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GEOMETRIC FINITENESS, QUASICONFORMAL STABILITY AND SURJECTIVITY OF THE BERS MAP FOR KLEINIAN GROUPS

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1. Introduction. The group Möb of all Möbius transformations acting on the extended complex plane $\hat{\boldsymbol{C}}$ is identified with the 3-dimensional complex Lie group $PSL(2, \boldsymbol{C})$. A discrete subgroup G of Möb is said to be Kleinian if its region of discontinuity $\Omega(G)$ in $\hat{\boldsymbol{C}}$ is not empty. $\Lambda(G) := \hat{\boldsymbol{C}} - \Omega(G)$ is called the limit set of G. If $\Lambda(G)$ contains infinitely many points, we say G is non-elementary. Throughout this paper, we denote by *G a finitely generated non-elementary Kleinian group* which may contain elliptic elements. For this G, we consider the following three conditions (A), (B) and (C), which are defined later in this section.

(A) G is geometrically finite.
(B) G is quasiconformally (QC) stable.
(C) The Bers map $\beta^* : B(\Omega(G), G) \to PH^1(G, \Pi)$ is surjective.

In §4 and §5 of this paper (Corollaries 1 and 2), we prove

$$(\mathrm{A}) \Rightarrow (\mathrm{B}) \Leftrightarrow (\mathrm{C})\,.$$

Stronger results for more restricted torsion-free Kleinian groups were obtained by Sullivan [16]. Concerning other known partial solutions to our problem, one may refer to §2.

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(A) Geometric finiteness is the most familiar criterion for Kleinian groups to be "good". G is said to be geometrically finite if the action of G as isometries on the hyperbolic space $\boldsymbol{H}^3$ has a finite-sided Dirichlet fundamental polyhedron. There is a well-known equivalent characterization by Beardon-Maskit (see [10, Chap. VI. C. 7]).

(B) To define quasiconformal (QC) stability, we choose a system of generators of $G = \langle g_1, \cdots, g_k \rangle$. All our arguments do not depend on the choice of generators. A homomorphism $\chi : G \to$ Möb is determined by the images of the generators $(\chi(g_1), \cdots, \chi(g_k))$, which satisfy relations arising from the relations satisfied by $g_1, \cdots, g_k$. In this sense, we represent by χ not only a homomorphism of G but also a point of the product manifold $(\text{Möb})^k$. Therefore the set $\mathrm{Hom}(G, \text{Möb})$ of homomorphisms $\chi : G \to$ Möb can be regarded as a subvariety of $(\text{Möb})^k$, which is an affine algebraic variety. Further, we denote by $\mathrm{Hom}_p(G, \text{Möb})$ the algebraic subvariety whose points correspond to parabolic homomorphisms $\chi: G \to \text{Möb}$, that is, the images of parabolic elements of G under χ are parabolic or the identity.

Let $M(G)$ be the unit ball in the Banach space of Beltrami differentials for G on $\hat{\boldsymbol{C}}$ with L^∞-norm $\|\ \ \|$. Note that by Sullivan's theorem [15, §V], the Beltrami differentials for G have essential support on $\Omega(G)$. We define a holomorphic map

$$\Phi_G: \text{Möb} \times M(G) \to (\text{Möb})^k$$

by

$$(\alpha, \mu) \mapsto (\alpha \circ w^\mu \circ g_1 \circ (\alpha \circ w^\mu)^{-1}, \cdots, \alpha \circ w^\mu \circ g_k \circ (\alpha \circ w^\mu)^{-1}),$$

where w^μ is the normalized QC automorphism of $\hat{\boldsymbol{C}}$ whose complex dilatation is μ. The image $\Phi_G(\text{Möb} \times M(G))$ is denoted by $\mathrm{Hom}_{qc}(G, \text{Möb})$. It consists of isomorphisms induced by QC automorphisms. Clearly $\mathrm{Hom}_{qc}(G, \text{Möb}) \subset \mathrm{Hom}_p(G, \text{Möb})$.

