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A KIND OF ASYMPTOTIC EXPANSION USING PARTITIONS

LEETSCH C. HSU AND WENCHANG CHU

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Abstract. It is shown that a certain algebraic identity involving a summation over partitions can be utilized to obtain a class of asymptotic expansions for large parameters. A number of special formulas related to some well-known number sequences and classical polynomials are presented as illustrative examples.

1. Introduction—Statement of problem. Throughout we will make use of formal power series (over the real or complex field) for which formal manipulations with ordinary addition, multiplication and substitution as well as formal differentiation are defined in the usual way. Logarithms and powers (with complex exponents) of a power series having a positive constant term are also defined formally. See Comtet's "Advanced Combinatorics" [4, §1.12, §3.5].

As usual, $\boldsymbol{Z}_+$ and $\boldsymbol{Z}_0$ denote respectively the sets of positive integers and of non-negative integers, and $\boldsymbol{C}$ the field of complex numbers.

Let $\phi(t)=\sum_{n\geqslant 0} a_n t^n$ be a formal power series with $a_0=\phi(0)=1$. Suppose that for any $\alpha\in\boldsymbol{C}$ with $\alpha\neq 0$ we have a formal power series expansion of the form

$$(1.1)\qquad (\phi(t))^\alpha=\sum_{n\geqslant 0}\left\{\begin{matrix}\alpha\\ n\end{matrix}\right\}t^n\,,\qquad \left\{\begin{matrix}\alpha\\ 0\end{matrix}\right\}=1\,,$$

where the Taylor coefficient $\left\{\begin{matrix}\alpha\\ n\end{matrix}\right\}$, written in contrast with the notation for the ordinary binomial coefficient $\binom{\alpha}{n}$ just for convenience, may be determined by use of formal differentiation, namely

$$(1.2)\qquad \left\{\begin{matrix}\alpha\\ n\end{matrix}\right\}=\frac{1}{n!}D_t^n(\phi(t))^\alpha\bigg|_{t=0}\,,\qquad D\equiv d/dt\,.$$

Certainly, $\left\{\begin{matrix}\alpha\\ n\end{matrix}\right\}$ may also be expressed using the notation of Bell polynomials.

Let $\lambda\in\boldsymbol{C}$. Our main problem concerned is how to determine an asymptotic expansion of the function $\left\{\begin{matrix}\lambda\alpha\\ n\end{matrix}\right\}$ in terms of λ as $|\lambda|\to\infty$. As a number of well-known special

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polynomials and sequences of numbers may be generated by expansions of the type (1.1), it may be expected that a general asymptotic formula for $\left\{ {\lambda\alpha \atop n} \right\}$ should be applicable to several special cases, so that some special approximation formulas of some interest may be derived.

2. An algebraic identity and its consequences. Let $\sigma(n)$ denote the set of partitions of n ($n \in \boldsymbol{Z}_+$), usually represented by $1^{k_1}2^{k_2}\cdots n^{k_n}$ with $k_1+2k_2+\cdots+nk_n=n$, k_i being the number of parts of size i. Moreover, we denote $\bar{k}:=(k_1, k_2, \cdots, k_n)$, $k=k_1+k_2+\cdots+k_n$, and $(\lambda)_k:=\lambda(\lambda-1)\cdots(\lambda-k+1)$ with $(\lambda)_0=1$ for $\lambda \in \boldsymbol{C}$, so that the multinomial coefficient $\binom{\lambda}{\bar{k}}$ may be defined by

$$\binom{\lambda}{\bar{k}} = (\lambda)_k \Big/ \prod_{i=1}^{n} k_i! \,. \tag{2.1}$$

Using the notation defined above we may now state and prove a propostion involving a useful algebraic identity (cf. [7]):

PROPOSITION. *Let* (1.1) *be given. Then for every* $\lambda \in \boldsymbol{C}$ *with* $\lambda \neq 0$ *we have*

$$\left\{ {\lambda\alpha \atop n} \right\} = \sum_{\sigma(n)} \binom{\lambda}{\bar{k}} \prod_{i=1}^{n} \left\{ {\alpha \atop i} \right\}^{k_i}, \tag{2.2}$$

where the summation is taken over all the partitions of n.

