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NECESSARY AND SUFFICIENT CONDITIONS FOR "ZERO CROSSING" IN INTEGRODIFFERENTIAL EQUATIONS*

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Abstract. Necessary and sufficient conditions are obtained for all solutions of a class of linear scalar neutral-integrodifferential equations to have at least one zero. An application to an "equilibrium level-crossing" of a logistic integrodifferential equation with infinite continuously distributed delay is briefly discussed.

Introduction. There has been increased activity recently in the investigation of oscillatory nature of neutral delay differential equations. A prominent result obtained in these investigations is that a necessary and sufficient condition for the oscillation of all solutions of an autonomous neutral delay differential equation is that the associated characteristic equation has no real roots; there is a growing literature on this aspect (for example see [1], [5]–[9], [13]–[15]).

The purpose of this article is to derive a necessary and sufficient condition for all solutions of neutral integrodifferential equations of the form

$$\frac{d}{dt}[x(t)-cx(t-\tau)]+a\int_0^\infty K(s)x(t-s)ds=0\,;\quad t>0 \tag{1.1}$$

to have at least one zero on $(-\infty, \infty)$. Solutions of (1.1) which have at least one zero on $(-\infty, \infty)$ are said to have "zero crossings"; on the other hand if there is a solution x of (1.1) such that either $x(t)>0$ on $(-\infty, \infty)$ or $x(t)<0$ on $(-\infty, \infty)$, then such a solution is said to have no "zero crossings" (sometimes these solutions are said to stay away from zero). For literature related to stability characteristics of neutral integrodifferential equations we refer to Kolmanovskii and Nosov [12].

As an application of a special case of our result, we shall consider briefly "equilibrium level-crossing" of the solutions of the logistic integrodifferential equation

$$\frac{dN(t)}{dt}=rN(t)\left[1-\frac{1}{C}\int_0^\infty K(s)N(t-s)ds\right] \tag{1.2}$$

where

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$$r, C \in (0, \infty), \quad K: [0, \infty) \mapsto [0, \infty), \quad \int_0^\infty K(s)ds = 1 .$$

The equation (1.2) will be supplemented with nonnegative initial values on $(-\infty, 0]$ such that $N(0) > 0$. We will be particularly interested in the existence of points t^* for which $N(t^*) = C$ where C denotes the positive equilibrium of (1.2).

2. Zero crossings. We shall derive a necessary and sufficient condition for all nontrivial solutions of (1.1) to have "zero crossings" (i.e. have at least one zero on $(-\infty, \infty)$). Our condition is based on the nature of the roots of the characteristic equation associated with (1.1) which is

$$\lambda(1 - ce^{-\lambda\tau}) + a\int_0^\infty K(s)e^{-\lambda s}ds = 0 . \tag{2.1}$$

The following result will be used in the proof of Theorem 2.2 below.

LEMMA 2.1. *Suppose* $K: [0, \infty) \mapsto [0, \infty)$ *and* $K \not\equiv 0$ *on some subinterval of* $[0, \infty)$; *let*

$$a \in (0, \infty); \quad c \in [0, 1); \quad \tau \in [0, \infty) .$$

If (2.1) *has no real roots, then there exists a positive number* m *such that*

$$a\int_0^\infty K(s)e^{\lambda s}ds > \lambda(1 - ce^{\lambda\tau}) + m ; \quad \lambda \in \boldsymbol{R} . \tag{2.2}$$

PROOF. Define F as follows:

$$F(\lambda) = \lambda(1 - ce^{-\lambda\tau}) + a\int_0^\infty K(s)e^{-\lambda s}ds . \tag{2.3}$$

We note from (2.3),

$$\left.\begin{aligned} &F(0) = a\int_0^\infty K(s)ds > 0 \\ &F(\lambda) \to \infty \quad \text{as} \quad \lambda \to \infty \\ &F(\lambda) \to \infty \quad \text{as} \quad \lambda \to -\infty \\ &F(\lambda) = 0 \quad \text{has no real roots} \end{aligned}\right\} \Rightarrow \inf_{\lambda \in R} F(\lambda) > 0 . \tag{2.4}$$