DEFINITION. G is said to be *quasiconformally* (QC) *stable* if there is an open neighborhood U of the origin $(g_1, \cdots, g_k)$ in $(\text{Möb})^k$ such that $U \cap \mathrm{Hom}_p(G, \text{Möb}) \subset \mathrm{Hom}_{qc}(G, \text{Möb})$.

REMARK 1. (1) One may define the QC stability equivalently by allowable homomorphisms instead of parabolic homomorphisms (cf. [3]).

(2) Elementary groups satisfy (A) and (B). But, since the following (C) is meaningless for them, we need to assume G is non-elementary.

(C) We now explain the cohomology of Kleinian groups and define β^*. Möb acts from the right on the vector space Π of quadratic polynomials via $pg(z) = p(gz)g'(z)^{-1}$ for $p \in \Pi$, $g \in \text{Möb}$, $z \in \boldsymbol{C}$. A mapping $\rho: G \to \Pi$ is called a cocycle if $\rho(g_1 \circ g_2) = \rho(g_1)g_2 + \rho(g_2)$ for g_1, $g_2 \in G$. The coboundary for $p \in \Pi$ is the cocycle ρ defined by $\rho(g) = pg - p$ for $g \in G$. The (Eichler) cohomology group $H^1(G, \Pi)$ is the group $Z^1(G, \Pi)$ of cocycles modulo the group $B^1(G, \Pi)$ of coboundaries. If $\rho \in Z^1(G, \Pi)$ satisfies $\rho|_{G_0} \in B^1(G_0, \Pi)$ for every parabolic cyclic subgroup G_0 of G, we say that ρ belongs to space $PZ^1(G, \Pi)$ of parabolic cocycles. We define the parabolic (Eichler) cohomology group to be $PH^1(G, \Pi) = PZ^1(G, \Pi)/B^1(G, \Pi)$.

Let $B(\Omega(G), G)$ be the Banach space of holomorphic quadratic differentials φ for G on $\Omega(G)$ with the norm

$$\|\varphi\|_B := \|\lambda^{-2}\varphi\|_{\Omega(G)} = \sup_{z \in \Omega(G)} \lambda^{-2}(z)|\varphi(z)|$$

finite, where $\lambda(z)$ is the Poincaré density defined on each component of $\Omega(G)$. Since G is finitely generated, $B(\Omega(G), G)$ is finite dimensional. The Bers map

$$\beta^*: B(\Omega(G), G) \to PH^1(G, \Pi)$$

is defined as follows: For $\varphi \in B(\Omega(G), G)$, let h be a potential for the canonical Beltrami differential $\lambda^{-2}\bar{\varphi}$, say,

$$h(z)=\frac{z(z-1)}{2\pi i}\iint_{\Omega(G)}\frac{\lambda^{-2}(\zeta)\bar{\varphi}(\zeta)}{\zeta(\zeta-1)(\zeta-z)}d\zeta \wedge d\bar{\zeta}\,.$$

Then

$$\rho g(z):=h(gz)g'(z)^{-1}-h(z) \qquad \text{for} \quad g \in G\,, \quad z \in \boldsymbol{C}\,,$$

belongs to Π. It can be verified that $\rho \in PZ^1(G, \Pi)$. We define $\beta^*\varphi$ as the cohomology class of ρ. The Bers map β^* is clearly anti-linear and is known to be injective (cf. [5, p. 170, Th. 2.4]).

2. A summary of known results. In this paper, we study the problem whether the conditions (A), (B) and (C) are equivalent to one another. We now summarize known results on this subject.

Gardiner and Kra [4, Th. 8.4] showed that (C) implies (B), and posed the problem whether the converse is true (see [6, Problems 5.1 & 3.1]). Later, Sakan supplemented their results and provided a necessary and sufficient condition for (C). Since his observation is useful for us, we briefly explain it.

