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BOUNDEDNESS OF THE BERGMAN PROJECTOR AND BELL'S DUALITY THEOREM

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Introduction. Suppose given a bounded domain Ω in $\boldsymbol{C}^n$ with smooth boundary, and a positive integer s. In case Ω is strictly pseudo-convex, Bell [3] has discussed the duality $W^sH(\Omega) \subset L^2H(\Omega) \subset W^sH(\Omega)^*$, where $W^sH(\Omega)$ denotes the space of holomorphic functions in Ω contained in the $L^2(\Omega)$ Sobolev space $W^s(\Omega)$ of order s. He has shown that $W^sH(\Omega)^* = W^{-s}H(\Omega)$ as a Banach space and that the natural isometry $\Lambda^s: W^sH(\Omega) \to W^sH(\Omega)^*$ is given by

$$\Lambda^s g(z) = (g, K(\cdot, z))_s \quad \text{for} \quad g \in W^sH(\Omega) \quad \text{and} \quad z \in \Omega ,$$

where $K(\cdot, \cdot)$ stands for the Bergman kernel. The purpose of the present paper is to observe that such a duality is most naturally stated in connection with the following regularity condition on the Bergman projector K:

$$(\mathrm{R})_0^s \qquad K: W_0^s(\Omega) \to W^sH(\Omega) \subset W^s(\Omega) \quad \text{is bounded} .$$

In particular, we shall show that $(\mathrm{R})_0^s$ is equivalent to that $W^sH(\Omega)^* = W_{\mathrm{cl}}^{-s}H(\Omega)$ as a Banach space, where $W_{\mathrm{cl}}^{-s}H(\Omega)$ denotes the closure of $L^2H(\Omega)$ in $W^{-s}(\Omega)$. Also an expression for Λ^s as above will be verified under the assumption $(\mathrm{R})_0^s$.

The regularity condition $(\mathrm{R})_0^s$ is the case without loss of derivatives of the so-called condition R due to Bell [2] (see also Bell-Ligocka [8]), while the condition R has been successfully used in the problem of extending a given biholomorphic mapping smoothly to the boundary, a problem which is expected to be solved affirmatively for pseudo-convex domains (see Fefferman [11]). In particular, Ligocka [23] has given a positive answer to the smooth extension problem for domains satisfying the condition R, see also Bell-Ligocka [8], Bell [4], Bell-Catlin [7], Diederich-Fornaess [10], and the references therein.

Observe that $(\mathrm{R})_0^s$ is satisfied if

$$(\mathrm{R})^s \qquad K: W^s(\Omega) \to W^sH(\Omega) \subset W^s(\Omega) \quad \text{is bounded} ,$$

which is another case of the condition R without loss of derivatives and

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is known to be satisfied by a fairly wide class of pseudo-convex domains including strictly pseudo-convex ones *via* Kohn's $\bar{\partial}$-Neumann theory in [17], [19], [21], see also Kohn-Nirenberg [22] and Folland-Kohn [12]. On the other hand, there is no known example of a domain satisfying the condition R but actually with loss of derivatives in the sense of either $(\mathrm{R})_0^s$ or $(\mathrm{R})^s$. Our results may be regarded as giving an example in which $(\mathrm{R})_0^s$ appears more naturally than $(\mathrm{R})^s$.

The duality between $W_{\mathrm{cl}}^{-s}H(\Omega)$ and $W^sH(\Omega)$ is realized by a pairing, which is a natural extension of the $L^2(\Omega)$ scalar product, but is different from the distributional one between $W^{-s}(\Omega)$ and $W_0^s(\Omega)$. Such a pairing is well-defined by virtue of a bounded right inverse Φ^s of K in $(\mathrm{R})_0^s$, an operator which is originally due to Bell [2], [3], [4]. It is somehow surprising that such an operator exists even without assuming $(\mathrm{R})_0^s$. As a consequence, $(\mathrm{R})_0^s$ turns out to be equivalent to either one of

$$(\mathrm{R}')_0^s \qquad KW_0^s(\Omega) \subset W^sH(\Omega)\,, \qquad\qquad (\mathrm{R}'')_0^s \qquad KW_0^s(\Omega) = W^sH(\Omega)\,.$$

This fact will be proved in Section 1, together with the well-definedness of the pairing above. Properties of the operator Φ^s will be further discussed in Appendix.

We shall state and prove our main results in Section 2, theorems on the duality and the isomorphism Λ^s. This duality may be compared with that between $W^{-s}(\Omega)$ and $W_0^s(\Omega)$, where the inverse of Λ^s corresponds to the Green operator for the zero Dirichlet problem associated with the Dirichlet integral $(\cdot\,,\cdot)_s$. As will be seen in Section 3, it turns out that the operator Λ^s admits an eigenfunction expansion, where the system of normalized eigenfunctions is complete and orthogonal in $L^2H(\Omega)$ and in $W^sH(\Omega)$ simultaneously. The corresponding Fourier series expansion will be possible for elements of $L^2H(\Omega)$, $W^sH(\Omega)$ and $W^sH(\Omega)^* = W_{\mathrm{cl}}^{-s}H(\Omega)$ under the assumption $(\mathrm{R})_0^s$.

An analogous argument is possible for the Szegö projector and the Szegö kernel. In Section 4, we shall show the corresponding theorems on the duality and the isomorphism.

The properties of the operator Φ^s needed in Section 1 have been proved by Bell [3], [4] in case Ω is pseudo-convex. In Appendix, we shall describe how his proof is modified to provide a proof in the general case. More properties of Φ^s will be also involved.

It should be mentioned that Steven Bell has announced and informed the author during the summer meeting at Oberwolfach, 1983, of the following results:

(1) If Ω is pseudo-convex, then $W_{\mathrm{cl}}^{-s}H(\Omega) = W^{-s}H(\Omega)$ for $0 < s \leqq +\infty$;

(2) If Ω is pseudo-convex and satisfies the condition R, then $W^{\infty}H(\Omega)$ and $W^{-\infty}H(\Omega)$ are mutually dual *via* a pairing which is a natural extension of the $L^2(\Omega)$ scalar product;

cf. Bell-Boas [24]. By virtue of (1), we do not need to consider the closure $W_{\mathrm{cl}}^{-s}H(\Omega)$ as far as pseudo-convex domains are concerned, while (2) is regarded as a Fréchet space version with loss of derivatives of our Banach (or Hilbert) space result.

