Tôhoku Math. J.
40 (1988), 413-423.

THE EXPONENT OF CONVERGENCE OF POINCARÉ SERIES ASSOCIATED WITH SOME DISCONTINUOUS GROUPS

Dedicated to Professor Tadashi Kuroda on his sixtieth birthday

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(Received February 24, 1987)

1. Introduction. Let $\boldsymbol{R}^{n+1}$ be the $(n+1)$-dimensional Euclidean space $(n \geqq 1)$. Each point of $\boldsymbol{R}^{n+1}$ is denoted by a column vector $v = {}^t(v_1, v_2, \cdots, v_{n+1})$, where t denotes the transpose. We put $|v| = \{\sum_{i=1}^{n+1}(v_i)^2\}^{1/2}$ and $x_{n+1}(v) = v_{n+1}$. Let $\boldsymbol{B}^{n+1} = \{v \in \boldsymbol{R}^{n+1}: |v| < 1\}$ and $\boldsymbol{H}^{n+1} = \{v \in \boldsymbol{R}^{n+1}: x_{n+1}(v) > 0\}$ be the open unit ball and the upper half space in $\boldsymbol{R}^{n+1}$, respectively. We denote by $S(x)$ the n-sphere in $\boldsymbol{R}^{n+1}$ with center at x and radius 1.

A Möbius transformation of $\boldsymbol{R}^{n+1} \cup \{\infty\}$ is, by definition, a composite of a finite number of inversions in $\boldsymbol{R}^{n+1} \cup \{\infty\}$ with respect to n-spheres or n-planes. Let Möb be the group of all the Möbius transformations of $\boldsymbol{R}^{n+1} \cup \{\infty\}$. We denote by $|\gamma'(x)|$ the $(n+1)$-th root of the absolute value of the determinant of the Jacobian matrix of $\gamma \in$ Möb at $x \in \boldsymbol{R}^{n+1} \setminus \{\gamma^{-1}(\infty)\}$.

An element $\gamma \in$ Möb with a fixed point at ∞ is of the form $\gamma(x) = \lambda A x + v$ for some $\lambda > 0$, $A \in O(n+1)$ and $v \in \boldsymbol{R}^{n+1}$, where $O(n+1)$ is the group of orthogonal matrices of degree $n+1$ (see [1, p. 20]). Next assume that $\gamma(\infty) \neq \infty$. Then, for the inversion σ with respect to $S(\gamma^{-1}(\infty))$, we have $\gamma \circ \sigma(\infty) = \infty$ so that $\gamma \circ \sigma(x) = \lambda A x + v$. Hence $\gamma(x) = \lambda A \sigma(x) + v$. Therefore $|\gamma'(x)| = \lambda / |x - \gamma^{-1}(\infty)|^2$ since $|\sigma'(x)| = 1/|x - \gamma^{-1}(\infty)|^2$. Let the center and the radius of the n-sphere $\{x \in \boldsymbol{R}^{n+1}: |\gamma'(x)| = 1\}$ be $\alpha(\gamma)$ and $\rho(\gamma)$, respectively. Then we have $\alpha(\gamma) = \gamma^{-1}(\infty)$ and $\rho(\gamma)^2 = \lambda$ so that

$$(1) \qquad |\gamma'(x)| = \rho(\gamma)^2 / |x - \alpha(\gamma)|^2 .$$

Further, let the interior and the exterior of the n-sphere be $I(\gamma)$ and $E(\gamma)$, respectively. Then, as in [1, p. 30],

$$(2) \qquad \gamma(E(\gamma)) = I(\gamma^{-1}) , \quad \gamma(I(\gamma)) = E(\gamma^{-1}) .$$

Let Möb$(\boldsymbol{B}^{n+1})$ be the subgroup of Möb whose elements map $\boldsymbol{B}^{n+1}$ onto itself. A subgroup Γ of Möb$(\boldsymbol{B}^{n+1})$ is said to be discontinuous if the orbit $\{\gamma(o)\}_{\gamma \in \Gamma}$ of the origin $o \in \boldsymbol{B}^{n+1}$ under Γ has no accumulation points in $\boldsymbol{B}^{n+1}$. Hence, for a discontinuous subgroup Γ, the set $\Lambda(\Gamma)$ of accumulation points of $\{\gamma(o)\}_{\gamma \in \Gamma}$ is contained in $\partial \boldsymbol{B}^{n+1}$. We call $\Lambda(\Gamma)$ the limit set of Γ. Let $\delta(\Gamma)$

be the exponent of convergence of the Poincaré series $\sum_{\gamma\in\Gamma}(1-|\gamma(o)|)^{s/2}$, that is,

$$\delta(\Gamma)=\inf\{s>0:\sum_{\gamma\in\Gamma}(1-|\gamma(o)|)^{s/2}<\infty\}\,.$$

In this paper we prove the following:

THEOREM. *Let Γ be a discontinuous subgroup of* Möb$(\boldsymbol{B}^{n+1})$ *with* $\#\Lambda(\Gamma)>2$ *and let $\xi_0\in\Lambda(\Gamma)$ be the unique fixed point of some transformation in Γ. If the group $\Gamma_{\xi_0}=\{\gamma\in\Gamma:\gamma(\xi_0)=\xi_0\}$ contains a free abelian group of rank $l(\geqq 1)$, then $\delta(\Gamma)$ is greater than l. Moreover, the lower bound l is the best possible.*

In the case of $n\leqq 2$, that is, in the case of a Kleinian group Γ acting on $\boldsymbol{H}^3$ with $\Lambda(\Gamma)\not\ni\infty$, Beardon [2] showed this result for the exponent of convergence of the series $\sum_{\gamma\in\Gamma\setminus\Gamma_\infty}\rho(\gamma)^s$. For the other properties concerning the exponent of convergence of the Poincaré series, see also Tukia [3, § E] and references quoted there.

In § 2, we give some preliminary lemmas on a Möbius transformation and in § 3, we give some properties of a discontinuous group mentioned in the above theorem. § 4 is devoted to showing some inequalities which are used in the proof in § 5 of the first half of the theorem. In § 6 we give an example of discontinuous groups which shows, in § 7, that the lower bound l is the best possible.

2. Preliminary lemmas. Let Möb$(\boldsymbol{H}^{n+1})$ be the subgroup of Möb whose elements map $\boldsymbol{H}^{n+1}$ onto itself. As in the introduction, each $\gamma\in$ Möb is written as $\gamma(x)=\lambda Ax+v$ or $\gamma(x)=\lambda A\sigma(x)+v$ for some $\lambda>0$, $A\in O(n+1)$ and $v\in\boldsymbol{R}^{n+1}$, where σ is the inversion with respect to $S(\gamma^{-1}(\infty))$. In particular, if $\gamma\in$ Möb$(\boldsymbol{H}^{n+1})$, then the following known lemma holds.

