

DECREASING NETS OF σ -ALGEBRAS AND THEIR APPLICATIONS TO ERGODIC THEORY

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Introduction. One of the important results in the classical ergodic theory is the following theorem of Rohlin and Sinai [10].

Let (X, \mathcal{B}, μ) be a Lebesgue probability space and let T be a measure preserving automorphism of it.

THEOREM A. *There exists a sub- σ -algebra $\mathcal{A} \subset \mathcal{B}$ such that*

- (a) $T^{-1}\mathcal{A} \subset \mathcal{A}$,
- (b) $\bigvee_{n=-\infty}^{+\infty} T^n\mathcal{A} = \mathcal{B}$,
- (c) $\bigcap_{n=-\infty}^{+\infty} T^n\mathcal{A} = \pi(T)$,
- (d) $h(T) = H(\mathcal{A} | T^{-1}\mathcal{A})$

where $\pi(T)$ and $h(T)$ denote the Pinsker σ -algebra and the entropy of T , respectively.

Every such σ -algebra is said to be perfect. Perfect σ -algebras have important applications to the investigations of mixing and spectral properties of automorphisms (cf. [9], [10]). Shimano [11], [12] investigated helices associated with a given perfect σ -algebra.

Theorem A has been generalized by the author in [3] as follows:

Let \mathbf{Z}^d denote the group of d -dimensional integers, o the null vector of \mathbf{Z}^d and \prec the lexicographical ordering of \mathbf{Z}^d , $d \geq 2$.

An ordered pair (A, B) of nonempty subsets of \mathbf{Z}^d is called a cut if $A \cup B = \mathbf{Z}^d$ and for every $g \in A$ and $h \in B$ it holds $g \prec h$.

A cut (A, B) is said to be a gap if A does not contain the greatest element and B does not contain the lowest element.

Let Φ be a \mathbf{Z}^d -action on (X, \mathcal{B}, μ) , i.e., Φ is a homomorphism of \mathbf{Z}^d into the group of all measure preserving automorphisms of (X, \mathcal{B}, μ) . We denote by Φ^g the automorphism of (X, \mathcal{B}, μ) which is the image of $g \in \mathbf{Z}^d$ under Φ .

The following result, formulated in [3] in terms of measurable partitions, is an analogue of Theorem A for \mathbf{Z}^d -actions.

THEOREM B. *There exists a sub- σ -algebra $\mathcal{A} \subset \mathcal{B}$ such that*

- (a₁) $\Phi^g \mathcal{A} \subset \mathcal{A} \quad \text{for } g \prec o,$
- (b₁) $\bigvee_{g \in \mathbb{Z}^d} \Phi^g \mathcal{A} = \mathcal{B},$
- (c₁) $\bigcap_{g \in \mathbb{Z}^d} \Phi^g \mathcal{A} = \pi(\Phi),$
- (d₁) $h(\Phi) = H(\mathcal{A} | \mathcal{A}^-) \quad \text{where } \mathcal{A}^- = \bigvee_{g \prec o} \Phi^g \mathcal{A},$
- (e₁) *for every gap (A, B) of \mathbb{Z}^d it holds*

$$\bigvee_{g \in A} \Phi^g \mathcal{A} = \bigcap_{g \in B} \Phi^g \mathcal{A},$$

where $\pi(\Phi)$ and $h(\Phi)$ denote the Pinsker σ -algebra and the entropy of Φ respectively.

Similarly as in the one-dimensional case \mathcal{A} is called a perfect σ -algebra of Φ . The essential difference between the one-dimensional and multidimensional concept of a perfect σ -algebra is contained in the condition (e₁) which one may call a continuity condition. It is shown in [5] that there exist σ -algebras satisfying (a₁)–(d₁) but not (e₁).

The paper [3] also contains applications of Theorem B. It would be interesting to know whether the results of Shimano have multidimensional analogues.

The definition of a perfect σ -algebra admits a more accessible form if we represent the considered action Φ by a d -tuple of natural automorphisms associated with Φ . For simplicity we will do this only in the case $d=2$.

Let T and S be automorphisms which are images under Φ of the vectors $(1, 0)$ and $(0, 1)$, respectively. Hence for $g = (m, n) \in \mathbb{Z}^2$ we have $\Phi^g = T^m \circ S^n$. Obviously, T and S commute. Then the conditions (a₁)–(e₁) may be written as follows:

- (a₂) $S^{-1} \mathcal{A} \subset \mathcal{A}, \quad T^{-1} \mathcal{A}_S \subset \mathcal{A},$
- (b₂) $\bigvee_{n=-\infty}^{+\infty} T^n \mathcal{A}_S = \mathcal{B},$
- (c₂) $\bigcap_{n=-\infty}^{+\infty} T^n \mathcal{A}_S = \pi(\Phi),$
- (d₂) $h(\Phi) = H(\mathcal{A} | S^{-1} \mathcal{A}),$
- (e₂) $\bigcap_{n=-\infty}^{+\infty} S^n \mathcal{A} = T^{-1} \mathcal{A}_S,$

where $\mathcal{A}_S = \bigvee_{n=-\infty}^{+\infty} S^n \mathcal{A}$.

Theorem B has been sharpened in [5] (see also [4]) in the following manner.

