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SOME RESULTS ON THE LOCAL MODULI OF NON-SINGULAR NORMALIZATIONS OF SURFACES WITH ORDINARY SINGULARITIES

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General notation. If Y, Z, D, L, $\mathscr{G}$ are a complex manifold, a subvariety of Y, a divisor on Y, a holomorphic line bundle over Y and a coherent analytic sheaf over Y, respectively, then we put

$\mathscr{O}_Y$: the structure sheaf of Y,

Ω_Y^p: the sheaf of germs of holomorphic p-forms on Y,

T_Y: the sheaf of germs of holomorphic vector fields on Y,

K_Y: the canonical divisor of Y,

$\mathscr{I}(Z)$: the sheaf of ideal of Z in $\mathscr{O}_Y$,

$T_Y(-Z)$: the subsheaf of T_Y consisting of those holomorphic vector fields on Y which vanish on Z,

$T_Y(\log Z)$: the subsheaf of T_Y consisting of the derivations of $\mathscr{O}_Y$ which send $\mathscr{I}(Z)$ into itself (we call this sheaf the *logarithmic tangent sheaf along* Z),

$[D]$: the line bundle determined by the divisor D,

$\mathscr{O}_Y(L)$: the sheaf of germs of local holomorphic cross-sections of L,

$\mathscr{O}_Y(L-Z)$: the subsheaf of $\mathscr{O}_Y(L)$ consisting of germs of those local holomorphic cross-sections of L which vanish on Z,

$\mathscr{O}_Y(L-2Z)$: the subsheaf of $\mathscr{O}_Y(L)$ consisting of germs of those local holomorphic cross-sections of L whose fiber coordinates vanish on Z together with their partial derivatives,

$$\Omega_Y^p(L) := \Omega_Y^p \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(L)\ ,\quad \Omega_Y^p(L-Z) := \Omega_Y^p \otimes_{\mathscr{O}_Y} \mathscr{O}_Y(L-Z)\ ,$$
$$\mathscr{O}_Z(L) := \mathscr{O}_Y(L)/\mathscr{I}(Z)\mathscr{O}_Y(L),$$
$$\mathscr{N}_Z := T_Y/T_Y(\log Z)\ ,\quad h^q(Y, \mathscr{G}) := \dim_C H^q(Y, \mathscr{G})\ .$$

Furthermore, if Z is non-singular, we put

$N_{Z/Y}$: the normal bundle of Z in Y, or the sheaf of germs of normal vectors on Z in Y.

If $f: Y_1 \to Y_2$ is a holomorphic map between complex manifolds, we put

$\mathscr{T}_{Y_1/Y_2}$: the cokernel of the natural sheaf homomorphism: $T_{Y_1} \to f^* T_{Y_2}$.

Throughout this paper we mean by a *surface* a compact irreducible analytic variety of dimension two, where an analytic variety means a reduced complex space. Diagrams in this paper are always commutative and exact unless otherwise explicitly mentioned.

Introduction. A surface S embedded in a compact threefold W is said to be *with ordinary singularities* if, for each singular point p of S, there exists on W a local coordinate (x, y, z) with center p such that; in a neighborhood of p, the surface is defined by one of the following three equations:

(1) $yz = 0$ (double point),
(2) $xyz = 0$ (triple point),
(3) $xy^2 - z^2 = 0$ (cuspidal point).

If we are given a non-singular algebraic surface X embedded in a complex projective space $\boldsymbol{P}^N(\boldsymbol{C})$ $(N \geqq 4)$, projecting X into a three dimensional linear subspace $\boldsymbol{P}^3(\boldsymbol{C}) \subset \boldsymbol{P}^N(\boldsymbol{C})$ by a *generic* linear projection, we get a surface S with ordinary singularities in $\boldsymbol{P}^3(\boldsymbol{C})$. We note that in this situation X is the normalization of S. In view of this well-known fact, Horikawa [3], Tsuboi [13] made attempts to compute the *number of moduli of deformations of the complex structures* of some algebraic surfaces X which are the normalizations of surfaces S with ordinary singularities in $\boldsymbol{P}^3(\boldsymbol{C})$ by computing the *number of effective parameters of maximal families of displacements* of S in $\boldsymbol{P}^3(\boldsymbol{C})$. The problem we encounter in this attempt is whether the so-called *connecting homomorphism*

$$\delta\colon H^0(S, \Phi_S) \to H^1(X, T_X)$$

is surjective, where Φ_S denotes the *sheaf of infinitesimal displacements* of the surface S in an ambient space (for the precise definition see [13] and [7]). In [13] we gave some sufficient conditions, expressed in terms of some sheaf cohomology concerning S, X and the ambient space W of S, for the connecting homomorphism δ to be surjective. In this paper we shall sharpen this result (Theorem 1.1). Making use of this result, we shall compute the *number of moduli* of certain algebraic surfaces *of general type* which are the normalizations of surfaces with ordinary singularities in the projective 3-space, *of type* (n, r_1, r_2, r_3) (Theorem 2.1).

1. Proof of the main theorem. Let S be a surface with ordinary singularities in a compact threefold W. We denote by X, Δ and Σt the normalization of S, the double curve of S and the set of triple points of S, respectively. We consider the following diagram:

$$
\begin{tikzcd}
\hat{W} \supset \hat{S} \arrow[dr, "u_2"] \arrow[ddrr, "\phi"] \arrow[rrr, "\mu"] \arrow[ddrr, bend right=40, "u"'] & & & X \arrow[ddl, "\psi"'] \arrow[dd, "\lambda"] \\
& W^* \arrow[dr, "u_1"] & & \\
& & W & S \arrow[l, hook', "\iota"']
\end{tikzcd}
\tag{1.1}
$$

where $u_1: W^* \to W$ is the blowing up along Σt; $u_2: \hat{W} \to W^*$ is the blowing up along the proper inverse image of Δ by the map u_1; $u: \hat{W} \to W$ is the composite of u_1 and u_2; $\hat{S}$ is the proper inverse image of S by the map u; $\phi: \hat{S} \to W$ is the restriction of u to $\hat{S}$; $\lambda: X \to S$ is the normalization of S; $\psi: X \to W$ is the composite of λ and the natural inclusion map $\iota: S \hookrightarrow W$. The map $\mu: \hat{S} \to X$ is as follows:

$\hat{S}$ is a desingularization of S, but not the normalization of S. There appear the exceptional curves of the first kind on $\hat{S}$, which correspond to the set Σt of triple points of S. The map $\mu: \hat{S} \to X$ is the blowing down of these exceptional curves.

In this situation, the following are known to hold (cf. [13]):

PROPOSITION 1.1. *There exists a commutative diagram of exact sequences of sheaves on W*

$$
\begin{tikzcd}
& 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T_{\hat{W}}(-\hat{S}) \arrow[r] \arrow[d] & u^*T_W \otimes_{\mathcal{O}_{\hat{W}}} \mathscr{I}(\hat{S}) \arrow[r] \arrow[d] & \mathscr{T}_{\hat{W}/W} \otimes_{\mathcal{O}_{\hat{W}}} \mathscr{I}(\hat{S}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{\hat{W}}(\log \hat{S}) \arrow[r] \arrow[d] & u^*T_W \arrow[r] \arrow[d] & \mathscr{T}'_{\hat{W}/W} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & T_{\hat{S}} \arrow[r] \arrow[d] & u^*T_W \otimes_{\mathcal{O}_{\hat{W}}} \mathcal{O}_{\hat{S}} \arrow[r] \arrow[d] & \mathscr{T}_{\hat{S}/W} \arrow[r] \arrow[d] & 0 \\
& 0 & 0 & 0 &
\end{tikzcd}
\tag{1.2}
$$

where the sheaf $\mathscr{T}'_{\hat{W}/W}$ is defined to be the cokernel of the canonical injective sheaf homomorphism $T_{\hat{W}}(\log \hat{S}) \to u^*T_W$.

PROPOSITION 1.2.

$$
\begin{aligned}
H^p(W, \Omega^1_W([S + K_W] - \Delta)) &\xrightarrow{\sim} H^p(\hat{W}, u^*\Omega^1_W \otimes_{\mathcal{O}_{\hat{W}}} \mathcal{O}_{\hat{W}}([\hat{S} + K_{\hat{W}}])) \\
&\simeq H^{3-p}(\hat{W}, u^*T_W \otimes_{\mathcal{O}_{\hat{W}}} \mathscr{I}(\hat{S}))
\end{aligned}
$$

for any integer $p \geqq 0$.

PROPOSITION 1.3.

$$H^p(\hat{W}, \mathscr{T}_{\hat{W}/W} \otimes_{\mathcal{O}_{\hat{W}}} \mathscr{I}(\hat{S})) = 0 \quad for \quad p \neq 1 .$$

PROPOSITION 1.4. *There exists a commutative diagram of exact sequences of cohomology groups*

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^0(\hat{W}, \mathscr{T}'_{\hat{W}/W}) \arrow[d] \arrow[dr, squiggly] & & \\
0 \arrow[r] & H^0(\Sigma\tilde{t}, N_{\Sigma\tilde{t}/X}) \arrow[r] \arrow[dr, squiggly] & H^0(\hat{S}, \mathscr{T}_{\hat{S}/W}) \arrow[r] \arrow[d] & H^0(X, \mathscr{T}_{X/W}) \arrow[r] & 0 \\
 & & H^1(\hat{W}, \mathscr{T}_{\hat{W}/W} \otimes_{\mathcal{O}_{\hat{W}}} \mathscr{I}(\hat{S})) \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where $\Sigma\tilde{t}$ denotes the inverse image of Σt by the map $\lambda: X \to S$.