From the arguments in [4], the Bers map β^* may be regarded as the differential at the orgin of the anti-holomorphic function from the unit ball of $B(\Omega(G), G)$ to the deformation variety up to conjugation, by the map which sends φ to the isomorphism induced by the normalized QC automorphism whose complex dilatation is $\lambda^{-2}\bar{\varphi}$. The target $PH^1(G, \Pi)=PZ^1(G, \Pi)/B^1(G, \Pi)$ may be considered as the "tangent space" of $\mathrm{Hom}_{\mathrm{p}}(G, \text{Möb})/\text{Möb}$ at the origin, though this statement is not quite correct because in general we do not know whether the origin is a regular point of the algebraic variety $\mathrm{Hom}_{\mathrm{p}}(G, \text{Möb})$ or not. By the implicit function theorem, β^* is surjective if and only if locally (I) $\mathrm{Hom}_{\mathrm{p}}(G, \text{Möb})$ coincides with $\mathrm{Hom}_{\mathrm{qc}}(G, \text{Möb})$, and (II) it is a complex analytic submanifold of $(\text{Möb})^k$. Here (I) is the same as (B), and Sakan [14, Lemma 2] remarked that (II) is equivalent to the following condition which was introduced by Bers [3, p. 32]:

DEFINITION. G is *quasi-stable* (conditionally stable) if for any open neighborhood N of the origin in $M(G)$, there exists an open neighborhood U of the orgin in $(\text{Möb})^k$ such that $U \cap \mathrm{Hom}_{\mathrm{qc}}(G, \text{Möb}) \subset \Phi_G(\text{Möb} \times N)$.

PROPOSITION 1 ([14, Th. 1]). *β^* is surjective if and only if G is both* QC *stable and quasi-stable.*

REMARK 2. Following Bers [3, p. 15], we say G to be *uniformly stable* if G is both QC stable and quasi-stable.

Another important result is a partial answer to our problem restricted to the so-called function groups. Namely, Nakada [13, Th. 3] showed that for function groups, (B), (C) and the following condition (A′) are equivalent to one another:

(A′) G is constructed from elementary groups and quasi-Fuchsian groups by applications of the combination theorems I and II finitely many times.

REMARK 3. The combination theorems I and II have several versions (cf. [10, Chap. VII]). Among them we adopt in this paper the form established in [9, p. 249], which is the same as [12, p. 700] and [17, p. 354]. They treat the cases where the amalgamated and the conjugated subgroups are either trivial, parabolic cyclic or elliptic cyclic.

On the other hand, for function groups, the equivalence (A) $\Leftrightarrow$ (A′) was shown, for instance, in [17, §2, Cor.]. Therefore we have the following:

PROPOSITION 2. *For function groups, that is, finitely generated non-elementary Kleinian groups with an invariant component of the region of discontinuity, the conditions* (A), (A′), (B) *and* (C) *are equivalent to one another.*

We sketch an outline of the proof for our later discussion in the proof of Lemma 1 in the next section. The equivalence (C) $\Leftrightarrow$ (A′) was shown in [12, Th. 5]. Both equivalences (A) $\Leftrightarrow$ (A′) and (C) $\Leftrightarrow$ (A′) are based on the following:

FACT 1 (Maskit [9. Th. 1]). *Every function group is constructed from elementary groups, quasi-Fuchsian groups and totally degenerate groups without accidental parabolic transformations* (APT) *by a finite number of applications of the combination theorems* I *and* II.

As we have mentioned, (C) $\Rightarrow$ (B) was proved by Gardiner-Kra. By the following two facts combined with Fact 1, we get (B) $\Rightarrow$ (A′), which completes the proof of this proposition:

FACT 2 (Nakada [13, Lemmas 5, 6]). *Let G be a finitely generated Kleinian group constructed from finitely generated Kleinian groups $\{G_1, \cdots, G_s\}$ by the combination theorems* I *and* II. *If G is* QC *stable, then each of $\{G_1, \cdots, G_s\}$ is* QC *stable.*

FACT 3. *A totally degenerate group without* APT *is not* QC *stable.*

Indeed, for such a group G, $\dim PH^1(G, \Pi) = 2 \dim B(\Omega(G), G)$ by Kra [5, p. 209], where $\dim B(\Omega(G), G) \neq 0$. Thus β^* is not surjective. By [4, p. 1058, Cor.], this implies G is not QC stable.