THEOREM C. *If $\mathcal{H} \subset \mathcal{B}$ is a sub- σ -algebra which is a factor of Φ , i.e., $\Phi^g \mathcal{H} = \mathcal{H}$, $g \in \mathbf{Z}^d$, then there exists a sub- σ -algebra $\mathcal{A} \supset \mathcal{H}$ satisfying (a₁), (b₁), (e₁) and*

$$(c_3) \quad \bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathcal{A} = \pi(\Phi | \mathcal{H}),$$

$$(d_3) \quad h(\Phi | \mathcal{H}) = H(\mathcal{A} | \mathcal{A}^-),$$

where $\pi(\Phi | \mathcal{H})$ and $h(\Phi | \mathcal{H})$ denote the relative Pinsker σ -algebra and the relative entropy of Φ with respect to \mathcal{H} (for the definitions see Section 2).

A sub- σ -algebra \mathcal{A} satisfying the properties given in Theorem C is said to be relatively perfect with respect to \mathcal{H} . It is clear that a relatively perfect σ -algebra with respect to the trivial σ -algebra is perfect.

Theorem C has been used in [5] (see also [4]) to give an axiomatic definition of the entropy of a \mathbf{Z}^d -action.

In this paper we use this theorem to show that the concept of a relative K -action given by Thouvenot [13] is an extension of the concept of a K -action in the sense of Kolmogorov. Using this fact, we prove that if \mathcal{H} is a factor having an independent complement \mathcal{H}^c such that the restriction of Φ to the space (X, \mathcal{H}^c, μ) is a K -action, then Φ is a relative K -action with respect to \mathcal{H} . We also show a formula for the direct product of relative Pinsker σ -algebras which implies that the product of relative K -actions is a relative K -action. This formula is an extension of that of Pollit [9] to \mathbf{Z}^d -actions. These results are obtained due to the property of the exchangeability of the order of taking suprema and intersections of nets of σ -algebras.

1. Decreasing nets of σ -algebras. Let (X, \mathcal{B}, μ) be a probability space, $\text{Sub } \mathcal{B}$ the family of all sub- σ -algebras of \mathcal{B} and $\mathcal{N} = \mathcal{N}(X)$ the trivial sub- σ -algebra. All equalities between sets, functions, transformations and σ -algebras are to be interpreted up to a set of measure zero. For $\mathcal{A} \in \text{Sub } \mathcal{B}$ we denote by $L^2(\mathcal{A})$ the subspace of $L^2(X, \mu)$ consisting of functions measurable with respect to \mathcal{A} . The conditional probability of a set $A \in \mathcal{B}$ with respect to \mathcal{A} is denoted by $\mu(A | \mathcal{A})$. For $f \in L^1(X, \mu)$ we put

$$Ef = \int_X f d\mu \quad \text{and} \quad \|f\| = E|f|.$$

Now, let P be a countable measurable partition of X and let \hat{P} be the sub- σ -algebra generated by P . We define the conditional entropy of P under \mathcal{A} as

$$H(P | \mathcal{A}) = E \left(- \sum_{A \in P} \mu(A | \mathcal{A}) \log \mu(A | \mathcal{A}) \right)$$

and the entropy of P as $H(P) = H(P | \mathcal{N})$.

If $\mathcal{C} \in \text{Sub } \mathcal{B}$, then we define the conditional entropy of \mathcal{C} under \mathcal{A} by the formula

$$H(\mathcal{C} | \mathcal{A}) = \sup H(P | \mathcal{A}),$$

where the supremum is taken over all countable measurable partitions P such that $\hat{P} \subset \mathcal{C}$ and $H(P) < \infty$.

It is easy to check that the last definition is equivalent to that of Jacobs [2].

If $\mathcal{A}_1, \mathcal{A}_2 \in \text{Sub } \mathcal{B}$, then the symbol $\mathcal{A}_1 \vee \mathcal{A}_2$ ($\mathcal{A}_1 \vee \mathcal{A}_2$) means the smallest algebra (σ -algebra) containing \mathcal{A}_1 and \mathcal{A}_2 .

In the sequel we use the following two elementary properties of the conditional probability.

Let $\mathcal{C} \in \text{Sub } \mathcal{B}$ be fixed.

(1) For every $\mathcal{A} \in \text{Sub } \mathcal{B}$ with $\mathcal{A} \supset \mathcal{C}$, $A \in \mathcal{B}$ and $C \in \mathcal{C}$, it holds

$$\mu(A \cap C | \mathcal{A}) = \mu(A | \mathcal{A}) \cdot \mathbf{1}_C.$$

(2) If $\mathcal{A}, \mathcal{D} \in \text{Sub } \mathcal{B}$ are such that $\mathcal{A} \vee \mathcal{D}$ and \mathcal{C} are independent, then for every $A \in \mathcal{A}$ it holds

$$\mu(A | \mathcal{C} \vee \mathcal{D}) = \mu(A | \mathcal{D}).$$

Let I be a countable set directed by an ordering relation $<$. A net $(\mathcal{B}_t)_{t \in I}$ ((\mathcal{B}_t) for short) in $\text{Sub } \mathcal{B}$ is said to be decreasing (resp. increasing) if $\mathcal{B}_s \supset \mathcal{B}_t$ (resp. $\mathcal{B}_s \subset \mathcal{B}_t$) for $s < t$.

