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ON THE RATIONAL STRUCTURES OF SYMMETRIC DOMAINS, II DETERMINATION OF RATIONAL POINTS OF CLASSICAL DOMAINS

Dedicated to the memory of Michio Kuga

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In Part I of this series, quoted hereafter as [Sa2], we gave an algebraic formulation of the Siegel domain realization of symmetric domains and applied it to the determination of "rational points" of symmetric domains with $\boldsymbol{Q}$-structure. To be more precise, let $\mathfrak{g}$ be a (real) semisimple Lie algebra of hermitian type defined over $\boldsymbol{Q}$, $\boldsymbol{Q}$-simple and of $\boldsymbol{Q}$-rank r_0. Let $\mathscr{D}$ be a symmetric domain associated with $\mathfrak{g}$, which (as a set) may be identified with the set of Cartan involutions of $\mathfrak{g}$. A point of $\mathscr{D}$ is called *rational* if the corresponding Cartan involution is defined over $\boldsymbol{Q}$. We showed in [Sa2] (Th. 3) that, if $r_0>0$, then the determination of rational points is essentially reduced to that for the "last" (i.e., the r_0-th) rational boundary components, which are always of classical type. By virtue of the isomorphisms between classical groups, it is known that all classical domains with $r_0=0$ are realized as a domain of type (I) (see §1 of this paper). The main purpose of this Part II is to give an actual determination of rational points in the case of domains of type (I).

The semisimple Lie algebra $\mathfrak{g}$, or the associated symmetric domain $\mathscr{D}$, is called *pure* if all $\boldsymbol{R}$-simple factors of $\mathfrak{g}$ are $\boldsymbol{R}$-isomorphic to one another. It is called *strictly pure* if all $\boldsymbol{Q}$-simple factors in the reductive part of the $\boldsymbol{Q}$-parabolic subalgebras corresponding to rational boundary components of $\mathscr{D}$ are pure. (Note that these two conditions are actually equivalent except for the case where $\mathfrak{g}$ is of type (D_l^m).) The results in [Sa2] (Lem. 3, Th. 3) imply that, if $\mathscr{D}$ has rational points, then $\mathfrak{g}$ is strictly pure. For the domains of tube type the converse of this is also true except for the case of domains of type (I), which is discussed in detail in this paper. A part of our results was obtained by K. Oiso in his Master thesis. It is given here in a refined form with a different proof.

Sections 1 and 2 are mostly of preliminary nature. In §1 we give a list of pure $\boldsymbol{Q}$-simple classical Lie algebras of hermitian type and in §2 summarize some basic facts on "unitary involutions" (i.e., involutions of the second kind) and hermitian forms. Then in §3 we explain a method to determine rational points of a domain $\mathscr{D}$ of type $(\mathrm{I}_{p,q})^m$ and give a necessary condition for the existence of rational points (Th. 4). The necessary and sufficient conditions for the existence of rational points (with a given CM-field) are given in §4 (Th. 5, 6, 7). Using these results, we discuss in §5 the rational points of classical domains in general. We also give in the concluding Remark a similar determination in the case of the exceptional domains.

Rational points of symmetric domains are a special case of the "isolated fixed points" (or "special points", "CM-points") in the sense of Shimura [Sh2, 3] (or Deligne [D]) which play an essential role in the theory of canonical models. Naturally our result has some relevance with the canonical model. Let G be a $\boldsymbol{Q}$-simple algebraic group of adjoint type with $\mathrm{Lie}\, G(\boldsymbol{R})=\mathfrak{g}$ and let $\mathscr{h}$ be a Hodge structure of $\mathfrak{g}$ defining the given complex structure of $\mathscr{D}$. Then, by a criterion of Deligne [D], it is easy to see that the field of definition $E(G, \mathscr{h})$ of the model $M_{\boldsymbol{C}}(G, \mathscr{h})$ (in the sense of [D]) reduces to $\boldsymbol{Q}$ if and only if the domain $\mathscr{D}$ is of tube type and $\mathfrak{g}$ is strictly pure. As mentioned above, this is certainly the case for the symmetric domains of tube type with $\boldsymbol{Q}$-rational points.

NOTATION. We use the standard notation $\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{Q}$, $\boldsymbol{Z}$, etc. For a positive real number α we put $\sqrt{\alpha}=\alpha^{1/2}>0$ and $\sqrt{-\alpha}=\sqrt{-1}\sqrt{\alpha}$. For a field F, $F^\times$ denotes the multiplicative group of non-zero elements of F; when F is a totally real number field, $F_+^\times$ is the subgroup of $F^\times$ consisting of totally positive elements. For a subgroup H of $F^\times$, the multiplicative equivalence class of $a\in F^\times$ modulo H is denoted as $a \pmod{^\times H}$. The multiplicative equivalence relation in $F^\times$ is often denoted as $\sim$ (see 2.1).

For a vector space V over F and a field extension F'/F, we write $V_{F'}$ for $V\otimes_F F'$ viewed as a vector space over F'. When F'/F is finite, we set $N(F'/F)=N_{F'/F}(F'^\times)$. The symbol $R_{F'/F}$ stands for the functor of restricting the ground field from F' to F.

For an algebra D over F, $\mathrm{tr}_{D/F}$ and $N_{D/F}$ (or simply tr and N) denote always the reduced trace and norm of D over F. The algebra of all $v\times v$ matrices with entries in D is denoted by $M_v(D)$. When D has an involution ρ of the second kind (or a "unitary involution" as we call it), the subspace of $M_v(D)$ consisting of hermitian matrices with respect to ρ is denoted by $\mathrm{Her}_v(D, \rho)$. The diagonal matrix with diagonal entries $a_1, \cdots, a_v$ is denoted by $\mathrm{diag}(a_1, \cdots, a_v)$; especially, $1_v=\mathrm{diag}(1, \cdots, 1)$ is the identity matrix of degree v. For $A\in M_v(D)$, ${}^tA$ stands for the transposed of A. For a (right) D-module V, the D-submodule generated by $v_i\in V$ $(1\le i\le m)$ is denoted by $\{v_1, \cdots, v_m\}_D$. The identity transformation of V is denoted by 1_V or simply 1 or id.

In §4, for $\alpha, \beta\in F^\times$, we denote by $(\alpha, \beta)_F$ the quaternion algebra D over F defined by

$$D=\{1, u, v, uv\}_F,$$
$$u^2=\alpha, \qquad v^2=\beta, \qquad uv=-vu.$$

We also set $D_-=\{x\in D \mid \bar{x}=-x\}$, $x\mapsto\bar{x}=\mathrm{tr}(x)-x$ denoting the canonical involution of D. The similarity relation (Brauer equivalence relation) between central simple algebras over F is written as $\sim$. Especially, $D\sim 1$ means that D splits over F.

For an algebraic number field F, we fix an imbedding $\sigma_1: F\to\boldsymbol{C}$ and consider F to be contained in $\boldsymbol{C}$. The complex conjugation of $\boldsymbol{C}$ is denoted by ρ_0. When F is totally real number field (imbedded in $\boldsymbol{R}$) and $\mathfrak{g}$ is a real Lie algebra defined over F, we denote by $\mathfrak{g}(F)$ the Lie algebra over F consisting of F-rational points in $\mathfrak{g}$; then $\mathfrak{g}=\mathfrak{g}(F)_{\boldsymbol{R}}$. By an abuse of notation, for any imbedding $\sigma: F\to\boldsymbol{R}$, we set $\mathfrak{g}^{\sigma}=\mathfrak{g}(F^{\sigma})_{\boldsymbol{R}}$ and call it a "conjugate" of $\mathfrak{g}$.

1. Classical symmetric domains.

1.1. The notation will be basically the same as in Part I ([Sa2]). Let $\mathfrak{g}$ be a real semisimple Lie algebra of hermitian type defined over $\boldsymbol{Q}$, which we assume to be non-compact, $\boldsymbol{Q}$-simple and of $\boldsymbol{Q}$-rank r_0 (≥ 0). We write $\mathfrak{g}$ in the form

$$\mathfrak{g}=R_{F/\boldsymbol{Q}}(\mathfrak{g}_1)=\bigoplus_{i=1}^{m}\mathfrak{g}_1^{\sigma_i}, \tag{1.1}$$

where F is a totally real number field of degree m, σ_1 ($=$id), $\cdots$, σ_m are distinct imbeddings of F into $\boldsymbol{R}$, and $\mathfrak{g}_1$ is an (absolutely) simple real Lie algebra of hermitian type defined over F. In view of [Sa2], Lemma 3, we assume further that $\mathfrak{g}$ is "pure", i.e., the following condition is satisfied:

(R1) All conjugates $\mathfrak{g}_1^{\sigma_i}$ ($1\leq i\leq m$) of $\mathfrak{g}_1$ are $\boldsymbol{R}$-isomorphic to $\mathfrak{g}_1$. (In particular, we have $\boldsymbol{R}$-rank $\mathfrak{g}_1^{\sigma_i}=\boldsymbol{R}$-rank $\mathfrak{g}_1=r_1>0$.)

The Lie algebra $\mathfrak{g}$ is called of "classical" type if $\mathfrak{g}_1$ is obtained from a (simple associative) algebra with involution, or equivalently from an ε-hermitian form. Under the assumption (R1), we know that there are the following seven cases.

($\mathrm{III}_{\nu/2}^{(1)}$) ($\nu$ even, ≥ 2) $\mathfrak{g}_1(F)=\mathfrak{sp}(\nu, F)=\mathfrak{sp}(A_1, F)$, where A_1 is a non-degenerate alternating bilinear form on F^{ν}. One has $r_1=r_0=\nu/2$.

($\mathrm{III}_{\nu}^{(2)}$) ($\nu\geq 1$) $\mathfrak{g}_1(F)=\mathfrak{su}(\nu, h, D/F)$, where D is a totally indefinite (central) division quaternion algebra over F and h is a non-degenerate D-hermitian form on D^{ν}. One has $r_1=\nu$ and $r_0=[\nu/2]$.

($\mathrm{IV}_{\nu-2}^{(1)}$) ($\nu\geq 5$) $\mathfrak{g}_1(F)=\mathfrak{so}(\nu, S_1, F)$, where S_1 is a non-degenerate symmetric bilinear form on F^{ν} with $\mathrm{sign}(S_1^{\sigma_i})=(\nu-2, 2)$ ($1\leq i\leq m$). One has $r_1=2$ and $r_0=1$ or 2 for $\nu=5$, 6 and $r_0=2$ for $\nu\geq 7$.

($\mathrm{IV}_{2\nu-2}^{(2)}$) ($\nu\geq 3$) $\mathfrak{g}_1(F)=\mathfrak{su}^-(\nu, h, D/F)$, where D is a totally indefinite division quaternion algebra over F and h is a non-degenerate D-skewhermitian form on D^{ν} such that $\mathfrak{g}_1^{\sigma_i}\cong\mathfrak{so}(2\nu-2, 2)$ ($1\leq i\leq m$). One has $r_1=2$ and $r_0=0$ or 1 for $\nu=3$ and $r_0=1$ for $\nu\geq 4$.

($\mathrm{II}_{\nu}^{(2)}$) ($\nu\geq 3$) $\mathfrak{g}_1(F)=\mathfrak{su}^-(\nu, h, D/F)$, where D is a totally definite quaternion algebra over F and h is a non-degenerate D-skewhermitian form on D^{ν}. One has $r_1=[\nu/2]$ and $r_0=[\nu/2]$ or $[\nu/2]-1$.

