ON A CONJECTURE OF FUCHS CONCERNING FACTORIZATION OF ENTIRE FUNCTIONS

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Abstract. In 1982, Fuchs raised the following conjecture: Suppose that $F$ is an entire function of finite order with at least one finite deficient value. Then $F$ is pseudo-prime. In this paper, we prove this conjecture under the additional condition that the number of the limiting directions of Julia directions of $F$ is finite.

1. Introduction. Following Gross [5] we say that a meromorphic function $F(z)$ has a factorization with factors $f(z)$ and $g(z)$, if

$$F(z) = f(g(z)),$$

where $f(z)$ is meromorphic and $g(z)$ is entire ($g$ may be meromorphic when $f$ is rational). $F(z)$ is said to be pseudo-prime if every factorization of the above form implies that $f(z)$ is rational or $g(z)$ is a polynomial.

Let $F(z)$ be a meromorphic function. A ray $J(\theta) := \{z: \arg z = \theta, 0 \leq \theta < 2\pi\}$ is called a Julia direction of $F(z)$ if, in any open sector containing the ray, $F(z)$ takes all finite values, with at most two finite exceptional values, infinitely often.

Goldstein [3] proved the following:

THEOREM A. Let $F(z)$ be an entire function of finite order such that $\delta(a, F) = 1$ for some $a \neq \infty$. Then $F(z)$ is pseudo-prime.

Recently Fuchs (cf. [11]) conjectured that the conclusion of Theorem A remains true under the weaker assumption $\delta(a, F) > 0$.

In this paper, we will prove:

THEOREM. Suppose that $F(z)$ is an entire function of finite lower order such that $\delta(a, F) > 0$ for some $a \neq \infty$. If the number of the limiting directions of Julia directions of $F(z)$ is finite, then $F(z)$ is pseudo-prime.

REMARK. In the original version of this paper the author used the condition that the number of Julia directions of $F(z)$ is finite. The weakened version is due to the referee.
2. **Notation.** Let \( \theta_1 < \theta_2 \) and \( 0 \leq r_1 < r < r_2 \leq +\infty \), we define

\[
\Gamma(\theta_1, \theta_2; r) = \{ z : \theta_1 < \arg z < \theta_2, |z| = r \},
\]

\[
\Omega(\theta_1, \theta_2) = \{ z : \theta_1 < \arg z < \theta_2 \},
\]

\[
\Omega(\theta_1, \theta_2; r) = \{ z : \theta_1 < \arg z < \theta_2, |z| < r \}
\]

and

\[
\Omega(\theta_1, \theta_2; r_1, r_2) = \{ z : \theta_1 < \arg z < \theta_2, r_1 \leq |z| \leq r_2 \}.
\]

Also, we denote by \( \widehat{E} \) the closure of a set \( E \) with respect to the plane \( |z| < +\infty \).

Furthermore, we write \( n(E, f=a) \) to be the number of roots with due count of multiplicity of the equation \( f = a \) in \( E \).

In addition, we assume that the reader is familiar with the standard notation of the Nevanlinna theory (see [6]).

3. **Known results.**

**Lemma 1** (cf. [2]). Meromorphic functions with more than one deficient value have a positive lower order.

The following lemma is the upper half part of Lemma 2 in [12].

**Lemma 2.** Suppose that \( F(z) \) is an entire function of finite lower order \( \mu \) such that \( \delta = \delta(a, F) > 0 \) for some \( a \neq \infty \). Set

\[
E(t) = \{ \theta : \log |F(te^{i\theta}) - a| \leq -\frac{\delta}{4} T(t, F), 0 \leq \theta < 2\pi \}.
\]

Then there exists a sequence \( \{ t_n \} \) of positive numbers tending to infinity such that

\[
\mes E(t_n) \geq K > 0,
\]

where \( K \) is an absolute constant depending only on \( \delta \) and \( \mu \).