In general, let Y be a compact complex manifold, Z a submanifold of Y, $f: \hat{Y} \to Y$ the blowing up of Y along the non-singular center Z, $\hat{Z}$ the proper inverse image of Z by f, and $g: \hat{Z} \to Z$ the restriction of the map $f: \hat{Y} \to Y$ to $\hat{Z}$. Then there exist the following exact sequences of sheaves on Y:

$$0 \longrightarrow T_{\hat{Y}} \longrightarrow f^* T_Y \longrightarrow \mathscr{T}_{\hat{Y}/Y} \longrightarrow 0 ; \tag{1.3}$$

$$0 \longrightarrow N_{\hat{Z}/\hat{Y}} \longrightarrow g^* N_{Z/Y} \longrightarrow \mathscr{T}_{\hat{Y}/Y} \longrightarrow 0 ; \tag{1.4}$$

from which follows the long exact sequence of cohomology groups:

$$\longrightarrow H^p(\hat{Y}, T_{\hat{Y}}) \longrightarrow H^p(Y, T_Y) \longrightarrow H^p(Z, N_{Z/Y}) \longrightarrow H^{p+1}(\hat{Y}, T_{\hat{Y}}) \longrightarrow \tag{1.5}$$

(cf. [11], [13, Corollary (1.2)]). Furthermore, we have an isomorphism

$$H^p(Y, T_Y(\log Z)) \simeq H^p(\hat{Y}, T_{\hat{Y}}(\log \hat{Z})) \tag{1.6}$$

for any non-negative integer p (cf. [13, Proposition (1.3)]). These facts will also be used in the following.

THEOREM 1.1. *In the same situation as above, we have the following:*

(a) *If $h^1(W, T_W) = h^1(W, \Omega^1_W([S + K_W] - \Delta)) = 0$, then the connecting homomorphism $\delta: H^0(S, \Phi_S) \to H^1(X, T_X)$ is surjective;*

(b) *In addition to* (a), *suppose $h^0(X, T_X) = 0$. Then we have*

$$h^1(X, T_X) = h^0(S, \Phi_S) - h^0(W, T_W) + h^0(W, T_W(\log S)) - h^2(W, \Omega^1_W([S + K_W] - \Delta)) .$$

PROOF. (a) We consider the following diagram of exact sequences of cohomology groups:

(1.7)

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & & H^0(\hat{S}, T_{\hat{S}}) \arrow[d] & & & \\
 & & H^0(X, T_X) \arrow[d] & & & \\
 & & H^0(\Sigma\tilde{t}, N_{\Sigma\tilde{t}/X}) \arrow[d, "g_1"] \arrow[dr, dashed, "g_3"] & & & \\
 {} \arrow[r] & H^1(\hat{W}, T_{\hat{W}}(\log \hat{S})) \arrow[r, "g_5"] \arrow[dr, dashed, "g_7"'] & H^1(\hat{S}, T_{\hat{S}}) \arrow[r, "g_2"] \arrow[d, "g_6"] & H^2(\hat{W}, T_{\hat{W}}(-\hat{S})) \arrow[r, "g_4"] & H^2(\hat{W}, T_{\hat{W}}(\log \hat{S})) \arrow[r] & {} \\
 & & H^1(X, T_X) \arrow[d] & & & \\
 & & 0 & & &
\end{tikzcd}$$

Here we obtain the vertical exact sequence by setting $\hat{Y} = \hat{S}$, $Y = X$, $Z = \Sigma\tilde{t}$ in (1.5); the horizontal exact sequence is the one associated to the vertical short exact sequence of sheaves on the left hand side in the diagram (1.2); and g_3, g_7 are defined by $g_3 := g_2 \circ g_1$, and $g_7 := g_6 \circ g_5$, respectively.

First, we prove the surjectivity of the map g_3 under the assumption $h^1(W, \Omega^1_W([S + K_W] - \Delta)) = 0$, which is the essential part of the proof of (a). Setting $\hat{Y} = \hat{S}$, $Y = X$ and $Z = \Sigma\tilde{t}$ in (1.3) and (1.4), we obtain

$$0 \longrightarrow T_{\hat{S}} \longrightarrow \mu^* T_X \longrightarrow \mathscr{T}_{\hat{S}/X} \longrightarrow 0 \tag{1.8}$$

$$0 \longrightarrow N_{\widehat{\Sigma\tilde{t}}/\hat{S}} \longrightarrow \mu^* N_{\Sigma\tilde{t}/X} \longrightarrow \mathscr{T}_{\hat{S}/X} \longrightarrow 0\ , \tag{1.9}$$

where $\widehat{\Sigma\tilde{t}}$ denotes the pull-back of $\Sigma\tilde{t}$ by the map $\mu\colon \hat{S} \to X$. Since $\mu\colon \hat{S} \to X$ is a blowing up, taking the direct images of (1.8) and (1.9) by the map μ, we obtain

$$0 \longrightarrow \mu_* T_{\hat{S}} \longrightarrow T_X \longrightarrow \mu_* \mathscr{T}_{\hat{S}/X} \longrightarrow 0\ ; \tag{1.10}$$

$$0 \longrightarrow N_{\Sigma\tilde{t}/X} \longrightarrow \mu_* \mathscr{T}_{\hat{S}/X} \longrightarrow 0\ ; \tag{1.11}$$

$$R^q \mu_* \mathscr{T}_{\hat{S}/X} = 0 \quad \text{for} \quad q \geqq 1\ . \tag{1.12}$$

Then we get the following commutative diagram:

$$\begin{tikzcd}
H^0(\hat{S}, \mathscr{T}_{\hat{S}/X}) \arrow[r, "g_{11}"] & H^1(\hat{S}, T_{\hat{S}}) \\
H^0(X, \mu_*\mathscr{T}_{\hat{S}/X}) \arrow[u, "g_9", "\wr"'] \arrow[r, "g_{12}"] & H^1(X, \mu_* T_{\hat{S}}) \arrow[u, "g_{10}", "\wr"'] \\
H^0(\Sigma\tilde{t}, N_{\Sigma\tilde{t}/X}) \arrow[u, "g_8", "\wr"'] \arrow[ur, dashed, "g_{13}"'] &
\end{tikzcd} \tag{1.13}$$

Here g_8 is the isomorphism derived from (1.11); g_9 is that derived from (1.12); g_{10} is the natural isomorphism resulting from the blowing up $\mu\colon \hat{S} \to X$; g_{11} is the homomorphism derived from (1.8); g_{12} is the one derived from (1.10); and g_{13} is so defined that the above diagram commutes. The composite map $g_{11} \circ g_9 \circ g_8$ in (1.13) is nothing but the map $g_1\colon H^0(\Sigma\tilde{t}, N_{\Sigma\tilde{t}/X}) \to H^1(\hat{S}, T_{\hat{S}})$ in (1.7). Hence by the commutativity of (1.13) we have

$$g_1 = g_{10} \circ g_{13} . \tag{1.14}$$

By the definition of the sheaf $\mathscr{T}_{\hat{S}/W}$, we have an exact sequence of sheaves

$$0 \longrightarrow T_{\hat{S}} \longrightarrow \phi^* T_W \longrightarrow \mathscr{T}_{\hat{S}/W} \longrightarrow 0 . \tag{1.15}$$

Taking the direct image of this by the blowing up $\mu\colon \hat{S} \to X$, we have

$$0 \longrightarrow \mu_* T_{\hat{S}} \longrightarrow \psi^* T_W \longrightarrow \mu_*\mathscr{T}_{\hat{S}/W} \longrightarrow 0 ; \tag{1.16}$$

$$R^q\mu_*\mathscr{T}_{\hat{S}/W} = 0 \quad \text{for} \quad q \geqq 1 . \tag{1.17}$$

Fitting (1.16), (1.10) and (1.11) together, we have a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & & \\
 & 0 \arrow[r] & \mu_* T_{\hat{S}} \arrow[r] \arrow[d] & \psi^* T_W \arrow[r] \arrow[d, equal] & \mu_*\mathscr{T}_{\hat{S}/W} \arrow[r] & 0 \\
 & 0 \arrow[r] & T_X \arrow[r] \arrow[d] & \psi^* T_W \arrow[r] & \mathscr{T}_{X/W} \arrow[r] & 0 \\
0 \arrow[r] & N_{\Sigma\tilde{t}/X} \arrow[r, "\simeq"] & \mu_*\mathscr{T}_{\hat{S}/X} \arrow[r] \arrow[d] & 0 & & \\
 & & 0 & & &
\end{tikzcd} \tag{1.18}$$

where the second horizontal exact sequence results from the definition of the sheaf $\mathscr{T}_{X/W}$. Chasing this diagram in a usual manner, we obtain the following exact sequence of sheaves:

$$
\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & N_{\Sigma\tilde{t}/X} \arrow[d, "\wr"'] & & & \\
0 \arrow[r] & \mu_*\mathscr{T}_{\hat{S}/X} \arrow[r] \arrow[d] & \mu_*\mathscr{T}_{\hat{S}/W} \arrow[r] & \mathscr{T}_{X/W} \arrow[r] & 0 \\
 & 0 & & &
\end{tikzcd}
\tag{1.19}
$$

Then we get the commutative diagram

$$
\begin{tikzcd}
H^0(\Sigma\tilde{t}, N_{\Sigma\tilde{t}/X}) \arrow[r, dashed, "g_{16}"] \arrow[d, "\wr"', "g_8"] & H^0(\hat{S}, \mathscr{T}_{\hat{S}/W}) \arrow[r, "g_{17}"] & H^1(\hat{S}, T_{\hat{S}}) \\
H^0(X, \mu_*\mathscr{T}_{\hat{S}/X}) \arrow[r, "g_{14}"] & H^0(X, \mu_*\mathscr{T}_{\hat{S}/W}) \arrow[r, "g_{15}"] \arrow[u, "\wr"] & H^1(X, \mu_* T_{\hat{S}}) \arrow[u, "\wr"', "g_{10}"]
\end{tikzcd}
\tag{1.20}
$$