(II_4–$\mathrm{IV}_6^{(2)}$) $\mathfrak{g}_1(F)=\mathfrak{su}^-(\nu, h, D/F)$, where D is a quaternion algebra over F such that $(D^{\sigma_i})_{\boldsymbol{R}}$ is division for $1\leq i\leq m_1$ and $\cong M_2(\boldsymbol{R})$ for $m_1+1\leq i\leq m$ with $1\leq m_1<m$ and h is a non-degenerate D-skewhermitian form on D^4 such that $\mathfrak{g}_1^{\sigma_i}\cong\mathfrak{so}(6, 2)$ for $m_1+1\leq i\leq m$. One has $r_1=2$ and $r_0=1$.

($\mathrm{I}_{p,q}^{(\delta)}$) ($p\geq q\geq 1, p+q=\delta\nu\geq 2$) $\mathfrak{g}_1(F)=\mathfrak{su}(\nu, h, D'/F'/F)$, where F'/F is a CM-field, D'

is a (central) division algebra over F' of degree δ with a "unitary involution" ρ (i.e., involution of the second kind) relative to F'/F and h is a non-degenerate (D', ρ)-hermitian form on D'^{v} such that $\mathfrak{g}_1^{\sigma_i} \cong \mathfrak{su}(p, q)$ for $1 \leq i \leq m$. One has $r_1 = q$ and $r_0 \leq q/\delta$ $(\leq v/2)$.

REMARK. In general, one has $r_0 \leq r_1/\delta' \leq v/2$ for a certain positive integer δ' $(=1, 2, \delta$ according to the case). A lower bound for r_0 is obtained from Theorems 1 and 2 below.

1.2. The following theorem is classical.

THEOREM 1. *Among the* $\mathbf{Q}$*-simple classical Lie algebras* $\mathfrak{g}$ *of hermitian type listed above, the "anisotropic" case (i.e., the case with* $r_0 = 0$*) occurs only in the following cases*:

$$(\mathrm{III}_1^{(2)}), \quad (\mathrm{IV}_4^{(2)}), \quad (\mathrm{II}_3^{(2)}), \quad (\mathrm{I}_{p,q}^{(\delta)}) \quad (v \leq 2 \text{ or } q < \delta).$$

This follows essentially from the Hasse principle or "local global principle" for isotropy (see, e.g., [Sc], p. 346–7, B1). For $(\mathrm{III}_{v/2}^{(1)})$ $(v \geq 2)$ and $(\mathrm{IV}_{v-2}^{(1)})$ $(v \geq 5)$, it is well known that one has always $r_0 > 0$. For $(\mathrm{III}_v^{(2)})$ $(v \geq 1)$, one has $r_0 = 0$ if and only if $v = 1$ ([Sc], p. 352, Ex. 1.8, (iii)). For $(\mathrm{IV}_{2v-2}^{(2)})$ and $(\mathrm{II}_v^{(2)})$ with $v \geq 4$, one has $r_0 > 0$ by [Sc], Lem. 10.3.5 and Th. 10.4.1, (i) (Kneser); and the same is also true for $(\mathrm{II}_4\text{–}\mathrm{IV}_6^{(2)})$. For $(\mathrm{I}_{p,q}^{(\delta)})$ with $v \geq 3$ and $q \geq \delta$, one has $r_0 > 0$ by [Sc], Th. 10.6.2 and p. 374, Rem. 6.3 and 6.4. Note that, when a place v is "non-decomposed" in F'/F (which is the case for all real places), the localization h_v is "isotropic" (in the sense of [Sc], p. 373), if and only if the corresponding F'_v-hermitian form of δv variables has Witt index $\geq \delta$ (Lemma 1 below).

It is well known that for the anisotropic cases in Theorem 1 one has the following isomorphisms:

$$(1.2) \qquad (\mathrm{III}_1^{(2)}) \cong (\mathrm{I}_{1,1}^{(\delta)}) \qquad (\delta = 1 \text{ or } 2),$$

$$(1.3) \qquad (\mathrm{IV}_4^{(2)}) \cong (\mathrm{I}_{2,2}^{(\delta)}) \qquad (\delta = 2 \text{ or } 4),$$

$$(1.3) \qquad (\mathrm{II}_3^{(2)}) \cong (\mathrm{I}_{3,1}^{(\delta)}) \qquad (\delta = 2 \text{ or } 4).$$

Thus all the classical anisotropic cases can be reduced to the unitary case $(\mathrm{I}_{p,q}^{(\delta)})$.

2. Unitary involutions and hermitian forms.

2.1 In order to fix the notation and terminology, we recall briefly some basic facts on hermitian forms pertinent to our considerations.

Let F'/F be a quadratic extension (in characteristic 0) with $\mathrm{Gal}(F'/F) = \{1, \rho_0\}$. Let D' be a central division algebra over F' of degree δ, i.e., $\dim_{F'} D' = \delta^2$, with a "unitary involution" ρ with respect to F'/F, (i.e., ρ is an F'-semilinear, involutive antiautomorphism of D'). Let V be a right D'-module of rank v and, for $\alpha \in D'$, let μ_α denote the right multiplication $v \mapsto v\alpha$ $(v \in V)$. By a "(D, ρ)-hermitian form" h on V, we mean an F-bilinear map $h\colon V \times V \to D'$ satisfying the conditions

$$h(v, v'\alpha) = h(v, v')\alpha\,, \tag{2.1}$$

$$h(v, v') = h(v', v)^\rho \tag{2.2}$$

for all $v, v' \in V$ and $\alpha \in D'$. It is easy to see that h can be identified with an F-linear map $V \mapsto V^*$ denoted also by h, satisfying the conditions

$$h \circ \mu_\alpha = {}^t\mu_{\alpha^\rho} \circ h\,, \tag{2.1$'$}$$

$$\langle h(v), v' \rangle = \langle h(v'), v \rangle^\rho \tag{2.2$'$}$$

for all $v, v' \in V$ and $\alpha \in D'$, the identification being made by the relation

$$\mathrm{tr}_{D'/F'}(h(v, v')) = \langle h(v), v' \rangle\,, \tag{2.3}$$

where $\mathrm{tr}_{D'/F'}$ is (as always) the reduced trace.

Let $A = \mathrm{End}(V/D')$ be the algebra of all D'-linear endomorphisms of V. Then A is a central simple algebra over F' and the F-linear map

$$\rho_h\colon x \mapsto h^{-1}\,{}^t x h \qquad (x \in A) \tag{2.4}$$

is a unitary involution of A with respect to F'/F. Clearly one has

$$h(x^{\rho_h}(v), v') = h(v, xv') \qquad (x \in A,\ v, v' \in V)\,. \tag{2.5}$$

One fixes a D'-basis $(e_1, \cdots, e_\nu)$ of V, which gives an isomorphism $M\colon A \xrightarrow{\sim} M_\nu(D')$. One denotes by $\mathrm{Her}_\nu(D', \rho)$ the space of ρ-hermitian matrices in $M_\nu(D')$; in particular, $\mathrm{Her}_1(D', \rho)$ is the space of ρ-invariant elements in D'. The hermitian form h is represented by a ρ-hermitian matrix $(h(e_k, e_l))_{1 \le k, l \le \nu}$. In what follows, we write $M(h) = (h(e_k, e_l)) \in \mathrm{Her}_\nu(D', \rho)$ and set $\det(h) = N(M(h)) \ (\mathrm{mod}^\times N(F'/F))$, where (the first) N denotes the reduced norm of $M_\nu(D')$ over F' and $N(F'/F)$ stands for $N_{F'/F}(F'^\times)$. We always assume that h is "non-degenerate", i.e., $\det(h) \neq 0$; then $\det(h)$ is an element of $F^\times/N(F'/F)$. The (multiplicative) equivalence relation in $F^\times$ $(\mathrm{mod}^\times N(F'/F))$ will be written as $\sim$.

2.2. Now let E'/E be another quadratic extension and D'' a central division algebra over E' with a unitary involution ρ' with respect to E'/E. Suppose there is given an imbedding $F' \subsetneq E'$ such that $F' \cap E = F$; then $D'_{E'} = D' \otimes_{F'} E'$ is a central simple algebra over E' and the involution ρ can naturally be extended to an involution of $D'_{E'}$, which is again denoted by ρ. Suppose that one has an E'-isomorphism $M_{E'}\colon D'_{E'} \xrightarrow{\sim} M_{\delta'}(D'')$. Then, as is well known, there exists an invertible element $A \in \mathrm{Her}_{\delta'}(D'', \rho')$ determined uniquely modulo $E^\times$ such that

$$M_{E'}(x^\rho) = A^{-1}\,{}^t M_{E'}(x)^{\rho'} A \qquad (x \in D'_{E'})\,. \tag{2.6}$$

Clearly x is ρ-invariant if and only if $AM_{E'}(x) \in \mathrm{Her}_{\delta'}(D'', \rho')$. More generally, for any positive integer ν one has an E'-isomorphism $M_{E'} \otimes \mathrm{id}_{M_\nu}\colon M_\nu(D'_{E'}) \xrightarrow{\sim} M_{\nu\delta'}(D'')$ and for $x \in M_\nu(D')$ one has $x \in \mathrm{Her}_\nu(D', \rho)$ if and only if

$$\tilde{M}_{E'}(x) = (A \otimes 1_\nu) \cdot (M_{E'} \otimes \mathrm{id}_{M_\nu})(x) \in \mathrm{Her}_{\nu\delta'}(D'', \rho') .$$

For a (D', ρ)-hermitian form h, we call $\tilde{M}_{E'}(M(h))$ the "matrix expression" of h in D'' (determined by M, $M_{E'}$ and A) and write $h \simeq \tilde{M}_{E'}(M(h))$. Clearly one has

$$\det(h) \sim N(A)^{-\nu} N(\tilde{M}_{E'}(M(h))) \qquad \text{in} \quad E^\times / N(E'/E) .$$

2.3. Now we assume that F is an algebraic number field of finite degree. For a place v of F, let F_v denote the completion of F at v. Then $F'_v = F_v \otimes_F F'$ is either isomorphic to $F_v \oplus F_v$ or a quadratic extension of F_v; accordingly one says that v is "decomposed" or "non-decomposed" in F'/F. One sets $D'_v = D' \otimes_F F_v$, $V_v = V \otimes_F F_v$, $\cdots$, etc. Then V_v is a vector space over F_v with a structure of free D'_v-module of rank ν. Any (D', ρ)-hermitian form h on V can naturally be extended to a (D'_v, ρ)-hermitian form $h_v \colon V_v \times V_v \to D'_v$, called the "localization" of h at v. The localization h_v is called "isotropic" if there exists a D'_v-basis $(e'_1, \cdots, e'_\nu)$ of V_v such that $h_v(e'_1, e'_1) = 0$ (see [Sc], p. 373).

It is well known (after Jacobson, cf. [Sc], Th. 10.2.2, (ii)) that, when v is non-decomposed, one has an isomorphism $M_v \colon D'_v \xrightarrow{\sim} M_\delta(F'_v)$, which gives a matrix expression of h in F'_v: $\tilde{M}_v(M(h)) = A_v M_v(h_v(e'_k, e'_l)) \in \mathrm{Her}_{\delta\nu}(F'_v/F_v)$. The following lemma is easily proved.

