)} = G_2, \tilde{W}_{(2)} = A_2 \times B_2$.

Then these exponents are

$$\begin{aligned} &W_{(1)}\colon 1, 2, 3\ , \quad W_{(2)}\colon 1, 1, 5\ ,\\ &\tilde{W}_{(1)}\colon 1, 5\ , \qquad \tilde{W}_{(2)}\colon 1, 2, 1, 3\ . \end{aligned}$$

Thus the sets of all the exponents of $W_{(1)}\times W_{(2)}$ and $\tilde{W}_{(1)}\times\tilde{W}_{(2)}$, and the sums of all the exponents of $W_{(1)}, \tilde{W}_{(1)}$ coincide each other. But the Coxeter graphs of $W_{(1)}\times W_2$, $\tilde{W}_{(1)}\times\tilde{W}_{(2)}$ are different.

EXAMPLE 2. Let $W_{(1)} = A_3\times A_1$, $\tilde{W}_{(1)} = A_2\times B_2$. Then the sets of all the exponents of $W_{(1)}$ and $\tilde{W}_{(1)}$ coincide. For any reflection group W, let $W_{(2)} = \tilde{W}_{(2)} = W$. Then $W_{(1)}\times W_{(2)}$ and $\tilde{W}_{(1)}\times\tilde{W}_{(2)}$ give the examples which satisfy the assumptions of Proposition 3.1.

4. **Proof of Proposition 2.2.** Proposition 2.2 can be proved in the similar manner as Theorem 4.3 in [U].

Let $\mathrm{Spec}_D(C_1) = \{\lambda_1 \leqq \lambda_2 \leqq \cdots\}$ be the spectrum of the fixed membrane problem for the domain C_1 in S^{n-1}, and $\{\Psi_i\}_{i=1}^{\infty}$ the complete basis of $L^2(C_1, d\omega)$ such that

$$\begin{cases} \Delta_S\Psi_i = \lambda_i\Psi_i & \text{in} \quad C_1\ ,\\ \Psi_i = 0 & \text{on} \quad \partial C_1\ . \end{cases} \tag{4.1}$$

Here $L^2(C_1, d\omega)$ is the space of all square integrable functions on C_1 with respect to the volume element $d\omega$ on S^{n-1}. For each λ_i in $\mathrm{Spec}_D(C_1)$, let L_{λ_i} be the differential operator on the open interval $(\varepsilon, \varepsilon_1)$ defined by

$$L_{\lambda_i} = -d^2/dr^2 - (n-1)\operatorname{Ct}(r)d/dr + \lambda_i \operatorname{Sn}(r)^{-2} . \tag{4.2}$$

Note that the differential equation in $(\varepsilon, \varepsilon_1)$

$$L_{\lambda_i}\Phi = \mu\Phi \tag{4.3}$$

is equivalent to the differential equation of Sturm-Liouville type:

$$\frac{d}{dr}\left(\operatorname{Sn}(r)^{n-1}\frac{d\Phi}{dr}\right) - \lambda_i \operatorname{Sn}(r)^{n-3}\Phi + \mu \operatorname{Sn}(r)^{n-1}\Phi = 0 . \tag{4.4}$$

Then we have:

LEMMA 4.1. *For arbitrary fixed constants $0 \leqq \alpha < \pi$, $0 < \beta \leqq \pi$, let us consider the boundary value problem* (4.4) *with the boundary conditions*

$$\begin{cases} (\sin\alpha)\operatorname{Sn}(\varepsilon)^{n-1}\Phi'(\varepsilon) - (\cos\alpha)\Phi(\varepsilon) = 0 , \\ (\sin\beta)\operatorname{Sn}(\varepsilon_1)^{n-1}\Phi'(\varepsilon_1) - (\cos\beta)\Phi(\varepsilon_1) = 0 . \end{cases} \tag{4.5}$$

Let $\{\mu_j^{\lambda_i}\}_{j=1}^{\infty}$ be the spectra of the boundary value problem (4.4) *and* (4.5), *$\Phi_j^{\lambda_i}$, $j = 1, 2, \cdots$, an eigenfunction on $(\varepsilon, \varepsilon_1)$ with the eigenvalue $\mu_j^{\lambda_i}$. Then $\{\Phi_j^{\lambda_i}\}_{j=1}^{\infty}$ is a complete basis of the space $L_2^2(\varepsilon, \varepsilon_1)$ of all square integrable functions on $(\varepsilon, \varepsilon_1)$ with respect to the volume element* $\operatorname{Sn}(r)^{n-1}dr$.