1. A bounded right inverse of the Bergman projector (preliminaries). Let Ω be a bounded domain in $\boldsymbol{C}^n$, $n \geqq 1$, with smooth boundary. Given a positive integer s fixed throughout, we denote by $W^s(\Omega)$ and $W_0^s(\Omega)$ the $L^2(\Omega)$ Sobolev space of order s and the closure of $C_0^\infty(\Omega)$ in $W^s(\Omega)$, respectively, equipped with a standard scalar product $(\cdot\,,\cdot)_s$ and the corresponding norm $\|\cdot\|_s$. The $L^2(\Omega)$ scalar product and norm will be denoted by $(\cdot\,,\cdot)_0$ and $\|\cdot\|_0$, respectively. Denoting by $\mathrm{Hol}\,(\Omega)$ the totality of holomorphic functions in Ω, we set

$$W^sH(\Omega) = W^s(\Omega)\cap \mathrm{Hol}\,(\Omega)\ , \qquad L^2H(\Omega) = L^2(\Omega)\cap \mathrm{Hol}\,(\Omega)\ ,$$

and regard them as Hilbert subspaces of $W^s(\Omega)$ and $L^2(\Omega)$, respectively.

Recall that the orthogonal projector $K\colon L^2(\Omega)\to L^2H(\Omega)\subset L^2(\Omega)$ is called the *Bergman projector* associated with Ω. We are concerned with the following regularity condition:

$$(\mathrm{R})_0^s \qquad K\colon W_0^s(\Omega)\to W^sH(\Omega)\subset W^s(\Omega) \quad \text{is bounded}\ .$$

It is somehow surprising that K has *a priori* a bounded right inverse even without assuming $(\mathrm{R})_0^s$ as follows.

LEMMA 1.1. *There exists a bounded linear operator $\Phi^s\colon W^sH(\Omega)\to W_0^s(\Omega)$ such that $K\Phi^s g = g$ for $g\in W^sH(\Omega)$.*

The proof of Lemma 1.1 will be given in Appendix with more properties of Φ^s. Let us here provide its immediate consequences, the first of which is related to the negative norm $\|\cdot\|_{-s}$ on the $L^2(\Omega)$ Sobolev space $W^{-s}(\Omega)$ of order $-s$.

LEMMA 1.2. *There exists a constant $C_s > 0$ such that*

$$|(f, g)_0| \leqq C_s\|f\|_{-s}\|g\|_s \quad \textit{for} \quad (f, g)\in L^2H(\Omega)\times W^sH(\Omega)\ .$$

PROOF. Since $(f, g)_0 = (f, K\Phi^s g)_0 = (f, \Phi^s g)_0$, it follows that C_s is realized by the operator norm of Φ^s in Lemma 1.1. q.e.d.

By virtue of Lemma 1.2, it is possible to generalize $(\cdot\,,\cdot)_0$ uniquely to a pairing $(\cdot\,,\cdot)_H$ on $W_{\mathrm{cl}}^{-s}H(\Omega)\times W^sH(\Omega)$, where $W_{\mathrm{cl}}^{-s}H(\Omega)$ stands for the closure of $L^2H(\Omega)$ in $W^{-s}(\Omega)$. Then,

$$|(f, g)_H| \leqq C_s\|f\|_{-s}\|g\|_s \quad \text{for} \quad (f, g) \in W^{-s}_{\mathrm{cl}}H(\Omega) \times W^s H(\Omega) ,$$

with the same constant C_s as in Lemma 1.2.

REMARK 1.1. Let $W^{-s}H(\Omega) = W^{-s}(\Omega) \cap \mathrm{Hol}(\Omega)$. In case Ω is strictly pseudo-convex, Bell [3] has proved the density statement

$$(\mathrm{D})^{-s} \qquad W^{-s}_{\mathrm{cl}}H(\Omega) = W^{-s}H(\Omega) ,$$

by using a stability property for the Bergman kernel due to Greene-Krantz [13]. However, it is not known whether $(\mathrm{D})^{-s}$ follows from $(\mathrm{R})^s_0$ or not.

Another consequence of Lemma 1.1 is in order. Though it will not be used in what follows, it may have its own interest.

PROPOSITION 1.3. *The condition* $(\mathrm{R})^s_0$ *is equivalent to either one of*:

$$(\mathrm{R}')^s_0 \qquad KW^s_0(\Omega) \subset W^s H(\Omega) , \qquad\qquad (\mathrm{R}'')^s_0 \qquad KW^s_0(\Omega) = W^s H(\Omega) .$$

PROOF. By Lemma 1.1, we have $KW^s_0(\Omega) \supset W^s H(\Omega)$, so that $(\mathrm{R}')^s_0$ and $(\mathrm{R}'')^s_0$ are equivalent. Obviously, $(\mathrm{R})^s_0$ implies $(\mathrm{R}')^s_0$. If $(\mathrm{R}')^s_0$ is satisfied, then the operator $K: W^s_0(\Omega) \to W^s H(\Omega)$ is closed, for K is bounded in $L^2(\Omega)$. Hence, $(\mathrm{R})^s_0$ follows from $(\mathrm{R}')^s_0$ by the closed graph theorem. q.e.d.