LEMMA 1. *The localization h_v is "isotropic" if and only if v is decomposed in F'/F and $\nu \geq 2$ or v is non-decomposed in F'/F and the matrix expression of h in F'_v has Witt index $\geq \delta$.*

2.4. Now assume further that F'/F is a CM-extension with a given CM-type in the sense of [Sa2], 1.2, i.e., F is a totally real number field of degree m, F'/F is a totally imaginary quadratic extension, and for each $1 \leq i \leq m$ one fixes an extension of $\sigma_i \colon F \subsetneq \boldsymbol{R}$ to an imbedding $F' \subsetneq \boldsymbol{C}$, denoted again by σ_i. (We always assume that σ_1 is the inclusion map.) Then, for each real place $v = \sigma_i$ of F, extended to a complex place of F', one can identify F'_v with $\boldsymbol{C}$ and D'_v with $(D'^{\sigma_i})_{\boldsymbol{C}} = D' \otimes_{F', \sigma_i} \boldsymbol{C}$. One fixes an isomorphism

$$M_i \colon (D'^{\sigma_i})_{\boldsymbol{C}} \xrightarrow{\sim} M_\delta(\boldsymbol{C}) ;$$

then there exists an invertible element $A_i \in \mathrm{Her}_\delta(\boldsymbol{C})$ such that

$$(2.7) \qquad M_i(x^{\sigma_i^{-1}\rho\sigma_i}) = A_i^{-1}\, {}^t\overline{M_i(x)}\, A_i \qquad (x \in D'^{\sigma_i}) .$$

The involution ρ is called "totally positive" if all A_i $(1 \leq i \leq m)$ can be taken to be positive definite. (This condition is equivalent to saying that $R_{F/\boldsymbol{Q}}(\rho)$ is a "positive involution" of $R_{F'/\boldsymbol{Q}}(D')$ in the sense of [W].) It is well known ([A], Ch. X) that, for a CM-field F'/F, a central division algebra D' over F' has a unitary involution with respect to F'/F if and only if D' can be expressed as a cyclic algebra (Z', σ, α), where $Z' = ZF'$ with a cyclic extension Z of F of degree δ not containing in F', σ is a generator of $\mathrm{Gal}(Z'/F') \simeq \mathrm{Gal}(Z/F)$, and α is an element of F' such that $\alpha^{1+\rho_0} \in N(Z/F)$, ρ_0 denoting the complex conjugation. Under this condition, the algebra D' has always a totally positive unitary involution with respect to F'/F. In what follows, we always assume that *the above condition is satisfied, ρ is totally positive and the A_i $(1 \leq i \leq m)$ are taken to be positive definite.* A (D', ρ)-hermitian form h, or a hermitian matrix $M(h) \in \mathrm{Her}_\nu(D', \rho)$, is called *positive* (resp., *totally positive*) if $\tilde{M}_1(M(h))$ is [resp., all $\tilde{M}_i(M(h)^{\sigma_i}) = (A_i \otimes 1_\nu)((M_i \otimes \mathrm{id})(M(h)^{\sigma_i}))$ $(1 \leq i \leq m)$ are] positive definite. In notation, we write $h > 0$ or $M(h) > 0$ (resp., $h \gg 0$ or $M(h) \gg 0$). More generally we define $\mathrm{sign}(h^{\sigma_i}) = (p_i, q_i)$ to be the signature of $\tilde{M}_i(M(h)^{\sigma_i})$. (Note that these notions are independent of the choices of M, M_i and A_i under the above assumptions.) Then one has:

THEOREM 2. *Let h be a non-degenerate (D', ρ)-hermitian form on V $(\cong D'^\nu)$ over a CM-field F'. Then h is isotropic if and only if the following two conditions are satisfied*:

(i) $\mathrm{Min}(p_i, q_i) \geq \delta$ *for* $1 \leq i \leq m$,

(ii) $\nu \geq 3$ *or* $\nu = 2$ *and* $\det(h) \sim (-1)^\delta$ $(\mathrm{mod}^\times N(F'/F))$.

PROOF. First suppose that h is isotropic. Then clearly $\nu \geq 2$ and the condition (i) is satisfied. When $\nu = 2$, h is necessarily hyperbolic, i.e.,

$$h \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

in D'. Hence $\det(h) \sim (-1)^\delta$.

Conversely, suppose that the conditions (i), (ii) are satisfied. When $\nu = 2$, one has by the assumption $p_i = q_i = \delta$ and $\det(h) \sim (-1)^\delta$. Hence the hermitian form h and the hyperbolic form of 2 variables have the same invariants (determinant and signatures), so that one has

$$h \simeq \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

([Sc], Cor. 10.6.6). Next suppose that $\nu \geq 3$. By the assumption, one has $\mathrm{Min}(p_i, q_i) \geq \delta$ for all $1 \leq i \leq m$. For a finite place $v = \mathfrak{p}$, which is non-decomposed in F'/F, let $r_\mathfrak{p}$ denote the Witt index of the matrix expression of h in $F'_\mathfrak{p}$ (defined by the imbedding $F'/F \subsetneq F'_\mathfrak{p}/F_\mathfrak{p}$). Then by the theory of hermitian forms in $\mathfrak{p}$-adic fields one has

$$r_\mathfrak{p} = \begin{cases} \dfrac{1}{2}(\delta\nu - 1) & \text{if } \delta\nu \text{ is odd}, \\[2ex] \dfrac{1}{2}\delta\nu & \text{if } \delta\nu \text{ is even and } \det(h_v) \sim -1, \\[2ex] \dfrac{1}{2}\delta\nu - 1 & \text{otherwise}. \end{cases} \tag{2.8}$$

In all cases, since $\nu \geq 3$, one has $r_{\mathfrak{p}} \geq \delta$. Hence by Lemma 1 the localizations $h_{\mathfrak{p}}$ are isotropic everywhere and so h is globally isotropic ([Sc], Th. 10.6.2). q.e.d.

3. Domains of type $(\mathrm{I}_{p,q}^{(\delta)})$ with rational points.

3.1. Let $F' = F(\sqrt{-\beta})$ be a CM-field (with the "standard" CM-type determined by $\sqrt{-\beta^{\sigma_i}} = \sqrt{-1}(\beta^{\sigma_i})^{1/2}$, see [Sa2], 1.3), D' a central division algebra over F' of degree δ with a totally positive unitary involution ρ with respect to F'/F, and h a non-degenerate (D', ρ)-hermitian form on $V \cong D'^{\nu}$ with $\delta\nu \geq 2$. The Lie algebra $\mathfrak{g}_1 = \mathfrak{su}(\nu, h, D'/F'/F)_{\boldsymbol{R}}$ is by definition a real Lie algebra defined over F such that

$$\mathfrak{g}_1(F) = \{x \in A = \mathrm{End}(V/D') \mid \mathrm{tr}_{A/F'}(x) = 0,\ x^{\rho_h} = -x\}\ . \tag{3.1}$$

Then $\mathfrak{g} = R_{\boldsymbol{R}/\boldsymbol{Q}}(\mathfrak{g}_1)$ is a real semisimple Lie algebra of hermitian type with a ***Q**-simple **Q**-structure*. $\mathfrak{g}$ satisfies (R1) if and only if one has $(p_i, q_i) = (p, q)$ or (q, p) $(p \geq q)$ for all $1 \leq i \leq m$, in which case $\mathfrak{g}$ is of type $(\mathrm{I}_{p,q}^{(\delta)})$ in the notation of 1.1. Replacing h by a suitable scalar multiple μh with $\mu \in F^{\times}$ if necessary, we will henceforce assume that $p_i = p$, $q_i = q$ for all $1 \leq i \leq m$. We will also assume that $\mathfrak{g}$ is non-compact, i.e., $q > 0$, unless otherwise expressed.

Let $\mathscr{D} = \mathscr{D}(V/D', h)$ be the symmetric domain associated with $\mathfrak{g}$, which we regard as the set of Cartan involutions of $\mathfrak{g}$. Let $\theta \in \mathscr{D}$ and $\theta = (\theta_i)$ with Cartan involutions θ_i of $\mathfrak{g}_1^{\sigma_i}$. Then θ is "rational" if and only if θ_1 is F-rational and one has $\theta_i = \theta_1^{\sigma_i}$ for all $1 \leq i \leq m$. It is classical (cf., e.g., [W]) that all F-rational Cartan involutions θ_1 of $\mathfrak{g}_1$ are obtained in the form $\theta_1 = -\rho' | \mathfrak{g}_1(F)$ with positive unitary involutions ρ' of A such that $[\rho_h, \rho'] = 0$. Such an involution ρ' can be written as $\rho' = \rho_{h'}$ with a positive (D', ρ)-hermitian form h' on V, and $\theta_1^{\sigma_i}$ is a Cartan involution of $\mathfrak{g}_1^{\sigma_i}$ for all i if and only if h' can be taken to be totally positive. The hermitian form h' is then uniquely determined modulo the multiplicative group of all totally positive elements in $F^{\times}$, which we denote by $F_+^{\times}$. It is easy to see that one has $[\rho_h, \rho_{h'}] = 0$ if and only if

$$(h^{-1}h')^2 = \lambda 1 \qquad \text{with} \quad \lambda \in F^{\times}\ . \tag{3.2}$$

This condition implies $h^{-1}h'h^{-1} = \lambda h'^{-1}$, whence one has $\lambda \in F_+^{\times}$. Clearly λ is uniquely determined by ρ' modulo $(F_+^{\times})^2$. We denote by $\mathscr{P}(V/D', \rho, h)$ the set of all totally positive (D', ρ)-hermitian forms h' on V satisfying (3.2). Then the correspondence $\theta \mapsto h'$ $(\mathrm{mod}\, F_+^{\times})$ gives a bijective correspondence between the set of rational points in $\mathscr{D}$ and $\mathscr{P}(V/D', \rho, h)/F_+^{\times}$.

LEMMA 2. *Let $h' \in \mathscr{P}(V/D', \rho, h)$ and $T = h^{-1}h'$. Then T is a D'-linear endomorphism of V having the following properties*:

$$T^2 = \lambda 1 \qquad \textit{with} \quad \lambda \in F_+^{\times} \quad \textit{and} \quad T^{\rho_h} = T\ .$$

The last equation follows from the relation

$$h(T^{\rho_h}v, v')=h(v, Tv')=h'(v, v')=h(Tv, v').$$

Now let $\theta=(\theta_1^{\sigma_i})$ be the Cartan involution corresponding to $h'\in\mathscr{P}(V/D', \rho, h)$ and $\mathfrak{k}_1$ the maximal compact subalgebra of $\mathfrak{g}_1$ corresponding to θ_1. Then for $x\in\mathfrak{g}_1$ one has

$$x\in\mathfrak{k}_1\Leftrightarrow\theta_1 x=x\Leftrightarrow x^{\rho_{h'}}=-x\Leftrightarrow[T, x]=0. \tag{3.3}$$

[According to our convention, we denote the $\boldsymbol{R}$-linear extension of ρ (resp. ρ_h or $\rho_{h'}$) to $D'_{\boldsymbol{C}}=D'\otimes_{F'}\boldsymbol{C}$ (resp. $A_{\boldsymbol{C}}=A\otimes_{F'}\boldsymbol{C}$) by the same letter.]

By Lemma 2, one has $(\sqrt{-\beta}\,T)^{\rho_h}=-\sqrt{-\beta}\,T$. On the other hand, one has

$$\mathrm{tr}_{A^{\sigma_i}/F'^{\sigma_i}}T^{\sigma_i}=(p-q)(\lambda^{\sigma_i})^{1/2} \qquad (1\leq i\leq m).$$

Hence, if $p>q$, one has $\sqrt{\lambda}\in F^\times$ and $\sqrt{\lambda}^{\,\sigma_i}=(\lambda^{\sigma_i})^{1/2}$, which means $\sqrt{\lambda}\in F_+^\times$. In general, one puts

$$T'=T-\frac{p-q}{p+q}\sqrt{\lambda}\,1_V. \tag{3.4}$$

Then one has $\sqrt{-\beta}\,T'\in\mathfrak{g}_1(F)$, and from the above one sees that the centralizer of $(\sqrt{-\beta}\,T')^{\sigma_i}$ in $\mathfrak{g}_1^{\sigma_i}$ coincides with the maximal compact subalgebra $\mathfrak{k}_1^{\sigma_i}$. Hence, if one puts

$$H_0=(H_{0,i}), \qquad H_{0,i}=\frac{1}{2}((\beta\lambda)^{\sigma_i})^{-1/2}(\sqrt{-\beta}\,T')^{\sigma_i}\in\mathfrak{g}_1^{\sigma_i}, \tag{3.5}$$

then H_0 is an "H-element" of $\mathfrak{g}$ and the matrix $(M_i\otimes\mathrm{id}_{M_v})(H_{0,i})$ is similar to

$$\mathrm{diag}\left(\frac{q}{p+q}\sqrt{-1}\,1_p, -\frac{p}{p+q}\sqrt{-1}\,1_q\right)$$

for all $1\leq i\leq m$. We will henceforth assume that the complex structure on $\mathscr{D}$ is compatible with this H-element. Then the point θ in $\mathscr{D}$ corresponding to H_0 is a rational point with CM-field $F''=F(\sqrt{-\beta\lambda})$ endowed with the standard CM-type. The set of all such rational points in $\mathscr{D}$ is denoted by $\mathscr{D}(F(\sqrt{-\beta\lambda})/F)$ (see [Sa], 1.3; 3.4).