**Lemma 3** (a modified version of [12, Lemma 3]). Let \( F(z) \) be an entire function of lower order \( \mu \) such that \( 0 < \mu < +\infty \). Suppose that there exist two finite complex numbers \( b_1 \) and \( b_2 \) \( (b_1 \neq b_2) \) such that

\[
\limsup_{r \to +\infty} \frac{\log^+ \left\{ \sum_{i=1}^2 n(\Omega(\theta', \theta''; r), F = b_i) \right\}}{\log r} = 0.
\]

If there are positive numbers \( \delta \) and \( B \), a finite complex value \( a_0 \) and a positive and sufficiently large number \( R_k \) such that

\[
\mes E > B
\]
for
\[ E = \{ \theta : \theta' < \theta < \theta'', \log |F(R_k e^{i\theta}) - a_0| \leq -\left(\delta/4\right)T(R_k, F) \}, \]
then for two positive numbers \( \varepsilon \) and \( Q, Q > 1 \), \( 0 < \varepsilon < \min\{B/4, (\theta'' - \theta')/4\} \),
\[ \log |F(z) - a_0| \leq -AT(R_k, F) \]
holds in the region
\[ \tilde{\Omega}(\theta' + \varepsilon, \theta'' - \varepsilon; 10^{-4Q} R_k, 10^{4Q} R_k), \]
where \( 0 < A < +\infty \) is a constant depending only on \( \delta, \varepsilon, B \) and \( Q \), i.e.
\[ A = \frac{\delta}{4 \left( 5 + 4 \log \frac{1}{h} \right)^{2N_1 + N_2}}, \quad N_1 = \left\lceil \frac{10\pi/\varepsilon}{} \right\rceil, \quad N_2 = \left\lceil \frac{20(10^{4Q} - 1)}{\varepsilon} \right\rceil + 4, \]
\[ h = B\varepsilon/8(2\varepsilon + 1)(10\pi + \varepsilon). \]

**Remark.** In [12] it is assumed that the order of \( F \) is finite. The referee showed to the author that this condition is unnecessary. Note that \( n_1 < R_k^{4+1} \) on p. 591 in [12] is unnecessary.

**Lemma 4 (cf. [1]).** Suppose that \( f(z) \) is a transcendental meromorphic function and \( g(z) \) is a transcendental entire function. Then
\[ \lim_{r \to +\infty} \frac{T(r, f(g))}{T(r, g)} = +\infty. \]

By virtue of Lemma 4 (as well as Pólya's lemma). We have the following result.

**Lemma 5 (cf. [9]).** Let \( f(z) \) be meromorphic with at most one pole and \( g(z) \) be entire. If the lower order of \( f(g) \) is finite, then either \( f \) is of zero lower order or \( g \) is a polynomial.

**Lemma 6 (cf. [8]).** Suppose that \( f(z) \) is a meromorphic function of lower order \( \mu \) with \( 0 \leq \mu < 1/2 \). If \( \delta(\infty, f) > 1 - \cos \pi \mu \), then
\[ \limsup_{r \to +\infty} \frac{\log^{+} \mu(r, f)}{T(r, f)} \geq C(\mu) \]
for some constant \( C(\mu) > 0 \), where \( \mu(r, f) = \min\{|f(z)|; |z| = r\} \).

**Lemma 7 (cf. [4]).** Let \( g \) be an entire function, \( f, F \) be meromorphic functions such that \( F = f(g) \). Suppose that \( L \) is a path tending to infinity such that \( F(z) \to 0 \) as \( z \to \infty \) along \( L \), and \( g(L) \) is bounded. Then \( g(z) \to z_0 \) as \( z \to \infty \) along \( L \), where \( z_0 \) is a zero of \( f \).
4. Properties of functions having no Julia directions in angles. Now we shall give a property of meromorphic functions without Julia directions in angles, which essentially belongs to Zhang [13].