If $\inf_{\lambda \in R} F(\lambda) = 0$, then there exists a sequence

$$\lambda_n \in (-\infty, \infty), \quad |\lambda_n| < \infty \text{ such that } F(\lambda_n) \to 0 \text{ as } n \to \infty.$$

Since the sequence $\{\lambda_n\}$ is bounded, there exists a convergent subsequence, say $\{\lambda_{n_k}\}$, such that

$$\lambda_{n_k} \to \lambda^* \quad \text{and} \quad F(\lambda_{n_k}) \to 0 \quad \text{as} \quad n_k \to \infty . \tag{2.5}$$

Since F is continuous in λ, it will follow that

$$F(\lambda_{n_k}) \to F(\lambda^*) = 0 \quad \text{as} \quad n_k \to \infty \tag{2.6}$$

and hence λ^* is a real root of F which is a contradiction. Thus there exists a positive number m such that

$$F(\lambda) = \lambda(1 - ce^{-\lambda\tau}) + a\int_0^\infty K(s)e^{-\lambda s}ds \geq m , \quad \lambda \in R \tag{2.7}$$

from which the result follows.

THEOREM 2.2. *Let* $a \in (0, \infty)$; $c \in [0, 1)$; $\tau \in [0, \infty)$ *and that* K *is eventually nonincreasing. Suppose* $K \not\equiv 0$ *on some subinterval of* $[0, \infty)$. *A necessary and sufficient condition for nontrivial solutions of* (1.1) *to have zero crossings is that the characteristic equation of* (1.1) *has no real roots.*

PROOF. The necessity of the condition is easily seen; for instance if $F(\lambda) = 0$ has a real root say $\mu \in (-\infty, \infty)$, then (1.1) has a solution of the form

$$x(t) = Ae^{\mu t} , \qquad A \in (-\infty, \infty)$$

which has no zero crossings. Thus the necessity of the condition follows.

The sufficiency part of the condition is proved as follows: we shall assume that (1.1) has a solution y which is strictly positive on $(-\infty, \infty)$ and then show that this will lead to a certain contradiction. The technique is similar to those used in the case of neutral differential equations with finite delays.

Let y be a positive solution of (1.1) on $(-\infty, \infty)$. Define a sequence $\{z_n\}$ as follows:

$$\begin{cases} z_0(t) = y(t) - cy(t-\tau) \\ z_{n+1}(t) = z_n(t) - cz_n(t-\tau) \end{cases} \quad t > -\infty . \tag{2.8}$$

It can be verified that (2.8) and (1.1) imply

$$\dot{z}_1(t) = -a\int_0^\infty K(s)z_0(t-s)ds$$

and furthermore

$$\dot{z}_n(t) = -a\int_0^\infty K(s)z_{n-1}(t-s)ds , \quad n = 1, 2, \cdots . \tag{2.9}$$

Since y is a positive solution of (1.1), we have from (1.1) that $y(t) - cy(t-\tau)$ is decreasing at t increases. Thus if $y(t) - cy(t-\tau)$ becomes zero or negative for some $t = t_0$, then for all $t>t_0$ we will have $y(t)-cy(t-\tau)>0$; as a consequence there will exist a $\delta>0$ such that

$$y(t)-cy(t-\tau)<-\delta\,, \quad \text{for} \quad t>t_1=t_0+1$$

and therefore for $c\in[0,1)$,

$$\begin{aligned} y(t)&<-\delta+cy(t-\tau)<-\delta+c[-\delta+cy(t-\tau)]\\ &<-\left(\frac{\delta}{1-c}\right)+c^n y(t-n\tau)\\ &=-\left(\frac{\delta}{1-c}\right)+y(t_0)\exp\left(\frac{t-t_0}{\tau}\ln(c)\right); \quad t=n\tau+t_0\,. \end{aligned}$$