Here g_{14} is the homomorphism of cohomology groups derived from (1.19); g_{15} and g_{17} are the ones derived from (1.16) and (1.15), respectively; g_8, g_{10} are the same as those in (1.13), the isomorphism $H^0(X, \mu_*\mathscr{T}_{\hat{S}/W}) \xrightarrow{\sim} H^0(\hat{S}, \mathscr{T}_{\hat{S}/W})$ in the middle is the one whose existence follows from (1.17); and g_{16} is so defined that the above diagram commutes. Then, taking into account how we derive (1.19) from (1.18), we can derive the following commutative diagram from (1.13) and (1.20):

$$
\begin{tikzcd}
H^0(\Sigma\tilde{t}, N_{\Sigma\tilde{t}/X}) \arrow[r, "g_{13}"] \arrow[d, "\wr"', "g_8"] & H^1(X, \mu_* T_{\hat{S}}) \\
H^0(X, \mu_*\mathscr{T}_{\hat{S}/X}) \arrow[r, "g_{14}"'] \arrow[ur, "g_{12}"] & H^0(X, \mu_*\mathscr{T}_{\hat{S}/W}) \arrow[u, "g_{15}"']
\end{tikzcd}
\tag{1.21}
$$

Using the assumption $h^1(W, \Omega^1_W([S + K_W] - \Delta)) = 0$, which is equivalent to $h^2(\hat{W}, u^* T_W \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{J}(\hat{S})) = 0$ (cf. Proposition 1.2), we get the following commutative diagram by (1.2):

$$
\begin{tikzcd}
H^0(\hat{S}, \mathscr{T}_{\hat{S}/W}) \arrow[r, "g_{18}"] \arrow[d, "g_{17}"] & H^1(\hat{W}, \mathscr{T}_{\hat{W}/W} \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{J}(\hat{S})) \arrow[d, "g_{19}"] \\
H^1(\hat{S}, T_{\hat{S}}) \arrow[r, "g_2"] & H^2(\hat{W}, T_{\hat{W}}(-\hat{S})) \arrow[d] \\
 & 0 .
\end{tikzcd}
\tag{1.22}
$$

We note that the homomorphism $H^0(\hat{S}, \mathscr{T}_{\hat{S}/W}) \to H^1(\hat{S}, T_{\hat{S}})$ in this diagram is the same g_{17} in (1.20), because (1.15) is identical with the horizontal short exact sequence at the bottom in (1.2). As a consequence we have

$$
\begin{aligned}
g_3 &= g_2 \circ g_1 && \text{(by definition)} \\
&= g_2 \circ (g_{10} \circ g_{13}) && ((1.14)) \\
&= g_2 \circ (g_{10} \circ (g_{15} \circ g_{14} \circ g_8)) && ((1.21)) \\
&= g_2 \circ (g_{17} \circ g_{16}) && ((1.20)) \\
&= (g_{19} \circ g_{18}) \circ g_{16} && ((1.22)) .
\end{aligned}
$$

The composite map $g_{18} \circ g_{16}$ is nothing but the isomorphism $H^0(\Sigma\tilde{t}, N_{\Sigma\tilde{t}/X}) \to H^1(\hat{W}, \mathscr{T}_{\hat{W}/W} \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{I}(\hat{S}))$ in Proposition 1.4, and g_{19} is surjective. Therefore we conclude that g_3 is surjective as desired.

The surjectivity of the map g_3 implies that of g_7 in (1.7). If $h^1(W, T_W) = 0$, the surjectivity of g_7 implies that of the connecting homomorphism δ: $H^0(S, \Phi_S) \to H^1(X, T_X)$. Indeed, as shown in [13], the connecting homomorphism δ is identical with the composite of the homomorphisms

$$
(1.23) \qquad H^1(S, \Phi_S) \longrightarrow H^1(W, T_W(\log S)) \underset{(1.6)}{\xrightarrow{\sim}} H^1(\hat{W}, T_{\hat{W}}(\log \hat{S}))
$$

and g_7 in (1.7), where the first homomorphism in (1.23) is the one derived from the exact sequence of sheaves

$$
(1.24) \qquad 0 \longrightarrow T_W(\log S) \longrightarrow T_W \longrightarrow \Phi_S \longrightarrow 0
$$

(cf. [13, (2.5)], [12, Proposition (1.2)]). If $h^1(W, T_W) = 0$, the first homomorphism in (1.23) is surjective. Therefore the connecting homomorphism δ is surjective. This completes the proof of (a).

(b) Besides the conditions

$$
h^1(W, T_W) = h^1(W, \Omega^1_W([S + K_W] - \Delta)) = 0 ,
$$

we assume $h^0(X, T_X) = 0$. Then by the vertical exact sequence in (1.7), we have

$$
\begin{aligned}
(1.25) \qquad h^1(X, T_X) &= h^1(\hat{S}, T_{\hat{S}}) - h^0(\Sigma\tilde{t}, N_{\Sigma\tilde{t}/X}) \\
&= h^1(\hat{S}, T_{\hat{S}}) - h^1(\hat{W}, \mathscr{T}_{\hat{W}/W} \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{I}(\hat{S})) ,
\end{aligned}
$$

where the second equality follows from Proposition 1.4. By the horizontal exact sequence in (1.7), we have

$$
(1.26) \qquad h^1(\hat{S}, T_{\hat{S}}) = h^2(\hat{W}, T_{\hat{W}}(-\hat{S})) + h^1(\hat{W}, T_{\hat{W}}(\log \hat{S})) - h^1(\hat{W}, T_{\hat{W}}(-\hat{S})) ,
$$

where we use the fact that g_4 in (1.7) is the zero map because of the surjectivity of g_3. Since $h^0(\hat{W}, \mathscr{T}_{\hat{W}/W} \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{I}(\hat{S})) = 0$ (cf. Proposition 1.3) and $h^2(\hat{W}, u^* T_W \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{I}(\hat{S})) = 0$ under the assumption $h^1(W, \Omega^1_W([S + K_W] -$

$\Delta)) = 0$ (cf. Proposition 1.2), by the long exact sequence of cohomology groups associated to the short exact sequence of sheaves at the top in (1.2), we have

$$\begin{aligned}(1.27)\quad & h^1(\hat{W}, \mathcal{T}_{\hat{W}/W} \otimes_{\mathcal{O}_{\hat{W}}} \mathcal{I}(\hat{S})) \\ &= h^2(\hat{W}, T_{\hat{W}}(-\hat{S})) + h^1(\hat{W}, u^* T_W \otimes_{\mathcal{O}_{\hat{W}}} \mathcal{I}(\hat{S})) - h^1(\hat{W}, T_{\hat{W}}(-\hat{S})) \\ &= h^2(\hat{W}, T_{\hat{W}}(-\hat{S})) + h^2(W, \Omega^1_W([S + K_W] - \Delta)) - h^1(\hat{W}, T_{\hat{W}}(-\hat{S}))\ ,\end{aligned}$$

where the second equality follows from Proposition 1.2. Substituting (1.26) and (1.27) into (1.25), we have

$$\begin{aligned}(1.28)\qquad h^1(X, T_X) &= h^1(\hat{W}, T_{\hat{W}}(\log \hat{S})) - h^2(W, \Omega^1_W([S + K_W] - \Delta)) \\ &= h^1(W, T_W(\log S)) - h^2(W, \Omega^1_W([S + K_W] - \Delta))\ ,\end{aligned}$$

where the second equality follows from (1.6). By (1.24) we have

$$(1.29)\qquad h^1(W, T_W(\log S)) = h^0(S, \Phi_S) - h^0(W, T_W) + h^0(W, T_W(\log S))\ ,$$

because $h^1(W, T_W) = 0$ by hypothesis. Then, by (1.28) and (1.29), we obtain the equality in (b). q.e.d.

For the terminology in the following corollary, we refer to [7] and [6].

COROLLARY 1.1. *Besides the conditions*

$$h^1(W, T_W) = h^1(W, \Omega^1_W([S + K_W] - \Delta)) = h^0(X, T_X) = 0\ ,$$

suppose that S belongs to an analytic family $\mathcal{S} = \bigcup_{t \in M} S_t$ of surfaces with ordinary singularities in W whose parameter space M is non-singular, and whose characteristic map

$$\sigma\colon T_0(M) \to H^0(S, \Phi_S)$$

at the point $0 \in M$ with $S_0 = S$ is surjective. Then the Kuranishi family of deformations of the complex structure of X is non-singular, and the number $m(X)$ of moduli of X is given by

$$\begin{aligned}m(X) &= h^1(X, T_X) \\ &= h^0(S, \Phi_S) - h^0(W, T_W) + h^0(W, T_W(\log S)) - h^2(W, \Omega^1_W([S + K_W] - \Delta))\ .\end{aligned}$$

PROOF. The normalizations X_t of S_t for $t \in M$ describe a family $\mathcal{X} = \bigcup_{t \in M} X_t$ of deformations of the complex structure of $X = X_0$. The characteristic maps of the families $\mathcal{S} = \bigcup_{t \in M} S_t$ and $\mathcal{X} = \bigcup_{t \in M} X_t$ at $0 \in M$ are related as

$$\begin{tikzcd} & H^1(X, T_X) \\ T_0(M) \arrow[ur, "\rho"] \arrow[dr, "\sigma"'] & \\ & H^0(S, \Phi_S) \arrow[uu, "-\delta"'] \end{tikzcd}$$

where δ is the so-called connecting homomorphism (cf. [3]). By Theorem 1.1 (a), δ is surjective, hence the characteristic map ρ is also surjective. Then, from the family $\mathscr{X} = \bigcup_{t\in M} X_t$, we can derive a family $\mathscr{X}' = \bigcup_{t\in M'} X_t$ of deformations of $X = X_0$, $0 \in M'$ such that M' is non-singular and the characteristic map ρ: $T_0(M') \to H^1(X, T_X)$ is bijective. From the condition $H^0(X, T_X) = 0$ it follows that $\dim H^1(X_t, T_{X_t})$ is independent on t, provided that t is sufficiently close to 0. Then the characteristic map ρ_t: $T_t(M') \to H^1(X_t, T_{X_t})$ is bijective at any point $t \in M'$ sufficiently close to 0. Therefore we conclude that $\mathscr{X}' = \bigcup_{t\in M'} X_t$ is the Kuranishi family of deformations of the complex structure of X. The formula for the number $m(X)$ of moduli follows from Theorem 1.1 (b). q.e.d.