Summing up, one has

THEOREM 3. *Let $\mathscr{D}=\mathscr{D}(V/D', h)$ be the symmetric domain associated with $\mathfrak{g}$. For $h'\in\mathscr{P}(V/D', \rho, h)$, let $T=h^{-1}h'$ and $T^2=\lambda 1$ with $\lambda\in F_+^\times$. Then, under the above assumption, $\theta=(\theta_1^{\sigma_i})$, $\theta_1=-\rho_{h'}|\mathfrak{g}$ is a rational point in $\mathscr{D}$ with CM-field $F''=F(\sqrt{-\beta\lambda})$ (endowed with the standard CM-type) and, for a fixed $\lambda\in F_+^\times$, the map $h'\mapsto\theta$ gives a bijective correspondence between $\{h'\in\mathscr{P}(V/D', \rho, h)\,|\,(h^{-1}h')^2=\lambda 1\}$ and $\mathscr{D}(F\sqrt{-\beta\lambda})/F)$.*

3.2. For $\lambda\in F_+^\times$, set $E=F(\sqrt{\lambda})$, $E'=EF'$, and $D'_{E'}=D'\otimes_{F'}E'$. We distinguish three cases:

1. The case where $\lambda\in(F^\times)^2$, i.e., $E=F$.

2. The case where $\lambda \notin (F^\times)^2$. In this case, one has $[E:F]=[E':F']=2$.

2.1. The case where $D'_{E'}$ remains division.

2.2. The case where $D'_{E'}$ is not division.

A rational point θ with CM-field $F(\sqrt{-\beta\lambda})$ will be called *of the first, second, and third type*, according as λ is in Case 1, 2.1, and 2.2.

REMARK. As mentioned in the Introduction, the notion of the rational points is a special case of the "isolated fixed points" in [Sh] and [M]. Actually, if one puts

$$j_0 = (((\beta\lambda)^{\sigma_i})^{-1/2}(\sqrt{-\beta}\,T)^{\sigma_i}) \in M_\nu(D_1)\otimes_{\boldsymbol{Q}} \boldsymbol{R} = \bigoplus_{i=1}^{m} M_\nu(D_1^{\sigma_i})_{\boldsymbol{C}},$$

then j_0 satisfies the condition in [M], 1.4 and the symmetric domain $\mathscr{D}$ (with the complex structure specified above) is identified with $\mathscr{H}_{j_0}$ in [M]. In the notation there, $C(j_0)$ is the commutor of T in $M_\nu(D')$ and hence

$$C(j_0)_{\boldsymbol{R}} \simeq \bigoplus^{m} (M_p(\boldsymbol{C}) \oplus M_q(\boldsymbol{C})).$$

Therefore $P=\tilde{C}(j_0)$ (the commutor of $C(j_0)$ in $M_\nu(D')$) coincides with the center of $C(j_0)$ (which assures that j_0 is an "isolated fixed point"). One has

$$P = \begin{cases} P_1 \oplus P_2\,, \quad P_1 \simeq P_2 \simeq F' & \text{in Case 1}\,, \\ P_1 = F'(T) \simeq E' & \text{in Case 2}\,. \end{cases}$$

In Case 2, E'/E is a CM-extension with $\mathrm{Gal}(E'/E)=\{1, \rho_0\}$. Let $\mathrm{Gal}(E'/F')=\{1, \tau\}$. We extend $\sigma_i: F' \subsetneq \boldsymbol{C}$ to an imbedding $E' \subsetneq \boldsymbol{C}$ (denoted by the same letter) by setting $\sqrt{\lambda}^{\,\sigma_i} = (\lambda^{\sigma_i})^{1/2}$ (σ_1 being the inclusion map). Then $\{\sigma_i|E, \tau\sigma_i|E\ (1 \le i \le m)\}$ is the set of distinct imbeddings of E into $\boldsymbol{R}$ and the standard CM-type of $E'=E(\sqrt{-\beta\lambda})$ is given by

$$\sqrt{-\beta\lambda}^{\,\sigma_i} = \sqrt{-\beta\lambda}^{\,\rho_0\tau\sigma_i} = \sqrt{-1}\,((\beta\lambda)^{\sigma_i})^{1/2}\,. \tag{3.6}$$

In Case 2.2, let $D'_{E'} \cong M_{\delta'}(D'_1)$ with a central division algebra D'_1 of degree $\delta_1 = \delta/\delta'$. Then a simple right ideal of $D'_{E'}$ is of D'_1-rank δ' and of D'-rank $2/\delta'$. Since $\delta'>1$, one has $\delta'=2$ and $\delta_1=\delta/2$.

The following necessary condition for the existence of rational points was obtained by K. Oiso by a different method (unpublished).

THEOREM 4. *Suppose that the symmetric domain $\mathscr{D}(V/D', h)$ has a rational point. Then, in addition to* (R1), *the following condition* (R2) *is satisfied.*
(R2) $\delta | q$ *or* $p=q$.

PROOF. By the assumption, $\mathscr{P}(V/D', \rho, h) \neq \varnothing$. Let $h' \in \mathscr{P}(V/D', \rho, h)$ and $T=h^{-1}h'$, $T^2=\lambda 1$ with $\lambda \in F_+^\times$. Then by the observation in 3.1, one has $p=q$ in Case 2. Hence it is enough to show that in Case 1 one has $\delta | q$. In this case, one has $\sqrt{\lambda} \in F^\times$ ($\in F_+^\times$ if $p>q$). Hence, for simplicity, replacing h by $\sqrt{\lambda}\,h$, we assume that $\lambda=1$. Put

$$V_{\pm}=\{v\in V \mid Tv=\pm v\}\ .$$

Then $V_{\pm}$ are D'-submodules and one has a direct sum decomposition $V=V_{+}\oplus V_{-}$. It is easy to see that the subspaces V_{+} and V_{-} are mutually orthogonal with respect to h and h'. Let $v_1=\operatorname{rank}_{D'}V_{+}$ and let $(e_1,\cdots,e_v)$ be an orthogonal D'-basis of V with respect to h such that $(e_1,\cdots,e_{v_1})$ is a D'-basis of V_{+}. Then one has a matrix expression of h in D' of the form

$$h\simeq \operatorname{diag}(a_1,\cdots,a_{v_1},a_{v_1+1},\cdots,a_v)$$

and hence

$$h'\simeq \operatorname{diag}(a_1,\cdots,a_{v_1},-a_{v_1+1},\cdots,-a_v)\ .$$

Since h' is totally positive, one should have $a_k\gg 0$ $(1\leq k\leq v_1)$ and $a_k\ll 0$ $(v_1+1\leq k\leq v)$. Thus one has

$$p=\delta v_1\ ,\qquad q=\delta(v-v_1)\ ,$$

which proves our assertion. q.e.d.

COROLLARY. *Suppose that $\mathscr{D}=\mathscr{D}(V/D',h)$ is not of tube type and has a last rational boundary component of positive dimension (i.e., $\mathscr{D}$ is "of type* (U2)", *see* 5.1). *Then $\mathscr{D}$ has no rational points.*

PROOF. Suppose that $\mathscr{D}$ has a rational point. Then by Theorem 4 and the assumptions the last rational boundary component $\mathscr{F}_0$ of $\mathscr{D}$ is of type $(\mathrm{I}^{(\delta)}_{p_0q_0})$ with $p_0>q_0>0$ and $\delta|q_0$. It then follows that $v_0=(p_0+q_0)/\delta\geq 3$, which contradicts Theorem 1. q.e.d.

3.3. Assuming the existence of rational points with CM-field $F(\sqrt{-\beta\lambda})$, we further consider Case 2. In the above notation, let $V_{E'}=V\otimes_{F'}E'$ and, extending T to an endomorphism of $V_{E'}$ by linearity, put

$$V_{\pm}=\{v\in V_{E'} \mid Tv=\pm\sqrt{\lambda}\,v\}\ .$$

Then one has a direct sum decomposition

$$V_{E'}=V_{+}\oplus V_{-}\ ,\qquad V_{-}=V_{+}^{\tau}\ ,$$

where the subspaces $V_{\pm}$ are $D'_{E'}$-invariant and mutually orthogonal with respect to (the natural extensions of) h and h'.

In Case 2.1 where $D'_{E'}$ is division, let $v_1=D'_{E'}$-rank V_{+}; then one has $v=2v_1$. Let $(e_1,\cdots,e_{v_1})$ be an orthogonal $D'_{E'}$-basis of V_{+} with respect to h; then one has

$$h|V_{+}\simeq \operatorname{diag}(a_1,\cdots,a_{v_1})\ ,\qquad a_k\in D'_{E'}\ . \tag{3.7}$$

Clearly $(e_1^{\tau},\cdots,e_{v_1}^{\tau})$ is an orthogonal $D'_{E'}$-basis of $V_{-}=V_{+}^{\tau}$ and one has

$$h|V_{-}\simeq \operatorname{diag}(a_1^{\tau},\cdots,a_{v_1}^{\tau})\ .$$

In the $D'_{E'}$-basis $(e_1, \cdots, e_{\nu_1}, e_1^\tau, \cdots, e_{\nu_1}^\tau)$ of $V_{E'}$ one then has

$$h' \simeq \mathrm{diag}(\sqrt{\lambda}\, a_1, \cdots, \sqrt{\lambda}\, a_{\nu_1}, -\sqrt{\lambda}\, a_1^\tau, \cdots, -\sqrt{\lambda}\, a_{\nu_1}^\tau).$$

Hence one has $\sqrt{\lambda}\, a_k \gg 0$ $(1 \le k \le \nu_1)$ and $p = q = \delta\nu_1$. It should be noted that one has

$$\det(h) \sim \lambda^{\delta\nu_1} \prod_{i=1}^{\nu_1} N(a_i)^{1+\tau} \quad \text{in} \quad F^\times/N(F'/F). \tag{3.8}$$

In Case 2.2 where $D'_{E'}$ is not division, one has $2|\delta$. The space V, viewed as a (T, D')-module, can be endowed with a structure of (right) $D'_{E'}$-module by setting $Tv = \sqrt{\lambda}\, v$ for $v \in V$. Since $D'_{E'}$-modules are completely reducible, the problem can be reduced to the case where V is irreducible. Then, the space V, being isomorphic to a simple right ideal of $D'_{E'}$, is of D'-rank one. Hence, let $V = \{e_1\}_{D'}$ and set $Te_1 = e_1 t_1$ with $t_1 \in D'$. Then $D'_{E'}$ contains matrix units e_{kl} $(k, l = 1, 2)$ such that

$$e_{11} = \frac{1}{2}(1 + \sqrt{\lambda}^{-1} t_1), \qquad e_{22} = \frac{1}{2}(1 - \sqrt{\lambda}^{-1} t_1). \tag{3.9}$$