**Lemma 8.** If a meromorphic function $F(z)$ has no Julia directions in $p$ angles $\Omega(\theta_{k_1}, \theta_{k_2})$ ($k = 1, \ldots, p$; $0 \leq \theta_{11} < \theta_{12} < \theta_{21} < \cdots < \theta_{p1} < \theta_{p2} < 2\pi$), then for any small $\varepsilon$ ($0 < \varepsilon < \min_{1 \leq k < p}(\theta_{k2} - \theta_{k1})/2$), there exist three distinct complex numbers $b_1$, $b_2$ and $b_3$ such that

$$\limsup_{r \to \infty} \frac{\log^+ \sum_{j=1}^{3} n(\bar{\Omega}, F=b_j)}{\log r} = 0$$

and the mutual spherical distances between $b_1$, $b_2$ and $b_3$ are $d$, $0 < d < 1/2$, where

$$\bar{\Omega} = \bigcup_{k=1}^{p} \bar{\Omega}(\theta_{k1} + \varepsilon, \theta_{k2} - \varepsilon; r).$$

**Proof.** Assume the conclusion is false. Then for all values $Z$, except possibly two values,

$$\limsup_{r \to \infty} \frac{\log^+ n(\bar{\Omega}, F=Z)}{\log r} > 0.$$ 

By the finite covering theorem, for any $\eta > 0$, there exists a half line $J(\theta) \in \bigcup_{k=1}^{p} \bar{\Omega}(\theta_{k1} + \varepsilon, \theta_{k2} - \varepsilon)$ such that the line measure of the set

$$\left\{ Z : \limsup_{r \to \infty} \frac{\log^+ n(\Omega(\theta - \eta, \theta + \eta; r), F=Z)}{\log r} > 0 \right\}$$

is positive. On the other hand, since $J(\theta)$ is not a Julia direction of $F(z)$, it follows that there exist a positive number $\eta'$ and three distinct values $\alpha$, $\beta$ and $\gamma$ such that the series

$$\sum_{n=1}^{\infty} |a_n(\eta', Z)|^{-\sigma}, \quad Z = \alpha, \beta, \gamma$$

converges for an arbitrary small number $\sigma > 0$, where $a_n(\eta', Z) (n = 1, 2, \ldots; |a_1(\eta', Z)| \leq |a_2(\eta', Z)| \leq \cdots)$ denote all zeros of $F(z)$ in $\Omega(\theta - \eta', \theta + \eta')$. According to a known result [10, p. 31], the series

$$\sum_{n=1}^{\infty} |a_n(\eta'', Z)|^{-\sigma}, \quad \eta'' < \eta'$$

converges for any value $Z$, except a set with zero line measure. Note that $\sigma$ can be arbitrarily small, thus for all $Z$, except a set with zero line measure,

$$\limsup_{r \to \infty} \frac{\log^+ n(\Omega(\theta - \eta'', \theta + \eta''; r), F=Z)}{\log r} = 0.$$
which is a contradiction. This completes the proof of Lemma 8.

5. Proof of Theorem. From Lemma 1 and the assumption we have $0 < \mu < +\infty$, where $\mu$ is the lower order of $F(z)$. By a well known classical result of Julia, $F(z)$ has at least one Julia direction. Let $m$ be the number of the limiting directions of Julia directions of $F(z)$. Then $0 < m < +\infty$. Without loss of generality, we assume that $a = 0$ and $m = 1$. For convenience, we suppose that the limiting direction is $J(0)$. Next we distinguish the proof into five steps.