3. Lemmas. By reconsidering Proposition 2 with the purpose of extending it to the general case, we get the following lemma which is a key to the proofs of our theorems:

LEMMA 1. *If G is* QC *stable, then every component subgroup H of G, i.e., the stabilizer of a component of $\Omega(G)$, satisfies the conditions* (A), (A′), (B) *and* (C). *Especially,*

H is quasi-stable.

PROOF. By the decomposition theorem of Abikoff-Maskit [2, Th. 1], every finitely generated Kleinian group is constructed from elementary groups, totally degenerate groups without APT and web groups by applications of the combination theorems finitely many times. By Fact 2 in §2, if G is QC stable, then each subgroup arising from the decomposition of G must be QC stable. Therefore by Fact 3, totally degenerate groups do not appear in the decomposition of G, that is, G is constructed from elementary groups and web groups. Since the operations of our combination theorems work on the component subgroups (see Remark 4 below for a more precise assertion), each component subgroup of G, which itself is regarded as a function group, satisfies (A′). By Proposition 2, it satisfies (A), (B) and (C). In particular, by Proposition 1, it is quasi-stable. □

REMARK 4. Let G be a Kleinian group constructed by, say, the combination theorem I, from Kleinian groups G_1 and G_2. We can describe each component subgroup of G in terms of the component subgroups of G_1 and G_2 as in the following proposition. It is a corollary to the proofs of the combination theorems.

PROPOSITION. (1) *Let G be a Kleinian group constructed by the combination theorem* I *from Kleinian groups G_1 and G_2 with the cyclic amalgamated subgroup H (we denote this condition by $G=G_1*^{\mathrm{I}}_H G_2$ for brevity). Then any component subgroup of G is conjugate with respect to some element of G to either*

$$\mathrm{Stab}_{G_i}(\Delta_i) \quad or \quad \mathrm{Stab}_{G_1}(\Omega_1) *^{\mathrm{I}}_H \mathrm{Stab}_{G_2}(\Omega_2)\,,$$

where each of Δ_i and Ω_i is a component of $\Omega(G_i)$ $(i=1, 2)$.

(2) *Let G be a Kleinian group constructed by the combination theorem* II *from a Kleinian group G' and a loxodromic transformation f with the cyclic conjugated subgroups H_1 and H_2 satisfying $fH_1f^{-1}=H_2$ (we denote this condition by $G=G' *^{\mathrm{II}}_{H_1} f$ for brevity). Then any component subgroup of G is conjugate with respect to some element of G to either*

$$\mathrm{Stab}_{G'}(\Delta')\,, \quad \mathrm{Stab}_{G'}(\Omega_1) *^{\mathrm{II}}_{H_1} (h \circ f) \quad \text{or} \quad \mathrm{Stab}_{G'}(\Omega_1) *^{\mathrm{I}}_{H_1} (f^{-1}\mathrm{Stab}_{G'}(\Omega_2) f)\,,$$

where each of Δ', Ω_1 and Ω_2 is a component of $\Omega(G')$ and h is an element of G'.

To construct a QC automorphism of $\hat{\mathbf{C}}$ satisfying our requirements, we first construct a QC homeomorphism on some components of $\Omega(G)$ compatible with the component subgroups, and then extend it to $\hat{\mathbf{C}}$ compatibly with G. This process is accomplished by an application of the so-called identity theorem of Maskit [8], and may be summarized as follows:

LEMMA 2. *Let G be a finitely generated Kleinian group, $\{\Delta_i\}$ $(i=1, \cdots, n)$ be a maximal collection of non-equivalent components of $\Omega(G)$, and H_i be the component subgroup of Δ_i for each $i=1, \cdots, n$. Suppose that $\chi \in \mathrm{Hom}(G, \text{Möb})$ and f_i $(i=1, \cdots, n)$ is a* QC *mapping of Δ_i into $\hat{\mathbf{C}}$ such that $f_i \circ h = \chi(h) \circ f_i$ for every $h \in H_i$. Then there is a* *mapping $\hat{f}$ of $\Omega(G)$ to $\hat{\mathbf{C}}$ which satisfies $\hat{f}\,|_{\Delta_i}=f_i$ and induces χ, i.e., $\hat{f}\circ g=\chi(g)\circ\hat{f}$ for every $g\in G$ on $\Omega(G)$. Especially, $\|\mu(\hat{f})\|_{\Omega(G)}=\max\{\|\mu(f_i)\|_{\Delta_i}\,|\,1\le i\le n\}$, that is, the L^∞-norm of the complex dilatation $\mu(\hat{f})(z)$ of $\hat{f}$ over $\Omega(G)$ is equal to the maximum taken over that of $\mu(f_i)(z)$ over Δ_i for all $i=1,\cdots,n$.*

PROOF. By assumption, any component Δ of $\Omega(G)$ is of the form $\Delta=g(\Delta_i)$ for some i and some $g\in G$. We define $\hat{f}$ on each component of $\Omega(G)$ by

$$\hat{f}=\chi(g)\circ f_i\circ g^{-1}\qquad\text{on}\quad g(\Delta_i)\,.$$

It is well defined. Indeed, if $g,\lambda\in G$ and $g(\Delta_i)=\lambda(\Delta_i)$, then $f_i\circ\lambda^{-1}\circ g=\chi(\lambda^{-1}\circ g)\circ f_i$. Hence we have $(\chi(\lambda)\circ f_i\circ\lambda^{-1})\circ(\chi(g)\circ f_i\circ g^{-1})^{-1}=\chi(\lambda)\circ f_i\circ\lambda^{-1}\circ g\circ f_i^{-1}\circ\chi(g)^{-1}=\chi(\lambda\circ\lambda^{-1}\circ g\circ g^{-1})=\mathrm{id}$. Similarly, we can check that $\hat{f}$ induces χ. □

LEMMA 3. *Assume that $\chi\in\mathrm{Hom}_{\mathrm{qc}}(G,\text{Möb})$, and let w be a* QC *automorphism of $\hat{\mathbf{C}}$ satisfying $\chi(g)=w\circ g\circ w^{-1}$ for every $g\in G$. Let $\hat{f}$ be a mapping of $\Omega(G)$ to $\hat{\mathbf{C}}$ such that the restriction of $\hat{f}$ to each component of $\Omega(G)$ is* QC, *and which induces χ. Then the following* (1) *and* (2) *hold*:

(1) *For each component Δ of $\Omega(G)$, $\hat{f}(\Delta)$ coincides with $w(\Delta)$. In particular, $\hat{f}$ is actually a* QC *homeomorphism of $\Omega(G)$ onto $w(\Omega(G))=\Omega(wGw^{-1})$.*

(2) *There is a* QC *automorphism F of $\hat{\mathbf{C}}$, having the property that $F\,|_{\Omega(G)}=\hat{f}$, F induces χ, and $\|\mu(F)\|_{\hat{\mathbf{C}}}=\|\mu(\hat{f})\|_{\Omega(G)}$.*

PROOF. (1) Let Δ be an arbitrary component of $\Omega(G)$ and f be the restriction of $\hat{f}$ to Δ. Set $u=w^{-1}\circ f$. It is a QC mapping of Δ satisfying $u\circ h=h\circ u$ for every h of the component subgroup H for Δ. We show $u(\Delta)=\Delta$ below, from which follows $f(\Delta)=w(\Delta)$.

It is obvious that $u(\Delta)$ is invariant under $uHu^{-1}=H$. $u(\Delta)$ has no intersection with $\Lambda(H)$, for if $u(\Delta)\cap\Lambda(H)$ were not empty, $\Delta\cap u^{-1}(\Lambda(H))=\Delta\cap\Lambda(u^{-1}Hu)=\Delta\cap\Lambda(H)$ would not be empty. Hence $u(\Delta)$ is contained in some component Δ_1 of $\Omega(H)$. Since the Riemann surface $u(\Delta)/H$, which is QC equivalent to Δ/H, is of finite type, the set of points of $\Delta_1-u(\Delta)$ is discrete in Δ_1 (cf. [10, Chap. II, F. 8]). If $\Delta_1-u(\Delta)\neq\varnothing$, $u(\Delta)$ is a domain with a puncture, hence so does the component Δ of $\Omega(G)$. This contradicts the fact that the limit set $\Lambda(G)$ of the non-elementary Kleinian group G has no isolated points. Therefore $\Delta_1=u(\Delta)$.

