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HOLOMORPHIC AUTOMORPHISMS OF CERTAIN CLASS OF DOMAINS OF INFINITE TYPE

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Abstract. We show that the automorphism group of a certain class of bounded Hartogs domains of infinite type of dimension two is compact.

1. Introduction and statement of results. An automorphism of a domain Ω in C^n is a biholomorphic mapping from Ω onto itself. The set of all automorphisms of Ω makes a group under composition. This group is called the automorphism group of Ω and denoted by $\mathrm{Aut}(\Omega)$.

The study of automorphism groups of domains in C^n has been attracting much attention lately in relation to the characterization of domains in $\mathbf{C}^n$. In 1977, Wong [Won] proved that any bounded strongly pseudo-convex domain in C^n with noncompact automorphism group is biholomorphically equivalent to the unit ball in C^n. The noncompactness of $\mathrm{Aut}(\Omega)$ means that there exist points $p\in\partial\Omega$ and $q\in\Omega$ and a sequence $\{F_n\}$ in $\mathrm{Aut}(\Omega)$ such that $F_n(q)\to p$ as $n\to\infty$ (p is called an orbit accumulation point). Rosay [Ros] removed the hypothesis of global strong pseudo-convexity and showed that Wong's theorem is true if $\partial\Omega$ is strongly pseudo-convex at p. For weakly pseudo-convex domains, Bedford and Pinchuk [BP] used the scaling technique to show that any bounded pseudo-convex domain of finite type is biholomorphically equivalent to a domain $\{(z_1, z_2)\in C^2 : |z_1|^2+|z_2|^{2m}<1\}$ for some integer m provided that the automorphism group of the domain is noncompact.

If we remove the *a priori* assumption of being finite type in Bedford-Pinchuk's result, the scaling technique does not work. So, it is natural to ask whether $\partial\Omega$ can be of infinite type at the orbit accumulation point. That $\partial\Omega$ is of infinite type at p means that the Levi form for Ω vanishes at p to the infinite order in the complex tangential direction (see the next section for a precise definition of infinite type). In this paper, we consider the following special kind of Hartogs domains

$$E_P=\{(z_1, z_2)\in C^2 : |z_1|^2+P(z_2, \bar{z}_2)<1\}\,, \tag{1.1}$$

where P is a subharmonic function with $P(0)=0$. We show that if the domain E_P is of

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infinite type, the $\mathrm{Aut}(E_P)$ is compact. If $P(z_2, \bar{z}_2)=\phi(|z_2|^2)$, then E_P is a Reinhardt domain and we can give a complete classification of the automorphism group by exploiting the circularity of the domain (cf. [Kan] [GK]).

Let us state the main result of this paper more precisely. Let $P(z)=P(z, \bar{z})$ be a smooth function satisfying the following conditions:

(C1) $P(z)>0$ if $z\neq 0$,

(C2) $(\partial^{\alpha+\beta}P/\partial z^{\alpha}\partial\bar{z}^{\beta})(0)=0$ for any non-negative integers α and β, and 0 is the only point with this property.

Assume that E_P is a bounded pseudo-convex domain. The pseudo-convexity of E_P is equivalent to the subharmonicity of P in a neighborhood of $\bar{E}_P$. The condition (C2) implies that E_P is of infinite type along the points $(e^{i\theta}, 0)\in\partial E_P$ and $(e^{i\theta}, 0)$ are the only points in ∂E_P where E_P is of infinite type (see the next section). Among functions satisfying the above conditions are $P(z)=C\exp(-1/|z|^2)$ as a radial function and $P(z)=C\exp(-1/(|z|^2+\varepsilon\Re(z^2)))$ as a nonradial one where C (resp. ε) is a large (resp. small) constant.