Step 1. According to Lemma 2, there exist a sequence $\{\theta_n\}$ of positive numbers tending to infinity and a set $E_n \subset [0, 2\pi)$ such that, for $\theta \in E_n$,

$$
\log |F(\theta_n e^{i\theta})| \leq -\frac{\delta}{4} T(\theta_n, F), \quad t_1 > r_0
$$

and

$$
\text{mes } E_n \geq K > 0,
$$

where $\delta = \delta(0, F) > 0$. Now we take a small number $\eta$ with $0 < \eta < \eta/32$. Then, by $m = 1$, the number of Julia directions of $F(z)$ in the complement of $\Omega(\eta, \eta)$ is finite. Thus there exist $q$ rays $J(\theta_k)$ ($\eta = \theta_1 < \cdots < \theta_{q-1} < \theta_q = 2\pi - \eta$) such that $F(z)$ has no Julia directions in the region $\bigcup_{k=1}^{q-1} \Omega(\theta_k, \theta_{k+1})$, where $1 < q < +\infty$ and $q$ depends on $\eta$. Put $\omega = \min_{1 \leq k \leq q} (\theta_{k+1} - \theta_k)$ and take a number $\epsilon$ such that $0 < \epsilon < \min\{\omega/32, K/32q, 1/8\}$ and define a sequence $\{\epsilon_j\}$ of positive numbers

$$
\epsilon_j = 2^{-(j+1)} \epsilon \quad (j = 0, 1, \ldots)
$$

By Lemma 8 applied to $F$ and $\epsilon_0$, there exist distinct finite complex numbers $b_1$ and $b_2$ such that

$$
\limsup_{r \to +\infty} \frac{\log^+ \{n(\tilde{\Omega}, F = b_1) + n(\tilde{\Omega}, F = b_2)\}}{\log r} = 0,
$$

where

$$
\tilde{\Omega} = \bigcup_{k=1}^{q} \Omega(\theta_k + \epsilon_0, \theta_{k+1} - \epsilon_0; r).
$$

Step 2. For any $n$, by (2) and the choice of $\epsilon_0$ there exists an integer $k(n)$ ($1 \leq k(n) \leq q - 1$) such that, for $\theta \in E_n^* = E_n \cap (\theta_{k(n)} + \epsilon_0, \theta_{k(n)+1} - \epsilon_0)$,

$$
\log |F(\theta_n e^{i\theta})| \leq -\frac{\delta}{4} T(\theta_n, F)
$$

and
Since \( q < +\infty \), there exists a sequence \( \{n_j\} \) tending to infinity such that

\[
(7) \quad k(n_1) = k(n_2) = \cdots = 1 \text{(say)}.
\]

Thus for

\[
(8) \quad \theta e_{n_j}^* = E_{n_j}^* \cap (\theta_1 + \varepsilon_0, \theta_2 - \varepsilon_0)
\]

we have

\[
(9) \quad \log |F(t_{n_j}e^{\theta})| \leq -\frac{\delta}{4} T(t_{n_j}, F)
\]

and

\[
(10) \quad \mes E_{n_j}^* \geq \frac{K}{2q}.
\]

In particular,

\[
(11) \quad \theta_2 - \theta_1 > \frac{K}{2q}.
\]

Step 3. In this step, we shall deal with the values of \( F(z) \) in the region \( \Omega(\theta_1 + \varepsilon, \theta_2 - \varepsilon) \).

For a chosen positive number \( Q (1 < Q) \) and any \( j \geq 2 \), there exists a non-negative integer \( m_j \) such that

\[
(12) \quad 10^{4Qm_j}t_{n_j-1} < t_{n_j} \leq 10^{4Q(m_j+1)}t_{n_j-1}.
\]

Now put

\[
(13) \quad R_{j,s} = 10^{-4Qs}t_{n_j} \quad (s = -1, 0, 1, \ldots, m_j + 1).
\]