Assume that $\Delta_1=u(\Delta)\neq\Delta$. H has two invariant components Δ and Δ_1. As is well known, H is then quasi-Fuchsian. Thus u is a QC homeomorphism of a quasi-disk Δ onto another quasi-disk Δ_1. Further, u has a homeomorphic extension $\tilde{u}$ of $\bar{\Delta}=\Delta\cup\Lambda(H)$ onto $\bar{\Delta}_1=\Delta_1\cup\Lambda(H)$. Let $z_0\in\Lambda(H)$ be an attractive fixed point of a loxodromic element h of H. Since $\lim_{n\to\infty}h^n(z)=z_0$ for any point z in $\hat{\mathbf{C}}$ except the repelling fixed point of h, and $u\circ h=h\circ u$, we have $\tilde{u}(z_0)=z_0$. Hence $\tilde{u}\,|_{\Lambda(H)}=\mathrm{id}$, for such attractive fixed points are dense in $\Lambda(H)$. Since the QC mapping u preserves orientation, this is a contradiction. Therefore we get $\Delta=u(\Delta)$.

(2) To $w^{-1}\circ\hat{f}$, we apply Maskit's identity theorem in [8]. Then there is a QC

automorphism J of $\hat{C}$, where $J|_{\Omega(G)}=w^{-1}\circ\hat{f}$ and $J|_{\Lambda(G)}=\mathrm{id}$. Set $F=w\circ J$. F is a QC automorphism of $\hat{C}$ such that $F|_{\Omega(G)}=\hat{f}$. Then it is obvious that $F\circ g=\chi(g)\circ F$ for every $g\in G$. The remaining assertion is a consequence of the theorem of Sullivan [15, p. 490], which guarantees that the Beltrami differentials for a finitely generated Kleinian group G do not have essential support on $\Lambda(G)$. $\square$

REMARK 5. The author learned a substantial part of Lemmas 2 and 3 from the original manuscript of Sakan [14].

4. A proof of (B)$\Leftrightarrow$(C). As we mentioned in §2, (C) is equivalent to uniform stability (Proposition 1 and Remark 2). Bers [3, p. 16] raised the conjecture that uniform stability is the same as QC stability (B). Therefore we know at this stage that his conjecture is just our problem (B)$\Leftrightarrow$(C). We can give an affirmative answer to it by proving the following theorem:

THEOREM 1. QC *stability implies quasi-stability.*

PROOF. Suppose G to be QC stable. By Lemma 1, every component subgroup of G is quasi-stable. Therefore a proof of Theorem 1 is reduced to the following:

LEMMA 4. *Let G be a finitely generated Kleinian group. If every component subgroup of G is quasi-stable, then G is quasi-stable.*

PROOF. Let $N_\varepsilon(G)=\{\mu\in M(G);\ \|\mu\|_{\hat{C}}<\varepsilon \text{ for } \varepsilon>0\}$. Let $\{\Delta_1,\cdots,\Delta_n\}$ be a system of conjugacy classes of the components of $\Omega(G)$, and $\{H_1,\cdots,H_n\}$ be their component subgroups. For a moment, we omit the index i, and denote each of $\{H_i\}$ by H. We fix generators of $G=\langle g_1,\cdots,g_k\rangle$, $H=\langle h_1,\cdots,h_m\rangle$, and fix words $h_j=w_j(g_1,\cdots,g_k)$ $(j=1,\cdots,m)$. We then let $(\text{Möb})^G$ (resp. $(\text{Möb})^H$) denote the product manifold $(\text{Möb})^k$ (resp. $(\text{Möb})^m$), and let r be the mapping of $(\text{Möb})^G$ to $(\text{Möb})^H$ defined by

$$r:(\lambda_1,\cdots,\lambda_k)\mapsto(w_1(\lambda_1,\cdots,\lambda_k),\cdots,w_m(\lambda_1,\cdots,\lambda_k))\,.$$

Then, r is continuous. For $\chi\in\mathrm{Hom}(G,\text{Möb})$, $r(\chi)$ represents the restriction of the homomorphism χ to H, hence $r(\chi)\in\mathrm{Hom}(H,\text{Möb})$. For each H_i $(i=1,\cdots,n)$, this map is denoted by r_i.