The following theorem will be used in §2 to compute the number of moduli of certain algebraic surfaces.

THEOREM 1.2. *Let W, S and X be the same as in the foregoing. If S is regular, i.e., $h^1(S, \Phi_S) = 0$ by definition, and if*

$$h^2(W, T_W) = h^0(W, \Omega^1_W([S + K_W] - \Delta)) = h^0(X, T_X) = 0\ ,$$

then we obtain $h^2(X, T_X) = 0$; hence the Kuranishi family of deformations of the complex structure of X is non-singular. Furthermore, the number $m(X)$ of moduli is given by

$$m(X) = h^1(X, T_X) = 10(p_a + 1) - c_1^2\ ,$$

where p_a, c_1 denote the arithmetic genus and the first Chern class of X, respectively.

PROOF. Since $h^2(\hat{W}, \mathscr{T}_{\hat{W}/W} \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{I}(\hat{S})) = h^3(\hat{W}, \mathscr{T}_{\hat{W}/W} \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{I}(\hat{S})) = 0$ (cf. Proposition 1.3), by the horizontal short exact sequence at the top in (1.2) we have

$$h^3(\hat{W}, T_{\hat{W}}(-\hat{S})) = h^3(\hat{W}, u^* T_W \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{I}(\hat{S}))\ ;$$

and we have

$$h^3(\hat{W}, u^* T_W \otimes_{\mathscr{O}_{\hat{W}}} \mathscr{I}(\hat{S})) = h^0(W, \Omega^1_W([S + K_W] - \Delta))$$

(cf. Proposition 1.2). But $h^0(W, \Omega^1_W([S + K_W] - \Delta)) = 0$ by hypothesis, hence

$$(1.30) \qquad h^3(\hat{W}, T_{\hat{W}}(-\hat{S})) = 0 .$$

Since $h^1(S, \Phi_S) = h^2(W, T_W) = 0$ by hypothesis, we have

$$h^2(W, T_W(\log S)) = 0$$

by (1.24). Hence

$$(1.31) \qquad h^2(\hat{W}, T_{\hat{W}}(\log \hat{S})) = 0$$

by (1.6). Applying (1.30) and (1.31) to the long exact sequence of cohomology groups associated to the vertical short exact sequence of sheaves on the left hand side in (1.2), we have $h^2(\hat{S}, T_{\hat{S}}) = 0$. Therefore we have $h^2(X, T_X) = 0$. Thus X satisfies $h^0(X, T_X) = h^2(X, T_X) = 0$. As a consequence we conclude that the number $m(X)$ of moduli of X is defined, and $m(X) = h^1(X, T_X)$ holds. The equality $h^1(X, T_X) = 10(p_a + 1) - 2c_1^2$ follows from the Riemann-Roch formula. q.e.d.

2. An example —surfaces of type (n, r_1, r_2, r_3)**—.** Throughout this section we denote the complex projective 3-space by P, and a point of P by $\xi = (\xi_0, \xi_1, \xi_2, \xi_3)$ in a fixed homogeneous coordinate system. We fix positive integers r_1, r_2, r_3 with $r_1 \geqq r_2 \geqq r_3$. Let S_1, S_2, S_3 be non-singular surfaces of respective orders r_1, r_2, r_3 in P, such that they intersect pairwise transversely, and are in general position at every point of $S_1 \cap S_2 \cap S_3$. We set

$$\Delta_1 := S_1 \cdot S_2 , \quad \Delta_2 := S_2 \cdot S_3 , \quad \Delta_3 := S_3 \cdot S_1 \quad \text{and}$$
$$\Delta := \Delta_1 + \Delta_2 + \Delta_3 .$$

Let f_i $(i = 1, 2, 3)$ be the homogeneous polynomial of degree r_i which defines the surface S_i. We choose and fix a positive integer $n \geqq 2r_1 + 2r_2$. For any homogeneous polynomials A, B, C, and D of respective degrees $n - r_1 - r_2 - r_3$, $n - 2r_1 - 2r_2$, $n - 2r_2 - 2r_3$, $n - 2r_3 - 2r_1$, we consider a surface S defined by the equation

$$(2.1) \qquad f := Af_1f_2f_3 + B(f_1f_2)^2 + C(f_2f_3)^2 + D(f_3f_1)^2 = 0 .$$

S is said to be *generic* if the following conditions are satisfied:

(1) S has only ordinary singularities and is non-singular outside of Δ;

(2) the normalization X of S is a minimal algebraic surface of general type.

We note that S satisfies the condition (1) if A, B, C and D are chosen sufficiently general. Indeed, by *Bertini's theorem* S is non-singular outside of Δ for generic A, B, C and D. The fact that the singularities of S along Δ are ordinary for generic A, B, C and D is proved as follows:

(i) Let $p \in \Delta$ be a point satisfying $f_1(p) = f_2(p) = f_3(p) = 0$. We may assume that $A(p)B(p)C(p)D(p) \neq 0$. We make the transformations of local coordinates

$$(f_1, f_2, f_3) \mapsto \Big(\frac{A}{\sqrt{BD}} \frac{X}{1 + X^2 + Y^2 + Z^2 + XYZ},$$
$$\frac{A}{\sqrt{BC}} \frac{Y}{1 + X^2 + Y^2 + Z^2 + XYZ}, \frac{A}{\sqrt{CD}} \frac{Z}{1 + X^2 + Y^2 + Z^2 + XYZ}\Big)$$

and

$$(X + YZ, Y + ZX, Z + XY) \to (X', Y', Z')$$

successively in a neighborhood of p. Then the equation $F = 0$ is transformed to $A'X'Y'Z' = 0$, where A' is a non-vanishing factor. Namely, the point p is a triple point.

(ii) Let $p \in \Delta$ be a point at which all of f_i, $i = 1, 2, 3$, do not vanish. We may assume that $f_1(p) = f_2(p) = 0$ and $f_3(p) \neq 0$. We write F as

$$F = (Bf_2^2 + Df_3^2)f_1^2 + (Af_3)f_1f_2 + (Cf_3^2)f_2^2 .$$

Since A, B, C and D are generic, we may assume that both $(Bf_2^2 + Df_3^2)$ and (Cf_3^2) do not vanish at p. Suppose $(Bf_2^2 + Df_3^2)(p) \neq 0$. Then F is written as

$$F = (Bf_2^2 + Df_3^2)\Big(f_1 + \frac{Af_3/2 + \sqrt{(Af_3/2)^2 - (Bf_2^2 + Df_3^2)(Cf_3^2)}}{Bf_2^2 + Df_3^2} f_2\Big)$$
$$\times\Big(f_1 + \frac{Af_3/2 - \sqrt{(Af_3/2)^2 - (Bf_2^2 + Df_3^2)(Cf_3^2)}}{Bf_2^2 + Df_3^2} f_2\Big)$$

in a neighborhood of p.

(ii)$_d$ If $(A^2/4 - DCf_3^2)(p) \neq 0$, then the transformation

$$f_1 + \frac{Af_3/2 + \sqrt{(Af_3/2)^2 - (Bf_2^2 + Df_3^2)(Cf_3^2)}}{Bf_2^2 + Df_3^2} f_2 \mapsto X ,$$
$$f_1 + \frac{Af_3/2 - \sqrt{(Af_3/2)^2 - (Bf_2^2 + Df_3^2)(Cf_3^2)}}{Bf_2^2 + Df_3^2} f_2 \mapsto Y$$

can be regarded as that of local coordinates. By this transformation the equation $F = 0$ is transformed to $(Bf_2^2 + Df_3^2)XY = 0$. Hence p is a double point.

(ii)$_c$ If $(A^2/4 - DCf_3^2)(p) = 0$, we make the transformation of local coordinates

$$\begin{aligned} \frac{(Af_3/2)^2 - (Bf_2^2 + Df_3^2)(Cf_3^2)}{(Bf_2^2 + Df_3^2)^2} &\mapsto X\,, \\ f_2 \qquad\qquad &\mapsto Y\,, \\ f_1 + \frac{Af_3/2}{Bf_2^2 + Df_3^2} f_2 \qquad &\mapsto Z \end{aligned}$$

in a neighborhood of p. Then the equation $F = 0$ is transformed to

$$(Bf_2^2 + Df_3^2)(Z + \sqrt{X}Y)(Z - \sqrt{X}Y) = (Bf_2^2 + Df_3^2)(Z^2 - XY^2) = 0\,.$$

Hence p is a cuspidal point.

Consequently, for generic A, B, C and D the surface S defined by $F = 0$ is a surface with ordinary singularities whose double curve is Δ.