THEOREM. *If E_P is as above, then any automorphism of E_P fixes the origin.*

COROLLARY. *If E_P is as above, then* $\mathrm{Aut}(E_P)$ *is compact.*

Note that unlike the radial case, E_P does not have a circularity. Proofs in this paper will consist of the following steps. Because of the invariance of type under automorphisms, we first show that certain directional derivatives of all orders of automorphisms of E_P vanish at the point of infinite type (Lemma 3.1). We then show that the rate of approach to the orbit accumulation point should be uniform if there exists any orbit accumulation point (Lemma 3.2). Finally we give some global arguments to finish up the proof. It is our hope that the arguments in this paper can be extended to a more general class of domains.

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2. Infinite type. In this section, we review Kohn's definition of type in [Koh]. Let Ω be a bounded domain in $\boldsymbol{C}^2$ with a smooth boundary and let r be a defining function for Ω, i.e., r is a smooth real-valued function in a neighborhood U of $\bar{\Omega}$ and $\nabla r(z)\neq 0$ if $z\in\partial\Omega$, and $\Omega=\{z\in U: r(z)<0\}$. Define a holomorphic tangent vector field L by

$$L=\frac{\partial r}{\partial z_2}\frac{\partial}{\partial z_1}-\frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2}. \tag{2.1}$$

The L is a basis for the maximal complex subspace of the tangent space to $\partial\Omega$ at each point. The Levi function λ for Ω is defined to be function such that $\mathscr{L}(L, \bar{L})=\lambda|L|^2$, where $\mathscr{L}$ is the Levi form for Ω. Kohn showed that if $\partial\Omega$ is of finite type at a point

$p \in \partial\Omega$, then

$$(L^{\alpha_1} \cdots L^{\alpha_m}\lambda)(p) \neq 0$$

for some $\alpha_1, \ldots, \alpha_m$, where each α_j is either 0 or 1, while $L^0 = L$ and $L^1 = \bar{L}$. Therefore, we take the following definition for infinite type.

DEFINITION. $\partial\Omega$ is said to be of infinite type at $p \in \partial\Omega$ if

$$(L^{\alpha_1} \cdots L^{\alpha_m}\lambda)(p) = 0 \tag{2.2}$$

for any $\alpha_1, \ldots, \alpha_m$ and for any $m = 0, 1, 2, \ldots$, where each α_j is either 0 or 1, while $L^0 = L$ and $L^1 = \bar{L}$.

Note that in the definition we may use any nonvanishing smooth complex tangent vector field instead of L since L generates such a vector field.

For our domain $E_P = \{(z_1, z_2) \in C^2 : |z_1|^2 + P(z_2, \bar{z}_2) < 1\}$,

$$L = \frac{\partial P}{\partial z_2} \frac{\partial}{\partial z_1} - \bar{z}_1 \frac{\partial}{\partial z_2} \tag{2.3}$$

and the Levi form is given by

$$\mathscr{L} = \begin{pmatrix} 1 & 0 \\ 0 & \Delta P(z_2)/4 \end{pmatrix}. \tag{2.4}$$

Hence we see, by straight forward computation, that the Levi function $\lambda(z_1, z_2)$ at (z_1, z_2) is

$$\lambda(z_1, z_2) = \frac{\left|\dfrac{\partial P}{\partial z_2}(z_2)\right|^2 + \dfrac{1}{4}\Delta P(z_2)|z_1|^2}{\left|\dfrac{\partial P}{\partial z_2}(z_2)\right|^2 + |z_1|^2}. \tag{2.5}$$

Note that because of (C1) and (C2)

$$(L^{\alpha_1} \cdots L^{\alpha_m}\lambda)(e^{i\theta}, 0) = (-1)^m e^{i(m-2k)\theta} \frac{\partial^m \lambda}{\partial z_2^k \partial \bar{z}_2^{m-k}}(e^{i\theta}, 0), \tag{2.6}$$

where k is the number of α_j's such that $\alpha_j = 0$. It then follows from (C1) and (C2) that

$$(L^{\alpha_1} \cdots L^{\alpha_m}\lambda)(e^{i\theta}, 0) = 0$$

for any $\alpha_1, \ldots, \alpha_m$ and for any $m = 0, 1, 2, \ldots$. Hence E_P is of infinite type at points $(e^{i\theta}, 0)$ for any θ. Moreover, $(e^{i\theta}, 0)$ are the only points of infinite type.

