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FIVE-DIMENSIONAL HOMOGENEOUS CONTACT MANIFOLDS AND RELATED PROBLEMS

DOMENICO PERRONE* and LIEVEN VANHECKE

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Abstract. We prove that a five-dimensional, compact, simply connected and homogeneous contact manifold is diffeomorphic to S^5 or $S^2 \times S^3$.

1. Introduction. Contact manifolds have been studied extensively. A first class of classical examples is provided by the tangent sphere bundles and a second class by the odd-dimensional spheres. As is well-known, Boothby and Wang extended this last class. They proved that every compact, simply connected, homogeneous contact manifold is a circle bundle over a homogeneous Hodge manifold and conversely, a compact Hodge manifold B has a contact manifold canonically associated with it as a circle bundle with B as a base space. Further, the contact structures on odd-dimensional spheres are not of the same type as those on tangent sphere bundles. One of the purposes of [6] was to study the circle bundles from the topological viewpoint in order to see when such manifolds were homeomorphic to tangent sphere bundles. In particular, the authors of [6] proved that a simply connected, compact and homogeneous contact manifold of dimension $4r+1$, $r>1$, is homeomorphic to the tangent sphere bundle of a manifold only when it is the Stiefel manifold $V_{2r+2,2}$.

Here we note that a contact manifold M is said to be homogeneous if there is a connected Lie group G acting transitively and effectively as a group of diffeomorphisms on M which leave the contact form invariant. In this context we note that it has been proved in [8] that the sphere is the only simply connected homogeneous contact manifold which can be equipped with an invariant contact metric of positive sectional curvature. Further, the first author proved in [13] that the sphere S^3 is the only compact simply connected three-dimensional manifold which admits a homogeneous contact structure.

In this note we concentrate on the five-dimensional case and complete the results of the first author [13], [14]. More specifically, we prove that any compact, simply connected, five-dimensional and homogeneous contact manifold is diffeomorphic to S^5 or $S^2 \times S^3$. This last manifold is, as is well-known, diffeomorphic to the Stiefel manifold $T^1(S^3)$. In addition, we also consider the non-simply connected case and further we prove two results about compact regular Sasakian manifolds. Finally, we give some new results about the three-dimensional case.

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2. Preliminaries. A contact manifold is a C^∞ $(2n+1)$-dimensional manifold M equipped with a global one-form η such that $\eta\wedge(d\eta)^n\neq 0$. It has an underlying *contact Riemannian structure* (also called a *contact metric structure*) (ξ, φ, η, g) where ξ is a vector field (called the characteristic field), φ a tensor field of type (1, 1) and g a Riemannian metric (called an *associated* metric). These structure tensors satisfy

$$\eta(\xi)=1\,,\qquad \varphi^2=-I+\eta\otimes\xi\,,\qquad \eta=g(\xi,\cdot)\,,\qquad d\eta=\phi\,,$$

where $\phi(X, Y)=g(X, \varphi Y)$ for all tangent vector fields X, Y.

If the almost complex structure J on $M\times\boldsymbol{R}$, defined by

$$J\left(X, f\frac{d}{dt}\right)=\left(\varphi X-f\xi, \eta(X)\frac{d}{dt}\right),$$

where f is a real-valued function, is integrable, then the contact structure is said to be *normal*. A normal contact Riemannian structure is called a *Sasakian structure*. Moreover, a Sasakian manifold whose Ricci tensor ρ has the form

$$\rho=ag+b\eta\otimes\eta$$

where $a=\tau/2n-1$ and $b=-\tau/2n+2n+1$ are constants (τ being the scalar curvature) is said to be an *η-Einstein manifold.* So, an η-Einstein manifold is an Einstein space if and only if $\tau=2n(2n+1)$.

A contact manifold M is said to be *regular* if any of its points has a cubical neighborhood such that each integral curve of the characteristic field ξ which passes through it does so only once. If M is *compact* and *regular* it is a principal circle bundle over a symplectic manifold B whose fundamental two-form has integral periods (a Hodge manifold). The corresponding fibration $\pi: M\rightarrow B$ is known as the *Boothby-Wang fibration* [6].