By applying Lemma 3 with

\[
R_k = R_{j,0}, \quad \theta' = \theta_1 + \varepsilon_0, \quad \theta'' = \theta_2 - \varepsilon_0, \quad \varepsilon = \varepsilon_1, \quad B = K/2q, \quad E = E_{n_j}^* \quad \text{and} \quad a_0 = 0,
\]

we deduce from (9) and (10) that

\[
(14) \quad \log |F(z)| \leq -A_1 T(R_{j,0}, F), \leq -A_1 T(R_{j,1}, F),
\]

where \( z \in \Omega(\theta_1 + \varepsilon_0 + \varepsilon_1, \theta_2 - \varepsilon_0 - \varepsilon_1; R_{j,1}, R_{j,-1}) \). \( 0 < A_1 < +\infty \) is a constant depending only on \( K, Q, q, \delta \) and \( \varepsilon_1 \). In particular, we have

\[
(15) \quad \log |F(z)| \leq -A_1 T(R_{j,1}, F)
\]

for \( z \in \Gamma(\theta_1 + \varepsilon_0 + \varepsilon_1, \theta_2 - \varepsilon_0 - \varepsilon_1; R_{j,1}) \). Now by the choice of \( \varepsilon \), we deduce from (3) and
(11) that, for any $s \geq 0$,
$$
e_0 + e_1 + \cdots + e_s = e(2^{-1} + 2^{-2} + \cdots + 2^{-s-1}) < \varepsilon$$
and
$$(\theta_2 - e_0 - e_1 - \cdots - e_s) - (\theta_1 + e_0 + e_1 + \cdots + e_s) \geq \theta_2 - \theta_1 - 2\varepsilon \geq K/4q.$$  
Thus, by applying Lemma 3 again with
$$B = \theta_2 - \theta_1 - 2(e_0 + e_1) > k/4q, \quad \theta' = \theta_1 + e_0 + e_1, \quad \theta'' = \theta_2 - e_0 - e_1,$$
we can verify that $\Phi(0, G) = \Phi(0, F) > 0$ and that $G$ and $F$ have the same lower order.

Note that
$$\Omega(\theta_1 + e, \theta_2 - e; t_{n_{j-1}}, t_n)$$
is contained in the set
$$\bigcup_{s=1}^{m_j+1} \Omega(\theta_1 + e_0 + \cdots + e_s, \theta_2 - e_0 - \cdots - e_s; R_{j,s}, R_{j,s-2}).$$
We conclude that, for $z \in \Omega(\theta_1 + e, \theta_2 - e; t_{n_{j-1}}, t_n)$,
$$\log |F(z)| \leq - \min_{1 \leq s \leq m_j+1} \{ A_s T(R_{j,s}, F) \} \leq 0,$$
i.e., $|F(z)| \leq 1$. Since $j$ is arbitrary, $F(z)$ is bounded on $\Omega(\theta_1 + e, \theta_2 - e)$. Hence, there exists an absolute constant $M > 0$ such that
$$|F(z)| \leq M, \quad z \in \Omega(\theta_1 + e, \theta_2 - e).$$

Step 4. In this step, we shall prove that $F(z)$ tends to zero in the set $\Omega(\theta_1 + 4e, \theta_2 - 4e)$.

Put
$$G(z) = zF(z).$$
Then we can verify that $\delta(0, G) = \delta(0, F) > 0$ and that $G$ and $F$ have the same lower order.
Now for sufficiently large \( r \) we determine a positive integer \( n \) such that

\[
2^{n-1} \leq r \leq 2^n.
\]

We define

\[
r_j = 2^j \quad (j = 1, \ldots, n).
\]

For any \( j \leq n \), we consider the mapping

\[
w = w_j(z) = \frac{(e^{-i\theta_j} \cdot z)^{\theta_j} - (r_j)^{\theta_j}}{(e^{-i\theta_j} \cdot z)^{\theta_j} + (r_j)^{\theta_j}},
\]

where \( \theta_j = (\theta_1 + \theta_2)/2 \) and \( \theta = \pi/((\theta_2 - \theta_1 - 2\epsilon)) \). Then the image of \( \Omega(\theta_1 + \epsilon, \theta_2 - \epsilon) \) in the \( w \)-plane is \( |w| \leq 1 \). Now for each \( z = te^{i\theta} \in \Omega(\theta_1 + 2\epsilon, \theta_2 - 2\epsilon; r_{j-1}, r_j) \) we have

\[
|w| = \left| \frac{\theta e^{i\theta(\phi - \rho_\theta)} - (r_j)^{\theta_j}}{\theta^2 e^{i(\phi - \rho_\theta)} + (r_j)^{\theta_j}} \right| = \left( 1 - \frac{4\theta e^{i\theta(\phi - \theta_\theta)} \cos \theta(\phi - \theta_\theta)}{t^2 + (r_j)^2 + 2\theta e^{i(\phi - \theta_\theta)} \cos \theta(\phi - \theta_\theta)} \right)^{1/2}.
\]