REMARK 1.2. The situation in Proposition 1.3 will be more clarified if we make Φ^s canonical as in [3]. Denoting by $W^s_0 H^\perp(\Omega)$ the null space of $K: W^s_0(\Omega) \to L^2 H(\Omega)$, we have the orthogonal decomposition $W^s_0(\Omega) = W^s_0 H^\perp(\Omega) \oplus W^s_0 H^\perp(\Omega)^\perp$, where $W^s_0 H^\perp(\Omega)^\perp$ stands for the orthogonal complement of $W^s_0 H^\perp(\Omega)$ in $W^s_0(\Omega)$. By using the orthogonal projector $P^s_0: W^s_0(\Omega) \to W^s_0 H^\perp(\Omega)^\perp \subset W^s_0(\Omega)$, we define

$$\Phi^s_{\mathrm{can}} = P^s_0 \Phi^s: W^s H(\Omega) \to W^s_0 H^\perp(\Omega)^\perp ,$$

which is bounded and injective. Moreover, setting

$$K^s_{\mathrm{can}} = K: W^s_0 H^\perp(\Omega)^\perp \to L^2 H(\Omega) ,$$

we have that $K^s_{\mathrm{can}} \Phi^s_{\mathrm{can}} g = g$ for $g \in W^s H(\Omega)$. Hence, $(\mathrm{R}')^s_0$ is equivalent to either one of the following conditions:

(1) Φ^s_{can} is surjective;

(2) Φ^s_{can} is a Banach space isomorphism;

(3) $K^s_{\mathrm{can}} W^s_0 H^\perp(\Omega)^\perp \subset W^s H(\Omega)$;

(4) $K^s_{\mathrm{can}}: W^s_0 H^\perp(\Omega)^\perp \to W^s H(\Omega)$ is a Banach space isomorphism.

REMARK 1.3. Observe that $(\mathrm{R})^s_0$ follows from

$$(\mathrm{R})^s \qquad K: W^s(\Omega) \to W^s H(\Omega) \subset W^s(\Omega) \quad \text{is bounded} .$$

It has been known (cf. Kohn [21]) that $(\mathrm{R})^s$ holds if the $\bar\partial$-Neumann problem for $(0, 1)$-forms is subelliptic. This fact is implicitly involved in the proof of the smoothness of the $\bar{\partial}$-Neumann solutions, see Kohn-Nirenberg [22] and Folland-Kohn [12]. In particular, $(\mathrm{R})^s$ is satisfied by a fairly wide class of pseudo-convex domains including strictly pseudo-convex ones, see Kohn [17], [19], [21].

Notice that $(\mathrm{R})^s$ always holds when $n = 1$, for the $\bar{\partial}$-Neumann problem reduces to the zero Dirichlet problem in this case.

2. A duality theorem and an isomorphism theorem. We are in a position to state and prove our main results. We shall reformulate Bell's results in [3], established for a strictly pseudo-convex domain Ω, in connection with the regularity condition $(\mathrm{R})_0^s$ on the Bergman projector K.

Let us begin with a duality theorem. We are concerned with the duality statement

$(*)^s$ $W_{\mathrm{cl}}^{-s}H(\Omega)$ and $W^sH(\Omega)$ are mutually dual as Banach spaces *via* the pairing $(\cdot\,,\cdot)_H$;

that is, the mappings $W_{\mathrm{cl}}^{-s}H(\Omega) \ni f \mapsto (f, \cdot)_H \in W^sH(\Omega)^*$ and $W^sH(\Omega) \ni g \mapsto \overline{(\cdot, g)_H} \in W_{\mathrm{cl}}^{-s}H(\Omega)^*$ are Banach space isomorphisms, where the dual space X^* of a Banach space X is regarded to consist of the totality of bounded conjugate linear functionals on X.

THEOREM 1 (duality theorem). *The conditions* $(\mathrm{R})_0^s$ *and* $(*)^s$ *are equivalent.*

The next theorem is concerned with an isomorphism, which is naturally related to the duality $(*)^s$. Let $K(z, w)$ for $z, w \in \Omega$ denote the *Bergman kernel* associated with Ω, which is the reproducing kernel associated with $L^2H(\Omega)$ and is related with the Bergman projector K by

$$Ku(z) = (u, K(\cdot, z))_0 \quad \text{for} \quad u \in L^2(\Omega) \quad \text{and} \quad z \in \Omega\,. \tag{2.1}$$

THEOREM 2 (isomorphism theorem). *If* $(\mathrm{R})_0^s$ *holds, then a Banach space isomorphism* $\Lambda^s\colon W^sH(\Omega) \to W_{\mathrm{cl}}^{-s}H(\Omega)$ *is given by*

$$\Lambda^s g(z) = (g, K(\cdot, z))_s \quad \textit{for} \quad g \in W^sH(\Omega) \quad \textit{and} \quad z \in \Omega\,, \tag{2.2}$$

which satisfies

$$(g, h)_s = (\Lambda^s g, h)_H \quad \textit{for} \quad g, h \in W^sH(\Omega)\,. \tag{2.3}$$

PROOF OF THEOREM 1. Since $W_0^s(\Omega)$ is dense in $L^2(\Omega)$ and K is bounded in $L^2(\Omega)$, it follows that $(\mathrm{R})_0^s$ implies the following density condition

$(\mathrm{D})^s$ $\qquad W^sH(\Omega)$ is dense in $L^2H(\Omega)$.

Let us assume $(\mathrm{D})^s$. Then, we may consider the duality

$$W^sH(\Omega) \hookrightarrow L^2H(\Omega) = L^2H(\Omega)^* \hookrightarrow W^sH(\Omega)^* , \tag{2.4}$$

where the second inclusion is given by the adjoint operator of the first one, so that both inclusions are dense and continuous with the operator norms $\leqq 1$. More precisely, by virtue of F. Riesz' theorem, there exists a linear surjective isometry $\Lambda^{-s}: W^sH(\Omega)^* \to W^sH(\Omega)$ such that

$$(\Lambda^{-s}f, g)_s = (f, g)_0 \quad \text{for} \quad (f, g) \in L^2H(\Omega) \times W^sH(\Omega) , \tag{2.5}$$

a relation which determines Λ^{-s} uniquely. Then, the duality (2.4) between $W^sH(\Omega)^*$ and $W^sH(\Omega)$ is realized by a pairing defined by

$$W^sH(\Omega)^* \times W^sH(\Omega) \ni (f, g) \mapsto (\Lambda^{-s}f, g)_s \in \boldsymbol{C} ,$$

which is a natural generalization of $(\cdot\,,\cdot)_0$. In particular, the norm $|||\cdot|||_{-s}$ on $W^sH(\Omega)^*$ satisfies and is uniquely determined by