We now prove a preliminary lemma regarding infinite type.

LEMMA 2.1. *Let ψ be a smooth increasing function on $(0, \varepsilon)$ for some ε. If ψ vanishes at 0 to infinite order, then for any $c > 1$,*

$$\limsup_{t\to 0} \frac{\psi(ct)}{\psi(t)} = \infty . \tag{2.7}$$

PROOF. Suppose that (2.7) does not hold. Then, there exist $t_0 > 0$ and $A > 1$ such that $\psi(ct) \leq A\psi(t)$ for any $t \leq t_0$. Hence, if m is large enough, then

$$\frac{\psi(t)}{t^m} \geq \frac{1}{A} \frac{\psi(ct)}{t^m} = \frac{c^m}{A} \frac{\psi(ct)}{(ct)^m} > \frac{\psi(ct)}{(ct)^m} .$$

By repeating this inequality, we have

$$\frac{\psi(t)}{t^m} > \frac{\psi(c^n t)}{(c^n t)^m} ,$$

for any positive integer n as long as $c^n t \leq t_0$. In particular, we have

$$\frac{\psi(t_0/c^n)}{(t_0/c^n)^m} > \frac{\psi(t_0)}{t_0^m} .$$

Letting $n \to \infty$, we have

$$\lim_{t\to 0} \psi^{(m)}(t) \geq \frac{m!\,\psi(t_0)}{t_0^m} > 0 ,$$

which contradicts our hypothesis. This completes the proof.

3. Proofs. We now give a proof of the main theorem of this paper. Each lemma corresponds to a step for the proof. As before, we let P be a smooth function satisfying the conditions (C1) and (C2) such that P is subharmonic in a neighborhood of $\bar{E}_P$ where

$$E_P = \{(z_1, z_2) \in \boldsymbol{C}^2 : |z_1|^2 + P(z_2) < 1\} .$$

Then, E_P is a pseudo-convex domain; of infinite type along $(e^{i\theta}, 0) \in \partial E_P$ and of finite type elsewhere. We let $W = \{(e^{i\theta}, 0) : |\theta| \leq \pi\}$. One can observe that any automorphism F of E_P maps W onto itself. In fact, since E_P is a complete Hartogs domain, the Bergmann projection on E_P maps $C^\infty(\overline{E_P})$ into $C^\infty(\overline{E_P})$ (cf. [BS, main theorem]). So, by a well-known theorem of Bell-Ligocka, any automorphism of E_P can be extended to a diffeomorphism of $\overline{E_P}$. We denote the extension also by F. This extended automorphism F maps W onto W.

LEMMA 3.1. *Any $F \in \mathrm{Aut}(E_P)$ is of the form*

$$F(z_1, z_2) = (\mu m(z_1), z_2 h(z_1, z_2)) ,$$

where m is a Möbius transformation of the unit disc Δ in $\boldsymbol{C}$, $|\mu| = 1$, and h is a holomorphic function on E_P.

PROOF. Let $F(z)=(f(z), g(z))$ be an automorphism of E_P. $F(W)=W$ implies that $g(z_1, 0)=0$ if $|z_1|=1$. By the maximum principle, we have $g(z_1, 0)=0$ if $|z_1|<1$. Moreover, $f(z_1, 0)$ is an automorphism of Δ. Hence, $f(z_1, 0)=\mu m(z_1)$, where $m(z_1)$ is a Möbius transformation of Δ and $|\mu|=1$. It remains to show that f is independent of z_2.