LEMMA 1. *If $\gamma\in$ Möb$(\boldsymbol{H}^{n+1})$, then $x_{n+1}(v)=0$ and $A=\begin{pmatrix}A_0 & 0\\ 0 & 1\end{pmatrix}$ for some $A_0\in O(n)$.*

Next we prove the following lemmas.

LEMMA 2. *Let γ_1 and γ_2 be elements of* Möb *satisfying $\gamma_1(\infty)\neq\infty$ and $\gamma_1\circ\gamma_2(\infty)\neq\infty$. Then $\rho(\gamma_1\circ\gamma_2)=\rho(\gamma_1)|\gamma_2'(\alpha(\gamma_1\circ\gamma_2))|^{-1/2}$ and $\alpha(\gamma_1\circ\gamma_2)=\gamma_2^{-1}(\alpha(\gamma_1))$.*

PROOF. From (1) we have

$$|(\gamma_1\circ\gamma_2)'(x)|=|\gamma_1'(\gamma_2(x))|\,|\gamma_2'(x)|=\{\rho(\gamma_1)^2/|\gamma_2(x)-\alpha(\gamma_1)|^2\}|\gamma_2'(x)|\,.$$

On the other hand, $|\gamma_2(x)-\alpha(\gamma_1)|^2=|\gamma_2'(x)|\,|\gamma_2'(\gamma_2^{-1}(\alpha(\gamma_1)))|\,|x-\gamma_2^{-1}(\alpha(\gamma_1))|^2$

(see [1, p. 19]). Therefore

$$|(\gamma_1 \circ \gamma_2)'(x)| = \rho(\gamma_1)^2/\{|\gamma_2'(\gamma_2^{-1}(\alpha(\gamma_1)))|\,|x - \gamma_2^{-1}(\alpha(\gamma_1))|^2\}\,,$$

from which we have the required equalities. q.e.d.

LEMMA 3. *Let* γ_1 *and* γ_2 *be elements of* Möb *satisfying* $\gamma_1(\infty) \neq \infty$, $\gamma_2(\infty) \neq \infty$, $\gamma_1 \circ \gamma_2(\infty) \neq \infty$ *and* $\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1}(\infty) \neq \infty$. *Then* $\rho(\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1}) = \rho(\gamma_2)|\alpha(\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1}) - \alpha(\gamma_1^{-1})|/|\alpha(\gamma_1) - \alpha(\gamma_2^{-1})|$.

PROOF. Lemma 2 and the identity $|\gamma_2'(x)| = |(\gamma_2^{-1})'(\gamma_2(x))|^{-1}$ show

$$\begin{aligned}\rho(\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1}) &= \rho(\gamma_1)|(\gamma_2 \circ \gamma_1^{-1})'(\alpha(\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1}))|^{-1/2} \\ &= \rho(\gamma_1)\{|\gamma_2'(\alpha(\gamma_1 \circ \gamma_2))|\,|(\gamma_1^{-1})'(\alpha(\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1}))|\}^{-1/2} \\ &= \rho(\gamma_1)|(\gamma_2^{-1})'(\alpha(\gamma_1))|^{1/2}|(\gamma_1^{-1})'(\alpha(\gamma_1 \circ \gamma_2 \circ \gamma_1^{-1}))|^{-1/2}\,.\end{aligned}$$

Now, by (1) and $\rho(\gamma) = \rho(\gamma^{-1})$, the last expression is equal to the one desired. q.e.d.

LEMMA 4. *Suppose* $\gamma \in$ Möb *satisfies* $\gamma(\infty) \neq \infty$ *and* $\mathrm{cl}(I(\gamma)) \cap \mathrm{cl}(I(\gamma^{-1})) = \emptyset$ *where* $\mathrm{cl}(I(\gamma^{\pm 1}))$ *is the closure of* $I(\gamma^{\pm 1})$. *Then* $\{I(\gamma^m)\}_{m=1}^{\infty}$ *and* $\{I(\gamma^{-m})\}_{m=1}^{\infty}$ *are decreasing sequences of sets with* $\lim_{m\to\infty} \rho(\gamma^{\pm m}) = 0$.

PROOF. Take a point $x \in I(\gamma)^c$. Then by (2), $\gamma(x) \in \mathrm{cl}(I(\gamma^{-1}))$ so that $\gamma(x) \in (\mathrm{cl}(I(\gamma)))^c$ by our assumption. Hence $|(\gamma^2)'(x)| = |\gamma'(\gamma(x))|\,|\gamma'(x)| < 1$, that is, $x \in (\mathrm{cl}(I(\gamma^2)))^c$. Thus $I(\gamma) \supset \mathrm{cl}(I(\gamma^2)) \supset I(\gamma^2)$. In the same manner, we have $I(\gamma^{-1}) \supset I(\gamma^{-2})$. Next assume that $I(\gamma) \supset I(\gamma^2) \supset \cdots \supset I(\gamma^m)$ and $I(\gamma^{-1}) \supset I(\gamma^{-2}) \supset \cdots \supset I(\gamma^{-m})$. Then for an element $x \in I(\gamma^m)^c$, we see $\gamma^m(x) \in \mathrm{cl}(I(\gamma^{-m})) \subset \mathrm{cl}(I(\gamma^{-1})) \subset (\mathrm{cl}(I(\gamma)))^c$ so that $|(\gamma^{m+1})'(x)| = |\gamma'(\gamma^m(x))|\,|(\gamma^m)'(x)| < 1$, that is, $x \in (\mathrm{cl}(I(\gamma^{m+1})))^c$. Therefore $I(\gamma^m) \supset I(\gamma^{m+1})$. Similarly we have $I(\gamma^{-m}) \supset I(\gamma^{-m-1})$.

Since $\mathrm{cl}(I(\gamma^2)) \subset I(\gamma)$, there exists a constant $c_1 > 1$ such that $|\gamma'(x)| \geqq c_1$ for all $x \in I(\gamma^2)$. Since $\alpha(\gamma^m) \in I(\gamma^m) \subset I(\gamma^2)$ for $m \geqq 2$, we have $|\gamma'(\alpha(\gamma^m))| \geqq c_1$ so that by Lemma 2, $\rho(\gamma^m) = \rho(\gamma^{m-1})|\gamma'(\alpha(\gamma^m))|^{-1/2} \leqq \rho(\gamma^{m-1})(c_1)^{-1/2}$. Thus $\lim_{m\to\infty} \rho(\gamma^m) \leqq \lim_{m\to\infty} (c_1)^{-(m-1)/2}\rho(\gamma) = 0$. Since $\rho(\gamma^m) = \rho(\gamma^{-m})$, we are done. q.e.d.