Note that \( r_{j-1} \leq t \leq r_j \) and \( \phi \leq \theta_2 - 2\epsilon \). Thus

\[
t^2 + (r_j)^2 + 2\theta e^{i(\phi - \theta_\theta)} \cos \theta(\phi - \theta_\theta) \leq 4(r_j)^2\theta,
\]

\[
4\theta e^{i(\phi - \theta_\theta)} > 4(r_{j-1})^{\theta_j} \cos \left( \frac{\pi}{2} - \theta_\epsilon \right) \geq \frac{8\theta e}{\pi} (r_{j-1})^{\theta_j}.
\]

Substituting these into (21) we obtain

\[
|w| \leq \left( 1 - \frac{2\theta e}{\pi} \frac{r_{j-1}}{r_j} \right)^{1/2} = \left( 1 - \frac{2\theta e}{\pi} \left( \frac{1}{2} \right)^{\theta} \right)^{1/2} \leq 1 - \frac{\theta e}{\pi} \left( \frac{1}{2} \right)^{\theta}.
\]

Let

\[
R = 1 - \frac{\theta e}{\pi} \left( \frac{1}{2} \right)^{\theta}.
\]

Then we see that the image of \( \Omega(\theta_1 + 2\epsilon, \theta_2 - 2\epsilon; r_{j-1}, r_j) \) in the \( w \)-plane is contained in the circle \( |z| \leq R < 1 \). Furthermore, we can derive from (20) that the inverse mapping of \( w = w_j(z) \) is

\[
z = z_j(w) = r e^{i\theta(\theta e - 1)w} \left( \frac{1 + w}{1 - w} \right)^{1/\theta}.
\]

Thus for \( |w| \leq (1 + R)/2 \), we have

\[
|z| \leq r_j \left( \frac{1 + (1 + R)/2}{1 - (1 + R)/2} \right)^{1/\theta} \leq r_j \left( \frac{4}{1 - R} \right)^{1/\theta} = 2r_j \left( \frac{4\pi}{\theta e} \right)^{1/\theta} \leq 4 \left( \frac{4\pi}{\theta e} \right)^{1/\theta}.
\]

Hence the inverse image of \( |w| \leq (1 + R)/2 \) is contained in the region
\[ \bar{\Omega}(\theta_1 + \varepsilon, \theta_2 - \varepsilon; 4(4\pi/\theta\varepsilon)^{1/\theta}r) . \]

Now put

\[ H_j(w) = G(z_j(w)) = z_j(w)F(z_j(w)) . \]

Then \( H_j(w) \) is holomorphic in \( |w| \leq 1 \). For two distinct and finite complex numbers \( x, y \) we denote by \( |x, y| \) the spherical distance between \( x \) and \( y \). It is easy to verify that

\[ \log^+ |x| + \log^+ |y| + \log \frac{1}{|x - y|} \leq 0, \]

From the Boutroux–Cartan Theorem [10], we have

\[ \prod_{j=1}^{n} |H_j(O), \alpha| \geq e^n \]

for any complex number \( \alpha \), except a set of \( \alpha \) which can be enclosed in a finite number of disks with the sum of total spherical radii not exceeding \( 2\varepsilon e < 1/4 \). The union of these disks is denoted by \( \gamma \).