Let D'_1 be the centralizer of $\{e_{kl}\ (k, l = 1, 2)\}$ in $D'_{E'}$. Then from the above D'_1 is a central division algebra of degree $\delta_1 = \delta/2$ with unitary involution with respect to E'/E and one has an E'-isomorphism $M_{E'}\colon D'_{E'} \xrightarrow{\sim} M_2(D'_1)$. Let ρ_1 be a totally positive involution of D'_1 with respect to E'/E. Then one has

$$M_{E'}(x^\rho) = A^{-1}\, {}^t M_{E'}(x)^{\rho_1} A \qquad (x \in D') \tag{3.10}$$

with $A \in \mathrm{Her}_2(D'_1)$, which one assumes to be totally positive. Let $h(e_1, e_1) = a_1$. Then one has $h'(e_1, e_1) = h(e_1, e_1 t_1) = a_1 t_1$ and so $a_1 t_1 = (a_1 t_1)^\rho = t_1^\rho a_1$. It follows that

$$M_{E'}(a_1) M_{E'}(t_1) = M_{E'}(t_1^\rho) M_{E'}(a_1) = A^{-1}\, {}^t M_{E'}(t_1)^{\rho_1} A M_{E'}(a_1).$$

Since

$$M_{E'}(t_1) = \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} \end{pmatrix},$$

setting $\tilde{M}_{E'}(\cdot) = A M_{E'}(\cdot)$, one has

$$\left[\tilde{M}_{E'}(a_1), \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & -\sqrt{\lambda} \end{pmatrix}\right] = 0.$$

It follows that the expressions of h and h' in D'_1 are of the form

$$h \simeq \tilde{M}_{E'}(a_1) = \begin{pmatrix} a'_1 & 0 \\ 0 & a''_1 \end{pmatrix}, \tag{3.11}$$

$$h' \simeq \tilde{M}_{E'}(a_1 t_1) = \begin{pmatrix} \sqrt{\lambda}\, a'_1 & 0 \\ 0 & -\sqrt{\lambda}\, a''_1 \end{pmatrix}.$$

Hence one has $\sqrt{\lambda}\, a'_1 \gg 0$ and $\sqrt{\lambda}\, a''_1 \ll 0$.

4. The necessary and sufficient conditions for the existence of rational points.

4.1. We retain the notation and the assumptions in §3. First we consider Case 1.

THEOREM 5. *The symmetric domain $\mathscr{D} = \mathscr{D}(V/D', h)$ has a rational point of the first type if and only if condition* (R1) *and the following condition* (R2.1) *are satisfied*:
(R2.1) $\delta | q$.

PROOF. The "only if" part was already shown in the proof of Theorem 4. By virtue of the result in [Sa2], 3.4, for the proof of the "if" part one may assume that $r_0 = 0$, i.e., h is anisotropic. If h is definite ($q = 0$), then $\mathscr{D}$ reduces to a point, which may be regarded as a rational point of the first type. (Note that in our case the special CM-field mentioned in [Sa2], 3.4, 2° coincides with $F(\sqrt{-\beta})$.) If h is anisotropic but indefinite ($q > 0$), then by (R2.1) and Theorem 2 one has $v = 2$, $p = q = \delta$ and $\det(h) \not\succ (-1)^{\delta}$. Take any $a_1 \in \mathrm{Her}_1(D', \rho)$ which is totally positive. By [Sc], Th. 10.6.9, one can then find $a_2 \in \mathrm{Her}_1(D', \rho)$ which is totally negative and satisfying the relation $N(a_2) = N(a_1)^{-1} \det(h)$. Then h and the hermitian form represented by the matrix $\mathrm{diag}(a_1, a_2)$ have the same invariants and hence are mutually equivalent ([Sc], Cor. 10.6.6). In other words, one has

$$h \simeq \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

for some D'-basis of V. Then the hermitian form h' on V defined by

$$h' \simeq \begin{pmatrix} a_1 & 0 \\ 0 & -a_2 \end{pmatrix}$$

in the same basis belongs to $\mathscr{P}(V/D', \rho, h)$ and gives a rational point in $\mathscr{D}$ of the first type. q.e.d.

4.2. Next, suppose that λ is in Case 2, i.e., $\lambda \in F_+^\times$, but $\notin (F^\times)^2$.

THEOREM 6. *Assume that λ is in Case* 2.1. *Then the symmetric domain $\mathscr{D} = \mathscr{D}(V/D', h)$ has a rational point of the second type with CM-field $F(\sqrt{-\beta\lambda})$ if and only if conditions* (R1), (R2.1) *and the following conditions* (R2.2) *and* (R3) *are satisfied*:
(R2.2) $p = q$.
(R3) $(-1)^{\delta v/2} \det(h) \in N(E/F) N(F'/F)$.

REMARK. Note that under condition (R2.2) one has $\delta v/2 = p$ and hence $p \equiv 0$ or $\equiv \delta/2 \pmod{\delta}$. Note also that, if one sets

$$D(h) = (-\beta, (-1)^{\delta v/2} \det(h))_F\,,$$

then condition (R3) is equivalent to saying that $F'' = F(\sqrt{-\beta\lambda})$ splits $D(h)$ (see Example

1 below).

PROOF. The "only if" part follows immediately from 3.3. To prove the "if" part, we may (hence will) again assume that h is anisotropic (and $\nu>0$). Then, by the assumptions, one has $\nu=2$, $p=q=\delta$ and there exists $\xi\in E^\times$ such that $(-1)^\delta\det(h)\sim\xi^{1+\tau}$ in $F^\times/N(F'/F)$. Here $\xi^{1+\tau}$ is totally positive and ξ may be replaced by $\xi\eta$ with any $\eta\in E^\times$ with $\eta^{1+\tau}=1$. Since the set $\{\eta\in E \mid \eta^{1+\tau}=1\}$ may be viewed as the set of F-rational points on a quadratic curve C defined over F, $C(F)=(R_{F/\boldsymbol{Q}}C)(\boldsymbol{Q})$, which is dense in $(R_{F/\boldsymbol{Q}}C)(\boldsymbol{R})$, one can choose ξ to be totally positive. (The author owes this argument to Y. Morita.) By [Sc], Th. 10.6.9, one can then find $a'_1\in\mathrm{Her}_1(D'_{E'},\rho)$ which is totally positive and satisfying $N(a'_1)\sim\xi$ in $E^\times/N(E'/E)$. Take $e'_1\in V_{E'}$ with $e'_1\neq e'^{\tau}_1$. Then (e'_1, e'^{τ}_1) is a $D'_{E'}$-basis of $V_{E'}$. Put $a_1=\sqrt{\lambda}^{-1}a'_1$ and let h_1 be a $(D'_{E'},\rho)$-hermitian form on $V_{E'}$ with matrix expression

$$h_1\simeq\begin{pmatrix} a_1 & 0\\ 0 & a_1^\tau\end{pmatrix}$$

in the basis (e'_1, e'^{τ}_1). Then one has $h_1^\tau=h_1$, i.e., h_1 comes from a (D',ρ)-hermitian form on V (which one denotes by the same letter h_1) by scalar extension. By the choice of a_1, one has

$$\mathrm{sign}(h_1^{\sigma_i})=(\delta,\delta)=\mathrm{sign}(h^{\sigma_i})\qquad(1\le i\le m)$$

and by (3.8)

$$\det(h_1)\sim\lambda^\delta N(\sqrt{\lambda}^{-1}a'_1)^{1+\tau}\sim(-1)^\delta\xi^{1+\tau}\sim\det(h)$$

in $F^\times/N(F'/F)$. Hence, by [Sc], Cor. 10.6.6, h and h_1 are equivalent over D'. This implies that there exists $e''_1\in V_{E'}$ such that $e''_1\neq e''^{\tau}_1$ and

$$h\simeq\begin{pmatrix} a_1 & 0\\ 0 & a_1^\tau\end{pmatrix}$$

in the $D'_{E'}$-basis (e''_1, e''^{τ}_1). Then the hermitian form h' on V defined by

$$h'\simeq\begin{pmatrix} a'_1 & 0\\ 0 & a'^{\tau}_1\end{pmatrix}$$

in the same basis belongs to $\mathscr{P}(V, D', \rho, h)$ and gives a rational point in $\mathscr{D}$ of the second type with CM-field $F(\sqrt{-\beta\lambda})$.

EXAMPLE 1: $(\mathrm{I}^{(1)}_{1,1})$. Let $\beta,\lambda\in F_+^\times$ and $F'=F(\sqrt{-\beta})$, $E=F(\sqrt{\lambda})$ be as above and take $\alpha\in F_+^\times$, $\notin N(F'/F)$. Let $D'=F'$ and $h=\mathrm{diag}(1,-\alpha)\in\mathrm{Her}_2(F'/F)$. Then $(p_i,q_i)=(1,1)$ $(1\le i\le m)$, h is anisotropic (Th. 2), and $D(h)=(\alpha,-\beta)_F$. Clearly, $D(h)$ is division and totally indefinite. It is easy to see that one has an F-isomorphism

$$\mathfrak{g}_1(F)=\mathfrak{su}(2,h,F'/F)\simeq\mathfrak{sl}(1,D(h))\,,\tag{4.1}$$

where $\mathfrak{sl}(1, D(h)) = D(h)_- = \{u \in D(h) \mid \mathrm{tr}_{D(h)/F}(u) = 0\}$. ((4.1) gives the isomorphism (1.2) with $\delta = 1$.) Conditions (R1), (R2.1), (R2.2) are clearly satisfied, and we are in Case 2.1. Condition (R3) is equivalent to saying that one has

$$\lambda = \xi^2 - \alpha(\eta_1^2 + \beta\eta_2^2) \qquad \text{with} \quad \xi, \eta_1, \eta_2 \in F, \tag{4.2}$$

or equivalently, $-\beta\lambda \in D(h)_-^2$. Thus we see that the corresponding domain $\mathscr{D}$ has rational points of the second type with CM-field $F'' = F(\sqrt{-\beta\lambda})$ if and only if F'' splits $D(h)$.

4.3. In order to treat the third case, we need more preparation. Suppose there is given a central division algebra D_1' of degree $\delta_1 = \delta/2$ over $E' = F'(\sqrt{\lambda})$ such that one has an E'-isomorphism $D'_{E'} \simeq M_2(D_1')$. Then, since $D_1' \sim D_1'^{\tau}$, one has an E'-isomorphism $\varphi: D_1' \tilde{\to} D_1'^{\tau}$. For $x_1 \in D_1'$, set $x_1^{[\tau]} = \varphi^{-1} x_1^{\tau}$. Then one has

$$x_1^{[\tau]^2} = \varphi^{-1}\varphi^{-\tau}(x_1) = f_1^{-1} x_1 f_1 \tag{4.3}$$

for some $f_1 \in D_1'^{\times}$, where one may (hence will) assume that $f_1^{[\tau]} = f_1$.

We fix an E'-isomorphism $M_{E'}: D'_{E'} \tilde{\to} M_2(D_1')$. Then one has

$$M_{E'}^{[\tau]}(\cdot) = \varphi^{-1} M_{E'}^{\tau}(\cdot) = C^{-1} M_{E'}(\cdot) C \tag{4.4}$$

with $C \in GL_2(D_1')$. It is easy to see that

$$CC^{[\tau]} = \gamma \begin{pmatrix} f_1 & 0 \\ 0 & f_1 \end{pmatrix} \tag{4.4a}$$

with $\gamma \in F'^{\times}$. We also fix a totally positive involution ρ_1 of D_1' with respect to E'/E and a totally positive element A in $\mathrm{Her}_2(D_1', \rho_1)$ satisfying (3.10). Put $\rho_1^{[\tau]} = [\tau]^{-1}\rho_1[\tau]$; then one has

$$x_1^{\rho_1^{[\tau]}} = b_1^{-1} x_1^{\rho_1} b_1 \tag{4.5}$$

with b_1 in $\mathrm{Her}_1(D_1', \rho_1)$. It follows from (4.3) and (4.5) that

$$b_1 b_1^{[\tau]} = \beta_1 f_1^{\rho_1} f_1 \tag{4.5a}$$

with $\beta_1 \in F'^{\times}$.