$$|||f|||_{-s} = \sup\{|(f, g)_0|;\ g \in W^sH(\Omega),\ \|g\|_s = 1\}$$

for $f \in L^2H(\Omega)$. On the other hand, by virtue of Lemma 1.2, the second inclusion in (2.4) extends uniquely to $W_{\mathrm{cl}}^{-s}H(\Omega) \hookrightarrow W^sH(\Omega)^*$, and $|||f|||_{-s} \leqq C_s\|f\|_{-s}$ for $f \in W_{\mathrm{cl}}^{-s}H(\Omega)$; moreover,

$$|||f|||_{-s} = \sup\{|(f, g)_H|;\ g \in W^sH(\Omega),\ \|g\|_s = 1\} .$$

Hence, recalling that $W^sH(\Omega)$ is a Hilbert space, which is reflexive, we see that $(*)^s$ holds if and only if $W^sH(\Omega)^* = W_{\mathrm{cl}}^{-s}H(\Omega)$ as a Banach space (possibly with different norms), which is further equivalent to the existence of a constant $C_s' > 0$ such that

$$\|f\|_{-s} \leqq C_s'|||f|||_{-s} \quad \text{for} \quad f \in L^2H(\Omega) . \tag{2.6}$$

Suppose that $(\mathrm{R})_0^s$ holds, so that $(\mathrm{D})^s$ is satisfied. Then, by using the duality (2.4), we have, for $(f, u) \in L^2H(\Omega) \times W_0^s(\Omega)$, that

$$|(f, u)_0| = |(f, Ku)_0| \leqq |||f|||_{-s}\|K\|_{s,s}\|u\|_s ,$$

where $\|K\|_{s,s}$ stands for the operator norm of K in $(\mathrm{R})_0^s$. Hence, recalling that $W^{-s}(\Omega)$ is defined by the duality

$$W_0^s(\Omega) \hookrightarrow L^2(\Omega) = L^2(\Omega)^* \hookrightarrow W_0^s(\Omega)^* = W^{-s}(\Omega) , \tag{2.7}$$

we get (2.6) with $C_s' = \|K\|_{s,s}$, obtaining $(*)^s$.

We next observe that $(*)^s$ implies $(\mathrm{D})^s$. In fact, if $(*)^s$ holds and $f \in L^2H(\Omega)$ satisfies $(f, g)_H = (f, g)_0 = 0$ for any $g \in W^sH(\Omega)$, then $f = 0 \in W_{\mathrm{cl}}^{-s}H(\Omega)$, obtaining $(\mathrm{D})^s$.

Suppose that $(*)^s$ is satisfied, so that $(\mathrm{D})^s$ is valid and that (2.6) holds with some $C_s' > 0$. Then, taking the duality (2.7) into account again, we have, for $(f, u) \in L^2H(\Omega) \times W_0^s(\Omega)$, that

$$|(f, Ku)_0| = |(f, u)_0| \leqq C_s' |||f|||_{-s} \| u \|_s .$$

Therefore, by virtue of the duality (2.4), we get $Ku \in W^sH(\Omega)$ and $\| Ku \|_s \leqq C_s' \| u \|_s$, obtaining $(\mathrm{R})_0^s$. (Incidentally, the least possible C_s' is attained by $\| K \|_{s,s}$.) q.e.d.

PROOF OF THEOREM 2. Let us begin with observing that $(\mathrm{R})_0^s$ implies

$$K(\cdot, z) \in W^sH(\Omega) \quad \text{for} \quad z \in \Omega , \tag{2.8}$$

so that the right hand side of (2.2) makes sense. In order to prove (2.8), we recall that $K(\cdot, \cdot)$ is sesqui-holomorphic and Hermitian symmetric. It then follows from (2.1) and the mean value property for harmonic functions that

$$K(w, z) = (\phi_z, K(\cdot, w))_0 = K\phi_z(w) \quad \text{for} \quad z, w \in \Omega ,$$

where $\phi_z \in C_0^\infty(\Omega)$ is radially symmetric around z and satisfies $(\phi_z, 1)_0 = 1$, an expression which is due to Bell [2], cf. also Kerzman [14]. Hence, $(\mathrm{R})_0^s$ implies (2.8).

It is easy to see that the desired isomorphism Λ^s is given by $\Lambda^s = (\Lambda^{-s})^{-1}$ with Λ^{-s} in the proof of Theorem 1, if we recall that $W_{\mathrm{cl}}^{-s}H(\Omega) = W^sH(\Omega)^*$ as a Banach space. In fact, (2.3) follows from (2.5). Setting $g = K(\cdot, z)$ in (2.5), we have, by (2.1), that

$$f(z) = (\Lambda^{-s}f, K(\cdot, z))_s \quad \text{for} \quad f \in L^2H(\Omega) \quad \text{and} \quad z \in \Omega ,$$

implying (2.2). q.e.d.

REMARK 2.1. Suppose that Ω is pseudo-convex. Then, the density condition $(\mathrm{D})^s$ is valid by virtue of Kohn's weighted $\bar{\partial}$-Neumann theory in [20]. More precisely, the space $W^\infty H(\Omega) = C^\infty(\bar{\Omega}) \cap \mathrm{Hol}(\Omega)$ is dense both in $W^sH(\Omega)$ and in $L^2H(\Omega)$, see Catlin [9].

REMARK 2.2. The duality (2.4) may be compared with (2.7), in which Λ^{-s} corresponds to the Green operator $\Lambda_0^{-s}\colon W^{-s}(\Omega) \to W_0^s(\Omega)$ for the zero Dirichlet problem associated with the Dirichlet integral $(\cdot, \cdot)_s$ on $W_0^s(\Omega) \times W_0^s(\Omega)$. The inverse $\Lambda_0^s = (\Lambda_0^{-s})^{-1}$ is realized by an elliptic partial differential operator, for one may integrate by parts, cf. (2.2).