Let (\mathcal{B}_t) be a decreasing net in $\text{Sub } \mathcal{B}$ and let $\mathcal{B}_t \supset \mathcal{C}$ for all $t \in I$. Proceeding in the same way as in the proof of Lemma 2 in [6] we have:

LEMMA 1. $\bigcap_{t \in I} \mathcal{B}_t = \mathcal{C}$ if and only if for every $B \in \bigvee_{t \in I} \mathcal{B}_t$ it holds

$$\limsup_{t \in I} \sup_{A \in \mathcal{B}_t} \|\mu(A \cap B | \mathcal{C}) - \mu(A | \mathcal{C}) \cdot \mu(B | \mathcal{C})\| = 0,$$

i.e., for any $\varepsilon > 0$ there exists $t_0 \in I$ such that

$$\|\mu(A \cap B | \mathcal{C}) - \mu(A | \mathcal{C}) \mu(B | \mathcal{C})\| < \varepsilon$$

for each $t < t_0$.

The following result is a sharpening of Theorem 2 in [6].

THEOREM 1. If (\mathcal{A}_t) is a decreasing net in $\text{Sub } \mathcal{B}$ such that $\bigvee_{t \in I} \mathcal{A}_t$ and \mathcal{C} are independent, then

$$\bigcap_{t \in I} (\mathcal{A}_t \vee \mathcal{C}) = \bigcap_{t \in I} \mathcal{A}_t \vee \mathcal{C}.$$

PROOF. We define

$$\mathcal{A}_\infty = \bigcap_{t \in I} \mathcal{A}_t, \quad \mathcal{C}_\infty = \mathcal{A}_\infty \vee \mathcal{C}$$

and

$$\mathcal{B}_t^\circ = \mathcal{A}_t \vee \mathcal{C}, \quad \mathcal{B}_t = \mathcal{A}_t \vee \mathcal{C}$$

for each $t \in I$. Let $s \in I$ be fixed. For any $B \in \mathcal{B}_s^\circ$, there exist sets F_1, \dots, F_q from \mathcal{A}_s and pairwise disjoint sets D_1, \dots, D_q from \mathcal{C} such that

$$B = \bigcup_{j=1}^q F_j \cap D_j.$$

First we shall prove that

$$(3) \quad \limsup_{t \in I} \sup_{A \in \mathcal{B}_t^\circ} \|\mu(A \cap B | \mathcal{C}_\infty) - \mu(A | \mathcal{C}_\infty) \cdot \mu(B | \mathcal{C}_\infty)\| = 0.$$

Let $t \in I$ be fixed. Similarly as above for any $A \in \mathcal{B}_t^\circ$ there exist sets E_1, \dots, E_p from \mathcal{A}_t and pairwise disjoint sets C_1, \dots, C_p from \mathcal{C} such that

$$A = \bigcup_{i=1}^p E_i \cap C_i.$$

It follows from (1), (2) and the independence assumption that

$$\begin{aligned} & \|\mu(A \cap B | \mathcal{C}_\infty) - \mu(A | \mathcal{C}_\infty) \cdot \mu(B | \mathcal{C}_\infty)\| \\ &= \left\| \sum_{i=1}^p \sum_{j=1}^q \{\mu(E_i \cap F_j \cap C_i \cap D_j | \mathcal{C}_\infty) - \mu(E_i \cap C_i | \mathcal{C}_\infty) \mu(F_j \cap D_j | \mathcal{C}_\infty)\} \right\| \\ &= \left\| \sum_{i=1}^p \sum_{j=1}^q \{\mu(E_i \cap F_j | \mathcal{A}_\infty) - \mu(E_i | \mathcal{A}_\infty) \cdot \mu(F_j | \mathcal{A}_\infty)\} \mathbf{1}_{C_i \cap D_j} \right\| \\ &\leq \sum_{i=1}^p \sum_{j=1}^q \|\mu(E_i \cap F_j | \mathcal{A}_\infty) - \mu(E_i | \mathcal{A}_\infty) \cdot \mu(F_j | \mathcal{A}_\infty)\| \mu(C_i \cap D_j) \\ &\leq \max_{1 \leq j \leq q} \sup_{E \in \mathcal{A}_t} \|\mu(E \cap F_j | \mathcal{A}_\infty) - \mu(E | \mathcal{A}_\infty) \cdot \mu(F_j | \mathcal{A}_\infty)\|. \end{aligned}$$

Therefore for every $B \in \mathcal{B}_s^\circ$ and $t \in I$ it holds

$$\begin{aligned} & \sup_{A \in \mathcal{B}_t^\circ} \|\mu(A \cap B | \mathcal{C}_\infty) - \mu(A | \mathcal{C}_\infty) \cdot \mu(B | \mathcal{C}_\infty)\| \\ &\leq \max_{1 \leq j \leq q} \sup_{E \in \mathcal{A}_t} \|\mu(E \cap F_j | \mathcal{A}_\infty) - \mu(E | \mathcal{A}_\infty) \cdot \mu(F_j | \mathcal{A}_\infty)\|. \end{aligned}$$