LEMMA 3. *One has*

$${}^t C^{\rho_1} A C A^{-[\tau]} = \varepsilon \begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix} \tag{4.6}$$

with $\varepsilon \in E^{\times}$.

PROOF. For $x \in D'_{E'}$, one has $x^{\rho\tau} = x^{\tau\rho}$. Hence, computing $M_{E'}^{[\tau]}(x^{\rho\tau}) = M_{E'}^{[\tau]}(x^{\tau\rho})$ in two different ways by (3.10), (4.4) and (4.5), one has

$$M_{E'}^{[\tau]}(x^{\rho\tau}) = A^{-[\tau]} \begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix}^{-1} {}^t C^{\rho_1}\, {}^t M_{E'}(x^{\tau})^{\rho_1}\, {}^t C^{-\rho_1} \begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix} A^{[\tau]},$$

and

$$M_{E'}^{[\tau]}(x^{\tau\rho}) = C^{-1}A^{-1}{}^tM_{E'}(x^{\tau})^{\rho_1}AC .$$

Hence one has

$$AC = \varepsilon\, {}^tC^{-\rho_1}\begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix} A^{[\tau]}$$

with $\varepsilon \in E'^{\times}$, which proves (4.6). This relation implies that $\varepsilon^{\rho_0} = \varepsilon$, i.e., $\varepsilon \in E^{\times}$. q.e.d.

From (4.6), using (4.5a), one obtains $\beta_1 = \varepsilon^{-1-\tau}\gamma^{1+\rho_0}$. Taking the norm of both sides of (4.4a), (4.5a) and (4.6), one has

(4.4b) $$N(C)^{1+\tau} = \gamma^{\delta} N_1(f_1)^2 ,$$

(4.5b) $$(\varepsilon^{\delta/2} N_1(b_1))^{1+\tau} = (\gamma^{\delta/2} N_1(f_1))^{1+\rho_0} ,$$

(4.6b) $$N(A)^{1-\tau} N(C)^{1+\rho_0} = \varepsilon^{\delta} N_1(b_1)^2 ,$$

where (and in what follows) N and N_1 denote the reduced norm of $M_2(D'_1)$ and D'_1, respectively. By (4.4b) one has $(N_1(\gamma f_1)^{-1}N(C))^{1+\tau} = 1$. Hence there exists $\eta \in E'^{\times}$, determined modulo $F'^{\times}$, such that

$$N_1(\gamma f_1)^{-1} N(C) = \eta^{1-\tau} .$$

Then by (4.5b) and (4.6b) one sees that

(4.7) $$\omega = N(A)(N_1(\varepsilon b_1))^{-1}\eta^{1+\rho_0} \in F^{\times} .$$

The proof of the following lemma is straightforward.

LEMMA 4. *The class of ω modulo $N(E/F)\cdot N(F'/F)$ is independent of the choices of D'_1, φ, f_1, ρ_1, $M_{E'}$, A, b_1 and C, and is uniquely determined only by D', ρ and E'.*

We call ω a *correcting factor* for $(\rho, E'/F')$. (In Case 2.1, one may consider that $\omega = 1$.)

4.4. For an actual computation, it will be convenient to take D'_1 and $M_{E'}$ in the following manner. Let $W = \{e_1\}_{D'}$ be an irreducible $D'_{E'}$-module. Then for $\xi \in E'$ one has $\xi e_1 = e_1 i(\xi)$ with an (F'-linear) imbedding $i\colon E' \hookrightarrow D'$. One can then construct matrix units e_{kl} ($k, l = 1, 2$) in $D'_{E'}$ satisfying (3.9) with $t_1 = i(\sqrt{\lambda})$. Setting $D'_1 = e_{11}D'_{E'}e_{11}$, one has an E'-isomorphism $M_{E'}\colon D'_{E'} \xrightarrow{\sim} M_2(D'_1)$ defined as follows: for $x \in D'_{E'}$ one has $M_{E'}(x) = (x_{ij})$ with $x_{ij} \in D'_1$ if and only if

$$x = x_{11} + x_{12}e_{12} + e_{21}x_{21} + e_{21}x_{22}e_{12} .$$

Since $e_{11}^{\tau} = e_{22}$, one has $D_1'^{\tau} = e_{22}D'_{E'}e_{22}$, and the map $\varphi\colon x_1 \mapsto e_{21}x_1e_{12}$ gives an E'-isomorphism $D'_1 \xrightarrow{\sim} D_1'^{\tau}$. One has

$$x_1^{[\tau]} = \varphi^{-1}x_1^{\tau} = e_{12}x_1^{\tau}e_{21} \qquad (x_1 \in D'_1) .$$

Also from $e_{11}^{\tau}=e_{22}$ one obtains

$$e_{21}^{\tau}=c_1e_{12}\,,\qquad e_{12}^{\tau}=e_{21}c_1^{-1}$$

with $c_1\in D_1'$, $c_1^{[\tau]}=c_1$. Hence one has (4.4) with

$$C=\begin{pmatrix}0 & c_1\\ 1 & 0\end{pmatrix}=C^{[\tau]}\,. \tag{4.4c}$$

Then by (4.4a) one has $c_1=\gamma f_1$ and hence $N(C)=(-1)^{\delta/2}N_1(\gamma f_1)$. It follows that one can choose η to be $=(\sqrt{\lambda})^{\delta/2}$ and hence

$$\omega=N(A)N_1(\varepsilon b_1)^{-1}\lambda^{\delta/2}\,. \tag{4.7a}$$

Lemma 5. *Suppose D_1', $M_{E'}$ and ω are chosen as above. Let $x\in\mathrm{Her}_1(D',\rho)$ and suppose that*

$$\tilde{M}_{E'}(x)=\begin{pmatrix}x_1' & 0\\ 0 & x_1''\end{pmatrix}$$

with x_1', $x_1''\in\mathrm{Her}_1(D_1',\rho_1)$. Then one has $x_1''=\varepsilon b_1x_1'^{[\tau]}$ and

$$N(x)=\omega^{-1}\lambda^{\delta/2}N_1(x_1')^{1+\tau}\,. \tag{4.8}$$

Proof. By (4.4) and (4.6) one has

$$\begin{pmatrix}x_1'^{[\tau]} & 0\\ 0 & x_1''^{[\tau]}\end{pmatrix}=\tilde{M}_{E'}(x)^{[\tau]}=(A^{[\tau]}C^{-1}A^{-1})\tilde{M}_{E'}(x)C$$

$$=\varepsilon^{-1}\begin{pmatrix}b_1 & 0\\ 0 & b_1\end{pmatrix}^{-1}{}^{t}C^{\rho_1}\begin{pmatrix}x_1' & 0\\ 0 & x_1''\end{pmatrix}C=\begin{pmatrix}\varepsilon^{-1}b_1^{-1}x_1'' & 0\\ 0 & \varepsilon^{-1}b_1^{-1}x_1'c_1\end{pmatrix},$$

whence one has $x_1''=\varepsilon b_1x_1'^{[\tau]}$. It follows from (4.7a) that

$$N(x)=N(A)^{-1}N_1(\varepsilon b_1)N_1(x_1')^{1+\tau}=\omega^{-1}\lambda^{\delta/2}N_1(x_1')^{1+\tau}\,,$$

which proves (4.8). q.e.d.

Example 2. The notation being as above, suppose further that the condition $t_1^{\rho}=t_1$ is satisfied. Then one obtains

$$e_{ii}^{\rho}=e_{ii}\qquad(i=1,2)\,,$$

$$e_{21}^{\rho}=d_1e_{12}\,,\qquad e_{12}^{\rho}=e_{21}d_1^{-1}$$

with $d_1\in D_1'$, $d_1^{\rho}=d_1$. It follows that $D_1'^{\rho}=D_1'$ so that one can set $\rho_1=\rho|D_1'$. Then one has (3.10) with

$$A=\begin{pmatrix}1 & 0\\ 0 & d_1\end{pmatrix}.$$

By (4.6) one obtains

$$d_1 = c_1^{\rho_1} c_1 d_1^{-[\tau]} = \varepsilon b_1$$

and hence $N(A) = N_1(d_1) = N_1(\varepsilon b_1)$. Therefore by (4.7a) one has

$$\omega = \lambda^{\delta/2} \sim (-1)^{\delta/2} \qquad (\mathrm{mod}^{\times} N(E/F)) . \tag{4.7b}$$

For instance, suppose that D' is given in the form of a cyclic algebra $D' = (Z'/F', \sigma, \gamma')$, $Z' = ZF'$ with a totally real Z. Then there exists an injection $\iota: Z' \to D'$ and $u \in D'$ such that

$$D' = \sum_{i=0}^{\delta-1} \iota(Z')u^i , \qquad u^{-1}\iota(\xi)u = \iota(\xi^{\sigma}) , \qquad u^{\delta} = \gamma' . \tag{4.9}$$

One can define a (totaly positive) unitary involution ρ with respect to F'/F by setting

$$\iota(\xi)^{\rho} = \iota(\xi^{\rho_0}) \quad \text{and} \quad u^{\rho} = \iota(\alpha)u^{-1} , \tag{4.10}$$

where α is a (totally positive) element of Z satisfying the condition $N_{Z/F}(\alpha) = N_{F'/F}(\gamma')$ (see the assumptions in 2.4). Since $\delta = 2\delta_1$, there is a unique totally real quadratic subextension E/F of Z/F. If $E = F(\sqrt{\lambda})$ and ρ are given in this manner, then $t_1 = \iota(\sqrt{\lambda})$ satisfies the condition $t_1^{\rho} = t_1$.

4.5. We retain the assumptions and the notation in 4.3. We obtain the following

THEOREM 7. *Assume that λ is in Case* 2.2. *Then the symmetric domain $\mathscr{D} = \mathscr{D}(V/D', h)$ has a rational point of the third type with CM-field $F(\sqrt{-\beta\lambda})$ if and only if conditions* (R1), (R2.2) *and the following condition* (R3′) *are satisfied*:

$$(-1)^{\delta\nu/2}\omega^{\nu} \det(h) \in N(E/F) \cdot N(F'/F) , \tag{R3′}$$

where ω is the correcting factor for $(\rho, E'/F')$.

REMARK. Note that the class of $\omega^{\nu}\det(h)$ is determined only by the involution ρ_h and E'. Note also that, as in the case of Theorem 6, condition (R3′) is equivalent to saying that $F'' = F(\sqrt{-\beta\lambda})$ splits the quaternion algebra similar to $D(h) \otimes_F (-\beta, \omega)_F^{\nu}$.

PROOF. The "only if" part follows from 3.3, Case 2.2 and Lemma 5. To prove the "if" part, suppose that conditions (R1), (R2.2) and (R3′) are satisfied. Without any loss of generality, we may further assume that h is anisotropic; then by Theorem 2 we have $\nu = 1$ or 2. We give a proof only in the first case, since the proof in the second case is similar.