3. Properties of discontinuous subgroups. Let Γ be a discontinuous subgroup of $\mathrm{M\ddot{o}b}(\boldsymbol{B}^{n+1})$ which satisfies the conditions stated in the Theorem in the introduction. Let τ be a Möbius transformation with $\tau(\boldsymbol{B}^{n+1}) = \boldsymbol{H}^{n+1}$, $\tau(\xi_0) = \infty$, $\tau(\infty) = -e_{n+1}$ and $\tau(o) = e_{n+1}$, where $e_{n+1} = {}^t(0, \cdots, 0, 1) \in \boldsymbol{H}^{n+1}$. We denote by $\{P_1, \cdots, P_l\}$ a system of free generators of the free abelian group of rank l contained in $\tau \circ \Gamma_{\xi_0} \circ \tau^{-1}$. We set $G = \tau \circ \Gamma \circ \tau^{-1} \subset \mathrm{M\ddot{o}b}(\boldsymbol{H}^{n+1})$ and $G_\infty = \tau \circ \Gamma_{\xi_0} \circ \tau^{-1}$. Note that G is a discontinuous subgroup of $\mathrm{M\ddot{o}b}(\boldsymbol{H}^{n+1})$, that is, $\{g(e_{n+1})\}_{g \in G}$ never accumulate in $\boldsymbol{H}^{n+1}$.

LEMMA 5. *There exists an element $g \in G \setminus G_\infty$ with* $\mathrm{cl}(I(g)) \cap \mathrm{cl}(I(g^{-1})) = \varnothing$.

PROOF. If a Möbius transformation $x \mapsto \lambda Bx + w$ has a unique fixed point at ∞, then $\lambda = 1$, for otherwise it has exactly two fixed points $(\lambda B - E_{n+1})^{-1}(-w)$ and ∞, where E_{n+1} is the unit matrix of degree $n+1$. Choose $g \in G_\infty$ which has a unique fixed point at ∞ and set $g(x) = Bx + w$. Let $h(x) = \lambda Ax + v$ be another such element in G_∞. Since $A, B \in O(n+1)$, we have $|g^{\pm 1}(x)| \leqq |x| + |w|$, $|h^m(x)| \leqq \lambda^m |x| + \sum_{k=0}^{m-1} \lambda^k |v|$ and $|h^{-m}(x)| \leqq \lambda^{-m}|x| + \sum_{k=0}^{m-1} \lambda^{-k-1}|v|$ for $m = 1, 2, 3, \cdots$. Now if $\lambda \neq 1$ (here we may assume that $\lambda < 1$), we have

$$|g \circ h^m \circ g^{-1} \circ h^{-m}(e_{n+1})| \leqq 1 + 2|w| + 2(1-\lambda)^{-1}|v|$$

for all $m = 1, 2, 3, \cdots$. On the other hand, $x_{n+1}(g \circ h^m \circ g^{-1} \circ h^{-m}(e_{n+1})) = 1$ by Lemma 1. Furthermore $\{g \circ h^m \circ g^{-1} \circ h^{-m}\}_{m=1}^{\infty}$ are mutually distinct, for if $g \circ h^m \circ g^{-1} \circ h^{-m} = \mathrm{id}$ for some m, then $g \circ h^m = h^m \circ g$ and g also fixes the finite fixed point $(\lambda A - E_{n+1})^{-1}(-v)$ of h. Therefore the orbit $\{g \circ h^m \circ g^{-1} \circ h^{-m}(e_{n+1})\}_{m=1}^{\infty}$ has an accumulation point in $\boldsymbol{H}^{n+1}$. This contradicts the discontinuity of G. Hence $\lambda = 1$ and we have

$$g(x) = Ax + v \quad (g \in G_\infty)\,. \tag{3}$$

Since $A = \begin{pmatrix} A_0 & 0 \\ 0 & 1 \end{pmatrix}$ and $x_{n+1}(v) = 0$ by Lemma 1, we see $x_{n+1}(g(e_{n+1})) = 1$ for all $g \in G_\infty$. Therefore the accumulation points of the orbit $\{g(e_{n+1})\}_{g \in G_\infty}$ consists of only one point $\{\infty\}$. Thus we have $G \supsetneqq G_\infty$ by the condition $\sharp \Lambda(G) > 2$.

Now we choose an element $g \in G \setminus G_\infty$. Since G is a discontinuous subgroup of $\mathrm{Möb}(\boldsymbol{H}^{n+1})$ we have $\lim_{m\to\infty} |P_1^m(x)| = \infty$ for $x \in \boldsymbol{H}^{n+1}$. Further, for $x \in \partial \boldsymbol{H}^{n+1}$, we see $P_1^m(x) + e_{n+1} = P_1^m(x + e_{n+1})$ by Lemma 1 so that $\lim_{m\to\infty} |P_1^m(x)| = \infty$ also for $x \in \partial \boldsymbol{H}^{n+1}$. Therefore

$$\begin{aligned} &\lim_{m\to\infty} \alpha(P_1^m \circ g \circ P_1^{-m} \circ g^{-1}) = \lim_{m\to\infty} g \circ P_1^m \circ g^{-1}(\infty) = \alpha(g^{-1}) \\ &\lim_{m\to\infty} |\alpha(g \circ P_1^m \circ g^{-1} \circ P_1^{-m})| = \lim_{m\to\infty} |P_1^m \circ g \circ P_1^{-m} \circ g^{-1}(\infty)| = \infty\,. \end{aligned} \tag{4}$$

Since $P_1^m \in G_\infty$, we have $|(P_1^m \circ g)'(x)| = |g'(x)|$ by (3) so that $\rho(P_1^m \circ g) = \rho(g)$. Therefore, by Lemma 2, (1) and $\rho(g) = \rho(g^{-1})$, we get

$$\begin{aligned} \rho(P_1^m \circ g \circ P_1^{-m} \circ g^{-1}) &= \rho(P_1^m \circ g)|(P_1^{-m} \circ g^{-1})'(\alpha(P_1^m \circ g \circ P_1^{-m} \circ g^{-1}))|^{-1/2} \\ &= \rho(g)|(g^{-1})'(\alpha(P_1^m \circ g \circ P_1^{-m} \circ g^{-1}))|^{-1/2} \\ &= |\alpha(P_1^m \circ g \circ P_1^{-m} \circ g^{-1}) - \alpha(g^{-1})|\,. \end{aligned}$$

Thus $\lim_{m\to\infty} \rho(g \circ P_1^m \circ g^{-1} \circ P_1^{-m}) = \lim_{m\to\infty} \rho(P_1^m \circ g \circ P_1^{-m} \circ g^{-1}) = 0$ by (4).