Furthermore, we can prove that the condition (2) is satisfied if $n \geqq r_1 + r_2 + 4$ and B, C, D are chosen sufficiently general.

DEFINITION 2.1. We call the generic surface S in the complex projective 3-space P which is defined by an equation of the form (2.1) a *surface of type* (n, r_1, r_2, r_3) *with ordinary singularities*. The non-singular normalization X of the surface S is called a *non-singular surface of type* (n, r_1, r_2, r_3).

Concerning a surface S of type (n, r_1, r_2, r_3) with ordinary singularities, we freely use the notation $S_1, S_2, S_3, f_1, f_2, f_3$ and $\Delta = \Delta_1 + \Delta_2 + \Delta_3$ below. For brevity we use the notation $\mathcal{O}_P(k)$, $\mathcal{O}_P(k - \Delta)$ and $\mathcal{O}_P(k - 2\Delta)$ instead of $\mathcal{O}_P([kE])$, $\mathcal{O}_P([kE] - \Delta)$ and $\mathcal{O}_P([kE] - 2\Delta)$, respectively, where E is a hyperplane in P and k is an integer. Furthermore, we use the following notation:

L_m: the vector space of homogeneous polynomials of degree m in $\xi_0, \xi_1, \xi_2, \xi_3$;

$L_m(-\Delta)$: the linear subspace of L_m consisting of those homogeneous polynomials of L_m which vanish on Δ;

$$C(m) := \dim_{\mathbf{C}} L_m = h^0(P, \mathcal{O}_P(m)) = (m + 1)(m + 2)(m + 3)/6\,.$$

PROPOSITION 2.1. *For any integer k there exists an exact sequence of sheaves*

$$\begin{aligned} 0 &\to \mathcal{O}_P(k-r_1-r_2-r_3)^{\oplus 2} \\ &\xrightarrow{\beta} \mathcal{O}_P(k-r_1-r_2) \oplus \mathcal{O}_P(k-r_2-r_3) \oplus \mathcal{O}_P(k-r_3-r_1) \xrightarrow{\alpha} \mathcal{O}_P(k-\Delta) \to 0\,. \end{aligned}$$

PROOF. The maps are defined as follows:

$$\alpha\colon (\phi_1, \phi_2, \phi_3) \mapsto f_1f_2\phi_1 + f_2f_3\phi_2 + f_3f_1\phi_3$$

for $(\phi_1, \phi_2, \phi_3) \in \mathcal{O}_P(k - r_1 - r_2) \oplus \mathcal{O}_P(k - r_2 - r_3) \oplus \mathcal{O}_P(k - r_3 - r_1)$;

$$\beta\colon (\psi_1, \psi_2) \mapsto (f_3\psi_1, f_1\psi_2, -f_2(\psi_1 + \psi_2))$$

for $(\psi_1, \psi_2) \in \mathcal{O}_P(k - r_1 - r_2 - r_3)^{\oplus 2}$, where we regard each f_i $(i = 1, 2, 3)$ as a global cross-section of the sheaf $\mathcal{O}_P(r_i)$. The proof of exactness is a simple calculation. q.e.d.

PROPOSITION 2.2. *For any integer k there exist exact sequences of sheaves*

(a) $0 \to \mathcal{O}_P(k-r_1-r_2-r_3) \xrightarrow{\beta} \mathcal{O}_P(k-2\Delta)$

$$\xrightarrow{\alpha} \mathcal{O}_{S_1}(k-2r_2-2r_3) \oplus \mathcal{O}_{S_2}(k-2r_3-2r_1) \oplus \mathcal{O}_{S_3}(k-2r_1-2r_2) \to 0\ ;$$

(b) $0 \to \mathcal{O}_P(k-2r_{i_2}-2r_{i_3}-r_{i_1}) \xrightarrow{\gamma_{i_1}} \mathcal{O}_P(k-2r_{i_2}-2r_{i_3}) \xrightarrow{R_{i_1}} \mathcal{O}_{S_{i_1}}(k-2r_{i_2}-2r_{i_3}) \to 0$

for any permutation (i_1, i_2, i_3) of $(1, 2, 3)$.

PROOF. (a) We set $\mathcal{O}_{S_i}(k - 2\Delta) := \mathcal{O}_P(k - 2\Delta)/\mathcal{O}_P(k - 2\Delta - S_i)$ for $i = 1, 2, 3$, where $\mathcal{O}_P(k - 2\Delta - S_i)$ denotes the subsheaf of $\mathcal{O}_P(k - 2\Delta)$ consisting of germs of those local cross-sections of $\mathcal{O}_P(k - 2\Delta)$ which vanish on S_i. Since $\Delta \cdot S_1 = \Delta_1 + \Delta_3$ and Δ_1, Δ_3 are defined on S_1 as the zero loci of homogeneous polynomials of respective degrees r_2, r_3, we have

$$\mathcal{O}_{S_1}(k - 2\Delta) \simeq \mathcal{O}_{S_1}(k - 2r_2 - 2r_3)\ .$$

Similarly, we have

$$\mathcal{O}_{S_2}(k - 2\Delta) \simeq \mathcal{O}_{S_2}(k - 2r_3 - 2r_1) \quad \text{and} \quad \mathcal{O}_{S_3}(k - 2\Delta) \simeq \mathcal{O}_{S_3}(k - 2r_1 - 2r_2)\ .$$

Taking these isomorphisms into account, we define α in the sequence (a) by

$$\begin{aligned} \phi \mapsto (\phi_{|S_1}, \phi_{|S_2}, \phi_{|S_3}) &\in \mathcal{O}_{S_1}(k - 2\Delta) \oplus \mathcal{O}_{S_2}(k - 2\Delta) \oplus \mathcal{O}_{S_3}(k - 2\Delta) \\ &\simeq \mathcal{O}_{S_1}(k - 2r_2 - 2r_3) \oplus \mathcal{O}_{S_2}(k - 2r_3 - 2r_1) \oplus \mathcal{O}_{S_3}(k - 2r_1 - 2r_2) \end{aligned}$$

for $\phi \in \mathcal{O}_P(k - 2\Delta)$, where $\phi_{|S_i}$ $(i = 1, 2, 3)$ denotes the restriction to S_i. We define the map β by

$$\psi \mapsto (f_1 f_2 f_3)\psi \quad \text{for} \quad \psi \in \mathcal{O}_P(k - r_1 - r_2 - r_3)\ .$$

Then the exactness follows from simple calculation.

(b) We define the map R_{i_1}, $(i_1 = 1, 2, 3)$ in the sequence (b) by restriction to S_{i_1}, and the map γ_{i_1} by

$$\psi \mapsto f_i \psi \quad \text{for} \quad \psi \in \mathcal{O}_P(k - 2r_{i_2} - 2r_{i_3} - r_{i_1})\ .$$

Then, obviously the sequence (b) is exact. q.e.d.

For the double curve Δ of a surface of type (n, r_1, r_2, r_3) in P with ordinary singularities, we consider the sheaf $\sum_{i=1}^{3} N_{\Delta_i}$ (direct sum) where

N_{Δ_i} $(i = 1, 2, 3)$ denotes the sheaf of normal vectors of Δ_i in P. The difference between $\mathscr{N}_\Delta := T_P/T_P(\log \Delta)$ and $\sum_{i=1}^3 N_{\Delta_i}$ is given by the following:

PROPOSITION 2.3. *There exists a natural exact sequence of sheaves*

$$0 \longrightarrow \mathscr{N}_\Delta \xrightarrow{\alpha} \sum_{i=1}^3 N_{\Delta_i} \xrightarrow{\beta} \mathscr{T}_{\Sigma t} \longrightarrow 0 ,$$

where $\mathscr{T}_{\Sigma t}$ is the sheaf with support Σt, the set of triple points of S $(=\Delta_1 \cap \Delta_2 \cap \Delta_3)$, and whose stalk at each point of Σt is isomorphic to $\boldsymbol{C}^3$.

PROOF. It suffices to prove the exactness at a triple point $p \in \Sigma t$. Let (x, y, z) be a system of local coordinates in a sufficiently small polycylindrical neighborhood of p in P such that

(1) S is defined by $xyz = 0$,

(2) $\Delta_1, \Delta_2, \Delta_3$ are defined by $y=z=0$, $z=x=0$, $x=y=0$, respectively.

Then we define $\alpha: \mathscr{N}_\Delta \to \sum_{i=1}^3 N_{\Delta_i}$ at p by

$$[\theta] \mapsto \left(\left(b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}\right)_{|\Delta_1} , \left(c\frac{\partial}{\partial z} + a\frac{\partial}{\partial x}\right)_{|\Delta_2} , \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right)_{|\Delta_3}\right)$$

for $\theta = a(\partial/\partial x) + b(\partial/\partial y) + c(\partial/\partial z) \in T_P(\log \Delta)_p$, where $[\theta]$ denotes the local holomorphic cross-section of the sheaf $\mathscr{N}_\Delta$ represented by θ, and $|\Delta_i$ $(i = 1, 2, 3)$ denotes the restriction to Δ_i. It is easy to see that this definition does not depend on the choice of a representative θ. We define the sheaf homomorphism $\beta: \sum_{i=1}^3 N_{\Delta_i} \to \boldsymbol{C}^3$ at p by

$$\phi \mapsto (\phi_2(0) - \psi_1(0) , \quad \psi_2(0) - \eta_1(0) , \quad \eta_2(0) - \phi_1(0))$$

for

$$\phi = \left(\phi_1(x)\frac{\partial}{\partial y} + \phi_2(x)\frac{\partial}{\partial z} , \quad \psi_1(y)\frac{\partial}{\partial z} + \psi_2(y)\frac{\partial}{\partial x} , \quad \eta_1(z)\frac{\partial}{\partial x} + \eta_2(z)\frac{\partial}{\partial y}\right)_p \in \left(\sum_{i=1}^3 N_{\Delta_i}\right)_p .$$

The exactness follows from simple calculation. q.e.d.