If t is large enough, then it will follow from $c\in[0,1)$ and the above that

$$y(t)\le-\left(\frac{\delta}{1-c}\right) \quad \text{for large enough } t\,, \tag{2.10}$$

which is impossible. Thus we conclude that

$$y(t)-cy(t-\tau)>0 \quad \text{for} \quad t\in(-\infty,\infty)$$

and similarly all $z_n>0$ for $t\in(-\infty,\infty)$, $n=0,1,2,3,\cdots$. We note from

$$\dot z_n(t)=-a\int_0^\infty K(s)z_{n-1}(t-s)ds \tag{2.11}$$

that

$$\dot z_n(t)-c\dot z_n(t-\tau)+a\int_0^\infty K(s)z_n(t-s)ds=0\,. \tag{2.12}$$

We can choose positive numbers (since $K\not\equiv0$ on some interval of $[0,\infty)$) α and β such that

$$\int_\alpha^\beta K(s)ds>0\,.$$

We have from (2.12),

$$\dot z_n(t)+a\int_\alpha^\beta K(s)z_n(t-s)ds\le0\,.$$

Since z_n is decreasing,

$$\dot z_n(t)+a\int_\alpha^\beta K(s)z_n(t-\alpha)ds\le0\,;$$

and hence

(2.13) $$\dot{z}_n(t)+Az_n(1-\alpha)\leq 0\,,\quad A=a\int_\alpha^\beta K(s)ds\,.$$

Integrating both sides of the inequality in (2.13) on $[t-\alpha/2, t]$,

$$z_n(t)-z_n\left(t-\frac{\alpha}{2}\right)+A\int_{t-\alpha/2}^{t} z_n(s-\alpha)ds\leq 0$$

leading to

$$z_n(t)-z_n\left(t-\frac{\alpha}{2}\right)+A\frac{\alpha}{2}z_n(t-\alpha)\leq 0$$

which implies

(2.14) $$A\frac{\alpha}{2}z_n(t-\alpha)\leq z_n\left(t-\frac{\alpha}{2}\right).$$

A similar integration of the inequality in (2.13) on $[t, t+\alpha/2]$ leads to

(2.15) $$A\frac{\alpha}{2}z_n\left(t-\frac{\alpha}{2}\right)\leq z_n(t)\,.$$

From (2.14) and (2.15),

(2.16) $$z_n(t-\alpha)<\frac{4}{(A\alpha)^2}z_n(t)\,.$$

Consider now the set Λ_n of real numbers defined by

(2.17) $$\Lambda_n=\{\lambda\geq 0\,|\,\dot{z}_n(t)+\lambda z_n(t)\leq 0 \text{ eventually for } t>0\}\,.$$

Clearly $\lambda=0\in\Lambda_n$ and Λ_n is nonempty; also Λ_n is a subinterval of $[0, \infty)$. The strategy of our proof is to show that the existence of a positive solution of (1.1) implies that the set Λ_n has the following contradictory properties P_1 and P_2 (by now this strategy seems to be a standard one);

$$\begin{cases} P_1: \text{the set } \Lambda_n \text{ is bounded.} \\ P_2: \lambda\in\Lambda_n\Rightarrow\lambda+m\in\Lambda_n \quad \text{where } m \text{ is as in Lemma 2.1.}\end{cases}$$

To establish P_1, we have to show the existence of an upper bound of Λ_n. Integrating both sides of

$$\dot{z}_n(t)+a\int_0^\infty K(s)z_n(t-s)ds=0$$

on $[t-\alpha, t]$,

$$z_n(t)-z_n(t-\alpha)+a\int_{t-\alpha}^{t}\left(\int_0^{\infty}K(s)z_n(u-s)ds\right)du=0\,,$$

implying that

$$a\int_{t-\alpha}^{t}\left[\int_{-\infty}^{0}z_n(s)K(u-s)ds+\int_0^{u}z_n(s)K(u-s)ds\right]du\le z_n(t-\alpha)\,.$$

Therefore for all large enough $t>0$,

$$a\int_{-\infty}^{0}z_n(s)K(t-s)ds+\int_0^{t}z_n(s)K(t-s)ds\le\frac{1}{\alpha}z_n(t-\alpha)\le\frac{4}{\alpha(A\alpha)^2}z_n(t)\,,$$

which is the same as

$$a\int_0^{\infty}z_n(t-s)K(s)ds\le\frac{4}{\alpha(A\alpha)^2}z_n(t)\quad\text{eventually}\,. \tag{2.18}$$

Now we have from (2.11) and (2.18),

$$0=\dot z_n(t)+a\int_0^{\infty}K(s)z_{n-1}(t-s)ds<\dot z_n(t)+\frac{4}{(A\alpha)^2\alpha}z_n(t) \tag{2.19}$$

which shows that $4/(A\alpha)^2\alpha$ does not belong to Λ_n. Thus the set Λ_n is bounded and hence the property P_1 holds.