Hence, again by (4), we have $\mathrm{cl}(I(P_1^m \circ g \circ P_1^{-m} \circ g^{-1})) \cap \mathrm{cl}(I(g \circ P_1^m \circ g^{-1} \circ P_1^{-m})) = \emptyset$ for all large m. q.e.d.

We set $\boldsymbol{Z}^l = \{\nu = (n_1, n_2, \cdots, n_l): n_i \in \boldsymbol{Z}\}$ and $(\boldsymbol{Z}^l)^* = \boldsymbol{Z}^l \setminus (0, \cdots, 0)$ where $\boldsymbol{Z}$ is the set of all integers. For $\nu = (n_1, n_2, \cdots, n_l) \in \boldsymbol{Z}^l$, we set $P_\nu = P_1^{n_1} \circ P_2^{n_2} \circ \cdots \circ P_l^{n_l}$. Since G is a discontinuous subgroup of $\mathrm{M\ddot{o}b}(\boldsymbol{H}^{n+1})$ and since $P_\nu(e_{n+1}) = P_\nu(o) + e_{n+1}$ by Lemma 1, $\lim_{|\nu|\to\infty} |P_\nu(o)| \geqq \lim_{|\nu|\to\infty}\{|P_\nu(e_{n+1})| - 1\} = \infty$, where $|\nu| = \max\{|n_i|: 1 \leqq i \leqq l\}$ for $\nu = (n_1, \cdots, n_l) \in \boldsymbol{Z}^l$, so that there exists a large number m_1 satisfying $|P_{m_1\nu}(o)| > 1$ for all $\nu = (n_1, \cdots, n_l) \in (\boldsymbol{Z}^l)^*$ where $m_1\nu = (m_1n_1, \cdots, m_1n_l)$.

Let $g \in G \setminus G_\infty$ be as in Lemma 5. Then by Lemma 4 we can choose a large number m_2 such that $\rho(g^{m_2}) < 1$.

Now we set $Q_\nu = P_{m_1\nu}$ and $g_0 = g^{m_2}$. Since $Q_\nu(x) = A_\nu x + Q_\nu(o)$ for some $A_\nu \in O(n+1)$ we may assume, by choosing m_1 sufficiently large, that $|Q_\nu(o)| > 1$ and

$$Q_\nu(x) \in E(g_0) \cap E(g_0^{-1}) \tag{5}$$

for all $x \in I(g_0) \cup I(g_0^{-1})$ and $\nu \in (\boldsymbol{Z}^l)^*$.

LEMMA 6. *Let g_0 and Q_ν be as above and let*

$$\hat{G} = \bigcup_{k=1}^{\infty}\{g_0 \circ Q_{\nu_1} \circ g_0 \circ \cdots \circ g_0 \circ Q_{\nu_k} \circ g_0 \colon \nu_1, \cdots, \nu_k \in (\boldsymbol{Z}^l)^*\}\ .$$

Then each element g of $\hat{G}$ is mutually distinct and satisfies $\alpha(g) \in I(g_0)$ and $g(\infty) \neq \infty$.

PROOF. Suppose that the equality

$$g_0 \circ Q_{\nu_1} \circ g_0 \circ \cdots \circ g_0 \circ Q_{\nu_p} \circ g_0 = g_0 \circ Q_{\mu_1} \circ g_0 \circ \cdots \circ g_0 \circ Q_{\mu_q} \circ g_0$$

holds for some $\nu_1, \cdots, \nu_p$ and $\mu_1, \cdots, \mu_q$ and assume that $\nu_1 = \mu_1, \cdots, \nu_{k-1} = \mu_{k-1}$ and $\nu_k \neq \mu_k$ for some $k \geqq 1$. Then $Q_{-\mu_q} \circ g_0^{-1} \circ \cdots \circ Q_{-\mu_{k+1}} \circ g_0^{-1} \circ Q_{-\mu_{k}+\nu_k} \circ g_0 \circ Q_{\nu_{k+1}} \circ \cdots \circ g_0 \circ Q_{\nu_p}$ is the identity mapping and fixes the point ∞, whereas no element of this form fixes ∞ by (2) and (5). This contradiction gives the first part of our assertion. Also by (2) and (5), we have the other assertions. q.e.d.

4. **Inequalities.** As is already seen in (3), P_i $(1 \leqq i \leqq l)$ is of the form $U_i x + a_i$. Hence, for an integer m, we have $P_i^m(x) = U_i^m x + b_i(m)$ where $b_i(m) = \sum_{k=0}^{m-1} U_i^k a_i$ for $m \geqq 0$ and $b_i(m) = \sum_{k=0}^{m-1} U_i^{-k-1}(-a_i)$ for $m < 0$. Since $Q_\nu(x) = P_1^{m_1n_1} \circ \cdots \circ P_l^{m_1n_l}(x)$ we see

$$Q_\nu(o) = b_1(m_1n_1) + \sum_{i=1}^{l-1} U_1^{m_1n_1} \cdots U_i^{m_1n_i}(b_{i+1}(m_1n_{i+1})) \tag{6}$$

for $\nu = (n_1, \cdots, n_l) \in (\boldsymbol{Z}^l)^*$.

For $\nu_1, \cdots, \nu_k \in (\boldsymbol{Z}^l)^*$, we denote by $Q(\nu_1, \cdots, \nu_k)$ the transformation $g_0 \circ Q_{\nu_1} \circ g_0 \circ \cdots \circ g_0 \circ Q_{\nu_k} \circ g_0$ of $\hat{G}$ in Lemma 6.

LEMMA 7. *There exists a positive constant* ε_1 *such that*

$$\rho(Q(\nu_1, \cdots, \nu_k)) \geqq \rho(g_0)\varepsilon_1^k \prod_{j=1}^{k} \left(\sum_{i=1}^{l} |n_{ij}|\right)^{-1},$$

where $\nu_j = (n_{1j}, \cdots, n_{lj})$ $(1 \leqq j \leqq k)$.