3. Fourier series expansion in terms of a doubly orthogonal system.

Suppose that the density condition $(\mathrm{D})^s$ holds. Noticing that the inclusions in (2.4) are compact, we may discuss the eigenfunction expansion of Λ^s in a standard manner. It will turn out that every element of $W^sH(\Omega)^*$ admits a Fourier series expansion in terms of a system of eigenfunctions of Λ^s. Taking account of Theorem 1, we shall have:

THEOREM 3 (expansion theorem). *If* $(\mathrm{R})_0^s$ *is satisfied, then there exists a complete orthonormal system* $\{h_j^s\}_j$ *of* $L^2H(\Omega)$, *which is also orthogonal in* $W^sH(\Omega)$, *such that*

$$f=\sum_j (f, h_j^s)_H h_j^s \quad for \quad f\in W_{\mathrm{cl}}^{-s}H(\Omega)\ ,$$

where the series in the right hand side converges in $W^{-s}(\Omega)$. *Moreover, every* h_j^s *is an eigenfunction of* Λ^s.

PROOF. Suppose that $(\mathrm{D})^s$ is satisfied. By using the inclusions in (2.4), we can realize Λ^{-s}: $W^sH(\Omega)^*\to W^sH(\Omega)$ as a linear operator Λ_L^{-s} in $L^2H(\Omega)$, which is compact and injective with dense range. By using (2.5), we have

$$(f, \Lambda_L^{-s}h)_0=(\Lambda_L^{-s}f, \Lambda_L^{-s}h)_s=(\Lambda_L^{-s}f, h)_0 \quad \text{for} \quad f, h\in L^2H(\Omega)\ ,$$

so that Λ_L^{-s} is self-adjoint and positive. Thus, the inverse $\Lambda_L^s=(\Lambda_L^{-s})^{-1}$ is a self-adjoint realization of Λ^s in $L^2H(\Omega)$.

Observe that Λ_L^{-s} admits an eigenfunction expansion

$$\Lambda_L^{-s}f=\sum_j \lambda_j^s(f, h_j^s)_0 h_j^s \quad \text{for} \quad f\in L^2H(\Omega)\ ,$$

with $\lambda_j^s\geqq\lambda_{j+1}^s>0$ and $\lim\lambda_j^s=0$, where $\{h_j^s\}_j$ is a complete orthonormal system of $L^2H(\Omega)$. Notice that

$$\Lambda_L^{-s}h_j^s=\lambda_j^s h_j^s\ , \qquad \Lambda_L^s h_j^s=(\lambda_j^s)^{-1}h_j^s\ ,$$

so that $h_j^s\in W^sH(\Omega)$. Then, by (2.5),

$$(h_j^s, h_k^s)_s=(\lambda_j^s)^{-1}\delta_{jk}\ , \qquad (\lambda_j^s)^{-1}=\|h_j^s\|_s^2\ .$$

That is, $\{h_j^s\}_j$ is orthogonal also in $W^sH(\Omega)$; thus, the name a *doubly orthogonal system*. Setting $g_j^s=(\lambda_j^s)^{1/2}h_j^s$, we obtain a complete orthonormal system $\{g_j^s\}_j$ of $W^sH(\Omega)$.

Every element $g\in L^2H(\Omega)$ admits a unique Fourier series expansion

$$g=\sum_j (g, h_j^s)_0 h_j^s \quad \text{with} \quad \|g\|_0^2=\sum_j |(g, h_j^s)_0|^2\ .$$

If $g\in W^sH(\Omega)$, then $(g, h_j^s)_0h_j^s=(g, g_j^s)_s g_j^s$, so that

$$\|g\|_s^2=\sum_j (\lambda_j^s)^{-1}|(g, h_j^s)_0|^2\ . \tag{3.1}$$

Moreover, given $g\in L^2H(\Omega)$, we see that $g\in W^sH(\Omega)$ if and only if the right hand side of (3.1) is finite. In fact, if we identify $L^2H(\Omega)$ with the space (l^2) of square summable complex sequences *via* the correspondence

$$L^2H(\Omega) \ni \sum_j a_j h_j^s \leftrightarrow \{a_j\}_j \in (l^2) ,$$

then $W^s H(\Omega)$ corresponds to

$$(w^s) = \{\{a_j'\}_j \in (l^2); \sum_j (\lambda_j^s)^{-1} |a_j'|^2 < +\infty\} .$$

The scalar products with the norms on $L^2H(\Omega)$ and $W^sH(\Omega)$ are transplanted on (l^2) and (w^s), respectively, as follows:

$$(\{a_j\}_j, \{b_j\}_j)_0 = \sum_j a_j \overline{b_j} , \qquad \| \{a_j\}_j \|_0^2 = \sum_j |a_j|^2 ,$$
$$(\{a_j'\}_j, \{b_j'\}_j)_s = \sum_j (\lambda_j^s)^{-1} a_j' \overline{b_j'} , \qquad \| \{a_j'\}_j \|_s^2 = \sum_j (\lambda_j^s)^{-1} |a_j'|^2 .$$

Let us consider the duality

$$(w^s) \hookrightarrow (l^2) = (l^2)^* \hookrightarrow (w^s)^* , \tag{3.2}$$

which is a copy of (2.4). Then, $(w^s)^*$ consists of the totality of complex sequences $\{a_j''\}_j$ such that

$$||| \{a_j''\}_j |||_{-s}^2 = \sum_j \lambda_j^s |a_j''|^2 < +\infty ,$$

and the duality (3.2) between $(w^s)^*$ and (w^s) is realized by a pairing defined by

$$(w^s)^* \times (w^s) \ni (\{a_j''\}_j, \{a_j'\}_j) \mapsto \sum_j a_j'' \overline{a_j'} \in \boldsymbol{C} .$$

In other words, every element $f \in W^sH(\Omega)^*$ admits a unique Fourier series expansion

$$f = \sum_j a_j'' h_j^s \quad \text{in} \quad W^sH(\Omega)^* \quad \text{with} \quad ||| f |||_{-s}^2 = \sum_j \lambda_j^s |a_j''|^2 . \tag{3.3}$$

Suppose that $(\mathrm{R})_0^s$ holds, so that $W^sH(\Omega)^* = W_{\mathrm{cl}}^{-s}H(\Omega)$ as a Banach space. Then, the Fourier series expansion in (3.3) converges in $W^{-s}(\Omega)$, and the coefficients are given by $a_j'' = (f, h_j^s)_H$. Therefore, the proof is finished. q.e.d.