PROOF. Denote $G(t)=(\phi(t))^\alpha$, and in accordance with (1.1) we have

$$(G(t))^\lambda = \sum_{n \geq 0} \left\{ {\lambda\alpha \atop n} \right\} t^n \,. \tag{2.3}$$

Let us define $f(u)=u^\lambda$ and $u=G(t)$, so that the Taylor coefficient $\left\{ {\lambda\alpha \atop n} \right\}$ may be determined by Faa di Bruno's formula for the higher derivatives of composite functions (cf. [4, §3.4]). Noticing that $G(0)=1$ and using (1.2) we find

$$f_k = [D_u^k f(u)]_{u=1} = (\lambda)_k \,, \qquad G_i = [D_t^i G(t)]_{t=0} = \left\{ {\alpha \atop i} \right\} i! \,.$$

Thus it follows that

$$\left\{ {\lambda\alpha \atop n} \right\} = \frac{1}{n!}[D_t^n f \circ G(t)]_{t=0} = \sum_{\sigma(n)} \frac{(\lambda)_k}{k_1! \cdots k_n!} \prod_{i=1}^{n} \left(\frac{G_i}{i!} \right)^{k_i} = \sum_{\sigma(n)} \binom{\lambda}{\bar{k}} \prod_{i=1}^{n} \left\{ {\alpha \atop i} \right\}^{k_i} .$$

This is what we desired.

Consequences of (2.2). (i) For the case $\phi(t)=1+t$ we have the binomial identity due to Chu [3]

$$\binom{xy}{n}=\sum_{\sigma(n)}\binom{x}{k}\prod_{i=1}^{n}\binom{y}{i}^{k_i}, \tag{2.4}$$

where $(x, y)\in \boldsymbol{C}^2$. It was observed by Chu that various special identities and binomial formulae due to Riordan, Carlitz, and Narayana, respectively, are all deducible from (2.4) as corollaries (cf. [3]).

(ii) For Pascal—T-triangle numbers $C_m(a, n)$ defined by

$$\phi^a(t):=(1+t+\cdots+t^{m-1})^a=\sum_{n\geqslant 0} C_m(a, n)t^n, \qquad (m\geqslant 2)$$

we have

$$C_m(ab, n)=\sum_{\sigma(n)}\binom{a}{k}\prod_{i=1}^{n} C_m^{k_i}(b, i), \tag{2.5}$$

where $(a, b)\in \boldsymbol{Z}_+^2$.

(iii) For the generalized Bernoulli numbers and Euler numbers defined respectively by the following (cf. [9])

$$\left(\frac{t}{e^t-1}\right)^m=\sum_{n\geqslant 0} B_n^{(m)}\frac{t^n}{n!}, \qquad \left(\frac{2e^t}{e^{2t}+1}\right)^m=\sum_{n\geqslant 0} E_n^{(m)}\frac{t^n}{n!},$$

we have the following identities

$$\frac{1}{n!}B_n^{(\alpha\beta)}=\sum_{\sigma(n)}\binom{\alpha}{k}\prod_{i=1}^{n}\left(\frac{1}{i!}B_i^{(\beta)}\right)^{k_i}, \tag{2.6}$$

$$\frac{1}{n!}E_n^{(\alpha\beta)}=\sum_{\sigma(n)}\binom{\alpha}{k}\prod_{i=1}^{n}\left(\frac{1}{i!}E_i^{(\beta)}\right)^{k_i}. \tag{2.7}$$

(iv) For the classical Gegenbauer-Humbert polynomials defined by (cf. [1], [6])

$$(1-mzt+t^m)^{-\lambda}=\sum_{n\geqslant 0} C_n^{(\lambda)}(z)t^n,$$

where $m=2, 3, \cdots$, and $\lambda\neq 0$ we have the identity

$$C_n^{(\alpha\beta)}(z)=\sum_{\sigma(n)}\binom{\alpha}{k}\prod_{i=1}^{n}(C_i^{(\beta)}(z))^{k_i}, \tag{2.8}$$

α and β being real numbers different from zero.

(v) For Lerch polynomials defined by (cf. [1])

$$\{1-z\log(1+t)\}^{-\lambda}=\sum_{n\geqslant 0} p_n^{(\lambda)}(z)t^n$$

we have

$$p_n^{(\alpha\beta)}(z)=\sum_{\sigma(n)}\binom{\alpha}{\bar{k}}\prod_{i=1}^{n}(p_i^{(\beta)}(z))^{k_i}. \tag{2.9}$$

(vi) It is familiar that the two kinds of Stirling numbers $S_1(n, k)$ and $S_2(n, k)$ can be defined by

$$\frac{(\log(1+t))^k}{k!}=\sum_{n\geqslant k}S_1(n,k)\frac{t^n}{n!}\quad\text{and}\quad\frac{(e^t-1)^k}{k!}=\sum_{n\geqslant k}S_2(n,k)\frac{t^n}{n!}.$$