To derive P_2, we define a sequence $\{\phi_n\}$ as follows:

$$\phi_n(t)=e^{\lambda t}z_n(t)\,,\quad \lambda\in\Lambda_n\,. \tag{2.20}$$

It is immediate that

$$\dot\phi_n(t)=e^{\lambda t}[\dot z_n(t)+\lambda z_n(t)]\le 0 \tag{2.21}$$

showing that ϕ_n is nonincreasing. We have from (2.11), (2.20), (2.21) and Lemma 2.1 that

$$\begin{aligned}\dot z_{n+1}(t)+(\lambda+m)z_{n+1}(t)&=-a\int_0^{\infty}K(s)z_n(t-s)ds+(\lambda+m)[z_n(t)-cz_n(t-\tau)]\\&=-a\int_0^{\infty}K(s)e^{-\lambda(t-s)}\phi_n(t-s)ds+(\lambda+m)e^{-\lambda t}\phi(t)-c(\lambda+m)e^{-\lambda(t-\tau)}\phi_n(t-\tau)\\&\le e^{-\lambda t}\left[-a\int_0^{\infty}K(s)e^{\lambda s}\phi_n(t)ds+(\lambda+m)\phi_n(t)-(\lambda+m)ce^{\lambda\tau}\phi_n(t)\right]\\&\le e^{-\lambda t}\phi_n(t)\left[-a\int_0^{\infty}K(s)e^{\lambda s}ds+\lambda(1-ce^{\lambda\tau})+m-mce^{\lambda\tau}\right]\le 0\end{aligned} \tag{2.22}$$

and therefore $\lambda+m\in\Lambda_n$ which establishes P_2. Since P_1 and P_2 together cannot hold for Λ_n, we have a contradiction. Thus (1.1) cannot have a positive solution on $(-\infty, \infty)$; since (1.1) is linear, in a similar way it cannot have a negative solution on $(-\infty, \infty)$. This completes the proof.

COROLLARY 2.3. *Assume that a, c, K, τ are as in Theorem 2.2. If*

$$a\int_0^\infty K(s)s\,ds > \frac{1-c}{e} \quad \text{or} \quad (a+c)\int_0^\infty K(s)ds > \frac{1}{e}, \tag{2.23}$$

then all nontrivial solutions of (1.1) *have zero crossings.*

PROOF. Suppose that (1.1) has a solution without zero-crossing on $(-\infty, \infty)$; by Theorem 2.2, the characteristic equation of (1.1) has a real root; that is

$$F(\lambda) = \lambda(1 - ce^{-\lambda\tau}) + a\int_0^\infty K(s)e^{-\lambda s}ds = 0 \tag{2.24}$$

has a real root; since

$$F(0) = a\int_0^\infty K(s)ds > 0\,, \quad F(\lambda) > 0 \quad \text{for} \quad \lambda \geq 0 \tag{2.25}$$

the real root has to be negative. Let $\lambda = -\mu$, $\mu > 0$ be such a root; then μ satisfies

$$\mu(1 - ce^{\mu\tau}) = a\int_0^\infty K(s)e^{\mu s}ds\,, \tag{2.26}$$

which implies that

$$1 - ce^{\mu\tau} = a\int_0^\infty sK(s)\frac{e^{\mu s}}{\mu s}ds$$

leading to

$$1 - c \geq ae\int_0^\infty sK(s)ds\,; \tag{2.27}$$

but this contradicts the first of (2.23).

We also have from (2.26) that

$$\mu = \mu ce^{\mu\tau} + a\int_0^\infty K(s)e^{\mu s}ds > c\mu + a\int_0^\infty K(s)e^{\mu s}ds$$

$$> ca\int_0^\infty K(s)e^{\mu s}ds + a\int_0^\infty K(s)e^{\mu s}ds = a(1+c)\int_0^\infty K(s)e^{\mu s}ds$$

leading to

$$1 > a(1+c)\int_0^\infty K(s)s\frac{e^{\mu s}}{\mu s}ds \geq a(1+c)e\int_0^\infty K(s)sds \tag{2.28}$$

and this contradicts the second of (2.23). Thus the conclusion of the corollary follows.