COROLLARY 2.1.

$$\dim H^0(\Delta, \mathscr{N}_\Delta) = C(r_1) + C(r_2) + C(r_3) - C(r_1 - r_2 - r_3) - 3 .$$

PROOF. By the exact sequence of sheaves in Proposition 2.3, we get the long exact sequence of cohomology groups

$$0 \longrightarrow H^0(\Delta, \mathscr{N}_\Delta) \xrightarrow{\hat{\alpha}} \sum_{i=1}^3 H^0(\Delta_i, N_{\Delta_i}) \xrightarrow{\hat{\beta}} \boldsymbol{C}^3_{\Sigma t} \longrightarrow \cdots . \tag{2.2}$$

Note that we can identify $\sum_{i=1}^3 H^0(\Delta_i, N_{\Delta_i})$ with the vector space

$$\sum_{i=1}^{3}\{(L_{r_i}/L_{r_i}(-\varDelta_i))\oplus(L_{r_{i+1}}/L_{r_{i+1}}(-\varDelta_i))\}\ ,$$

where we set $r_4=r_1$. We denote this vector space by V. For $\phi\in L_r$ ($r=r_1, r_2, r_3$) we denote by $\phi_{|\varDelta_i}$ ($i=1, 2, 3$) the corresponding element of $L_r/L_r(-\varDelta_i)$. Then the above map

$$\hat{\beta}\colon \sum_{i=1}^{3} H^0(\varDelta_i, N_{\varDelta_i})\to \boldsymbol{C}^3_{\Sigma t}$$

is given by

$$\begin{aligned}&(\phi_{1|\varDelta_1}\oplus\phi_{2|\varDelta_1})\oplus(\psi_{1|\varDelta_2}\oplus\psi_{2|\varDelta_2})\oplus(\eta_{1|\varDelta_3}\oplus\eta_{2|\varDelta_3})\\ &\qquad\mapsto\sum_{p\in\Sigma t}(\phi_2(p)-\psi_1(p),\ \psi_2(p)-\eta_1(p),\ \eta_2(p)-\phi_1(p))\in\sum_{p\in\Sigma t}\boldsymbol{C}^3_p\ ,\end{aligned}$$

where $\phi_1, \eta_2\in L_{r_1}$, $\phi_2, \psi_1\in L_{r_2}$, $\psi_2, \eta_1\in L_{r_3}$. Therefore by the exactness of (2.2) we can identify $H^0(\varDelta, \mathcal{N}_\varDelta)$ with the vector subspace V_1 of V consisting of the elements

$$(\phi_{1|\varDelta_1}\oplus\phi_{2|\varDelta_1})\oplus(\psi_{1|\varDelta_2}\oplus\psi_{2|\varDelta_2})\oplus(\eta_{1|\varDelta_3}\oplus\eta_{2|\varDelta_3})$$

of V which satisfy

$$\phi_2(p)-\psi_1(p)=\psi_2(p)-\eta_1(p)=\eta_2(p)-\phi_1(p)=0\quad\text{for any}\quad p\in\Sigma t\ . \tag{2.3}$$

We note that Σt coincides with the common zero locus of the homogeneous polynomials f_1, f_2, f_3, and any point $p\in\Sigma t$ has multiplicity one. Then, in view of (2.3) we can apply *generalized Max Nöether's theorem* in [4] to the polynomials $\phi_2-\psi_1$, $\psi_2-\eta_1$, $\eta_2-\phi_1$. As a result we infer that $\phi_2-\psi_1$, $\psi_2-\eta_1$, $\eta_2-\phi_1$ are of the form

$$\begin{aligned}\phi_2-\psi_1&=a_1f_1+a_2f_2+a_3f_3\ ,\\ \psi_2-\eta_1&=b_1f_1+b_2f_2+b_3f_3\ ,\\ \eta_2-\phi_1&=c_1f_1+c_2f_2+c_3f_3\ ,\end{aligned}$$

where $a_1, \cdots, c_3$ are homogeneous polynomials of appropriate degrees. We set

$$\begin{aligned}\Phi&:=\phi_2-a_1f_1=\psi_1+a_2f_2+a_3f_3\ ,\\ \Psi&:=\psi_2-b_2f_2=\eta_1+b_1f_1+b_3f_3\ ,\\ H&:=\eta_2-c_3f_3=\phi_1+c_1f_1+c_2f_2\ .\end{aligned}$$

We define

$$\hat{\gamma}\colon L_{r_1}\oplus L_{r_2}\oplus L_{r_3}\to V \tag{2.4}$$

by $(\phi, \psi, \eta)\mapsto(\phi_{|\varDelta_1}\oplus\psi_{|\varDelta_1})\oplus(\psi_{|\varDelta_2}\oplus\eta_{|\varDelta_2})\oplus(\eta_{|\varDelta_3}\oplus\phi_{|\varDelta_3})$, for $(\phi, \psi, \eta)\in L_{r_1}\oplus L_{r_2}\oplus L_{r_3}$. Then we have

$$\hat{\gamma}(H, \Phi, \Psi) = (\phi_{1|\Delta_1} \oplus \phi_{2|\Delta_1}) \oplus (\psi_{1|\Delta_2} \oplus \psi_{2|\Delta_2}) \oplus (\eta_{1|\Delta_3} \oplus \eta_{2|\Delta_3}) .$$

This shows image $\hat{\gamma} = V_1$. Therefore we have

$$\begin{aligned} \dim H^0(\Delta, \mathcal{N}_\Delta) &= \dim V_1 = \dim \operatorname{image} \hat{\gamma} \\ &= \dim(L_{r_1} \oplus L_{r_2} \oplus L_{r_3}) - \dim \ker \hat{\gamma} = C(r_1) + C(r_2) + C(r_3) - \dim \ker \hat{\gamma} . \end{aligned}$$

Since

$$\ker \hat{\gamma} = \{(\lambda f_1 + A f_2 f_3, \mu f_2, c f_3) \mid \lambda, \mu, c \in C, A \in L_{r_1 - r_2 - r_3}\} ,$$

we have $\dim \ker \hat{\gamma} = C(r_1 - r_2 - r_3) + 3$. q.e.d.

PROPOSITION 2.4. *Let S be a surface of type (n, r_1, r_2, r_3) with ordinary singularities. Then S belongs to a maximal analytic family $\mathcal{S} = \cup_{t \in M} S_t$ of surfaces in P with ordinary singularities such that*

(a) *the parameter space M is non-singular and*

(b) *the characteristic map*

$$\sigma_t^{\mathcal{S}}: T_t(M) \to H^0(S_t, \Phi_{S_t})$$

is surjective at any point $t \in M$.

PROOF. We define $m_0(\Delta)$ to be the smallest integer m_0 such that

$$H^1(P, \mathcal{O}_P(k - 2\Delta)) = 0 \quad \text{for} \quad k \geqq m_0 .$$

By Theorem 8 in [9] it suffices to show that

(i) $n \geqq m_0(\Delta)$ and

(ii) Δ belongs to an analytic family $\mathfrak{f} = \cup_{t \in M_1} \Delta_t$ of locally trivial displacements of Δ in P such that

(a′) the parameter space M_1 is non-singular,

(b′) the *characteristic map* $\sigma^{\mathfrak{f}}: T_0(M_1) \to H^0(\Delta, \mathcal{N}_\Delta)$ at the point $0 \in M_1$ with $\Delta_0 = \Delta$ is surjective.

Strictly speaking, Theorem 8 in [9] treats only the case where a double curve Δ is non-singular, hence we can not apply that theorem directly to our case. But, as shown in [14], a *characteristic map* $\sigma^{\mathfrak{f}}: T_0(M) \to H^0(\Delta, \mathcal{N}_\Delta)$ can also be defined, even if Δ is singular. By direct calculation we can easily prove that for an analytic family $\mathcal{S} = \cup_{t \in M} S_t$ of surfaces with ordinary singularities in P such that $S = S_0$ for $0 \in M$, the following diagram is commutative:

$$\begin{array}{ccc} & & H^0(S, \Phi_S) \\ & \overset{\sigma^{\mathcal{S}}}{\nearrow} & \\ T_0(M) & & \Big\downarrow \# \\ & \underset{\sigma^{\mathfrak{f}}}{\searrow} & \\ & & H^0(\Delta, \mathcal{N}_\Delta) \end{array} ,$$

where $\sigma^{\mathfrak{f}}$ is the characteristic map at the point $0 \in M$ of the family $\mathfrak{f} = \bigcup_{t \in M} \Delta_t$ of the double curve Δ_t of each S_t, $t \in M$, and $\sharp$ is the map induced by the fundamental exact sequence

$$0 \longrightarrow \mathcal{O}_S([S] - 2\Delta) \longrightarrow \Phi_S \longrightarrow \mathcal{N}_\Delta \longrightarrow 0 \tag{2.5}$$

(cf. [9, Theorem 4] and [12, Proposition (1.1)]). Therefore, by the same arguments as in the proof of Theorem 8 in [9], we can generalize that theorem to the case where a double curve Δ may be singular.