Each H_i $(i=1,\cdots,n)$ is quasi-stable by assumption. Hence for $N_\varepsilon(H_i)$, there is a neighborhood U_i of the origin in $(\text{Möb})^{H_i}$ such that $U_i\cap\mathrm{Hom}_{\mathrm{qc}}(H_i,\text{Möb})\subset\Phi_{H_i}(\text{Möb}\times N_\varepsilon(H_i))$. We choose a neighborhood U of the origin in $(\text{Möb})^G$ so that $r_i(U)$ is contained in U_i for all $i=1,\cdots,n$. For this U, we show that $U\cap\mathrm{Hom}_{\mathrm{qc}}(G,\text{Möb})\subset\Phi_G(\text{Möb}\times N_\varepsilon(G))$. Let χ be an arbitrary point of $U\cap\mathrm{Hom}_{\mathrm{qc}}(G,\text{Möb})$, and w be a QC automorphism of $\hat{C}$ which induces χ. Let $\chi_i=r_i(\chi)$ be the restriction of χ to H_i. Then $\chi_i\in U_i\cap\mathrm{Hom}_{\mathrm{qc}}(H_i,\text{Möb})\subset\Phi_{H_i}(\text{Möb}\times N_\varepsilon(H_i))$. Hence there exists a QC automorphism $f_i\colon\hat{C}\to\hat{C}$ which satisfies $\|\mu(f_i)\|_{\hat{C}}<\varepsilon$ and induces χ_i. We need this f_i only on Δ_i and not on the complement of Δ_i. By applying Lemmas 2 and 3, we get a

QC automorphism F of $\hat{C}$ which satisfies $\| \mu(F) \|_{\hat{C}} < \varepsilon$ and induces χ. Therefore χ is in $\Phi_G(\text{Möb} \times N_\varepsilon(G))$. □□

REMARK 6. One can find criteria for quasi-stability of function groups in [11, Th. 4].

COROLLARY 1. (B) *and* (C) *are equivalent.*

REMARK 7. Since (C) is preserved under the combination theorems [12, Th. 3, 4 & Lemmas 10, 11, 12], so is (B) by the above corollary. That is, a Kleinian group constructed from QC stable Kleinian groups by the combination theorems is also QC stable. This result was stated by Abikoff [1, Th. 3, 4].

5. A proof of (A) ⇒ (B). In this section, we prove (A) ⇒ (B). If G is torsion-free, Marden [7] used 3-dimensional methods and proved this result. In order to reduce the problem to the case of torsion-free Kleinian groups, we first show the following Theorem 2. It gives an affirmative answer to the problem which was raised in [6, Problem 3.3]. The implication (A) ⇒ (B) then follows as a corollary to Theorem 2.

THEOREM 2. *Let G be a finitely generated Kleinian group, and Γ be its subgroup of finite index. If Γ is* QC *stable, then so is G.*

PROOF. As in the proof of Lemma 4, we fix a system of generators of Γ and their words constructed from $\{g_1, \cdots g_k\}$. Let r': $(\text{Möb})^G \to (\text{Möb})^\Gamma$ be the mapping defined by the words. By the QC stability of Γ, there is a neighborhood V of the origin in $(\text{Möb})^\Gamma$ such that $V \cap \text{Hom}_p(\Gamma, \text{Möb}) \subset \text{Hom}_{qc}(\Gamma, \text{Möb})$.