PROOF. Lemma 2 gives

$$\rho(Q(\nu_1, \cdots, \nu_j)) = \rho(Q(\nu_1, \cdots, \nu_{j-1}))|(Q_{\nu_j} \circ g_0)'(\alpha(Q(\nu_1, \cdots, \nu_j)))|^{-1/2}$$

for $1 \leqq j \leqq k$, where we assume that $\rho(Q(\nu_0)) = \rho(g_0)$. Since $|(Q_{\nu_j} \circ g_0)'(x)| = |g_0'(x)| = \rho(g_0)^2|x - \alpha(g_0)|^{-2}$, we have

$$(7) \qquad \rho(Q(\nu_1, \cdots, \nu_j)) = \rho(Q(\nu_1, \cdots, \nu_{j-1}))\{|\alpha(Q(\nu_1, \cdots, \nu_j)) - \alpha(g_0)|/\rho(g_0)\}.$$

Also from Lemma 2, $\alpha(Q(\nu_1, \cdots, \nu_j)) = g_0^{-1} \circ Q_{-\nu_j}(\alpha(Q(\nu_1, \cdots, \nu_{j-1})))$. Let $g_0^{-1}(x) = \lambda A\sigma(x) + v$, where σ is the inversion with respect to $S(g_0(\infty))$. Then, for $\xi = Q_{-\nu_j}(\alpha(Q(\nu_1, \cdots, \nu_{j-1})))$, we have

$$\begin{aligned} |\alpha(Q(\nu_1, \cdots, \nu_j)) - \alpha(g_0)| &= |g_0^{-1}(\xi) - g_0^{-1}(\infty)| = \lambda|\sigma(\xi) - \sigma(\infty)| \\ &= \rho(g_0^{-1})^2|\sigma(\xi) - \sigma(\infty)|. \end{aligned}$$

Since $\sigma(\xi) = g_0(\infty) + (\xi - g_0(\infty))|\xi - g_0(\infty)|^{-2} = \sigma(\infty) + (\xi - \alpha(g_0^{-1}))|\xi - \alpha(g_0^{-1})|^{-2}$, we see

$$(8) \qquad |\alpha(Q(\nu_1, \cdots, \nu_j)) - \alpha(g_0)| = \rho(g_0^{-1})^2|\xi - \alpha(g_0^{-1})|^{-1}.$$

On the other hand, since ξ is rewritten as $A_{-\nu_j}(\alpha(Q(\nu_1, \cdots, \nu_{j-1}))) + Q_{-\nu_j}(o)$ for $A_{-\nu_j} \in O(n+1)$ and since $\alpha(Q(\nu_1, \cdots, \nu_{j-1})) \in I(g_0)$ by Lemma 6, it holds that

$$\begin{aligned} |\xi - \alpha(g_0^{-1})| &\leqq |A_{-\nu_j}(\alpha(Q(\nu_1, \cdots, \nu_{j-1})))| + |Q_{-\nu_j}(o)| + |\alpha(g_0^{-1})| \\ &\leqq \{|\alpha(g_0)| + \rho(g_0)\} + |Q_{-\nu_j}(o)| + |\alpha(g_0^{-1})|. \end{aligned}$$

Since $|Q_\nu(o)| > 1$, the last expression above is bounded by $c_2|Q_{-\nu_j}(o)|$ for some constant $c_2 > 0$. Hence, by (6), $|\xi - \alpha(g_0^{-1})| \leqq c_2 \sum_{i=1}^{l} |b_i(-m_1 n_{ij})| \leqq c_2 m_1 \sum_{i=1}^{l} |n_{ij}|\,|a_i|$ so that, together with (7), (8) and $\rho(g_0) = \rho(g_0^{-1})$, we have the desired inequality for the constant $\varepsilon_1 = \rho(g_0)/c_2 m_1(\max\{|a_i|: 1 \leqq i \leqq l\})$. q.e.d.

Let $\sum_\nu$ be the summation over $\nu \in (\boldsymbol{Z}^l)^*$ and let $\zeta(s) = \sum_{k=1}^{\infty} k^{-s}$.

LEMMA 8. *For any positive number s, it holds that*

$$\sum_{k=1}^{\infty} [\sum_{\nu_1} \cdots \sum_{\nu_k} \{\rho(Q(\nu_1, \cdots, \nu_k))\}^s] \geqq \rho(g_0)^s \sum_{k=1}^{\infty} \{\varepsilon_1^s l^{-s} \zeta(s - l + 1)\}^k.$$

PROOF. From Lemma 7 we have

$$\sum_{k=1}^{\infty}\Big[\sum_{\nu_1}\cdots\sum_{\nu_k}\{\rho(Q(\nu_1,\cdots,\nu_k))\}^s\Big] \geqq \rho(g_0)^s\sum_{k=1}^{\infty}\Big[\varepsilon_1^{ks}\Big\{\sum_{\nu_1}\cdots\sum_{\nu_k}\Big(\prod_{j=1}^{k}\Big(\sum_{i=1}^{l}|n_{ij}|\Big)^{-s}\Big)\Big\}\Big]$$
$$= \rho(g_0)^s\sum_{k=1}^{\infty}\Big\{\varepsilon_1^{ks}\prod_{j=1}^{k}\Big(\sum_{\nu}\Big(\sum_{i=1}^{l}|n_i|\Big)^{-s}\Big)\Big\} = \rho(g_0)^s\sum_{k=1}^{\infty}\Big\{\varepsilon_1^{s}\Big(\sum_{\nu}\Big(\sum_{i=1}^{l}|n_i|\Big)^{-s}\Big)\Big\}^k,$$

where $\nu = (n_1, \cdots, n_l) \in (\boldsymbol{Z}^l)^*$. By considering $X(k) = \{\nu = (n_1, \cdots, n_l) \in (\boldsymbol{Z}^l)^* : |n_i| \leqq k$ for $1 \leqq i \leqq l$ and $|n_i| = k$ for at least one $i\}$ for a natural number k, we obtain

$$\sum_{\nu}\Big(\sum_{i=1}^{l}|n_i|\Big)^{-s} = \sum_{k=1}^{\infty}\sum_{\nu\in X(k)}\Big(\sum_{i=1}^{l}|n_i|\Big)^{-s}$$
$$\geqq l^{-s}\sum_{k=1}^{\infty}(\#X(k))k^{-s} \geqq l^{-s}\sum_{k=1}^{\infty}(k^{l-1})k^{-s} = l^{-s}\zeta(s-l+1).$$

Thus we have our lemma. q.e.d.

5. Proof of the first half of the Theorem. Let $\hat{\Gamma} := \tau^{-1}\circ\hat{G}\circ\tau$ and let $\delta(\hat{\Gamma})$ be the exponent of convergence of $\sum(1-|\gamma(o)|)^{s/2}$, where the summation is taken over $\gamma\in\hat{\Gamma}$. Then $\delta(\Gamma) \geqq \delta(\hat{\Gamma})$ so that, to prove our theorem, it suffices to show that $\delta(\hat{\Gamma}) > l$.