Finally, a Sasakian manifold is said to be a *homogeneous Sasakian manifold* if the structure tensors are invariant under the group of isometries acting transitively on the manifold. Further, a Sasakian manifold is said to be a *locally φ-symmetric space* [15] if and only if each Kähler manifold which is a base space of a *local* fibering, is a Hermitian locally symmetric space. A compact and simply connected locally φ-symmetric space is a globally φ-symmetric space and in this case it is a principal S^1-bundle over a Hermitian symmetric space.

We refer to [4], [20] for more details about contact geometry.

3. Homogeneous contact manifolds. We start with the main result of this note. All manifolds are supposed to be connected.

THEOREM 1. *Let (M, η) be a compact, simply connected, five-dimensional and homogeneous contact manifold. Then M is diffeomorphic to S^5 or $S^2\times S^3$. In both cases it is a globally φ-symmetric space with respect to the underlying invariant Sasakian* *structure.*

PROOF. Since (M, η) is homogeneous, it follows from [6] (see also [7]) that the contact structure is regular. Further, the base space B of the Boothby-Wang fibration $\pi: M \to B$ is a compact simply connected homogeneous Kähler manifold of complex dimension two. Moreover, η defines a connection on M whose curvature form is $\pi^*\Omega = d\eta$, where Ω is the fundamental two-form of the Kähler manifold B. The underlying homogeneous Sasakian structure (ξ, φ, η, g) on M is determined by

$$g = \pi^*h + \eta \otimes \eta ,$$

where h is the Kähler metric of B.

Next, it follows from the explicit classification given in [2], [10] that any compact, simply connected, homogeneous, four-dimensional Kähler manifold is symmetric and hence, B is either $CP^2(\lambda)$ or $CP^1(\lambda_1) \times CP^1(\lambda_2)$, where the complex projective spaces $CP^n(\lambda)$ are endowed with the Fubini-Study metric. This implies that $(M, \xi, \varphi, \eta, g)$ is a globally φ-symmetric space.

When $B = CP^2(\lambda)$, we get at once from the result in [13], [14] that M is diffeomorphic to S^5.

In the other case, we have for $[\Omega]$ in the cohomology group $H^2(B, Z)$ that

$$[\Omega] = k[\pi_1^*\Omega_1] + l[\pi_2^*\Omega_2]$$

where π_i denotes the projection $CP^1(\lambda_1) \times CP^1(\lambda_2) \to CP^1(\lambda_i)$, $i = 1, 2$. Ω_i is a harmonic two-form on $CP^1(\lambda_i)$ such that $[\Omega_i] \in H^2(CP^1, Z)$ and k, l are integers. Hence, M is a $P_{k,l}$ [17], [3, p. 471] and since M is simply connected, k and l are relatively prime. Moreover, $P_{k,l}$ is diffeomorphic to $S^3 \times S^3/S^1$, which is diffeomorphic to $S^2 \times S^3$ [17], [1].

This completes the proof of the theorem.

REMARKS. 1. It is well-known that $S^2 \times S^3$ is diffeomorphic to the Stiefel manifold $T^1(S^3)$.

2. A classification of simply connected and complete, five-dimensional globally φ-symmetric spaces is given in [11]. Kowalski communicated to the authors that a part of Theorem 1 may be derived from that classification.

3. For the construction of examples of φ-symmetric structures on $S^2 \times S^3$ we refer to [19] and for an Einstein metric on $S^2 \times S^3$ see [16].