REMARK 3.1. Under the condition $(\mathrm{D})^s$, the spaces $W^sH(\Omega)$ and $W^sH(\Omega)^*$ are characterized by decreasing and growth conditions, respectively, on the Fourier coefficients relative to the eigenvalues of the operator Λ^s. This fact may be compared with a characterization of the L^2 Sobolev spaces defined on the torus, where $-|\alpha|^2$ is the eigenvalue of the Laplacian associated with the eigenfunctions $e^{i\alpha\cdot x}$.

4. The case of the Szegö projector. An argument similar to that in the preceding sections is also possible for the Szegö projector in place of the Bergman projector. In order to describe it, we begin with recalling the definitions of the Szegö kernel and the Szegö projector.

Let $L_b^2H(\Omega)$ denote the totality of holomorphic functions in Ω possessing the L^2 boundary values, equipped with the $L^2(\partial\Omega)$ scalar product. Recall that $L_b^2H(\Omega)$ admits the reproducing kernel $K_b(z, w)$ for $z, w \in \Omega$, which is called the *Szegö kernel* associated with Ω. Correspondingly, the orthogonal projector K_b: $L^2(\partial\Omega) \to L^2H_b(\partial\Omega) \subset L^2(\partial\Omega)$, where $L^2H_b(\partial\Omega) = L_b^2H(\Omega)|_{\partial\Omega}$, is called the *Szegö projector* associated with Ω.

Setting $W^sH_b(\partial\Omega) = W^s(\partial\Omega) \cap L^2H_b(\partial\Omega)$, let us consider the following condition:

$(\mathrm{R}_b)^s$ $\qquad K_b\colon W^s(\partial\Omega) \to W^sH_b(\partial\Omega) \subset W^s(\partial\Omega)$ is bounded ,

implying that

$(\mathrm{D}_b)^s$ $\qquad W^sH_b(\partial\Omega)$ is dense in $L^2H_b(\partial\Omega)$.

The situation here is much simpler than that before, for $W_0^s(\partial\Omega) = W^s(\partial\Omega)$ so that $W^{-s}(\partial\Omega) = W^s(\partial\Omega)^*$. Hence, we have obviously that

$$|\langle f, g\rangle| \leqq \|f\|_{-s}\|g\|_s \quad \text{for} \quad (f, g) \in W_{\mathrm{cl}}^{-s}H_b(\partial\Omega) \times W^sH_b(\partial\Omega) ,$$

where $W_{\mathrm{cl}}^{-s}H_b(\partial\Omega)$ denotes the closure of $L^2H_b(\partial\Omega)$ in $W^{-s}(\partial\Omega)$, and $\langle\cdot, \cdot\rangle$ stands for the sesqui-linear distribution pairing on $W^{-s}(\partial\Omega) \times W^s(\partial\Omega)$. Then, as in Section 2, we first have:

THEOREM 1^b (duality theorem). *The condition* $(\mathrm{R}_b)^s$ *is satisfied if and only if*

$(*_b)^s$ $\qquad$ *$W_{\mathrm{cl}}^{-s}H_b(\partial\Omega)$ and $W^sH_b(\partial\Omega)$ are mutually dual as Banach spaces via the pairing $\langle\cdot, \cdot\rangle$.*

The proof is analogous to that of Theorem 1, and we shall not repeat it here. Let us only remark that if $(\mathrm{R}_b)^s$ is satisfied, then there exists a constant $C_s' > 0$ such that

$$|||f|||_{-s} \leqq \|f\|_{-s} \leqq C_s'|||f|||_{-s} \quad \text{for} \quad f \in L^2H_b(\partial\Omega) ,$$

where $|||\cdot|||_{-s}$ denotes the norm on $W^sH_b(\partial\Omega)^*$, and the least possible C_s' is attained by the operator norm $\|K_b\|_{s,s}$ of K_b in $(\mathrm{R}_b)^s$.

Corresponding to Theorem 2, we next have:

THEOREM 2^b (isomorphism theorem). *If* $(\mathrm{R}_b)^s$ *holds, then a Banach space isomorphism* Λ_b^s: $W^sH_b(\partial\Omega) \to W_{\mathrm{cl}}^{-s}H_b(\partial\Omega)$ *is given by*

$$\Lambda_b^s g(z) = (g, K_b(\cdot, z))_s \quad \textit{for} \quad g \in W^sH_b(\partial\Omega) \quad \textit{and} \quad z \in \Omega ,$$

which satisfies

$$(g, h)_s = \langle \Lambda_b^s g, h\rangle \quad \textit{for} \quad g, h \in W^sH_b(\partial\Omega) .$$

Again, the previous proof applies, if we notice that $(\mathrm{R}_b)^s$ implies

$$K_b(\cdot, z) \in W^s H_b(\partial\Omega) \quad \text{for} \quad z \in \Omega .$$

This fact is a consequence of the following expression due to Kerzman-Stein [15]:

$$K_b(\cdot, z) = K_b P(z, \cdot) \in K_b C^\infty(\partial\Omega) ,$$

where $P(z, \zeta)$ for $(z, \zeta) \in \Omega \times \partial\Omega$ stands for the ordinary Poisson kernel; in fact, for $z, w \in \Omega$,

$$K_b(w, z) = \int_{\partial\Omega} P(z, \zeta) K_b(w, \zeta) dS(\zeta) = [K_b P(z, \cdot)](w) ,$$

where dS denotes the surface element of $\partial\Omega$.

It will be now obvious that we have:

THEOREM 3^b (expansion theorem). *If* $(\mathrm{R}_b)^s$ *is satisfied, then there exists a complete orthonormal system* $\{h_j^s\}_j$ *of* $L^2 H_b(\partial\Omega)$, *which is also orthogonal in* $W^s H_b(\partial\Omega)$, *such that*

$$f = \sum_j \langle f, h_j^s \rangle h_j^s \quad \textit{for} \quad f \in W_{\mathrm{cl}}^{-s} H_b(\partial\Omega) ,$$

where the series in the right hand side converges in $W^{-s}(\partial\Omega)$. *Moreover, every* h_j^s *is an eigenfunction of* Λ_b^s.