By Proposition 2.2 and *Bott's theorem* concerning the cohomology groups $H^p(\boldsymbol{P}^n, \Omega^q_{P^n}(k))$ in [1], we obtain

$$H^1(P, \mathcal{O}_P(k - 2\Delta)) = 0 \quad \text{for any integer } k .$$

Hence it follows that $m_0(\Delta) = -\infty$, and so (i) holds. (ii) is proved as follows:

Let $\Delta_1, \Delta_2, \Delta_3, f_1, f_2, f_3$ be the same as before. In the following we regard a homogeneous polynomial of degree k in variables $\xi_0, \xi_1, \xi_2, \xi_3$ as a point of $\boldsymbol{C}^{C(k)}$ by assigning its coefficients. For $i = 1, 2, 3$ we denote by $f_i(\xi, t_i)$ the homogeneous polynomials of degree r_i in the variables $\xi_0, \xi_1, \xi_2, \xi_3$ which corresponds to a point $t_i \in \boldsymbol{C}^{C(r_i)}$. We set

$$\begin{aligned}
&\tilde{f}_i(\xi, t_i) := f_i(\xi, t_i) + f_i(\xi) \qquad (i = 1, 2, 3) ; \\
&N := C(r_1) + C(r_2) + C(r_3) ; \\
&M_1 := \{t = (t_1, t_2, t_3) \in \boldsymbol{C}^N \mid |t| < \varepsilon\} \qquad (\varepsilon\text{: a positive number}) ; \\
&\mathfrak{f} := \{(\xi, t) \in P \times M_1 \mid \tilde{f}_1(\xi, t_1)\tilde{f}_2(\xi, t_2) = \tilde{f}_2(\xi, t_2)\tilde{f}_3(\xi, t_3) = \tilde{f}_3(\xi, t_3)\tilde{f}_1(\xi, t_1) = 0\} .
\end{aligned}$$

We denote by $\varpi\colon \mathfrak{f} \to M_1$ the restriction of the canonical projection $\mathrm{Pr}_{M_1}\colon P \times M_1 \to M_1$ to $\mathfrak{f}$. Then, in our terminology $\varpi\colon \mathfrak{f} \to M_1$ is an analytic family of *locally trivial displacements* of the double curve Δ of S in P (cf. [14, Definition 8.1]) provided that the positive number ε is sufficiently small. We claim that the characteristic map $\sigma^{\mathfrak{f}}\colon T_0(M_1) \to H^0(\Delta, \mathcal{N}_\Delta)$ at the origin $0 \in M_1$ of the family $\varpi\colon \mathfrak{f} \to M_1$ is surjective. In order to prove this we consider the same vector space V as in the proof of Corollary 2.1. Then, as shown there, we can identify $H^0(\Delta, \mathcal{N}_\Delta)$ with a vector subspace V_1 of V. Under this identification we wish to clarify how the characteristic map $\sigma^{\mathfrak{f}}\colon T_0(M_1) \to V_1$ is described explicitly. We take an open covering $\{U_\alpha\}_{0 \leqq \alpha \leqq 3}$ of P, where $U_\alpha := \{\xi = (\xi_0, \xi_1, \xi_2, \xi_3) \in P \mid \xi_\alpha \neq 0\}$. We set

$$X_i^\alpha(t) = \tilde{f}_i(\xi/\xi_\alpha, t) \quad \text{for} \quad 1 \leqq i \leqq 3 , \quad 0 \leqq \alpha \leqq 3 .$$

Then $(X_1^\alpha(t), X_2^\alpha(t), X_3^\alpha(t), t)$ may be regarded as a system of local coordinates on $U \times M_1$. For any $(\partial/\partial t) \in T_0(M_1)$ we set

$$\theta_\alpha := \sum_{i=1}^{3} \frac{\partial X_i^\alpha}{\partial t}(0) \frac{\partial}{\partial X_i^\alpha(0)} \qquad (0 \leqq \alpha \leqq 3) .$$

Then by definition

$$\sigma^{\mathfrak{f}}\left(\frac{\partial}{\partial t}\right) = \{Q_\alpha(\theta_\alpha)\}_{0 \leqq \alpha \leqq 3} \in H^0(\Delta, \mathcal{N}_\Delta) , \tag{2.6}$$

where Q_α denotes the map $\Gamma(U_\alpha, T_P) \to \Gamma(U_\alpha \cap \Delta, \mathcal{N}_\Delta)$ induced by the natural projection of sheaves $T_P \to \mathcal{N}_\Delta$. By the definition of $\alpha: \mathcal{N}_\Delta \to \sum_{i=1}^{3} N_{\Delta_i}$ in Proposition 2.3, the element of $V_1 \subset V$ which corresponds to the one in 2.6 by the identifications $\sum_{i=1}^{3} H^0(\Delta_i, N_{\Delta_i}) = V$ and $H^0(\Delta, \mathcal{N}_\Delta) = V_1$ is

$$\left(\frac{\partial \tilde{f}_1}{\partial t}(\xi, 0)_{|\Delta_1} \oplus \frac{\partial \tilde{f}_2}{\partial t}(\xi, 0)_{|\Delta_1}\right) \oplus \left(\frac{\partial \tilde{f}_2}{\partial t}(\xi, 0)_{|\Delta_2} \oplus \frac{\partial \tilde{f}_3}{\partial t}(\xi, 0)_{|\Delta_2}\right) \oplus \left(\frac{\partial \tilde{f}_3}{\partial t}(\xi, 0)_{|\Delta_3} \oplus \frac{\partial \tilde{f}_1}{\partial t}(\xi, 0)_{|\Delta_3}\right) , \tag{2.7}$$

since $X_i^\alpha(t) = \tilde{f}_i(\xi/\xi_\alpha, t)$ for $1 \leqq i \leqq 3$, $0 \leqq \alpha \leqq 3$. This element is nothing but $\sigma^{\mathfrak{f}}(\partial/\partial t)$ if we consider the characteristic map $\sigma^{\mathfrak{f}}$ to be one from $T_0(M)$ to V_1. Suppose an element $v \in V_1$ is given. Then, as shown in the proof of Corollary 2.1 there exists an element $(H, \Phi, \Psi) \in L_{r_1} \oplus L_{r_2} \oplus L_{r_3}$ such that $\hat{\gamma}((H, \Phi, \Psi)) = v$, where $\hat{\gamma}: L_{r_1} \oplus L_{r_2} \oplus L_{r_3} \to V$ is the same map as in (2.4). We can choose tangent vectors $(\partial/\partial t_1) \in T_0(\boldsymbol{C}^{G(r_1)})$, $(\partial/\partial t_2) \in T_0(\boldsymbol{C}^{G(r_2)})$, $(\partial/\partial t_3) \in T_0(\boldsymbol{C}^{G(r_3)})$ so that

$$\frac{\partial \tilde{f}_1}{\partial t_1}(\xi, 0) = H(\xi) , \quad \frac{\partial \tilde{f}_2}{\partial t_2}(\xi, 0) = \Phi(\xi) , \quad \frac{\partial \tilde{f}_3}{\partial t_3}(\xi, 0) = \Psi(\xi) .$$

We set

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} \in T_0(M_1) = \sum_{i=1}^{3} T_0(\boldsymbol{C}^{G(r_i)}) .$$

Then, by (2.7) we have

$$\sigma^{\mathfrak{f}}\left(\frac{\partial}{\partial t}\right) = \hat{\gamma}((H, \Phi, \Psi)) = v .$$

Consequently, we conclude that the characteristic map $\sigma^{\mathfrak{f}}: T_0(M) \to H^0(\Delta, \mathcal{N}_\Delta)$ is surjective. This completes the proof of Proposition 2.4.

As in [9], a surface S with ordinary singularities in a compact threefold W is said to be *regular* if $H^1(S, \Phi_S) = 0$. Concerning the regularity of a surface of type (n, r_1, r_2, r_3) in P with ordinary singularities, we obtain the following:

PROPOSITION 2.5. *Let S be a surface of type (n, r_1, r_2, r_3) with ordinary singularities. We assume that $n \geqq 2r_1 + 2r_2 + r_3 - 3$. Then S is regular if and only if both of the following two conditions are satisfied:*

(a) $r_1 \leqq 3$;

(b) $C(r_1) + C(r_2) + C(r_3) + C(r_1 - r_2 - r_3) - C(r_1 - r_2) - C(r_1 - r_3) - C(r_2 - r_3) - \delta_{r_1, r_2} - \delta_{r_2, r_3} - \delta_{r_3, r_1} - 3 = 3r_1 r_2 r_3$.

PROOF. By Proposition 2.2, *Bott's theorem* and the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_P \longrightarrow \mathcal{O}_P([S] - 2\Delta) \longrightarrow \mathcal{O}_S([S] - 2\Delta) \longrightarrow 0 \tag{2.8}$$

with $\mathcal{O}_S([S] - 2\Delta) := \mathcal{O}_P([S] - 2\Delta)/\mathcal{O}_P([S] - S)$, we have $h^\nu(S, \mathcal{O}_S([S] - 2\Delta)) = 0$ for $\nu = 1, 2$. Then $H^1(S, \Phi_S) \simeq H^1(\Delta, \mathcal{N}_\Delta)$ by (2.5). Hence S is regular if and only if $H^1(\Delta, \mathcal{N}_\Delta) = 0$. On the other hand, by Proposition 2.3 there exists an exact sequence of cohomogy groups

$$\begin{aligned} 0 \longrightarrow H^0(\Delta, \mathcal{N}_\Delta) \xrightarrow{\hat{\alpha}} \sum_{i=1}^{3} H^0(\Delta_i, \mathcal{N}_{\Delta_i}) \xrightarrow{\hat{\beta}} C^3_{\Sigma t} \\ \longrightarrow H^1(\Delta, \mathcal{N}_\Delta) \longrightarrow \sum_{i=1}^{3} H^1(\Delta_i, N_{\Delta_i}) \longrightarrow 0 \,. \end{aligned} \tag{2.9}$$

From this it follows that $H^1(\Delta, \mathcal{N}_\Delta) = 0$ if and only if both of the following two conditions are satisfied:

(a′) $H^1(\Delta_i, N_{\Delta_i}) = 0$ for $i = 1, 2, 3$;

(b′) $\dim \operatorname{image} \hat{\beta} = 3r_1 r_2 r_3$ $(= \dim C^3_{\Sigma t})$.