LEMMA 9. *There exists a constant $\varepsilon_2 > 0$ such that $(1-|\gamma(o)|)^{1/2} \geqq \varepsilon_2\rho(\tau\circ\gamma\circ\tau^{-1})$ for all $\gamma\in\hat{\Gamma}$.*

PROOF. Let $\gamma\in\hat{\Gamma}$ and let $\gamma = \tau^{-1}\circ g\circ\tau$ for $g = g_0\circ Q_{\nu_1}\circ g_0\circ\cdots\circ g_0\circ Q_{\nu_k}\circ g_0 \in \hat{G}$. As in [1, p. 29, (43)], we have $1-|\gamma(o)|^2 = |(\gamma^{-1})'(o)|$ so that $(1-|\gamma(o)|^2)^{1/2} = \rho(\gamma^{-1})/|\alpha(\gamma^{-1})|$ by (1). Since $-e_{n+1}\in E(g_0)$ we see $g(-e_{n+1})\in I(g_0^{-1})$ and we have $|\alpha(\gamma^{-1})| = |\gamma(\infty)| = |\tau^{-1}(g(-e_{n+1}))| \leqq c_3$ for some constant $c_3 > 0$. Hence, using $1+|\gamma(o)| < 2$, we have $(1-|\gamma(o)|)^{1/2} \geqq \rho(\gamma^{-1})/\sqrt{2}\,c_3$.

Also since $-e_{n+1}\in E(g_0^{-1})$ we see $g^{-1}(-e_{n+1})\in I(g_0)$ and we have $\gamma^{-1}(\infty) = \tau^{-1}(g^{-1}(-e_{n+1})) \neq \infty$. Moreover, $g(\infty)\neq\infty$ by Lemma 6. Thus, applying Lemma 3, we have $\rho(\gamma^{-1}) = \rho(g^{-1})|\gamma(\infty)-\xi_0|/|e_{n+1}+g^{-1}(\infty)|$. Since $g^{-1}(\infty)\in I(g_0)$, we get $|g^{-1}(\infty)| \leqq |\alpha(g_0)| + \rho(g_0)$. On the other hand, the facts $\xi_0 = \tau^{-1}(\infty)$ and $\gamma(\infty) = \tau^{-1}(g(-e_{n+1}))\in\tau^{-1}(I(g_0^{-1}))$ imply $|\gamma(\infty)-\xi_0| \geqq c_4$ for some constant $c_4 > 0$. Hence $\rho(\gamma^{-1}) \geqq c_4\rho(g^{-1})/\{|\alpha(g_0)|+\rho(g_0)+1\}$. Thus, by $\rho(g) = \rho(g^{-1})$, we have our inequality for $\varepsilon_2 = c_4/\sqrt{2}\,c_3\{|\alpha(g_0)|+\rho(g_0)+1\}$. q.e.d.

Now Lemmas 6, 8 and 9 show

$$\sum_{\gamma\in\hat{\Gamma}}(1-|\gamma(o)|)^{s/2} \geqq \varepsilon_2^s\sum_{g\in\hat{G}}\{\rho(g)\}^s$$
$$\geqq \varepsilon_2^s\rho(g_0)^s\sum_{k=1}^{\infty}\{\varepsilon_1^s l^{-s}\zeta(s-l+1)\}^k.$$

Hence, if $s > \delta(\hat{\Gamma})$, then $\sum_{k=1}^{\infty} \{\varepsilon_1^s l^{-s} \zeta(s-l+1)\}^k < \infty$. Consequently we have

$$(9) \qquad \zeta(s-l+1) < (l/\varepsilon_1)^s$$

for all $s > \delta(\hat{\Gamma})$. On the other hand, if s tends to l then $\zeta(s-l+1)$ tends to ∞. Hence there exists a t_0 $(>l)$ such that $\zeta(t-l+1) \geqq (l/\varepsilon_1)^t$ for all t, $l < t \leqq t_0$. Therefore by (9), $\delta(\hat{\Gamma}) \geqq t_0 > l$.

6. A discontinuous group. We give an example which shows that the lower bound l in our theorem is the best possible. The construction of the following is similar to that in [2].

Let $\{e_i\}_{i=1}^{n+1}$ be the standard basis of $\boldsymbol{R}^{n+1}$ and let θ be a positive number with $\theta \geqq 3$. We define Möbius transformations $P_1, \cdots, P_l$ $(1 \leqq l \leqq n)$ and g_0 by $P_i(x) = x + \theta e_i$ $(1 \leqq i \leqq l)$ and $g_0(x) = {}^t(-x_1, \cdots, -x_n, x_{n+1})/|x|^2$ for $x = (x_1, \cdots, x_{n+1})$. Let $G(\theta)$ be the group generated by $\{P_1, \cdots, P_l, g_0\}$. Then, by the same argument as in [2], $G(\theta)$ is a discontinuous subgroup of $\mathrm{M\ddot{o}b}(\boldsymbol{H}^{n+1})$.

For $\nu = (n_1, \cdots, n_l) \in \boldsymbol{Z}^l$ we denote the element $P_1^{n_1} \circ \cdots \circ P_l^{n_l}$ by P_ν. Let

$$\hat{G}(\theta) = \bigcup_{k=1}^{\infty} \{g_0 \circ P_{\nu_1} \circ g_0 \circ \cdots \circ g_0 \circ P_{\nu_k} \circ g_0 \colon \nu_1, \cdots, \nu_k \in (\boldsymbol{Z}^l)^*\} .$$

Since $I(g_0) = I(g_0^{-1})$ and since $P_\nu(I(g_0)) \subset E(g_0)$ for $\nu \in (\boldsymbol{Z}^l)^*$, we see, by the same argument as in the proof of Lemma 6, that each element $g \in \hat{G}(\theta)$ is mutually distinct and satisfies $\alpha(g) \in I(g_0)$ and $g(\infty) \neq \infty$.

Since $g_0^2 = \mathrm{id}$, we have

$$G(\theta) = \{P_{\nu_1} \circ g_0 \circ P_{\nu_2} \circ g_0 \circ \cdots \circ P_{\nu_{k-1}} \circ g_0 \circ P_{\nu_k} \colon k \geqq 2,\ \nu_1, \cdots, \nu_k \in \boldsymbol{Z}^l\} ,$$

so that $\hat{G}(\theta) \cup \{g_0, \mathrm{id}\}$ is a complete system of representatives of the double coset space $G' \backslash G(\theta) / G'$, where $G' = \{P_\nu \colon \nu \in \boldsymbol{Z}^l\}$. If $g \in \hat{G}(\theta) \cup \{g_0\}$, then no element of the double coset $G'gG'$ fixes ∞. Hence $G_\infty(\theta) := \{g \in G(\theta) \colon g(\infty) = \infty\}$ is the same as G'.