COROLLARY 2.4. *Let a, c, K be as in Theorem 2.2; let H: $[0, \infty)\mapsto[0, \infty)$ be such that*

$$c\int_0^\infty H(s)ds < 1 .$$

Then a necessary and sufficient condition for all solutions of

$$\frac{d}{dt}\left[x(t) - c\int_0^\infty H(s)x(t-s)ds\right] + a\int_0^\infty K(s)x(t-s)ds = 0 \tag{2.29}$$

to have zero crossings is that the characteristic equation

$$\lambda\left(1 - c\int_0^\infty H(s)e^{-\lambda s}ds\right) + a\int_0^\infty K(s)e^{-\lambda s}ds = 0 \tag{2.30}$$

has no real roots.

PROOF. Details are exactly similar to those of Theorem 2.2 and hence are omitted.

COROLLARY 2.5. *Assume that a, c, τ, K, H are as in Corollary 2.4. If*

$$a\int_0^\infty K(s)sds > \frac{1}{e}\left(1 - c\int_0^\infty H(s)ds\right), \tag{2.31}$$

then all nontrivial solutions of (2.29) have zero crossings.

PROOF. Details are similar to those of Corollary 2.3 and hence are omitted.

3. An application to level crossing. We shall now consider briefly the nonlinear logistic integrodifferential equation

$$\frac{dN(t)}{dt} = rN(t)\left[1 - \frac{1}{C}\int_0^\infty K(s)N(t-s)ds\right] \tag{3.1}$$

where $r, C \in (0, \infty)$ and

$$\begin{cases} N(s) = \phi(s) \geq 0, \quad s \in (-\infty, 0]; \quad \phi(0) > 0; \\ K: [0, \infty)\mapsto[0, \infty); \quad \displaystyle\int_0^\infty K(s)ds = 1 \end{cases} \tag{3.2}$$

and derive sufficient conditions for all positive solutions of (3.1) to cross the positive equilibrium level in the sense that there exists a $t^* \in (-\infty, \infty)$ for which

$$N(t^*)-C=0\,; \tag{3.3}$$

solutions satisfying (3.3) can be called equilibrium crossing or "level-crossing" for short. The integrodifferential equation (3.1) represents a generalisation of the familiar delay logistic equation

$$\frac{dx(t)}{dt}=rx(t)\left[1-\frac{x(t-\tau)}{C}\right]$$

with a finite discrete delay τ. The following result shows that if the linear variational system corresponding to the positive equilibrium of (3.1) has zero crossings, then the nonlinear system (3.1) has level crossings.

THEOREM 3.1. *Let $r, C\in(0,\infty)$; suppose K is not identically zero on some subinterval of $[0,\infty)$ and that K is eventually nonincreasing. If*

$$r\int_0^\infty K(s)e^{\lambda s}ds>\lambda \quad \textit{for} \quad \lambda\in(0,\infty)\,, \tag{3.4}$$

then all positive solutions of (3.1) have level crossings.

PROOF. First we note that every solution of (3.1)–(3.2) satisfies

$$N(t)>0 \quad \text{for} \quad t>0\,. \tag{3.5}$$

We let

$$N(t)=C[1+u(t)] \tag{3.6}$$

in (3.1) and derive that u is governed by

$$\frac{du(t)}{dt}+r[1+u(t)]\int_0^\infty K(s)u(t-s)ds=0\,. \tag{3.7}$$

It is easily seen from (3.5) and (3.6) that the problem of level crossing of N about C is equivalent to that of zero crossing of u where u satisfies (3.7).

As in Section 2, one can show the existence of a positive number m such that

$$r\int_0^\infty K(s)e^{\lambda s}ds\geq\lambda+m \quad \text{for} \quad \lambda\in R\,. \tag{3.8}$$

Now suppose that (3.7) has a solution without zero crossing; for instance let $u(t)>0$ for $t\in(-\infty,\infty)$. It follows from (3.7) that u is decreasing on R and therefore

$$u(t)\to L\geq 0 \quad \text{as} \quad t\to\infty\,.$$

One can show that $L=0$, since otherwise u can become negative eventually for large t, contradicting the positivity of u on R.