Hence using Lemma 1 we get (3). Now, let $B \in \bigvee_{t \in I} \mathcal{B}_t$ and let $\varepsilon > 0$ be arbitrary. Then there exists $s \in I$ and a set $B_\varepsilon \in \mathcal{B}_s^\circ$ such that $\mu(B \div B_\varepsilon) < \varepsilon/5$. It follows from (3) that there exists $t_0 \in I$ such that for $t < t_0$ and any $E \in \mathcal{B}_t^\circ$ it holds

$$(4) \quad \|\mu(E \cap B_\varepsilon | \mathcal{C}_\infty) - \mu(E | \mathcal{C}_\infty) \cdot \mu(B_\varepsilon | \mathcal{C}_\infty)\| < \varepsilon/5.$$

Let $t < t_0$ and let $A \in \mathcal{B}_t$. Then there exists $A_\varepsilon \in \mathcal{B}_t^\circ$ such that $\mu(A \div A_\varepsilon) < \varepsilon/5$. It follows from (4) and basic properties of the conditional probability that

$$\| \mu(A \cap B | \mathcal{C}_\infty) - \mu(A | \mathcal{C}_\infty) \cdot \mu(B | \mathcal{C}_\infty) \| \leq 2\mu(A \div A_\varepsilon) + 2\mu(B \div B_\varepsilon) + \varepsilon/5 < \varepsilon.$$

Thus for every $B \in \bigvee_{t \in I} \mathcal{B}_t$ it holds

$$\limsup_{t \in I} \sup_{A \in \mathcal{B}_t} \| \mu(A \cap B | \mathcal{C}_\infty) - \mu(A | \mathcal{C}_\infty) \cdot \mu(B | \mathcal{C}_\infty) \| = 0.$$

Using again Lemma 1 we obtain the desired result.

REMARK 1. It is worth noting that Weizsacker [14] characterized decreasing sequences (\mathcal{A}_n) in Sub \mathcal{B} for which it holds

$$\bigcap_{n=0}^\infty (\mathcal{A}_n \vee \mathcal{C}) = \bigcap_{n=0}^\infty \mathcal{A}_n \vee \mathcal{C}.$$

Now, let (X, \mathcal{B}, μ) and $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ be probability spaces and let $(X \times \tilde{X}, \mathcal{B} \otimes \tilde{\mathcal{B}}, \mu \times \tilde{\mu})$ be the direct product.

COROLLARY 1. If (\mathcal{A}_t) is a decreasing net in Sub \mathcal{B} and $\tilde{\mathcal{C}} \in \text{Sub } \tilde{\mathcal{B}}$, then

$$\bigcap_{t \in I} (\mathcal{A}_t \otimes \tilde{\mathcal{C}}) = \bigcap_{t \in I} \mathcal{A}_t \otimes \tilde{\mathcal{C}}.$$

PROOF. It is easy to see that the above equality is valid for $\tilde{\mathcal{C}} = \mathcal{N}(\tilde{X})$. Let now $\tilde{\mathcal{C}}$ be arbitrary. It follows from Theorem 1 that

$$\bigcap_{t \in I} (\mathcal{A}_t \otimes \tilde{\mathcal{C}}) = \bigcap_{t \in I} (\mathcal{A}_t \otimes \mathcal{N}(\tilde{X}) \vee \mathcal{N}(X) \otimes \tilde{\mathcal{C}}) = \bigcap_{t \in I} (\mathcal{A}_t \otimes \mathcal{N}(\tilde{X})) \vee (\mathcal{N}(X) \otimes \tilde{\mathcal{C}}) = \bigcap_{t \in I} \mathcal{A}_t \otimes \tilde{\mathcal{C}}.$$

REMARK 2. The results given in Corollary 1 and the following lemma are announced in [9] without proofs.

LEMMA 2. For arbitrary $\mathcal{A} \in \text{Sub } \mathcal{B}$ and $\tilde{\mathcal{A}} \in \text{Sub } \tilde{\mathcal{B}}$ it holds $(\mathcal{A} \otimes \tilde{\mathcal{B}}) \cap (\mathcal{B} \otimes \tilde{\mathcal{A}}) = \mathcal{A} \otimes \tilde{\mathcal{A}}$.

PROOF. It is enough to show that

$$L^2(\mathcal{A} \otimes \tilde{\mathcal{B}} \cap \mathcal{B} \otimes \tilde{\mathcal{A}}) \subset L^2(\mathcal{A} \otimes \tilde{\mathcal{A}}).$$

There exist orthonormal basis $(f_\alpha)_{\alpha \in I}$ in $L^2(X, \mu)$, $(g_\beta)_{\beta \in J}$ in $L^2(\tilde{X}, \tilde{\mu})$ and subsets $I_0 \subset I$, $J_0 \subset J$ such that $(f_\alpha)_{\alpha \in I_0}$ is an orthonormal basis in $L^2(\mathcal{A})$, $(g_\beta)_{\beta \in J_0}$ is an orthonormal basis in $L^2(\tilde{\mathcal{A}})$.

We put $h_{\alpha\beta} = f_\alpha \cdot g_\beta$, $(\alpha, \beta) \in I \times J$. It is clear that the sets $(h_{\alpha\beta}, \alpha \in I_0, \beta \in J)$, $(h_{\alpha\beta}, \alpha \in I, \beta \in J_0)$ and $(h_{\alpha\beta}, \alpha \in I_0, \beta \in J_0)$ are orthonormal basis in $L^2(\mathcal{A} \otimes \tilde{\mathcal{B}})$, $L^2(\mathcal{B} \otimes \tilde{\mathcal{A}})$ and $L^2(\mathcal{A} \otimes \tilde{\mathcal{A}})$, respectively.