Let $r(z_1, z_2)=|z_1|^2+P(z_2)-1$. Then $r\circ F$ is also a defining function for E_P, since F can be extended to a diffeomorphism of $\overline{E_P}$. For notational convenience, we put $Q(z)=(P\circ g)(z)$. Note that

$$\frac{\partial^{\alpha+\beta}Q}{\partial z_2^\alpha \partial \bar{z}_2^\beta}(e^{i\theta}, 0)=0 \tag{3.1}$$

for any $\alpha\geq 0$ and $\beta\geq 0$, since P vanishes to infinite order at 0 and $g(e^{i\theta}, 0)=0$. In terms of the defining function $r\circ F$, the Levi function for E_P is given by

$$\lambda(z)=\frac{\varphi(z)\left|\frac{\partial P}{\partial z_2}(z_2)\right|^2-2\Re\left(z_1\psi(z)\frac{\partial P}{\partial z_2}(z_2)\right)+|z_1|^2\theta(z)}{\left|\frac{\partial P}{\partial z_2}(z_2)\right|^2+|z_1|^2}, \tag{3.2}$$

where

$$\varphi(z)=\left|\frac{\partial f}{\partial z_1}\right|^2+\frac{1}{4}\Delta_1 Q,$$

$$\psi(z)=\frac{\partial f}{\partial z_1}\frac{\partial \bar{f}}{\partial \bar{z}_2}+\frac{\partial^2 Q}{\partial z_1\partial \bar{z}_2},$$

$$\theta(z)=\left|\frac{\partial f}{\partial z_2}\right|^2+\frac{1}{4}\Delta_2 Q,$$

and Δ_j is the Laplacian with respect to z_j, $j=1, 2$. Since E_P is of infinite type along W, we have

$$((\bar{L}L)^n\lambda)(e^{i\theta}, 0)=0$$

for any nonnegative integer n. Recall that

$$L=\frac{\partial P}{\partial z_2}\frac{\partial}{\partial z_1}-\bar{z}_1\frac{\partial}{\partial z_2}.$$

It follows from (C1), (C2) and (3.1) that

$$0=((\bar{L}L)^n\lambda)(e^{i\theta}, 0)=4^{-n}(\Delta_2^n\lambda)(e^{i\theta}, 0)=4^{-n}(\Delta_2^n\theta)(e^{i\theta}, 0)=\left|\frac{\partial^{n+1}f}{\partial z_2^{n+1}}(e^{i\theta}, 0)\right|^2$$

for any $n=0, 1, 2, \ldots$. It then follows from the maximum principle that f is independent of z_2. This completes the proof.

LEMMA 3.2. *Let* $F(z_1, z_2) = (\mu m(z_1), z_2 h(z_1, z_2))$ *be an automorphism of* E_P. *Then,* $|h(e^{i\theta}, 0)| = 1$ *for any* θ.

PROOF. Since r and $r \circ F$ are defining functions for E_P, there exist positive constants A and B such that

$$Ar(z) \le (r \circ F)(z) \le Br(z)$$

for all z near ∂E_P. Recall that $r(z_1, z_2) = |z_1|^2 + P(z_2) - 1$. Fix θ and let $z_1 = e^{i\theta}$. Then, for any z_2 near 0, we have

$$AP(z_2) \le P(z_2 h(e^{i\theta}, z_2)) \le BP(z_2)\,. \tag{3.3}$$

Suppose that $|h(e^{i\theta}, 0)| > 1$. Then, by the second inequality in (3.3), we have

$$\int_{|z_2| \le \delta} P(z_2 h(e^{i\theta}, z_2))\, dV(z_2) \le B \int_{|z_2| \le \delta} P(z_2)\, dV(z_2)$$

for any δ small enough. We make a change of variables $w = g(e^{i\theta}, z_2) = z_2 h(e^{i\theta}, z_2)$ and let $z_2 = g^{-1}(w)$ be its inverse. Then, the real Jacobian $|J_{\mathbf{R}} g^{-1}|$ of the change of variables $z_2 = g^{-1}(w)$ is bounded below by a positive constant C for $|z_2| < \delta$ if δ is small enough. Therefore, we have

$$\begin{aligned} C \int_{|g^{-1}(w)| \le \delta} P(w)\, dV(w) &\le \int_{|g^{-1}(w)| \le \delta} P(w)\, |J_{\mathbf{R}} g^{-1}|\, dV(w) \\ &\le \int_{|z_2| \le \delta} P(z_2 h(e^{i\theta}, z_2))\, dV(z_2) \\ &\le B \int_{|z_2| \le \delta} P(z_2)\, dV(z_2)\,. \end{aligned}$$