In order to apply (2.2) we have to consider expansions instead

$$\left\{\frac{\log(1+t)}{t}\right\}^k=k!\sum_{n\geqslant 0}\frac{S_1(n+k,k)}{(n+k)!}t^n,\qquad\left\{\frac{e^t-1}{t}\right\}^k=k!\sum_{n\geqslant 0}\frac{S_2(n+k,k)}{(n+k)!}t^n$$

so that $\phi(t)=\log(1+t)/t$ and $\psi(t)=(e^t-1)/t$ satisfy the required condition $\phi(0)=\psi(0)=1$. Consequently, we obtain a pair of partition identities as follows

$$\frac{S_r(\alpha\beta+n,\alpha\beta)}{(\alpha\beta+n)_n}=\sum_{\sigma(n)}\binom{\alpha}{\bar{k}}\prod_{i=1}^{n}\left\{\frac{S_r(\beta+i,\beta)}{(\beta+i)_i}\right\}^{k_i}, \tag{2.10}$$

where $r=1, 2$ and $(\alpha,\beta)\in \boldsymbol{Z}_+^2$.

Formulas (2.5)–(2.7) and (2.10) may be used to investigate some divisibility properties of these special numbers involved.

3. A general asymptotic expansion formula. We will find an asymptotic formula, by the aid of (2.2), for the Taylor coefficient $\left\{{\lambda\alpha \atop n}\right\}$ in which λ becomes large. Denote by $\sigma(n, k)$ the subset of $\sigma(n)$ consisting of the partitions of n with k parts. Then (2.2) may be rewritten in the form

$$\left\{{\lambda\alpha \atop n}\right\}=\sum_{k=1}^{n}\binom{\lambda}{k}\sum_{\sigma(n,k)}\binom{k}{\bar{k}}\prod_{i=1}^{n}\left\{{\alpha \atop i}\right\}^{k_i}. \tag{3.1}$$

Corresponding to $k=n, n-1, \cdots$, successively, a short table for the partitions of n may be displayed as in the Table. Accordingly, for $n\geqslant 2$ we have a two-term asymptotic expansion

$$\left\{{\lambda\alpha \atop n}\right\}=\frac{(\lambda)_n}{n!}\left\{{\alpha \atop 1}\right\}^n+\frac{(\lambda)_{n-1}}{(n-2)!}\left\{{\alpha \atop 1}\right\}^{n-2}\left\{{\alpha \atop 2}\right\}+O((\lambda)_{n-2}).$$

In order to attain a general asymptotic expansion, let us denote for every j $(0\leqslant j<n)$,

$$W(\alpha,n,j)=\sum_{\sigma(n,n-j)}\prod_{i=1}^{n}\frac{1}{k_i!}\left\{{\alpha \atop i}\right\}^{k_i}, \tag{3.2}$$

TABLE

k	k_1	k_2	k_3	k_4	$\cdot$	k_n
n	n	0	0	0	$\cdot\cdot$	0
$n-1$	$n-2$	1	0	0	$\cdot\cdot$	0
$n-2$	$n-3$	0	1	0	$\cdots$	0
$n-2$	$n-4$	2	0	0	$\cdots$	0
$n-3$	$n-4$	0	0	1	$\cdot$	0
$n-3$	$n-5$	1	1	0	$\cdot\cdot$	0
$n-3$	$n-6$	3	0	0	$\cdot\cdot$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$

where the summation runs over all the partitions $\bar{k}:=(k_1, k_2, \cdots, k_n)$ of n such that $k_1+2k_2+\cdots+nk_n=n$ with $k_1+k_2+\cdots+k_n=n-j$. Then, by reformulating the right-hand-side of (3.1) in accordance with $k=n, n-1, n-2, \cdots$, we get a complete asymptotic expansion of $\left\{\begin{matrix}\lambda\alpha\\ n\end{matrix}\right\}$ which may be stated by the following:

THEOREM. *For every given $m<n$ we have an asymptotic expansion of the form*

$$\frac{1}{(\lambda)_n}\left\{\begin{matrix}\lambda\alpha\\ n\end{matrix}\right\}=\sum_{j=0}^{m}\frac{W(\alpha, n, j)}{(\lambda-n+j)_j}+O\left(\left(\frac{1}{\lambda}\right)^{m+1}\right). \tag{3.3}$$