Choose \( \alpha \notin \gamma \) such that \( \alpha \) satisfies (23). By the first fundamental theorem we deduce from (22) and (18) that

\[ n(\bar{\Omega}(\theta_1 + 2\varepsilon, \theta_2 - 2\varepsilon; r_j, r_j), \alpha) \leq n(R, H_j(w) = \alpha) \]

\[ \leq \frac{1}{\log(1 + R) - \log 2R} \int_{R}^{(1 + R)/2} n(t, H_j(w) = \alpha) \frac{dt}{t} \]

\[ \leq \frac{1}{\log(1 + R) - \log 2R} \left\{ T\left( \frac{1 + R}{2}, H_j(w) - \alpha \right) + \log \frac{1}{|H_j(O) - \alpha|} \right\} \]

\[ \leq \frac{1}{\log(1 + R) - \log 2R} \left\{ \log^+ M\left( \frac{1 + R}{2}, H_j(w) \right) + \log 2 + \log^+ |\alpha| + \log \frac{1}{|H_j(O) - \alpha|} \right\} \]

\[ \leq \frac{1}{\log(1 + R) - \log 2R} \left\{ \log^+ M\left( \bar{\Omega}(\theta_1 + \varepsilon, \theta_2 - \varepsilon; 4(4\pi/\theta\varepsilon)^{1/\theta}r), zF(z) \right) + \log 2 + \log \frac{1}{|H_j(O) - \alpha|} \right\} \]

\[ \leq D\left\{ \log r + \log \frac{1}{|H_j(O), \alpha|} + C \right\} , \]

where
Hence

\[ n \{ \Omega(\theta_1 + 2\varepsilon, \theta_2 - 2\varepsilon; r), G(z) = \alpha \} \]

\[ \leq \sum_{j=1}^{n} n \{ \Omega(\theta_1 + 2\varepsilon, \theta_2 - 2\varepsilon; r_j), G(z) = \alpha \} + O(1) \]

\[ \leq D \left\{ n \log r + \log \left( \prod_{j=1}^{n} |H(O, \alpha)| \right)^{-1} + nC \right\} + O(1) . \]

Now from (19) we have \( n \leq (\log 2)^{-1} \log r + 1 . \) Therefore, by (23),

\[ n \{ \Omega(\theta_1 + 2\varepsilon, \theta_2 - 2\varepsilon; r), G(z) = \alpha \} \]

\[ \leq \frac{D}{\log 2} \left\{ (\log r)^2 + \left( \log 2 + C + \log \frac{1}{\varepsilon} \right) \log r + \log 2 \left( C + \log \frac{1}{\varepsilon} \right) \right\} + O(1) , \]

which results in

\[ \lim_{r \to +\infty} \frac{\log^{+} n \{ \Omega(\theta_1 + 2\varepsilon, \theta_2 - 2\varepsilon; r), G(z) = \alpha \}}{\log r} = 0 , \]

where \( \alpha \not\in (\gamma) \) and \( \alpha \) satisfies (23). Obviously, there are infinitely many such complex values \( \alpha \).

Now we deduce from (9) and \( T(r, G) \leq \log r + T(r, F) \) that

\[ \log |G(t_{nj}e^{i\theta})| \leq \left( 1 + \frac{\delta}{4} \right) \log t_{nj} - \frac{\delta}{4} T(t_{nj}, G) , \quad \theta \in E_{nj}^{*} . \]

Since the lower order \( \mu \) of \( G \) is positive, we have

\[ (1 + \delta/4) \log t_{nj} = o(T(t_{nj}, G)) . \]

Thus, for sufficiently large \( j \),

\[ \log |G(t_{nj}e^{i\theta})| \leq -\frac{\delta}{5} T(t_{nj}, G) , \quad \theta \in E_{nj}^{*} . \]

By the same reasoning as in Step 3 we conclude that, with Lemma 8 replace by (24), the function \( G(z) \) is bounded in the region \( \Omega(\theta_1 + 4\varepsilon, \theta_2 - 4\varepsilon) \). Hence

\[ \lim_{z \to \infty} F(z) = 0 . \]