Let $f \in L^2(\mathcal{A} \otimes \tilde{\mathcal{B}} \cap \mathcal{B} \otimes \tilde{\mathcal{A}})$ and let $c_{\alpha\beta}$ denote the Fourier coefficient of f with

respect to $h_{\alpha\beta}$, i.e.,

$$c_{\alpha\beta} = E(f \cdot \bar{h}_{\alpha\beta}), \quad (\alpha, \beta) \in I \times J.$$

Since $f \in L^2(\mathcal{A} \otimes \tilde{\mathcal{B}}) \cap L^2(\mathcal{B} \otimes \tilde{\mathcal{A}})$ we have $c_{\alpha\beta} = 0$ for $\alpha \notin I_0, \beta \in J$ or $\alpha \in I, \beta \notin J_0$, and f has the following Fourier expansions:

$$\sum c_{\alpha\beta} \cdot h_{\alpha\beta} = f = \sum c_{\alpha\beta} \cdot h_{\alpha\beta}$$

where the first sum is taken over all $(a, b) \in I_0 \times J$ and the second over all $(\alpha, \beta) \in I \times J_0$. Hence $c_{\alpha\beta} = 0$ for $(\alpha, \beta) \notin I_0 \times J_0$ and so $f = \sum c_{\alpha\beta} \cdot h_{\alpha\beta}$, where the sum is taken over all $(\alpha, \beta) \in I_0 \times J_0$. This means that $f \in L^2(\mathcal{A} \otimes \tilde{\mathcal{A}})$ and the lemma is proved.

COROLLARY 2. *If (\mathcal{A}_t) and $(\tilde{\mathcal{A}}_t)$ are decreasing nets in $\text{Sub } \mathcal{B}$ and $\text{Sub } \tilde{\mathcal{B}}$, respectively, then*

$$\bigcap_{t \in I} (\mathcal{A}_t \otimes \tilde{\mathcal{A}}_t) = \bigcap_{t \in I} \mathcal{A}_t \otimes \bigcap_{t \in I} \tilde{\mathcal{A}}_t.$$

PROOF. It follows from Corollary 1 that

$$\bigcap_{t \in I} (\mathcal{A}_t \otimes \tilde{\mathcal{A}}_t) \subset \bigcap_{t \in I} (\mathcal{A}_t \otimes \tilde{\mathcal{B}}) = \bigcap_{t \in I} \mathcal{A}_t \otimes \tilde{\mathcal{B}}$$

and similarly

$$\bigcap_{t \in I} (\mathcal{A}_t \otimes \tilde{\mathcal{A}}_t) \subset \mathcal{B} \otimes \bigcap_{t \in I} \tilde{\mathcal{A}}_t.$$

Hence using Lemma 2 we get

$$\bigcap_{t \in I} (\mathcal{A}_t \otimes \tilde{\mathcal{A}}_t) \subset \bigcap_{t \in I} \mathcal{A}_t \otimes \bigcap_{t \in I} \tilde{\mathcal{A}}_t.$$

Since the opposite inclusion is obvious, we get the desired result.

REMARK 3. It is not difficult to show that Corollary 2 is also valid for increasing nets of sub- σ -algebras.

We will also use in the sequel a property analogous to that given in Corollary 2, for increasing nets of σ -algebras.

REMARK 4. If (\mathcal{A}_t) and $(\tilde{\mathcal{A}}_t)$ are increasing nets in $\text{Sub } \mathcal{B}$ and $\text{Sub } \tilde{\mathcal{B}}$ respectively, then

$$\bigvee_{t \in I} (\mathcal{A}_t \otimes \tilde{\mathcal{A}}_t) = \bigvee_{t \in I} \mathcal{A}_t \otimes \bigvee_{t \in I} \tilde{\mathcal{A}}_t,$$

as can be easily proved.

2. Relative Kolmogorov Z^d -actions. Investigating measure preserving automor-

phisms of a Lebesgue space with the strong Pinsker property, Thouvenot has introduced in [13] an interesting class of factors of automorphisms—the so called entropy maximal factors. These factors have been also objects of investigations of Ornstein [8]. Lind used them in [7] for studying skew products on compact groups.

Now we recall the definition of these factors. Let T be an automorphism of a Lebesgue space (X, \mathcal{B}, μ) with $h(T) < \infty$.

A factor $H_T = \bigvee_{n=-\infty}^{+\infty} T^n H$ of T , where H is a finite partition of X , is said to be entropy maximal if for every finite partition P of X , the conditions

$$P_T \supset H_T \quad \text{and} \quad h(P, T) = h(H, T)$$

imply $P_T = H_T$.

Instead of saying that H_T is entropy maximal one says in [13] that T is a relative K -system with respect to H_T .

At first glance the concept of a relative K -system seems to have no connection with the traditional meaning of a Kolmogorov system (automorphism) for which there should exist some special exhaustive σ -algebras for the automorphism.

However, we will show that such a connection exists, not only for single automorphisms with finite entropy, but also for arbitrary \mathbb{Z}^d -actions. In order to do so we introduce some concepts concerning \mathbb{Z}^d -actions.