Let Δ be an arbitrary component of $\Omega(G)$. Δ is also a component of $\Omega(\Gamma)$, because $[G: \Gamma] < \infty$ guarantees $\Lambda(G) = \Lambda(\Gamma)$. By Ahlfors' finiteness theorem, $\Delta/\text{Stab}_G(\Delta)$ and $\Delta/\text{Stab}_\Gamma(\Delta)$ are Riemann surfaces of finite type. Let A_1(resp. A_2) be the hyperbolic area of $\Delta/\text{Stab}_G(\Delta)$ (resp. $\Delta/\text{Stab}_\Gamma(\Delta)$). Then $A_2 = A_1 \times [\text{Stab}_G(\Delta)\colon \text{Stab}_\Gamma(\Delta)]$. Since both A_1 and A_2 are positive and finite, we have $[\text{Stab}_G(\Delta)\colon \text{Stab}_\Gamma(\Delta)] < \infty$. Further, by Lemma 1, the component subgruoup $\text{Stab}_\Gamma(\Delta)$ of Γ is geometrically finite. Since an extension of a geometrically finite Kleinian group with finite index is also geometrically finite [10, Chap. VI, E. 6], we know that $\text{Stab}_G(\Delta)$ is geometrically finite.

Let $\{\Delta_1, \cdots, \Delta_n\}$ be a system of conjugacy classes of components of $\Omega(G)$, and $\{H_1, \cdots, H_n\}$ be their component subgroups. By the above consideration, each H_i $(i = 1, \cdots, n)$ is geometrically finite. It follows from Proposition 2 that H_i is QC stable: there is a neighborhood U_i of the origin in $(\text{Möb})^{H_i}$ such that $U_i \cap \text{Hom}_p(H_i, \text{Möb}) \subset \text{Hom}_{qc}(H_i, \text{Möb})$. Now we take a neighborhood U of the origin in $(\text{Möb})^G$ so that $r'(U) \subset V$ and $r_i(U) \subset U_i$ for $i = 1, \cdots, n$, where r_i is the same mapping as in the proof of Lemma 4.

Let χ be a point of $U \cap \text{Hom}_p(G, \text{Möb})$. By the definition of U, $r'(\chi) \in \text{Hom}_{qc}(\Gamma, \text{Möb})$ and $r_i(\chi) \in \text{Hom}_{qc}(H_i, \text{Möb})$. The latter implies that there is a QC homeomorphism f_i of $\hat{C}$ which satisfies $\chi(h_i) = f_i \circ h_i \circ f_i^{-1}$ for every $h_i \in H_i$. By Lemma

2, we then get a mapping $\hat{f}$ of $\Omega(G)$ satisfying $\hat{f}|_{\Delta_i}=f_i$ and $\hat{f}\circ g=\chi(g)\circ\hat{f}$ for every $g\in G$. We denote $r'(\chi)$ by χ_Γ. Clearly $\hat{f}\circ\gamma=\chi_\Gamma(\gamma)\circ\hat{f}$ for every $\gamma\in\Gamma$ on $\Omega(G)=\Omega(\Gamma)$. Since $\chi_\Gamma\in\mathrm{Hom}_{\mathrm{qc}}(\Gamma, \text{Möb})$, applying Lemma 3, we see that there is a QC automorphism F of $\hat{\boldsymbol{C}}$ such that $F|_{\Omega(G)=\Omega(\Gamma)}=\hat{f}$. Therefore F turns out to satisfy $\chi(g)=F\circ g\circ F^{-1}$ for every $g\in G$. It means that $\chi\in\mathrm{Hom}_{\mathrm{qc}}(G, \text{Möb})$. Since χ is an arbitrary point of $U\cap\mathrm{Hom}_{\mathrm{p}}(G, \text{Möb})$, we have completed the proof. □

COROLLARY 2. (A) *implies* (B).

PROOF. A finitely generated Kleinian group G has a torsion-free subgroup Γ of finite index by Selberg's well-known lemma, and Γ is geometrically finite since G is geometrically finite (see [10, Chap. VI, F. 3]). For a torsion-free Kleinian group, the assertion was already proved by Marden [7, §9]. Hence Γ is QC stable. By Theorem 2, G is QC stable. □

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