REMARK 4.1. Corresponding to Proposition 1.3, we have, without change of the proof, that $(\mathrm{R}_b)^s$ is equivalent to either one of:

$$(\mathrm{R}_b')^s \qquad K_b W^s(\partial\Omega) \subset W^s H_b(\partial\Omega) , \qquad\qquad (\mathrm{R}_b'')^s \qquad K_b W^s(\partial\Omega) = W^s H_b(\partial\Omega) .$$

We also have the orthogonal decomposition $W^s(\partial\Omega) = W^s H_b^\perp(\partial\Omega) \oplus W^s H_b^\perp(\partial\Omega)^\perp$, where $W^s H_b^\perp(\partial\Omega)$ denotes the null space of $K_b \colon W^s(\partial\Omega) \to L^2 H_b(\partial\Omega)$, and $W^s H_b^\perp(\partial\Omega)^\perp$ stands for the orthogonal complement of $W^s H_b^\perp(\partial\Omega)$ in $W^s(\partial\Omega)$. Then, $(\mathrm{R}_b')^s$ holds if and only if $K_b \colon W^s H_b^\perp(\partial\Omega)^\perp \to W^s H_b(\partial\Omega)$ is a Banach space isomorphism.

REMARK 4.2. It is easy to see that $(\mathrm{R}_b)^s$ holds if the $\bar\partial_b$-Neumann problem is subelliptic, as in the case of the Bergman projector stated in Remark 1.3. In particular, $(\mathrm{R}_b)^s$ is satisfied by a strictly pseudo-convex domain in C^n with $n \geqq 3$, see Kohn [18] and Folland-Kohn [12].

Notice that $(\mathrm{R}_b)^s$ always holds when $n = 1$, a fact which is seen, for instance, from an expression of the Szegö projector in terms of the Cauchy projector due to Kerzman-Stein [16].

Appendix. PROPERTIES OF THE OPERATOR Φ^s. Let us here prove Lemma 1.1 with more properties of Φ^s. After reviewing a construction of Φ^s, we shall state and prove these properties of Φ^s. Remarks will be added finally.

A.1. CONSTRUCTION OF Φ^s The operator Φ^s given here is a revised version, due to Barrett [1], of the one given originally by Bell [2], [3], [4].

Let $r \in C^\infty(\boldsymbol{C}^n; \boldsymbol{R})$ be a smooth defining function of Ω, that is, $\Omega = \{z \in \boldsymbol{C}^n; r(z) < 0\}$ and $dr \neq 0$ on $\partial\Omega$. We cover $\bar{\Omega}$ by relatively open sets $U_0, U_1, \cdots, U_n$ in such a way that $U_0 \Subset \Omega$ and $r_i \neq 0$ on U_i for $1 \leqq i \leqq n$, where $r_i = \partial r/\partial z_i$. Let $\{\phi_j\}_j$ be a smooth partition of unity subordinate to the covering $\{U_j\}_j$ of $\bar{\Omega}$, that is, $\phi_j \in C_0^\infty(U_i)$ and $\sum \phi_j = 1$. We then define Φ^s by setting

$$\Phi^s u = \phi_0 u + \frac{r^s}{s!} \sum_{i=1}^n (T_i')^s(\phi_i u) , \tag{A.1}$$

where T_i' is the formal adjoint of T_i defined by $T_i u = (\overline{r_i})^{-1}\partial u/\partial \bar{z}_i$, so that $T_i' u = -\partial(u/r_i)/\partial z_i$.

If one wants Φ^s to be in a more intrinsic form, then one may also set

$$\Phi^s u = \phi_0 u + \frac{r^s}{s!}\left[\vartheta \frac{\sigma(\bar{\partial}, dr)}{|dr|^2/2}\right]^s [(1 - \phi_0)u] , \tag{A.1$'$}$$

where ϑ denotes the formal adjoint of $\bar{\partial}$, and $\sigma(\bar{\partial}, dr)$ stands for the principal symbol of $\bar{\partial}$ in the direction dr.

A.2. PROPERTIES OF Φ^s. Assuming Ω to be pseudo-convex, Bell has proved Lemma 1.1 in [3], [4]; he also mentioned in [4] that the assumption of pseudo-convexity is superfluous. Since no literature on Lemma 1.1 is found for a general domain, we shall describe how Bell's proof in [3], [4] is modified to provide a proof in the general case. The properties of Φ^s in Lemma 1.1 are involved in the following:

PROPOSITION A.1. *The differential operator Φ^s in* (A.1) *or* (A.1)$'$ *satisfies the following properties, where s' is supposed to be a non-negative integer*:

$$\Phi^s \colon W^{s'}H(\Omega) \to W^{s'}(\Omega) \quad \textit{is bounded} ; \tag{A.2}$$

$$\Phi^s W^{s'}H(\Omega) \subset W_0^{s'}(\Omega) \quad \textit{for} \quad s' \leqq s ; \tag{A.3}$$

$$K\Phi^s g = g \quad \textit{for} \quad g \in L^2H(\Omega) ; \tag{A.4}$$

$$K\Phi^s u = Ku \quad \textit{for} \quad u \in W^s(\Omega) . \tag{A.5}$$

A.3. PROOF OF PROPOSITION A.1 IN THE PSEUDO-CONVEX CASE. The following argument is essentially due to Bell [3], [4]. Suppose that Ω is pseudo-convex. Then, $W^\infty H(\Omega)$ is dense in $W^{s'}H(\Omega)$ for any s', see Remark 2.1. Given $h \in W^\infty H(\Omega)$, we have, integrating by parts, that

(A.6) $$(\Phi^s u, h)_0 = (\Phi^{s-1}u, h)_0 = \cdots = (u, h)_0 \quad \text{for} \quad u \in W^s(\Omega)\ ,$$

which remains valid for $h \in L^2H(\Omega)$, obtaining (A.5). By a similar use of integration by parts,

$$\|\Phi^s g\|_{s'} \leqq C_{s,s'}\|g\|_{s'} \quad \text{with some constant} \quad C_{s,s'} > 0\ ,$$

first for $g \in W^\infty H(\Omega)$ and then for $g \in W^{s'}H(\Omega)$, implying (A.2). Since $\Phi^s W^\infty H(\Omega) \subset C^\infty(\bar{\Omega}) \cap W_0^s(\Omega)$, (A.3) follows from (A.2). As in (A.6), we have $(\Phi^s g, h)_0 = (g, h)_0$, first for $g, h \in W^\infty H(\Omega)$ and then for $g, h \in L^2H(\Omega)$ by virtue of (A.2), obtaining (A.4). Therefore, the proof is finished in case Ω is pseudo-convex.