THEOREM 2. *Let (M, η) be a compact, five-dimensional and homogeneous contact manifold. Then M is covered by S^5 or $S^2 \times S^3$ with α leaves where $\alpha = \operatorname{card} \pi_1(M)$. Moreover, M is locally φ-symmetric with respect to the underlying invariant Sasakian structure.*

PROOF. Since (M, η) is a homogeneous contact manifold, it follows from [7] that M is a homogeneous space for a transitive compact semi-simple Lie group G and moreover, it is the total space of a principal circle bundle over a simply connected compact homogeneous Hodge manifold B. According to [6], B is a simply connected compact homogeneous Kähler manifold. Since B has real dimension four, we may conclude as in Theorem 1 that B is Hermitian symmetric. Consequently, M is a locally φ-symmetric space with respect to the underlying homogeneous Sasakian structure. Next, following the proof of Theorem 6.3 in [15], we conclude that the universal covering space $\tilde{M}$ of $M=G/K$, K being the isotropy subgroup of a point $p_0 \in M$, is a globally φ-symmetric space. Finally, the fundamental group $\pi_1(M)$ is finite abelian (see [7, p. 348]) and therefore $\tilde{M}$ is a covering with α leaves. Since M is compact, $\tilde{M}$ is also compact. Hence, $\tilde{M}$ is a compact, globally φ-symmetric space and then the result follows from Theorem 1.

Next, we derive some additional results.

THEOREM 3. *Let M be a compact regular Sasakian manifold with constant scalar curvature and non-negative sectional curvature. Then M is a locally φ-symmetric space. When M is in addition simply connected, then it is globally φ-symmetric.*

PROOF. Let (B, h) denote the base space of the Boothby-Wang fibration of the Sasakian manifold. The sectional curvatures of (M, g) and (B, h) are related by

$$K(X^*, Y^*)=K(X, Y)\circ\pi-3\{\phi(X^*, Y^*)\}^2,$$

where X^*, Y^* are the horizontal lifts of X, Y (see for example [12], [14]). Hence, the sectional curvatures of (B, h) are non-negative. Moreover, the scalar curvatures are related by

$$\tau(g)=\tau(h)-4$$

and so, $\tau(h)$ is constant. Then Theorem 1.1 of [9] implies that (B, h) is locally symmetric and so (M, g) is locally φ-symmetric. When M is simply connected, it is globally φ-symmetric.

As corollaries we get:

COROLLARY 4. *Let M be a compact regular η-Einstein manifold of dimension ≥ 5 with non-negative sectional curvature. Then M is locally φ-symmetric.*

COROLLARY 5. *A compact, simply connected, regular Sasakian manifold of dimension five with constant scalar curvature and non-negative sectional curature is diffeomorphic to S^5 or $S^2 \times S^3$.*

PROOF. This follows at once from Theorem 1 and Theorem 3.

This corollary extends, in the five-dimensional case, Theorem 1 of [8].

We note that it has been proved in [18] that any five-dimensional, compact, Sasakian Einstein space with non-negative sectional curvature is locally φ-symmetric.

In the simply connected case it is globally φ-symmetric and then Theorem 1 implies that it must be diffeomorphic to S^5 or $S^2 \times S^3$.

We finish this note with some remarks on *three-dimensional contact manifolds*.

1. Let (M, η) be a three-dimensional, compact, homogeneous contact manifold. In a way similar to that in Theorem 2 and using Remark 3.1 of [13] we get that M is covered by S^3 with α leaves, where α is again the number of elements in $\pi_1(M)$. Moreover, M is locally φ-symmetric with respect to the underlying invariant Sasakian structure.

2. Let (M, η, g) be a three-dimensional homogeneous Sasakian manifold. Then it is complete and has constant scalar curvature. Hence, it is locally φ-symmetric [18] and following Theorem 6.3 of [15] it is locally isomorphic to a globally φ-symmetric space. Then, (M, η, g) is locally isometric to one of the spaces given in Theorem 11 of [5].

Note that a compact, three-dimensional, homogeneous contact manifold admits a homogeneous Sasakian structure [6] but, if it is non-compact, we do not know if it admits such a structure.

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UNIVERSITÀ DEGLI STUDI DI LECCE
DIPARTIMENTO DI MATEMATICA
VIA ARNESANO
73100 LECCE
ITALY

KATHOLIEKE UNIVERSITEIT LEUVEN
DEPARTMENT OF MATHEMATICS
CELESTIJNENLAAN 200B
B-3001 LEUVEN
BELGIUM