Let $P(\nu_1, \cdots, \nu_i) = g_0 \circ P_{\nu_1} \circ g_0 \circ \cdots \circ g_0 \circ P_{\nu_i} \circ g_0 \in \hat{G}(\theta)$. Then

$$\begin{aligned} |\alpha(P(\nu_1, \cdots, \nu_i))| &= |g_0 \circ P_{-\nu_i}(\alpha(P(\nu_1, \cdots, \nu_{i-1})))| \\ &= |P_{-\nu_i}(\alpha(P(\nu_1, \cdots, \nu_{i-1})))|^{-1} \\ &= \left| \alpha(P(\nu_1, \cdots, \nu_{i-1})) - \theta \sum_{j=1}^{l} n_{ji} e_j \right|^{-1} \end{aligned}$$

for $\nu_i = (n_{1i}, \cdots, n_{li}) \in (\boldsymbol{Z}^l)^*$. Since $\alpha(P(\nu_1, \cdots, \nu_{i-1})) \in I(g_0)$, we have $|\alpha(P(\nu_1, \cdots, \nu_{i-1}))| < 1$. Therefore $|\alpha(P(\nu_1, \cdots, \nu_i))| \leqq \{\theta |\sum_{j=1}^{l} n_{ji} e_j|/2\}^{-1}$ by $\theta \geqq 3$ and $|\sum_{j=1}^{l} n_{ji} e_j| > 1$. Now, by Lemma 2, $\rho(P(\nu_1, \cdots, \nu_i)) = \rho(P(\nu_1, \cdots, \nu_{i-1})) |\alpha(P(\nu_1, \cdots, \nu_i))|$ so that, for the summation over $g \in \hat{G}(\theta)$, we have

$$\begin{aligned}\sum \{\rho(g)\}^s &= \sum_{j=1}^{\infty} [\sum_{\nu_1} \cdots \sum_{\nu_j} \{\rho(P(\nu_1, \cdots, \nu_j))\}^s] \\ &= \sum_{j=1}^{\infty} \left[\sum_{\nu_1} \cdots \sum_{\nu_j} \left\{\prod_{i=1}^{j} |\alpha(P(\nu_1, \cdots, \nu_i))|\right\}^s\right] \\ &\leqq \sum_{j=1}^{\infty} \left\{(\theta/2)^{-s} \sum_{\nu} \left(\left|\sum_{i=1}^{l} n_i e_i\right|^{-s}\right)\right\}^j\end{aligned}$$

where $\sum_\nu$ is the summation defined in § 4. Let $X(k)$ be the set in the proof of Lemma 8. Then $\{\sum_{i=1}^l n_i e_i : (n_1, \cdots, n_l) \in X(k)\}$ consists of lattice points in the l-dimensional Euclidean space $\boldsymbol{R}^l$ satisfying $|n_i| \leqq k$ $(1 \leqq i \leqq l)$ and $|n_i| = k$ for at least one $i \in \{1, \cdots, l\}$. Therefore $|\sum_{i=1}^l n_i e_i| \geqq k$ for all $(n_1, \cdots, n_l) \in X(k)$. Hence

$$\begin{aligned}\sum_{\nu} \left|\sum_{i=1}^{l} n_i e_i\right|^{-s} &\leqq \sum_{k=1}^{\infty} \sum_{\nu \in X(k)} k^{-s} = \sum_{k=1}^{\infty} (\#X(k))k^{-s} \\ &\leqq \sum_{k=1}^{\infty} \{2l(2k+1)^{l-1}\}k^{-s} \leqq l2^{2l-1}\zeta(s-l+1)\end{aligned}$$

so that we have, for the summation $\sum$ over $g \in \hat{G}(\theta)$,

$$\sum \{\rho(g)\}^s \leqq \sum_{j=1}^{\infty} \{l2^{s+2l-1}\theta^{-s}\zeta(s-l+1)\}^j . \tag{10}$$

Let s_0 be an arbitrary number with $s_0 > l$ and let θ_0 $(\geqq 3)$ be such a number that $\theta_0^{s_0} > l2^{s_0+2l-1}\zeta(s_0 - l + 1)$. Then the right hand side of (10) converges for $s = s_0$.

Let $h_0(x) = g_0(x + e_1)$. Applying Lemma 3 to $g \in G(\theta_0) \setminus G_\infty(\theta_0)$ and h_0, we have $\rho(h_0 \circ g \circ h_0^{-1}) = \rho(g)\{|g(\infty) + e_1| \, |g^{-1}(-e_1) + e_1|\}^{-1}$. Hence for an element $P_\mu \circ g \circ P_\nu$ of the double coset $G_\infty(\theta_0) g G_\infty(\theta_0)$ $(g \in \hat{G}(\theta_0) \cup \{g_0\})$,

$$\rho(h_0 \circ P_\mu \circ g \circ P_\nu \circ h_0^{-1}) = \rho(g)\{|g(\infty) + P_\mu(e_1)| \, |g^{-1} \circ P_{-\mu}(-e_1) + P_{-\nu}(e_1)|\}^{-1} ,$$

where we used $\rho(P_\mu \circ g \circ P_\nu) = \rho(g)$. Since $g \in \hat{G}(\theta_0) \cup \{g_0\}$, we see $g(\infty) \in I(g_0^{-1}) = I(g_0)$ and $g^{-1} \circ P_{-\mu}(-e_1) \in I(g_0)$. Furthermore, since $\theta_0 \geqq 3$, we have $|P_\nu(e_1)| \geqq 2$ for all $\nu \in (\boldsymbol{Z}^l)^*$. Hence $|P_\mu(e_1) + g(\infty)| \, |P_{-\nu}(e_1) + g^{-1} \circ P_{-\mu}(-e_1)| \geqq \{|P_\mu(e_1)|/2\}\{|P_{-\nu}(e_1)|/2\}$ so that $\rho(h_0 \circ P_\mu \circ g \circ P_\nu \circ h_0^{-1}) \leqq 4\rho(g)\{|P_\mu(e_1)| \, |P_{-\nu}(e_1)|\}^{-1}$ for all $\mu, \nu \in (\boldsymbol{Z}^l)^*$. Because of $s_0 > l$ we see $\sum_\nu |P_\nu(e_1)|^{-s_0} < \infty$. Therefore we have the following for some constant c_5:

$$\begin{aligned}&\sum \sum_{\mu} \sum_{\nu} \{\rho(h_0 \circ P_\mu \circ g \circ P_\nu \circ h_0^{-1})\}^{s_0} + \sum_{\mu} \sum_{\nu} \{\rho(h_0 \circ P_\mu \circ g_0 \circ P_\nu \circ h_0^{-1})\}^{s_0} \\ &\qquad \leqq c_5(\sum \{\rho(g)\}^{s_0} + \{\rho(g_0)\}^{s_0}) ,\end{aligned} \tag{11}$$

where $\sum$ means the summation over $g \in \hat{G}(\theta_0)$. On the other hand, by Lemma 2 and (1), $\rho(h_0 \circ P_\nu \circ h_0^{-1}) = |(h_0^{-1})'(\alpha(h_0 \circ P_\nu \circ h_0^{-1}))|^{-1/2} = |\alpha(h_0 \circ P_\nu \circ h_0^{-1}) - \alpha(h_0^{-1})| = |P_{-\nu}(o)|^{-1}$ so that

$$\sum_{\nu} \{\rho(h_0 \circ P_\nu \circ h_0^{-1})\}^{s_0} = \sum_{\nu} |P_\nu(o)|^{-s_0} < \infty \ . \tag{12}$$

Since $\hat{G}(\theta_0) \cup \{g_0, \mathrm{id}\}$ is a complete system of representatives for the double coset space $G_\infty(\theta_0)\backslash G(\theta_0)/G_\infty(\theta_0)$ and since $G_\infty(\theta_0) = \{P_\nu: \nu \in \boldsymbol{Z}^l\}$, the summation $\sum \{\rho(h_0 \circ g \circ h_0^{-1})\}^{s_0}$ over $g \in G(\theta_0) \setminus \{\mathrm{id}\}$ is equal to the sum on the left hand sides of (11) and (12). Hence it converges by the inequality (10).

7. Proof of the second half of the Theorem. We set $G_0 = h_0 \circ G(\theta_0) \circ h_0^{-1} \subset \mathrm{M\ddot{o}b}(\boldsymbol{H}^{n+1})$. Let τ be a Möbius transformation with $\tau(\boldsymbol{B}^{n+1}) = \boldsymbol{H}^{n+1}$ and $\tau(\infty) = -e_{n+1}$, and let $\Gamma_0 = \tau^{-1} \circ G_0 \circ \tau$. Then Γ_0 is a discontinuous subgroup of $\mathrm{M\ddot{o}b}(\boldsymbol{B}^{n+1})$ which satisfies the hypothesis in our Theorem for $\xi_0 = \tau^{-1} \circ h_0(\infty)$. Now, as in the proof of Lemma 9, $(1 - |\gamma(o)|)^{1/2} \leqq \rho(\gamma^{-1})/|\alpha(\gamma^{-1})| \leqq \rho(\gamma)$ for $\gamma \in \Gamma_0 \setminus \{\mathrm{id}\}$.

Let $g \in G(\theta_0)$. Then g is written as $g = P_\mu \circ g_1 \circ P_\nu$ for some $\mu, \nu \in \boldsymbol{Z}^l$ and $g_1 \in \hat{G}(\theta_0) \cup \{g_0, \mathrm{id}\}$ so that each element of $G(\theta_0) \setminus \{\mathrm{id}\}$ does not fix $-e_1$ and $-e_1 \pm e_{n+1}$.

Since $-e_1$ and $-e_1 - e_{n+1}$ are not fixed by $g \in G(\theta_0) \setminus \{\mathrm{id}\}$, each element, different from the identity, of G_0 and Γ_0 does not fix ∞. Hence Lemma 3 gives $\rho(\gamma) = \rho(g)|\alpha(\gamma) - \alpha(\tau)|/|\alpha(\tau^{-1}) - \alpha(g^{-1})|$ for $\gamma \in \Gamma_0 \setminus \{\mathrm{id}\}$ and $g = \tau \circ \gamma \circ \tau^{-1} \in G_0 \setminus \{\mathrm{id}\}$. Since $\alpha(g^{-1}) = g(\infty) \in \partial \boldsymbol{H}^{n+1}$ and since $\alpha(\tau^{-1}) = \tau(\infty) = -e_{n+1}$ we have $|\alpha(\tau^{-1}) - \alpha(g^{-1})| \geqq 1$. Therefore $\rho(\gamma) \leqq \rho(g)\{|\alpha(\gamma)| + 1\}$.

By the discontinuity of $G(\theta_0)$ and the fact $g(-e_1 + e_{n+1}) \neq -e_1 + e_{n+1}$ for $g \in G(\theta_0) \setminus \{\mathrm{id}\}$, there exists a constant $c_6 > 0$ such that $|g(-e_1 + e_{n+1}) - (-e_1 + e_{n+1})| \geqq c_6$ for all $g \in G(\theta_0) \setminus \{\mathrm{id}\}$. Therefore $|g(-e_1 - e_{n+1}) - (-e_1 - e_{n+1})| \geqq c_6$, since the left hand side is the same as $|g(-e_1 + e_{n+1}) - (-e_1 + e_{n+1})|$ by Lemma 1. Furthermore, $\tau^{-1} \circ h_0(-e_1 - e_{n+1}) = \tau^{-1}(-e_{n+1}) = \infty$ so that we have $|\tau^{-1} \circ h_0(g(-e_1 - e_{n+1}))| \leqq c_7$ for some constant c_7. Hence $|\alpha(\gamma)| = |\tau^{-1} \circ h_0(g(-e_1 - e_{n+1}))| \leqq c_7$ for $\gamma = \tau^{-1} \circ h_0 \circ g^{-1} \circ h_0^{-1} \circ \tau \in \Gamma_0 \setminus \{\mathrm{id}\}$.

Thus $(1 - |\gamma(o)|)^{1/2} \leqq \rho(\gamma) \leqq \rho(g)\{|\alpha(\gamma)| + 1\} \leqq \rho(g)(c_7 + 1)$ so that

$$\sum_{\gamma} (1 - |\gamma(o)|)^{s/2} \leqq (1 + c_7)^s \sum_{g} \{\rho(g)\}^s ,$$

where γ and g run over $\Gamma_0 \setminus \{\mathrm{id}\}$ and $G_0 \setminus \{\mathrm{id}\}$, respectively. As proved in § 6, the sum $\sum \{\rho(g)\}^{s_0}$ over $g \in G_0 \setminus \{\mathrm{id}\}$ converges so that the left hand side of the above inequality is finite for $s = s_0$. This implies $\delta(\Gamma_0) \leqq s_0$ and we have the second half of our Theorem.

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