PROOF. See [P, p. 508] or [Y, p. 109, Theorem 1].

Now for the complete basis $\{\Psi_i\}_{i=1}^{\infty}$ of $L^2(C_1, d\omega)$ satisfying (4.1), and the eigenfunctions $\Phi_j^{\lambda_i}$, $j = 1, 2, \cdots$, of (4.4) and (4.5) on $(\varepsilon, \varepsilon_1)$ with the eigenvalues $\mu_j^{\lambda_i}$, define C^∞ functions $\Phi_j^{\lambda_i} \otimes \Psi_i$ on D_ε by

$$\Phi_j^{\lambda_i} \otimes \Psi_i(\exp(r\omega)) = \Phi_j^{\lambda_i}(r)\Psi_i(\omega) , \quad r \in (\varepsilon, \varepsilon_1), \ \omega \in C_1 .$$

Then the functions $\Phi_j^{\lambda_i} \otimes \Psi_i$ on D_ε satisfy, by (1.3),

$$\Delta(\Phi_j^{\lambda_i} \otimes \Psi_i) = L_{\lambda_i}\Phi_j^{\lambda_i} \otimes \Psi_i = \mu_j^{\lambda_i}\Phi_j^{\lambda_i} \otimes \Psi_i \quad \text{in} \quad D_\varepsilon ,$$

and the following boundary conditions:

$$\begin{cases} \Phi_j^{\lambda_i} \otimes \Psi_i = 0 & \text{on} \quad \{\exp(r\omega);\ \varepsilon < r < \varepsilon_1,\ \omega \in \partial C_1\} , \\ (\sin\alpha)\operatorname{Sn}(\varepsilon)^{n-1}\dfrac{\partial}{\partial n}(\Phi_j^{\lambda_i} \otimes \Psi_i) - (\cos\alpha)\Phi_j^{\lambda_i} \otimes \Psi_i = 0 , & \text{on} \quad \exp(\varepsilon C_1) , \quad \text{and} \\ (\sin\beta)\operatorname{Sn}(\varepsilon_1)^{n-1}\dfrac{\partial}{\partial n}(\Phi_j^{\lambda_i} \otimes \Psi_i) - (\cos\beta)\Phi_j^{\lambda_i} \otimes \Psi_i = 0 , & \text{on} \quad \exp(\varepsilon_1 C_1) . \end{cases}$$

Moreover we have:

LEMMA 4.2. *$\{\Phi_j^{\lambda_i} \otimes \Psi_i;\ i, j = 1, 2, \cdots\}$ is a complete basis of $L^2(D_\varepsilon)$. Here $L^2(D_\varepsilon)$ is the space of all square integrable functions on D_ε with respect to the volume element dv of (M, g) (see* 1.2)).

PROOF. It can be proved by the same way as Lemma 4.2 in [U], due to Lemma 4.1.

Due to Lemma 4.2, if we choose $\alpha = 0$ and $\beta = \pi$ (resp. $\rho = \mathrm{Sn}(\varepsilon)^{1-n}\cot\alpha = \mathrm{Sn}(\varepsilon_1)^{1-n}\cot\beta$), as in Lemma 4.1, then the set $\{\mu_j^{\lambda_i}; i, j = 1, 2, \cdots\}$ gives the spectra $\mathrm{Spec}_D(D_\varepsilon)$ (resp. $\mathrm{Spec}_{M_\rho}(D_\varepsilon)$). Thus we prove (i) of Proposition 2.2. We can prove (ii) in the similar manner as (i) making use of Lemma 4.1 for $\alpha = \beta = \pi/2$.

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