Step 5. Suppose that \( F(z) \) is not pseudo-prime. Then there exist a transcendental
meromorphic function \( f(z) \) and a transcendental entire function \( g(z) \) such that

\begin{equation}
F(z) = f(g(z)).
\end{equation}

Thus by Lemma 5, \( f(z) \) is of zero lower order. Also, \( f(z) \) has at most one pole, since \( F(z) \) is entire. Hence \( \delta(\infty, f) = 1 \). By this and Lemma 6 there exists a sequence \( \{u_n\} \) with \( u_n \to +\infty \) as \( n \to \infty \) such that

\begin{equation}
\min_{|z|=u_n} |f(z)| \to +\infty.
\end{equation}

Now take a connected path \( L \) running to infinity and having the following properties:

(i) \( L \) contains \( \Gamma(\theta_1 + 4\varepsilon, \theta_2 - 4\varepsilon; \tau_n) \) \( (j = 1, \ldots) \);
(ii) \( L \subseteq \Omega(\theta_1 + 4\varepsilon, \theta_2 - 4\varepsilon) \).

From (8), (9) and (10) we have, for \( \theta \in \mathcal{E}_n \) \( (\theta_1 + 4\varepsilon, \theta_2 - 4\varepsilon) \cap E_n \),

\begin{equation}
\log |F(z)| \leq -\frac{\delta}{4} T(t_n, F), \quad z = (t_n)e^{i\theta}
\end{equation}

and

\begin{equation}
\text{mes} \mathcal{E}_n \geq \frac{K}{4q}.
\end{equation}

By (25), (26) and (27), \( g(L) \) must be bounded. Hence Lemma 7 asserts that \( g(z) \to z_0 \) as \( z \to \infty \) along \( L \), where \( z_0 \) is a zero of \( f(z) \). Thus there exists an integer \( j_0 \geq 1 \) such that, for \( j \geq j_0 \),

\begin{equation}
|g(z) - z_0| < \varepsilon, \quad \theta \in \mathcal{E}_n \quad \text{and} \quad z = (t_n)e^{i\theta}.
\end{equation}

Now, if \( f(z) \) has a zero of order \( m \) \( (m \geq 1) \) at \( z_0 \), then there is a constant \( c > 0 \) such that

\[ |f(z)| \geq c|z - z_0|^m \quad \text{if} \quad |z - z_0| < \varepsilon. \]

Combining this with (30) we obtain

\[ |F(z)| = |f(g(z))| \geq c|g(z) - z_0|^m \quad \text{if} \quad |z - z_0| < \varepsilon. \]

So, for \( \theta \in \mathcal{E}_n \) \( (j \geq j_0) \) and \( z = (t_n)e^{i\theta} \), we have

\[ m \cdot \log^+ \frac{1}{|g(z) - z_0|} \geq -\log |F(z)| + \log c \geq \frac{\delta}{4} T(t_n, F) + \log c. \]

It follows from (29) and the first fundamental theorem that

\[ m \cdot T(t_n, g) + O(1) \geq m \cdot m \left( \frac{1}{g(z_0)} \right) \geq \frac{m}{2\pi} \int_{E_n} \log^+ \frac{1}{|g(t_n)e^{i\theta} - z_0|} d\theta \]

\[ \geq \frac{\text{mes} \mathcal{E}_n}{2\pi} \left( \frac{\delta}{4} T(t_n, F) + \log c \right) \geq \frac{K\delta}{32q\pi} T(t_n, F) + \frac{K}{8q\pi} \log c. \]
This contradicts Lemma 4. The proof is completed.

**Final Remark.** Niino [7] proved another kind of result: If an entire function \( f \) belongs to some family \( \delta(\lambda, \mu) \) and entire function \( g \) is of order \( \lambda \) and lower order \( \mu \), then \( \delta(a, f(g)) = 0 \) for any \( a \) in \( \mathbb{C} \).

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**References**