Usually, when applying (3.3) to specific problems, it is required to compute $W(\alpha, n, j)$ explicitly. In particular, we have, according to the table given previously,

$$W(\alpha, n, 0)=\frac{1}{n!}\left\{\begin{matrix}\alpha\\ 1\end{matrix}\right\}^n, \qquad W(\alpha, n, 1)=\frac{1}{(n-2)!}\left\{\begin{matrix}\alpha\\ 1\end{matrix}\right\}^{n-2}\left\{\begin{matrix}\alpha\\ 2\end{matrix}\right\},$$

$$W(\alpha, n, 2)=\frac{1}{(n-3)!}\left\{\begin{matrix}\alpha\\ 1\end{matrix}\right\}^{n-3}\left\{\begin{matrix}\alpha\\ 3\end{matrix}\right\}+\frac{1}{2!\cdot(n-4)!}\left\{\begin{matrix}\alpha\\ 1\end{matrix}\right\}^{n-4}\left\{\begin{matrix}\alpha\\ 2\end{matrix}\right\}^2,$$

$$W(\alpha, n, 3)=\frac{1}{(n-4)!}\left\{\begin{matrix}\alpha\\ 1\end{matrix}\right\}^{n-4}\left\{\begin{matrix}\alpha\\ 4\end{matrix}\right\}+\frac{1}{(n-5)!}\left\{\begin{matrix}\alpha\\ 1\end{matrix}\right\}^{n-5}\left\{\begin{matrix}\alpha\\ 2\end{matrix}\right\}\left\{\begin{matrix}\alpha\\ 3\end{matrix}\right\}$$

$$+\frac{1}{3!\cdot(n-6)!}\left\{\begin{matrix}\alpha\\ 1\end{matrix}\right\}^{n-6}\left\{\begin{matrix}\alpha\\ 2\end{matrix}\right\}^3.$$

4. Applications of (3.3)—Examples.

EXAMPLE 1 (Asymptotics of Stirling numbers). Notice that (2.10) is a particular case of (2.2) with substitutions $\lambda\to\alpha$, $\alpha\mapsto\beta$ and

$$\left\{\begin{matrix}\beta\\ i\end{matrix}\right\}=\frac{S_r(\beta+i, \beta)}{(\beta+i)_i}, \qquad r=1, 2.$$

Now in the formula (2.10) let the substitutions $\alpha \mapsto \lambda$ and $\beta \mapsto m$ be made. Then using (3.3) we obtain, for λ large,

$$\frac{S_r(\lambda m+n, \lambda m)}{(\lambda)_n(\lambda m+n)_n} = \sum_{j=0}^{s} \frac{W_r(m, n, j)}{(\lambda-n+j)_j} + O\left(\left(\frac{1}{\lambda}\right)^{s+1}\right), \tag{4.1}$$

where $s<n$ and $W_r(m, n, j)$ $(r=1, 2)$ are given by

$$W_r(m, n, j) = \sum_{\sigma(n, n-j)} \prod_{i=1}^{n} \frac{1}{k_i!}\left(\frac{S_r(m+i, m)}{(m+i)_i}\right)^{k_i}. \tag{4.2}$$

Clearly (4.1) is a unified asymptotic formula for the two kinds of Stirling numbers. In particular, for $m=1$ it gives asymptotic expansions for $S_1(\lambda+n, \lambda)$ and $S_2(\lambda+n, \lambda)$, respectively. In these instances the computation of $W_r(m, n, j)$ will become more simplified since $S_1(i+1, 1)=(-1)^i i!$ and $S_2(i+1, 1)=1$ $(i=1, 2, \cdots)$.

REMARK. The asymptotic expansion of $S_2(\lambda+n, \lambda)$ implied by (4.1) is essentially equivalent to the old one given by Hsu [8] in 1948. (Cf. David and Barton [5], and Moser and Wyman [10]). A useful generalization of the asymptotic formula for Stirling functions of the second kind has been achieved very recently by Butzer and Hauss [2].

EXAMPLE 2 (Asymptotics of generalized Bernoulli and Euler polynomials). Let us start with the familiar definitions (cf. [6])

$$e^{xt}\left(\frac{t}{e^t-1}\right)^m = \sum_{n\geq 0} \frac{B_n^{(m)}(x)}{n!} t^n, \qquad e^{xt}\left(\frac{2}{e^t+1}\right)^m = \sum_{n\geq 0} \frac{E_n^{(m)}(x)}{n!} t^n.$$