Similarly one can show that if (3.7) has a solution u such that $u(t)<0$ on $(-\infty, \infty)$, then u will be nondecreasing (since $1+u(t)>0$) on $(-\infty, \infty)$, implying that

$$u(t) \to l \leq 0 \quad \text{as} \quad t \to \infty$$

and one can show that $l=0$.

We now rewrite (3.7) in the form

$$\frac{du(t)}{dt} + p(t)\int_0^\infty K(s)u(t-s)ds = 0 \tag{3.9}$$

where

$$p(t) = r[1+u(t)] > 0 \quad \text{on} \quad (-\infty, \infty)$$

and note

$$p(t) = r[1+u(t)] \to r \quad \text{as} \quad t \to \infty \tag{3.10}$$

whenever u denotes a solution of (3.9) without zero crossing.

Suppose (3.7) has a solution u satisfying $u(t)>0$ on $(-\infty, \infty)$. Define a set Λ_u as follows:

$$\Lambda_u = \{\lambda \geq 0 \mid \dot{u}(t) + \lambda u(t) \leq 0 \quad \text{eventually for all large } t\}\,. \tag{3.11}$$

It is easily seen that $\lambda = 0 \in \Lambda_u$ and that Λ_u is a subinterval of $[0, \infty)$. The rest of the proof is accomplished by showing that the set Λ_u has the contradictory properties P_1 and P_2 (as in the proof of Theorem 2.2):

P_1: Λ_u is bounded

P_2: $\lambda \in \Lambda_u \Rightarrow \lambda + m \in \Lambda_u$ where m is as in (3.8).

The derivation of the properties P_1 and P_2 are similar to that in the proof of Theorem 2.2 and we shall be brief. In fact we omit the derivation of P_1 and proceed with the derivation of P_2. Define a ϕ as follows:

$$\phi(t) = e^{\lambda t}u(t)\,, \quad \lambda \in \Lambda_u\,. \tag{3.12}$$

We note from (3.12) and (3.11) that

$$\dot{\phi}(t) = e^{\lambda t}[\dot{u}(t) + \lambda u(t)] \leq 0 \quad \text{eventually} \tag{3.13}$$

and so ϕ is nonincreasing eventually and ϕ is positive. We have from (3.9)

$$\begin{aligned} \dot{u}(t) + (\lambda+m)u(t) &= -p(t)\int_0^\infty K(s)u(t-s)ds + (\lambda+m)u(t) \\ &= -p(t)\int_0^\infty K(s)e^{-\lambda(t-s)}\phi(t-s)ds + (\lambda+m)e^{-\lambda t}\phi(t) \end{aligned} \tag{3.14}$$

$$\leq e^{-\lambda t}\left[-p(t)\int_0^\infty K(s)e^{\lambda s}\phi(t-s)ds+(\lambda+m)\phi(t)\right]$$

$$\leq e^{-\lambda t}\phi(t)\left[-p(t)\int_0^\infty K(s)e^{\lambda s}ds+(\lambda+m)\right]$$

$$\leq e^{-\lambda t}\phi(t)\left[-\left(\liminf_{t\to\infty} p(t)\right)\int_0^\infty K(s)e^{\lambda s}ds+(\lambda+m)\right]$$

$$\leq e^{-\lambda t}\phi(t)\left[-r\int_0^\infty K(s)e^{\lambda s}ds+(\lambda+m)\right]\leq 0 \tag{3.15}$$

showing that $\lambda+m\in\Lambda_u$ and hence the validity of P_2 follows. If (3.6) has a solution $u(t)<0$ on $(-\infty, \infty)$, then one can consider $-u(t)$ and repeat the proof, since in this case also $p(t)=r[1-u(t)]\to r$ increasing monotonically as $t\to\infty$. This completes the proof.

We remark that the condition (3.4) means that the characteristic equation

$$z+r\int_0^\infty K(s)e^{-zs}ds=0 \tag{3.16}$$

associated with the linear autonomous integrodifferential equation

$$\frac{dx(t)}{dt}+r\int_0^\infty K(s)x(t-s)ds=0 \tag{3.17}$$

has no real roots; in other words we have shown that if all the solutions of (3.17) have zero crossings, then all the positive solutions of (3.1) have level crossings.

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