By simple calculation we can see that the condition (a′) is equivalent to (a). By (2.9) and Corollary 2.1,

$$\begin{aligned} \dim \operatorname{image} \hat{\beta} &= \sum_{i=1}^{3} \dim H^0(\Delta_i, N_{\Delta_i}) - \dim H^0(\Delta, \mathcal{N}_\Delta) \\ &= C(r_1) + C(r_2) + C(r_3) + C(r_1 - r_2 - r_3) - C(r_1 - r_2) \\ &\quad - C(r_1 - r_3) - C(r_2 - r_3) - \delta_{r_1, r_2} - \delta_{r_2, r_3} - \delta_{r_3, r_1} - 3 \,. \end{aligned}$$

Hence the condition (b′) is identical with (b). q.e.d.

THEOREM 2.1. *Let X be a non-sigular surface of type (n, r_1, r_2, r_3), and let S be the surface with oridinary singularities in P corresponding to X. Then:*

(a) *Except for those of types* (6, 1, 1, 1), (7, 2, 1, 1), (8, 2, 2, 1), (8, 2, 2, 2), *we obtain*

$$h^1(P, \mathcal{O}_P([S + K_P] - \Delta)) = 0 \,.$$

Hence the connecting homomorphism $\delta: H^0(S, \Phi_S) \to H^1(X, T_X)$ is surjective, and the Kuranishi family of deformations of the complex structure of X is non-singular. The number $m(X)$ of moduli is given by

$$\begin{aligned} m(X) &= h^0(P, \mathcal{O}_P(n-2\Delta)) - 1 + h^0(\Delta, \mathcal{N}_\Delta) - h_0(P, T_P) \\ &= C(n-r_1-r_2-r_3)+C(n-2r_2-2r_3)-C(n-2r_2-2r_3-r_1) \\ &\quad + C(n-2r_3-2r_1)-C(n-2r_3-2r_1-r_2)+C(n-2r_1-2r_2) \\ &\quad - C(n-2r_1-2r_2-r_3)+C(r_1)+C(r_2)+C(r_3)-C(r_1-r_2-r_3)-19 . \end{aligned}$$

(b) *As to those of types* (6, 1, 1, 1) *and* (7, 2, 1, 1) *we obtain*

$$h^0(X, T_X) = h^2(X, T_X) = 0 .$$

Hence its Kuranishi family of deformations is also non-singular. The number $m(X)$ *of moduli is given by*

$$m(X) = \begin{cases} 34 & (6, 1, 1, 1) \\ 42 & (7, 2, 1, 1) . \end{cases}$$

PROOF. (a) Applying $\otimes\Omega_p^1$ to the exact sequence of sheaves in Proposition 2.1 and setting $k = n-4$, we obtain the following exact sequence of sheaves:

$$\begin{aligned} & 0 \longrightarrow \Omega_P^1(n-4-r_1-r_2-r_3)^{\oplus 2} \\ (2.10) \qquad & \longrightarrow \Omega_P^1(n-4-r_1-r_2) \oplus \Omega_P^1(n-4-r_2-r_3) \oplus \Omega_P^1(n-4-r_3-r_1) \\ & \longrightarrow \Omega_P^1((n-4)-\Delta) \longrightarrow 0 . \end{aligned}$$

Note that

$$\left.\begin{matrix} n-4-r_1-r_2 \neq 0 \\ n-4-r_2-r_3 \neq 0 \\ n-4-r_3-r_1 \neq 0 \end{matrix}\right\} \Leftrightarrow \begin{cases} (n, 2, 2, 2) , & n \geqq 9 \\ (n, 2, 2, 1) , & n \geqq 9 \\ (n, 2, 1, 1) , & n \geqq 8 \\ (n, 1, 1, 1) , & n \geqq 7 . \end{cases}$$

Therefore, taking the long exact sequence of cohomology groups associated to (2.10) we have

$$h^1(P, \Omega_P^1([S+K_P]-\Delta)) = h^1(P, \Omega_P^1((n-4)-\Delta)) = 0$$

except for the surfaces S of types (6, 1, 1, 1), (7, 2, 1, 1), (8, 2, 2, 1), (8, 2, 2, 2). Hence by Theorem 1.1 (a) the connecting homomorphism δ: $H^0(S, \Phi_S) \to H^1(X, T_X)$ is surjective for the surfaces in case (a) of the theorem. By Proposition 2.4 S belongs to a maximal analytic family $\mathcal{S} = \cup_{t\in M} S_t$ of surfaces in P with ordinary singularities which satisfies the conditions in Corollary 1.1. Therefore by Corollary 1.1 we conclude that the Kuranishi family of deformations of X is non-singular for the surfaces X in case (a) of the theorem, and the number $m(X)$ of moduli is given by

$$m(X) = h^0(S, \Phi_S) - h^0(P, T_P) + h^0(P, T_P(\log S)) - h^2(P, \Omega_P^1([S+K_P]-\Delta)) .$$

We have $h^0(P, T_P) = 15$. By classifying the structure of the non-singular normalizations of the surfaces with ordinary singularities defined by the equation (2.1), the following turns out: if X is of general type, then the order of S in P is not less than five. Then the *logarithmic Kodaira dimension* $\bar{\kappa}(P - S)$ is equal to three. Therefore by Theorem 6 and the corollary to Proposition 4 in [5], we have $h^0(P, T_P(\log S)) = 0$. By (2.10) and *Bott's theorem*

$$h^2(P, \Omega_P^1([S + K_P] - \Delta)) = h^2(P, \Omega_P^1((n - 4) - \Delta)) = 0 .$$

By Proposition 2.2, *Bott's theorem* and (2.8), we have $h^1(S, \mathcal{O}_S([S]-2\Delta)) = 0$. Then by (2.5) we have

$$\begin{aligned} h^0(S, \Phi_S) &= h^0(S, \mathcal{O}_S([S] - 2\Delta)) + h^0(\Delta, \mathcal{N}_\Delta) \\ &= h^0(P, \mathcal{O}_P([S] - 2\Delta)) - 1 + h^0(\Delta, \mathcal{N}_\Delta) . \end{aligned}$$

Therefore the number $h^0(S, \Phi_S)$ is calculated by Proposition 2.2, *Bott's theorem* and Corollary 2.1. Consequently, we obtain the formula for $m(X)$ for the surfaces in case (a) of the theorem.

(b) As to the surfaces X of types (6, 1, 1, 1) and (7, 2, 1, 1), by (2.10) and *Bott's theorem* we derive

$$h^0(P, \Omega_P^1([S + K_P] - \Delta)) = h^0(P, \Omega_P^1((n - 4) - \Delta)) = 0 .$$

By Proposition 2.5 they are regular in P. Therefore by Theorem 1.2 we have $h^2(X, T_X) = 0$; hence their Kuranishi families of deformations of the complex analytic structures are non-singular, and the number $m(X)$ of moduli is given by

$$m(X) = 10(p_a + 1) - c_1^2 .$$

By the classical formula (cf. [10]) for p_a and c_1^2 of the non-singular normalizations of the surfaces with ordinary singularities in P we can calculate the number $m(X)$ of moduli of the surfaces of types (6, 1, 1, 1) and (7, 2, 1, 1). q.e.d.

REFERENCES

[1] R. BOTT, Homogeneous vector bundles, Ann. of Math. 66 (1966), 203–248.

[2] E. HORIKAWA, On deformations of holomorphic maps I, J. Math. Soc. Japan, 25 (1973), 372–396.

[3] E. HORIKAWA, On the number of moduli of certain algebraic surfaces of general type, J. Fac. Sci. Univ. Tokyo 22 (1975), 67–78.

[4] S. IITAKA, Max Nöther's theorem on a regular projective algebraic variety, J. Fac. Sci. Univ. Tokyo, Sec. I, Vol. 8 (1966), 129–137.

[5] S. IITAKA, On logarithmic Kodaira dimension of algebraic varieties, in Complex Analysis and Algebraic Geometry (W. L. Baily, Jr. and T. Shioda, eds.), Iwanami, Tokyo, 1977.

[6] K. KODAIRA AND D. C. SPENCER, On deformations of complex analytic structures I, Ann. of Math. 67 (1958), 328-401.
[7] K. KODAIRA, A theorem of completeness for analytic systems of surfaces with ordinary singularities, Ann. of Math. 74 (1961), 591-627.
[8] K. KODAIRA, On the structure of compact complex analytic surfaces, I, Amer. J. Math. 86 (1964), 751-798.
[9] K. KODAIRA, On characteristic systems of families of surfaces with ordinary singularities in a projective space, Amer. J. Math. 87 (1965), 227-256.
[10] K. KODAIRA, The theory of algebraic surfaces, Seminar Notes, No. 20, Department of Mathematics, Tokyo University, 1968 (in Japanese).
[11] I. R. PORTEOUS, Blowing up Chern classes, Proc. Cambridge Phil. Soc. 56 (1960), 118-124.
[12] S. TSUBOI, On the sheaves of holomorphic vector fields on surfaces with ordinary singularities in a projective space I, Sci. Rep. Kagoshima Univ. 25 (1976), 1-26.
[13] S. TSUBOI, On the number of moduli of non-singular normalizations of surfaces with ordinary singularities, ibid. 32 (1983), 23-46.
[14] S. TSUBOI, Deformations of locally stable holomorphic maps and locally trivial displacements of analytic subvarieties with ordinary singularities, ibid. 35 (1986), 9-90.

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