Since $|h(e^{i\theta}, 0)| > 1$, there exists a constant $c > 1$ such that $\{w : |w| \le c\delta\} \subset \{w : |g^{-1}(w)| \le \delta\}$ for any δ small enough. Hence, we have

$$C \int_{|w| \le c\delta} P(w)\, dV(w) \le B \int_{|w| \le \delta} P(w)\, dV(w)\,. \tag{3.4}$$

Let

$$\psi(\delta) = \int_{|w| \le \delta} P(w)\, dV(w) = \delta^2 \int_{|w| \le 1} P(\delta w)\, dV(w)\,.$$

Then, we see from (C2) that ψ vanishes to infinite order at 0. On the other hand, (3.4) implies that

$$\limsup_{\delta \to 0} \frac{\psi(c\delta)}{\psi(\delta)} \le \frac{B}{C}\,.$$

This contradicts Lemma 2.1. Hence $|h(e^{i\theta}, 0)| \le 1$. We can use the first inequality in (3.3) to show that $|h(e^{i\theta}, 0)| \ge 1$. Hence, $|h(e^{i\theta}, 0)| = 1$. This completes the proof.

LEMMA 3.3. *Let $F=(f, g)$ be an automorphism of E_P. Then $f(z)=\mu z_1$ for some μ with $|\mu|=1$.*

PROOF. By Lemmas 3.1 and 3.2, $f(z_1, z_2)=\mu m(z_1)$ and $g(z_1, z_2)=z_2 h(z_1, z_2)$ with $|h(e^{i\theta}, 0)|=1$ where $|\mu|=1$ and m is a Möbius transformation. Assume that $\mu=1$ without loss of generality. Suppose that $m(0)\neq 0$. Put $D_\varepsilon=\{(0, z_2): |z_2|\le\varepsilon\}$. Let $F^n=F\circ\cdots\circ F$ be the n times iteration of F, and let $F^n=(G_n, H_n)$. Then, $G_n(z_1, z_2)=m^n(z_1)$ and hence $|G_n(0, z_2)|\to 1$ as $n\to\infty$ for any z_2 and $G_n(0, z_2)$ is independent of z_2. Therefore, we have

$$\int_{F^n(D_\varepsilon)} dV(z_2)=\int_{H_n(D_\varepsilon)} dV(z_2)\to 0 \qquad \text{as} \quad n\to\infty$$

for any $\varepsilon>0$. Let

$$g(z_1, z_2)=\nu z_2+\sum_{n=2}^{\infty} a_n(z_1) z_2^n$$

in some neighborhood of $(0, 0)$ containing D_ε for a small ε. Then we have $|\nu|=1$ by Lemma 3.2. We can show by induction that

$$H_n(z_1, z_2)=\nu^n z_2+\sum_{n=2}^{\infty} b_n(z_1) z_2^n$$

for some holomorphic functions b_n. Hence,

$$\begin{aligned}\int_{H_n(D_\varepsilon)} dV(z_2)&=\int_{D_\varepsilon}\left|\frac{\partial H_n}{\partial z_2}(0, z_2)\right|^2 dV(z_2)\\ &\ge \varepsilon^2\pi|\nu|^{2n}+\sum_{n=2}^{\infty} n\pi\varepsilon^{2n}|b_n(0)|^2\ge\varepsilon^2\pi,\end{aligned}$$

which is a contradiction. Thus, $m(0)=0$ and the proof is complete.

Theorem and Corollary follow from Lemma 3.3.

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