Let Φ be a \mathbb{Z}^d -action on a Lebesgue space (X, \mathcal{B}, μ) . Let $\mathcal{H} \in \text{Sub } \mathcal{B}$ be a factor of Φ . For a countable measurable partition P of X with $H(P) < \infty$ we put

$$h(P, \Phi | \mathcal{H}) = H(P | P^- \vee \mathcal{H}),$$

where $P^- = \hat{P}^-$.

For a given factor $\mathcal{C} \supset \mathcal{H}$ we define

$$h(\Phi / \mathcal{C} | \mathcal{H}) = \sup h(P, \Phi | \mathcal{H}),$$

where the supremum is taken over all partitions P with $\hat{P} \subset \mathcal{C}$ and $H(P) < \infty$.

By the relative entropy of Φ with respect to \mathcal{H} we mean $h(\Phi | \mathcal{H}) = h(\Phi / \mathcal{B} | \mathcal{H})$.

The smallest sub- σ -algebra containing all sub- σ -algebras \hat{P} , where P is a countable measurable partition such that $H(P) < \infty$ and $h(P, \Phi | \mathcal{H}) = 0$ is called the relative Pinsker σ -algebra with respect to \mathcal{H} and is denoted by $\pi(\Phi | \mathcal{H})$.

It is clear that the sub- σ -algebra $\pi(\Phi | \mathcal{H})$ is a factor of Φ with $h(\Phi / \pi(\Phi | \mathcal{H}) | \mathcal{H}) = 0$.

We shall use in the sequel the following two results.

LEMMA 3 (cf. [5]). *For every factor $\mathcal{C} \supset \mathcal{H}$ it holds*

$$h(\Phi / \mathcal{C}) = h(\Phi / \mathcal{H}) + h(\Phi / \mathcal{C} | \mathcal{H}).$$

DEFINITION 1. A σ -algebra $\mathcal{A} \in \text{Sub } \mathcal{B}$ is said to be exhaustive if it satisfies the properties (a₁), (b₁) and (e₁) of perfect σ -algebras.

LEMMA 4. *If $\mathcal{A} \supset \mathcal{H}$ is exhaustive, then*

$$\bigcap_{g \in \mathbb{Z}^d} \Phi^g \mathcal{A} \supset \pi(\Phi | \mathcal{H}).$$

If, in addition, $H(\mathcal{A} | \mathcal{A}^-) = h(\Phi | \mathcal{H}) < \infty$, then

$$\bigcap_{g \in \mathbb{Z}^d} \Phi^g \mathcal{A} = \pi(\Phi | \mathcal{H}).$$

PROOF. The first part of the lemma is proved in [5]. Now let us suppose $h(\Phi | \mathcal{H}) < \infty$. Let P, Q be countable measurable partitions of X with $H(P) < \infty$, $H(Q) < \infty$, $\hat{P} \subset \mathcal{A}$ and $\hat{Q} \subset \bigcap_{g \in \mathbb{Z}^d} \Phi^g \mathcal{A}$. The following equality is a relativized version of the Pinsker formula for \mathbb{Z}^d -actions.

$$(5) \quad h(P \vee Q, \Phi | \mathcal{H}) = h(Q, \Phi | \mathcal{H}) + H(P | P^- \vee Q_\Phi \vee \mathcal{H}),$$

where $Q_\Phi = \bigvee_{g \in \mathbb{Z}^d} \Phi^g Q$. The proof of (5) is analogous to that given in [1] in the case $\mathcal{H} = \mathcal{N}$. Using (5) and the inclusions $\hat{P} \subset \mathcal{A}$, $Q_\Phi \subset \mathcal{A}$, $\mathcal{H} \subset \mathcal{A}$ we get

$$\begin{aligned} h(\Phi | \mathcal{H}) &\geq h(P \vee Q, \Phi | \mathcal{H}) = h(Q, \Phi | \mathcal{H}) + H(P | P^- \vee Q_\Phi \vee \mathcal{H}) \\ &\geq h(Q, \Phi | \mathcal{H}) + H(P | \mathcal{A}^-). \end{aligned}$$

Hence

$$h(\Phi | \mathcal{H}) \geq h(Q, \Phi | \mathcal{H}) + H(\mathcal{A} | \mathcal{A}^-).$$

It follows from our assumption that $h(Q, \Phi | \mathcal{H}) = 0$, i.e., $\hat{Q} \subset \pi(\Phi | \mathcal{H})$. Thus we have shown the inclusion

$$\bigcap_{g \in \mathbb{Z}^d} \Phi^g \mathcal{A} \subset \pi(\Phi | \mathcal{H})$$

which completes the proof.

Now we formulate an extension of the definition of Thouvenot to \mathbb{Z}^d -actions in our notation.

DEFINITION 2. A \mathbb{Z}^d -action Φ is called a relative Kolmogorov action (K-action for short) with respect to a factor \mathcal{H} of Φ (or \mathcal{H} is entropy maximal) if for every factor $\mathcal{C} \supset \mathcal{H}$ with $h(\Phi | \mathcal{C} | \mathcal{H}) = 0$ it holds $\mathcal{C} = \mathcal{H}$.