We choose D'_1, $M_{E'}$ and ω as explained in 4.4. In the case $\nu = 1$, one has $p = q = \delta/2$ by (R2.2). By (R3′) there exists $\xi \in E^{\times}$ such that

$$(-1)^{\delta/2}\omega \det(h) \sim \xi^{1+\tau} \quad (\mathrm{mod}^{\times} N(F'/F)) .$$

Then one has $\varepsilon^{\delta/2}\xi^{1+\tau}\gg 0$. As in the proof of Theorem 6 one may assume that $\xi^{\sigma_i}>0$ for all $1\le i\le m$. Then by [Sc], Th. 10.6.9, one can find $a'\in \mathrm{Her}_1(D'_1, \rho_1)$ such that $N_1(a')=(\sqrt{\lambda})^{-\delta/2}\xi$ and $a'^{\sigma_i}>0$, $\varepsilon^{\sigma_i}a'^{[\tau]\sigma_i}<0$ for $1\le i\le m$. Put $a''=\varepsilon b_1 a'^{[\tau]}$ and find $x\in D'_{E'}$ such that $\tilde{M}_{E'}(x)=\mathrm{diag}(a', a'')$. Then it is easy to check that $x^\tau=x^\rho=x$, i.e., $x\in \mathrm{Her}_1(D', \rho)$. Moreover, one has $\mathrm{sign}(x^{\sigma_i})=(\delta/2, \delta/2)$ and by Lemma 5

$$N(x)=\omega^{-1}\lambda^{\delta/2}N_1(a')^{1+\tau}=(-1)^{\delta/2}\omega^{-1}\xi^{1+\tau}.$$

Let $V=\{e_1\}_{D'}$ and let h_1 be a (D', ρ)-hermitian form on V with $h_1(e_1, e_1)=x$. Then by [Sc], Cor. 10.6.6, h and h_1 are equivalent over D'. This means that there exists $e'_1\in V$ such that $h(e'_1, e'_1)=x$. Then the (D', ρ)-hermitian form h' defined by $h'(e'_1, e'_1)=xt_1$ with $t_1=i(\sqrt{\lambda})$ belongs to $\mathscr{P}(V, D', \rho, h)$ and gives a rational point in $\mathscr{D}$ of the third type with CM-field $F(\sqrt{-\beta\lambda})$. q.e.d.

4.6. We consider here a special case where δ_1 is *odd*. In this case, one has

$$D'=D'_0\otimes_{F'}D'_2, \qquad D'_{0E'}\simeq D'_1, \qquad D'_{2E'}\sim 1, \tag{4.11}$$

where D'_0 and D'_2 are central division algebras over F' of degree δ_1 and 2, respectively. We follow the notation of 4.4 and, in doing so, choose e_{ij} in $D'_{2E'}$. Then one has $c_1\in F'$, i.e., one may set $c_1=\gamma$, $f_1=e_{11}$. This implies that $[\tau]^2=1$, i.e., $\{1, \varphi\}$ is a 1-cocycle, which defines an F'-form of D'_1. (It is easy to see that, conversely, if f_1 can be taken to be $=1$, then δ_1 is odd.) One denotes by ψ the E'-isomorphism $D'_1 \xrightarrow{\sim} D'_{0E'}$ defined by $\psi(x_1)=x_0 \Leftrightarrow x_1=x_0\otimes e_{11}$ $(x_1\in D'_1, x_0\in D'_{0E'})$. Then one has

$$\psi(x_1)=x_1+\varphi(x_1), \qquad \psi(x_1^{[\tau]})=\psi(x_1)^\tau \qquad (x_1\in D'_1). \tag{4.12}$$

The F'-form of D'_1 mentioned above is given by $D'_0\otimes e_{11}$.

By [A], Ch. X, the algebras D'_i $(i=0, 2)$ have a totally positive unitary involution ρ'_i $(i=0, 2)$ with respect to F'/F. One defines the involution ρ_1 of D'_1 by setting $x_1^{\rho_1}=\psi^{-1}(\psi(x_1)^{\rho'_0})$ for $x_1\in D'_1$. Then by (4.12) one has $\rho_1^{[\tau]}=\rho_1$. Hence one may set $b_1=e_{11}$.

On the other hand, let $x_2\mapsto \bar{x}_2$ denote the canonical involution of the quaternion algebra D'_2. Then the map $x_2\mapsto \bar{x}_2^{\rho'_2}$, being a semilinear involutive automorphism of D'_2, defines an F-form $D_2(\rho'_2)$ of D'_2:

$$D_2(\rho'_2)=\{x_2\in D'_2 \mid \bar{x}_2^{\rho'_2}=x_2\}. \tag{4.13}$$

From $D_2(\rho'_2)_{E'}=D'_{2E'}\sim 1$, one can conclude that

$$D_2(\rho'_2)\sim D_2\otimes_F D_3,$$

where D_2 and D_3 are central quaternion algebras over F such that $D_{2E}\sim 1$ and $D_{3F'}\sim 1$. Then one has $D'_2=D_2(\rho'_2)_{F'}\simeq D_{2F'}$, which means that D_2 is also an F-form of D'_2. We regard D_2 as contained in D'_2 and denote by ρ_2 the (not totally positive) unitary involution of D'_2 such that $D_2=D_2(\rho_2)$. Since $D_{2E}\sim 1$, one may choose e_{ij} in D_{2E}. Then one has $\gamma \in F$ and

$$D_2 = (\lambda, \gamma)_F .$$

For $x = x_0 \otimes x_2$ ($x_0 \in D'_0$, $x_2 \in D'_2$) let

$$x^\rho = a^{-1}(x_0^{\rho'_0} \otimes x_2^{\rho_2})a$$

with $a \in D'$, $a^\rho = -a$. Then by the definitions one has

$$M_{E'}(x_0^{\rho'_0} \otimes x_2^{\rho_2}) = \psi^{-1}(x_0)^{\rho_1} \cdot J^{-1\,t}M_{E'}(x_2)^{\rho_0}J$$

with

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Hence one has

$$M_{E'}(x^\rho) = A^{-1\,t}M_{E'}(x)^{\rho_1}A$$

with $A = JM_{E'}(a) \in \mathrm{Her}_2(D'_1, \rho_1)$. Since ρ is totally positive, one can choose a so that A is totally positive. From $a^\tau = a$ one has $A^{[\tau]} = JC^{-1}AC$. Hence it follows from Lemma 3 and (4.4c) that $\varepsilon = -\gamma$ (and $b_1 = e_{11}$). Therefore by (4.7a) one obtains

(4.7c) $$\omega = N(a)N_1(\varepsilon b_1)^{-1}\lambda^{\delta_1} \sim \gamma N(a) \pmod{^\times N(E/F)} .$$

In Example 3 below, we need the following lemma, which we state in a little more general situation.

LEMMA 6. *Let ρ_2 and ρ'_2 be unitary involutions of D'_2 with respect to F'/F and let*

$$x^{\rho'_2} = a_2^{-1}x^{\rho_2}a_2 \qquad \textit{for} \quad x \in D'_2$$

with $a_2 \in D'^{\times}_2$, $a_2^{\rho_2} = -a_2$. Then one has

(4.14) $$D_2(\rho'_2) \sim D_2(\rho_2) \otimes_F (-\beta, -N_2(a_2))_F ,$$

N_2 denoting the reduced norm of D'_2.

COROLLARY. *A unitary involution ρ'_2 is totally positive if and only if $D_2(\rho'_2)$ is totally definite.*

This follows from Lemma 6 by taking ρ_2 as specified above.

PROOF OF LEMMA 6. One writes a_2 and $x \in D'_2$ in the form

$$a_2 = \alpha_2\sqrt{-\beta} + u' \qquad \text{with} \quad \alpha_2 \in F,\ u' \in D_2(\rho_2)_- ,$$

$$x = x' + \sqrt{-\beta}x'' \qquad \text{with} \quad x',\ \ x'' \in D_2(\rho_2) .$$

Then an easy computation shows that one has

$$\bar{x}^{\rho_2'}=x \Leftrightarrow \bar{x}^{\rho_2}a_2=a_2x \Leftrightarrow \begin{cases}[u', x']=2\alpha_2\beta x'' \\ u'x''+x''u'=0 .\end{cases}$$

When $u'=0$, the assertion of the lemma is trivial. Hence, assuming $u'\neq 0$, one sets

$$D_2(\rho_2)=\{1, u', u'', u'u''\}_F=(\gamma', \gamma'')_F$$

with $u'^2=\gamma'$, $u''^2=\gamma''$ and $u'u''=-u''u'$. Then one has

$$D_2(\rho_2')=\{1, u', \alpha_2\beta u''+\sqrt{-\beta}u'u'', \alpha_2\beta u'u''+\gamma'\sqrt{-\beta}u''\}_F=(\gamma', (\alpha_2^2\beta+\gamma')\beta\gamma'')_F .$$

Since $-N_2(a_2)\beta=(\alpha_2^2\beta+\gamma')\beta$ is in $N(F(\sqrt{-\beta\gamma'}))$, one has

$$D_2(\rho_2')\sim(\gamma', \gamma'')_F\otimes_F(-\beta, -N_2(a_2))_F ,$$

which proves our assertion. q.e.d.

Example 3: $(\mathrm{I}_{1,1}^{(2)})$. The notation being as above, consider the case $\delta_1=1$, i.e., $D'=D_2'$, and $\nu=1$. Let $h\simeq(a_1)$ with $a_1^\rho=a_1$, $(p_i, q_i)=(1, 1)$ $(1\leq i\leq m)$. Then one has

$$\mathfrak{g}_1(F)=\mathfrak{su}(1, h, D'/F'/F)=\{x\in D' \mid x^{\rho_h}=\bar{x}=-x\} ,$$

where $x^{\rho_h}=a_1^{-1}x^\rho a_1=(aa_1)^{-1}\bar{x}^{\rho_2}aa_1$. Thus one has

$$\mathfrak{g}_1(F)=D(\rho_h)_-=\mathfrak{sl}(1, D(\rho_h)) , \tag{4.15}$$

(which gives the isomorphism (1.2) with $\delta=2$). By Lemma 6 one has

$$D(\rho_h)\sim D(\rho_2)\otimes(-\beta, -N(aa_1))_F ,$$

which (or (4.15)) shows that $D(\rho_h)$ is totally indefinite. Since $D(\rho_2)=(\lambda, \gamma)_F$, it follows from (4.7c) that

$$D(\rho_h)_{F''}\sim(-\beta, \omega)_{F''}\otimes_{F''}D(h)_{F''} .$$

Thus, in this case, condition (R3′) is equivalent to saying that F'' splits $D(\rho_h)$.

5. Classification.

5.1. In general, let $\mathscr{D}$ be a $\boldsymbol{Q}$-irreducible symmetric domain of $\boldsymbol{Q}$-rank r_0 satisfying the condition (R1). Such a domain $\mathscr{D}$ is classified into the following four types ([Sa2], 3.4):

(T1) The case where $\mathscr{D}$ is of tube type and the "last" (i.e., the r_0-th) rational boundary components reduce to a point.

(T2) The case where $\mathscr{D}$ is of tube type and the last rational boundary components are of positive dimension.

(U1) The case where $\mathscr{D}$ is not of tube type and the last rational boundary components reduce to a point.

(U2) The case where $\mathscr{D}$ is not of tube type and the last rational boundary components are of positive dimension.

For the domains of classical type listed in 1.1 we obtain the following classification:

(T1): $(\mathrm{III}_{\nu/2}^{(1)})$, $(\mathrm{III}_{\nu}^{(2')})$ (ν even, ≥ 2, $r_0=\nu/2$), $(\mathrm{IV}_{\nu-2}^{(1)})$ ($\nu\geq 5$, $r_0=1, 2$), $(\mathrm{II}_{\nu}^{(2')})$ (ν even, ≥ 4, $r_0=\nu/2$), $(\mathrm{I}_{p,p}^{(\delta)})$ ($r_0=p/\delta=\nu/2\geq 1$).