A.4. A CUT-OFF FUNCTION. If Ω is not necessarily pseudo-convex, then we have to be more careful on the boundary integrals arising from integration by parts. We then need a cut-off function $\tilde{\chi}_\varepsilon$, depending on a small parameter $\varepsilon > 0$, which translates the growth rate of the "boundary integrals" to negative powers of ε.

Given $\varepsilon > 0$ small, let χ_ε denote the characteristic function of the set $\Omega_\varepsilon = \{z \in \Omega;\ \text{distance}\ (z, \partial\Omega) > \varepsilon\}$. In order to mollify χ_ε, we choose a non-negative function $\phi \in C_0^\infty(\boldsymbol{C}^n)$ such that $\int \phi dV = 1$ and $\operatorname{supp}(\phi) \subset \{z \in \boldsymbol{C}^n;\ |z| < 1\}$, where dV denotes the standard volume element of $\boldsymbol{C}^n$. Setting $\phi_\varepsilon(z) = \varepsilon^{-2n}\phi(z/\varepsilon)$, we define $\tilde{\chi}_\varepsilon$ by

$$\tilde{\chi}_\varepsilon(z) = \int \chi_\varepsilon(z-w)\phi_\varepsilon(w)dV(w) = \int \phi_\varepsilon(z-w)\chi_\varepsilon(w)dV(w)\ .$$

Then, $\tilde{\chi}_\varepsilon$ satisfies $\tilde{\chi}_\varepsilon \in C_0^\infty(\Omega)$, $0 \leqq \tilde{\chi}_\varepsilon \leqq 1$ and $\tilde{\chi}_\varepsilon = 1$ on $\Omega_{2\varepsilon}$. Moreover, for a non-zero multi-index $\alpha \in \boldsymbol{Z}_+^{2n}$,

(A.7) $$\operatorname{supp}(D^\alpha\tilde{\chi}_\varepsilon) \subset \Omega\backslash\Omega_{2\varepsilon}\ , \qquad r^{|\alpha|}|D^\alpha\tilde{\chi}_\varepsilon| \leqq C_\alpha\ ,$$

with some constant $C_\alpha > 0$ independent of ε, where $D = (D_1, \cdots, D_{2n})$ denotes the standard differentiation with respect to the real coordinates in $\boldsymbol{C}^n$ divided by the imaginary unit $(-1)^{1/2}$.

A.5. PROOF OF PROPOSITION A.1. In order to prove (A.5), we take $(u, h) \in W^s(\Omega) \times L^2H(\Omega)$. Taking account of (A.7), we have, integrating by parts, that

$$(\tilde{\chi}_\varepsilon(\Phi^s u - \Phi^{s-1}u), h)_0 \to 0 \quad \text{as} \quad \varepsilon \to 0\ ,$$

obtaining (A.5). By a similar use of integration by parts, we get

$$\varlimsup_{\varepsilon\to 0} \int_\Omega \tilde{\chi}_\varepsilon \Big|\sum_{|\alpha|\leqq s'} D^\alpha\Phi^s g\Big|^2 dV \leqq C_{s,s'}\|g\|_{s'}^2\ ,$$

for $g \in W^{s'}H(\Omega)$, with some constant $C_{s,s'} > 0$, implying (A.2). Taking

(A.2) into account, we obtain (A.4) as in the proof of (A.5). In order to prove (A.3), it suffices to show that

$$(\Phi^s g, D^\alpha v)_0 = (D^\alpha \Phi^s g, v)_0 \quad \text{for} \quad (g, v) \in W^{s'}H(\Omega) \times C^\infty(\bar{\Omega}) ,$$

whenever $|\alpha| \leqq s' \leqq s$, a fact which follows from

$$(\tilde{\chi}_\varepsilon \Phi^s g, D^\alpha v)_0 - (\tilde{\chi}_\varepsilon D^\alpha \Phi^s g, v)_0 \to 0 \quad \text{as} \quad \varepsilon \to 0 .$$

Therefore, the proof is completed.

A.6. MISCELLANEOUS REMARKS. Since $\Phi^s\colon W^{s+s'}(\Omega) \to W_0^{s'}(\Omega)$ is continuous, it follows from (A.5) that the following two conditions are equivalent:

$(\mathrm{R})^\infty$ $$K\colon C^\infty(\bar{\Omega}) \to C^\infty(\bar{\Omega}) \quad \text{is continuous ;}$$

$(\mathrm{R})_0^\infty$ $$K\colon W_0^\infty(\Omega) = \bigcap_{s'>0} W_0^{s'}(\Omega) \to C^\infty(\bar{\Omega}) \quad \text{is continuous ,}$$

either of which is called the *condition* R. The equivalence of $(\mathrm{R})^\infty$ and $(\mathrm{R})_0^\infty$ has been observed by Barrett [1], see also Bell-Boas [6].

By (A.5), we have $KW_0^{s'}(\Omega) \supset KW^{s+s'}(\Omega)$, so that

$$\bigcap_{s'>0} KW_0^{s'}(\Omega) = \bigcap_{s'>0} KW^{s'}(\Omega) \supset W^\infty H(\Omega) .$$

Then, it may be natural to ask whether

(A.8) $$KW_0^\infty(\Omega) \supset W^\infty H(\Omega) ,$$

which is actually valid. Assuming Ω to be strictly pseudo-convex, Bell has proved (A.8) in [3], where he must have used (A.2) with $s' = s - 1$. Subsequently, an elegant proof of (A.8) in the general case has been given by Bell-Catlin [7], which is based on the fact that

(A.9) $$(\Delta u, h)_0 = 0 \quad \text{for} \quad (u, h) \in W_0^2(\Omega) \times L^2H(\Omega) ,$$

where Δ stands for the real Laplacian. For an another application of (A.9) in a context similar to that in [7], see Bell [5].

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