It follows immediately from Lemma 3 that in the case $d = 1$ and $h(\Phi) < \infty$ our definition reduces to that of Thouvenot. It is clear that in the absolute case ($\mathcal{H} = \mathcal{N}$) it coincides with the definition of a \mathbb{Z}^d -action ($d \geq 1$) with completely positive entropy and therefore (see [10] for $d = 1$ and [3] for $d \geq 2$) with the definition of a K-action.

THEOREM 2. Φ is a relative K-action with respect to \mathcal{H} if and only if there exists an exhaustive sub- σ -algebra $\mathcal{A} \supset \mathcal{H}$ with $\bigcap_{g \in \mathbb{Z}^d} \Phi^g \mathcal{A} = \mathcal{H}$.

PROOF. Since $h(\Phi / \pi(\Phi | \mathcal{H}) | \mathcal{H}) = 0$ our assumption implies $\pi(\Phi | \mathcal{H}) = \mathcal{H}$. It fol-

lows from Theorem C that there exists an exhaustive sub- σ -algebra $\mathcal{A} \supset \mathcal{H}$ with

$$\bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathcal{A} = \pi(\Phi | \mathcal{H}).$$

Now, let us suppose $\mathcal{A} \supset \mathcal{H}$ is an exhaustive sub- σ -algebra with

$$\bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathcal{A} = \mathcal{H}.$$

It follows from Lemma 4 that

$$\bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathcal{A} \supset \pi(\Phi | \mathcal{H}),$$

i.e., $\pi(\Phi | \mathcal{H}) = \mathcal{H}$. If $\mathcal{C} \supset \mathcal{H}$ is a factor such that $h(\Phi | \mathcal{C} | \mathcal{H}) = 0$, then $\mathcal{C} \subset \pi(\Phi | \mathcal{H})$ and therefore $\mathcal{C} \subset \mathcal{H}$, i.e., $\mathcal{C} = \mathcal{H}$. Thus Φ is a relative K-action with respect to \mathcal{H} .

COROLLARY 1. *Every \mathbf{Z}^d -action Φ is a relative K-action with respect to the Pinsker σ -algebra $\pi(\Phi)$.*

It is enough to take as \mathcal{A} in Theorem 2 an arbitrary perfect σ -algebra of Φ .

It is shown in [3] that if $h(\Phi) = 0$ then \mathcal{B} is the only exhaustive σ -algebra. Therefore in this case there are no nontrivial factors with respect to which Φ is a relative K-action.

Now, let Φ be a \mathbf{Z}^d -action with $h(\Phi) > 0$.

COROLLARY 2. *If \mathcal{H} is a factor such that there exists a factor \mathcal{H}^c independent of \mathcal{H} , $\mathcal{H} \vee \mathcal{H}^c = \mathcal{B}$ and the action Φ restricted to the space (X, \mathcal{H}^c, μ) is a K-action, then Φ is a relative K-action with respect to \mathcal{H} .*

PROOF. It follows from Theorem 2 that there exists $\mathcal{C} \in \text{Sub } \mathcal{B}$ which is exhaustive in (X, \mathcal{H}^c, μ) and such that

$$(6) \quad \bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathcal{C} = \mathcal{N}.$$

Let $\mathcal{A} = \mathcal{C} \vee \mathcal{H}$. It is clear that

$$\Phi^g \mathcal{A} \subset \mathcal{A} \quad \text{for } g <_o \quad \text{and} \quad \bigvee_{g \in \mathbf{Z}^d} \Phi^g \mathcal{A} = \mathcal{B}.$$

Let a cut (A, B) of \mathbf{Z}^d be a gap. Applying Theorem 1 in the preceding section to the directed set $(B, <)$ and to the net $(\Phi^g \mathcal{A})_{g \in B}$ we get

$$\bigcap_{g \in B} \Phi^g \mathcal{A} = \bigcap_{g \in B} (\Phi^g \mathcal{C} \vee \mathcal{H}) = \bigcap_{g \in B} \Phi^g \mathcal{C} \vee \mathcal{H} = \bigvee_{g \in A} \Phi^g \mathcal{C} \vee \mathcal{H} = \bigvee_{g \in A} \Phi^g \mathcal{A}.$$

This means that \mathcal{A} is an exhaustive sub- σ -algebra. In the same way using (6), we get

$$\bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathcal{A} = \bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathcal{C} \vee \mathcal{H} = \mathcal{H} .$$

It follows from Theorem 2 that Φ is a relative K-action with respect to \mathcal{H} .

Now, let (X, \mathcal{B}, μ) , $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$ be Lebesgue probability spaces and let $\Phi, \tilde{\Phi}$ be \mathbf{Z}^d -actions on (X, \mathcal{B}, μ) and $(\tilde{X}, \tilde{\mathcal{B}}, \tilde{\mu})$, respectively. We denote by $\Phi \times \tilde{\Phi}$ the product action of Φ and $\tilde{\Phi}$, i.e.,

$$(\Phi \times \tilde{\Phi})^g(x, y) = (\Phi^g x, \tilde{\Phi}^g y), \quad (x, y) \in X \times \tilde{X}, \quad g \in \mathbf{Z}^d .$$

Our next aim is to show the following:

THEOREM 3. *For any factors \mathcal{H} and $\tilde{\mathcal{H}}$ of Φ and $\tilde{\Phi}$, respectively, it holds*

$$\pi(\Phi \times \tilde{\Phi} | \mathcal{H} \times \tilde{\mathcal{H}}) = \pi(\Phi | \mathcal{H}) \otimes \pi(\tilde{\Phi} | \tilde{\mathcal{H}}) .$$

PROOF. Let \mathcal{A} and $\tilde{\mathcal{A}}$ be relatively perfect sub- σ -algebras for Φ and $\tilde{\Phi}$ with respect to \mathcal{H} and $\tilde{\mathcal{H}}$, respectively. It is clear by Theorem C that

$$\mathcal{H} \otimes \tilde{\mathcal{H}} \subset (\Phi \times \tilde{\Phi})^g(\mathcal{A} \otimes \tilde{\mathcal{A}}) \subset \mathcal{A} \otimes \tilde{\mathcal{A}}, \quad g < o$$

and

$$\bigvee_{g \in \mathbf{Z}^d} (\Phi \times \tilde{\Phi})^g(\mathcal{A} \otimes \tilde{\mathcal{A}}) = \mathcal{B} \otimes \tilde{\mathcal{B}} .$$

Let (A, B) be a cut of \mathbf{Z}^d which is a gap. It follows from Corollary 2 to Theorem 1 and Remark 4 that

$$\begin{aligned} \bigcap_{g \in B} (\Phi \times \tilde{\Phi})^g(\mathcal{A} \otimes \tilde{\mathcal{A}}) &= \bigcap_{g \in B} (\Phi^g \mathcal{A} \otimes \tilde{\Phi}^g \tilde{\mathcal{A}}) = \bigcap_{g \in B} \Phi^g \mathcal{A} \otimes \bigcap_{g \in B} \tilde{\Phi}^g \tilde{\mathcal{A}} \\ &= \bigvee_{g \in A} \Phi^g \mathcal{A} \otimes \bigvee_{g \in A} \tilde{\Phi}^g \tilde{\mathcal{A}} = \bigvee_{g \in A} (\Phi \times \tilde{\Phi})^g(\mathcal{A} \otimes \tilde{\mathcal{A}}) . \end{aligned}$$

This means that $\mathcal{A} \otimes \tilde{\mathcal{A}}$ is exhaustive. Therefore using Lemma 4 and the fact that \mathcal{A} and $\tilde{\mathcal{A}}$ are relatively perfect, we get

$$\pi(\Phi \times \tilde{\Phi} | \mathcal{H} \otimes \tilde{\mathcal{H}}) \subset \bigcap_{g \in \mathbf{Z}^d} (\Phi \times \tilde{\Phi})^g(\mathcal{A} \otimes \tilde{\mathcal{A}}) = \bigcap_{g \in \mathbf{Z}^d} \Phi^g \mathcal{A} \otimes \bigcap_{g \in \mathbf{Z}^d} \tilde{\Phi}^g \tilde{\mathcal{A}} = \pi(\Phi | \mathcal{H}) \otimes \pi(\tilde{\Phi} | \tilde{\mathcal{H}}) .$$

In order to show the opposite inclusion let us suppose that $E \in \pi(\Phi | \mathcal{H})$ and $\tilde{E} \in \pi(\tilde{\Phi} | \tilde{\mathcal{H}})$. We consider the partitions P and \tilde{P} of X and \tilde{X} respectively such that

$$P = \{E, E^c\}, \quad \tilde{P} = \{\tilde{E}, \tilde{E}^c\}$$

where $E^c = X \setminus E$ and $\tilde{E}^c = \tilde{X} \setminus \tilde{E}$. It follows from the definition of the relative Pinsker σ -algebra that $h(P, \Phi | \mathcal{H}) = 0$ and $h(\tilde{P}, \tilde{\Phi} | \tilde{\mathcal{H}}) = 0$. It is easy to check that

$$h(P \times \tilde{P}, \Phi \times \tilde{\Phi} | \mathcal{H} \otimes \tilde{\mathcal{H}}) = h(P, \Phi | \mathcal{H}) + h(\tilde{P}, \tilde{\Phi} | \tilde{\mathcal{H}})$$

where $P \times \tilde{P}$ is the partition of $X \times \tilde{X}$ defined by

$$P \times \tilde{P} = \{E \times \tilde{E}, E^c \times \tilde{E}, E \times \tilde{E}^c, E^c \times \tilde{E}^c\}.$$

Therefore we have $h(P \times \tilde{P}, \Phi \times \tilde{\Phi} | \mathcal{H} \otimes \tilde{\mathcal{H}}) = 0$ which implies $A \times \tilde{A} \in \pi(\Phi \times \tilde{\Phi} | \mathcal{H} \otimes \tilde{\mathcal{H}})$. Hence, by the definition of a product σ -algebra, we get the desired inclusion.

COROLLARY. *If Φ and $\tilde{\Phi}$ are relative K -actions with respect to \mathcal{H} and $\tilde{\mathcal{H}}$, respectively, then $\Phi \times \tilde{\Phi}$ is a relative K -action with respect to $\mathcal{H} \otimes \tilde{\mathcal{H}}$.*

Assuming $\mathcal{H} = \mathcal{N}(X)$ and $\tilde{\mathcal{H}} = \mathcal{N}(\tilde{X})$ we get an extension of the Pollit formula (cf. [9]) to Z^d -actions.

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