(T2): $(\mathrm{III}_{\nu}^{(2')})$ (ν odd, ≥ 1, $r_0=(\nu-1)/2$), $(\mathrm{IV}_{2\nu-2}^{(2)})$ ($\nu\geq 3$, $r_0=0, 1$), $(\mathrm{II}_{\nu}^{(2')})$ (ν even, ≥ 4, $r_0=\nu/2-1$), $(\mathrm{II}_4\text{–}\mathrm{IV}_6^{(2')})$ ($r_0=1$), $(\mathrm{I}_{p,p}^{(\delta)})$ ($p/\delta=\nu/2$, $r_0=[(\nu-1)/2]\geq 0$).

(U1): $(\mathrm{II}_{\nu}^{(2')})$ (ν odd, ≥ 3, $r_0=(\nu-1)/2$), $(\mathrm{I}_{p,q}^{(\delta)})$ ($p>q\geq 1$, $r_0=q/\delta<\nu/2$).

(U2): $(\mathrm{II}_{\nu}^{(2')})$ (ν odd, ≥ 3, $r_0=(\nu-3)/2$), $(\mathrm{I}_{p,q}^{(\delta)})$ ($p>q\geq 1$, $r_0<q/\delta<\nu/2$).

5.2. Let $\mathscr{F}_0$ be a last rational boundary component of $\mathscr{D}$ and let $\mathfrak{g}_{\mathscr{F}_0}^{(1)}=R_{F/\boldsymbol{Q}}(\mathfrak{g}_{\mathscr{F}_0,1}^{(1)})$ (with $\mathfrak{g}_{\mathscr{F}_0,1}^{(1)}$ defined over F) be the semisimple hermitian part of the reductive quotient of the parabolic subalgebra corresponding to $\mathscr{F}_0$. (In the notation of [Sa2], $\mathfrak{g}_{\mathscr{F}_0}^{(1)}$ is $\boldsymbol{Q}$-isomorphic to $\mathfrak{g}_{\kappa}^{(11)}$ for $\kappa\in\mathscr{K}_{\mathscr{F}_0}$.) $\mathfrak{g}_{\mathscr{F}_0}^{(1)}$ is the Lie algebra of the semisimple $\boldsymbol{Q}$-group acting on $\mathscr{F}_0$. For the domain of type (T1), $\mathfrak{g}_{\mathscr{F}_0}^{(1)}$ reduces to $\{1\}$. For the domain of type (U1), $\mathfrak{g}_{\mathscr{F}_0}^{(1)}$ is of type $(\mathrm{I}_{\nu_0\delta,0}^{(\delta)})$ with $\nu_0\geq 1$. As was shown in [Sa2], *the domain $\mathscr{D}$ of type* (T1) *has rational points with any CM-field and CM-type and the one of type* (U1) *has rational points with a particular CM-field and CM-type. The domain of type* (U2) *has no rational points* (by 1.2 and Cor. to Th. 4).

As for the domain of type (T2) we have the following three cases.

(a) $(\mathrm{III}_{\nu}^{(2')})$ (ν odd, ≥ 1), $(\mathrm{IV}_{2\nu-2}^{(2)})$ ($\nu\geq 3$, $r_0=1$).

(b) $(\mathrm{II}_{\nu}^{(2')})$ (ν even, ≥ 4, $r_0=\nu/2-1$), $(\mathrm{II}_4\text{–}\mathrm{IV}_6^{(2')})$ ($r_0=1$).

(c) $(\mathrm{IV}_4^{(2')})$ ($r_0=0$) ($\simeq(\mathrm{I}_{2,2}^{(\delta)})$ ($\delta=2, 4$)), $(\mathrm{I}_{p,p}^{(\delta)})$ ($p=\delta\nu/2$, $r_0=[(\nu-1)/2]$).

In the case (a), $\mathscr{F}_0$ is of type $(\mathrm{III}_1^{(2')})$, and one has

$$\mathfrak{g}_{\mathscr{F}_0}^{(1)}(\boldsymbol{Q})\cong\mathfrak{g}_{\mathscr{F}_0,1}^{(1)}(F)=\mathfrak{sl}(1, D)=\{x\in D\,|\,\mathrm{tr}_{D/F}x=0\} \tag{5.1}$$

with "the" given totally indefinite division quaternion algebra D over F. Hence *the domain $\mathscr{D}$ has rational points with any CM-field which splits D.*

In the case (b), $\mathscr{F}_0$ is again of type (III_1), but the Lie algebra $\mathfrak{g}_{\mathscr{F}_0}^{(1)}$ has compact factors. In fact, $\mathfrak{g}_{\mathscr{F}_0}^{(1)}$ has $2m$ or $3m$ simple factors (over $\boldsymbol{R}$), but has only m non-compact simple factors. To be more precise, let $h=\mathrm{diag}(a_1, \cdots, a_\nu)$, $a_i^2=\alpha_i\in F$, and $\Delta=(-1)^\nu\det(h)=\prod_{i=1}^{\nu}\alpha_i$. Then the action of the Galois group $\mathrm{Gal}(\bar{\boldsymbol{Q}}/F)$ on the root diagram of $\mathfrak{g}_{\mathscr{F}_0,1}^{(1)}$ is non-trivial (i.e., the arrow prevails in Figure 1) if and only if one has $\Delta\not\sim 1$ $(\mathrm{mod}^{\times}(F^{\times})^2)$.

When $\Delta\sim 1$, $\mathfrak{g}_{\mathscr{F}_0}^{(1)}$ has two $\boldsymbol{Q}$-simple factors corresponding to the "last" two vertices of the root diagram of $\mathfrak{g}_{\mathscr{F}_0,1}^{(1)}$; they are given (in $\boldsymbol{Q}$-rational points) by $\mathfrak{sl}(1, D_i)$ $(i=1, 2)$,

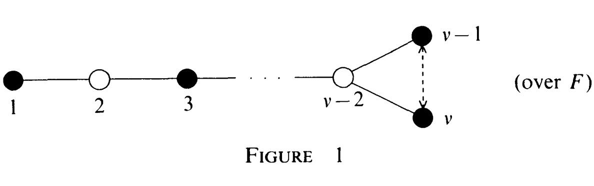

FIGURE 1

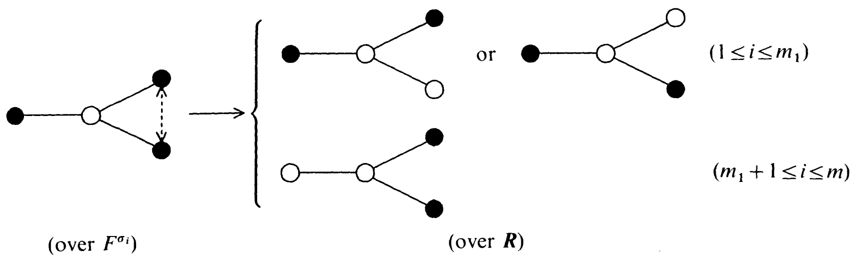

FIGURE 2

where D_i's are (central division) quaternion algebras over F such that $D_1 \otimes D_2 \sim D$. (The numeration of the vertices of the root diagram is as shown in Figure 1.) When $\Delta \not\sim 1$, $\mathfrak{g}^{(1)}_{\mathscr{F}_0}$ has instead a $\boldsymbol{Q}$-simple factor given by $R_{F(\sqrt{\Delta})/F}(\mathfrak{sl}(1, D_1'))$, where D_1' is a (central division) quaternion algebra over $F(\sqrt{\Delta})$. This $\boldsymbol{Q}$-simple factor is not pure, having m compact and m non-compact simple factors (over $\boldsymbol{R}$). For $(\mathrm{II}_4^{(2')})$ and $(\mathrm{II}_4\text{–}\mathrm{IV}_6^{(2')})$ $(r_0=1)$, one has one more $\boldsymbol{Q}$-simple factor corresponding to the "first" vertex of the root diagram of $\mathfrak{g}^{(1)}_{\mathscr{F}_{0,1}}$, which is given by $\mathfrak{sl}(1, D)$. For $(\mathrm{II}_4^{(2')})$ this $\boldsymbol{Q}$-simple factor is compact. But for $(\mathrm{II}_4\text{–}\mathrm{IV}_6^{(2')})$ it is not pure, having m_1 compact and $m-m_1$ non-compact simple factors (over $\boldsymbol{R}$). [The relation of the root diagrams for $(\mathrm{II}_4\text{–}\mathrm{IV}_6^{(2')})$ over F^{σ_i} $(1 \leq i \leq m)$ and over $\boldsymbol{R}$ is shown in Figure 2.] Thus one sees that, *except for the special case* $(\mathrm{II}_\nu^{(2')})$ $(r_0=\nu/2-1)$ *with* $\Delta \sim 1$ *for which one of the* D_i*'s, say* D_1*, is totally definite* (hence the other, D_2, is totally indefinite), (all) the non-compact $\boldsymbol{Q}$-simple factor(s) of $\mathfrak{g}^{(1)}_{\mathscr{F}_0}$ is (are) not pure, and hence *the domain* $\mathscr{D}$ *has no rational points. In that special case, where* D_2 *is totally indefinite*, $\mathfrak{g}^{(1)}_{\mathscr{F}_0}$ has, along with one or two compact $\boldsymbol{Q}$-simple factor(s), a pure non-compact $\boldsymbol{Q}$-simple factor, which is of type $(\mathrm{III}_1^{(2')})$; hence $\mathscr{D}$ *has rational points with CM-fields which split* D_2. In particular, the case $(\mathrm{II}_4^{(2')})$ with D_1 totally definite reduces to the case (a), $(\mathrm{IV}_6^{(2')})$.

In the case (c), according as $p \equiv 0$ or $\delta/2 \pmod{\delta}$, $\mathscr{F}_0$ is of type $(\mathrm{I}^{(\delta)}_{\delta,\delta})$ $(\nu=2)$ or $(\mathrm{I}^{(\delta)}_{\delta/2,\delta/2})$ $(\nu=1)$. *In the first case, the domain* $\mathscr{D}$ *has rational points of the first and second type* (Th. 5 and 6). In particular, the case $(\mathrm{I}^{(1)}_{1,1})$ reduces to the case (a), $(\mathrm{III}_1^{(2')})$ (Example 1). *In the second case, the domain* $\mathscr{D}$ *has no rational points of the first and second type, but has rational points of the third type with CM-field* $F(\sqrt{-\beta\lambda})$ *for which condition* (R3′) *is satisfied* (Th. 7). In particular, the case $(\mathrm{I}^{(2)}_{1,1})$ reduces to the case (a), $(\mathrm{III}_1^{(2')})$ (Example 3).

REMARK. For the domains of exceptional type, satisfying (R1), we have the following possibilities for $\mathfrak{g}_1$. We use the notation in the list of Tits [T].

(T1) $(E^{28}_{7,3})$ $(r_0=3)$.

(T2) $({}^3D^9_{4,1})$, $({}^6D^9_{4,1})$ $(r_0=1)$, $(E^{31}_{7,2})$ $(r_0=2)$.

(U1) $({}^2E^{16'}_{6,2})$ $(r_0=2)$.

(U2) $({}^2E^{35}_{6,1})$ $(r_0=1)$.

The fact that these are the only possibilities in the list of Tits follows from a kind of Hasse principle given by Harder ([H], Satz 4.3.3). Among the domains of type (T2), the domain $\mathscr{D}$ corresponding to $(E_{7,2}^{31})$ has the last rational boundary component $\mathscr{F}_0$ of type $(\mathrm{III}_1^{(2)})$ (as in the above case (a)). Hence $\mathscr{D}$ has rational points. For the one corresponding to $({}^3D_{4,1}^9)$ or $({}^6D_{4,1}^9)$ the Lie algebra $\mathfrak{g}_{\mathscr{F}_0}^{(1)}$ is $\boldsymbol{Q}$-simple but not pure, having $2m$ compact and m non-compact factors (similarly to the case (b) above). Hence $\mathscr{D}$ has no rational points.

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