Obviously one may take $\phi_1(t)=e^t$ and $\phi_2(t)=t/(e^t-1)$ or $\phi_2(t)=2/(e^t+1)$ so that $\phi_1(0)=\phi_2(0)=1$, and that (2.2) applies to the function of (2.3): $G(t):=\phi_1^x(t)\phi_2^m(t)$ with $\phi(t):=(G(t))^{1/\alpha}$. Consequently for every $\lambda \in \boldsymbol{Z}_+$ we see that (2.2) implies the following identities

$$\frac{1}{n!} B_n^{(\lambda m)}(\lambda x) = \sum_{\sigma(n)} \binom{\lambda}{k} \prod_{i=1}^{n} \left\{\frac{1}{i!} B_i^{(m)}(x)\right\}^{k_i}, \tag{4.3}$$

$$\frac{1}{n!} E_n^{(\lambda m)}(\lambda x) = \sum_{\sigma(n)} \binom{\lambda}{k} \prod_{i=1}^{n} \left\{\frac{1}{i!} E_i^{(m)}(x)\right\}^{k_i}. \tag{4.4}$$

Thus for fixed n and $x \in \boldsymbol{C}$ and for $\lambda \to \infty$ we have an asymptotic expansion of $B_n^{(\lambda m)}(\lambda x)$ as follows:

$$\frac{B_n^{(\lambda m)}(\lambda x)}{n!(\lambda)_n} = \sum_{j=0}^{s} \frac{W(m, n, j; x)}{(\lambda-n+j)_j} + O\left(\left(\frac{1}{\lambda}\right)^{s+1}\right), \tag{4.5}$$

where $s<n$ and $W(m, n, j; x)$ is given by

$$W(m, n, j; x)=\sum_{\sigma(n,n-j)}\prod_{i=1}^{n}\frac{1}{k_i!}\left\{\frac{1}{i!}B_i^{(m)}(x)\right\}^{k_i}. \tag{4.6}$$

Notice that (4.3) is an algebraic identity in x. Thus if one requires to find an asymptotic expansion of $B_n^{(\lambda m)}(t)$ for fixed $t\in \boldsymbol{C}$, one needs only to put $\lambda x=t$ in (4.5) so that the left-hand-side of (4.5) becomes $B_n^{(\lambda m)}(t)/n!(\lambda)_n$ which may be computed approximately through estimations of $B_i^{(m)}(t/\lambda)$ $(i=1, 2, \cdots, n)$.

Similar treatment as given above applies to $E_n^{(\lambda m)}(\lambda x)$ and $E_n^{(\lambda m)}(t)$ for large λ.

EXAMPLE 3 (Asymptotics of Laguerre polynomials). It is known that Laguerre polynomials $L_n^{(\alpha)}(z)$ may be defined by the following

$$(1-t)^{-(\alpha+1)}\exp\left(\frac{zt}{t-1}\right)=\sum_{n\geq 0}L_n^{(\alpha)}(z)t^n,$$

where α is real and $\alpha>-1$. This function may be embedded in (1.1) with $(\phi(t))^{\alpha}:=G(t)=(1/(1-t))^{\alpha+1}\cdot\exp(zt/(t-1))$, (cf. (2.3)). Consequently an application of (2.2) yields an identity of the form

$$L_n^{(\lambda\alpha+\lambda-1)}(\lambda z)=\sum_{\sigma(n)}\binom{\lambda}{\bar{k}}\prod_{i=1}^{n}\{L_i^{(\alpha)}(z)\}^{k_i}. \tag{4.7}$$

Moreover, making use of (3.3) we obtain an asymptotic expansion of the form

$$\frac{1}{(\lambda)_n}L_n^{(\lambda\alpha+\lambda-1)}(\lambda z)=\sum_{j=0}^{m}\frac{W(\alpha, n, j; z)}{(\lambda-n+j)_j}+O\left(\left(\frac{1}{\lambda}\right)^{m+1}\right), \tag{4.8}$$

where $m<n$ and $W(\alpha, n, j; z)$ is defined by

$$W(\alpha, n, j; z)=\sum_{\sigma(n,n-j)}\prod_{i=1}^{n}\frac{1}{k_i!}\{L_i^{(\alpha)}(z)\}^{k_i}. \tag{4.9}$$

In particular, if the asymptotic value of $L_n^{(\lambda\alpha+\lambda-1)}(t)$ is required, then the coefficients $W(\alpha, n, j; z)$ contained in (4.8) should be replaced by $W(\alpha, n, j; t/\lambda)$ which may be computed via asymptotic estimation of $L_i^{(\alpha)}(t/\lambda)$, $(i=1, 2, \cdots, n)$.

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Institute of Applied Mathematics
Dalian University of Technology
Dalian 116024
People's Republic of China

Fachbereich Mathematik
Institut für Mathematik (WE2)
Freie Universität Berlin
WE2, Arnimallee 3
1000 Berlin 33
Federal Republic of Germany