Tôhoku Math. J.
48 (1996), 601–612

OSCILLATOR AND PENDULUM EQUATION ON PSEUDO-RIEMANNIAN SPACES

WOLFGANG KÜHNEL AND HANS-BERT RADEMACHER

(Received August 17, 1995, revised March 21, 1996)

Abstract. We study conformal vector fields on pseudo-Riemannian manifolds which are locally gradient fields. This is closely related with a certain differential equation for the Hessian of a real function. We obtain global solutions of the oscillator and peandulum equation for the Hessian of this function on a pseudo-Riemannian manifold, generalizing previous results by M. Obata, Y. Tashiro, and Y. Kerbrat. In particular, it turns out that the pendulum equation characterizes a certain conformal type of metrics carrying a conformal vector field with infinitely many zeros.

1. Introduction. Conformal mappings and conformal vector fields are classical topics in geometry. Essential conformal vector fields on Riemannian spaces were studied by Obata, Lelong-Ferrand and Alekseevskii [A1], [La2]. Conformal gradient fields are essentially solutions of the differential equation $\nabla^2\varphi=(\Delta\varphi/n)\cdot g$. This equation was studied since the 1920's by Brinkmann, Fialkow, Yano, Tashiro, Kerbrat and others. In the Riemannian case the results are quite complete. In the pseudo-Riemannian case a systematic approach has started in our previous paper [KR2] including a conformal classification theorem.

A classical result by Obata and Tashiro characterizes the standard sphere as the only complete Riemannian manifold admitting a non-constant solution of the equation $\nabla^2\varphi=-c^2\varphi g$ for a non-zero constant c. This is nothing but the classical *harmonic oscillator equation*. In Section 3 we study the following generalization: given a function h: $\boldsymbol{R}\to\boldsymbol{R}$, the conformal gradient field equation $\nabla^2\varphi+h(\varphi)\cdot g=0$ imposes very strong conditions on the underlying pseudo-Riemannian manifold. We give analogous results for the case of the equation of the general *undamped oscillator*. This illustrates how a metric can be modeled within a conformal class by a second order differential equation. The metric in this case is completely determined by the equation and the choice of a constant of integration which is essentially the energy of the undamped oscillator. Similarly, for small energy, the pendulum equation on a pseudo-Riemannian manifold determines the metric uniquely. In the Riemannian case it is conformal to the standard sphere whereas in the case of an indefinite metric it is conformal to a noncompact manifold $M(\boldsymbol{Z})$. This manifold carries a conformal gradient field with infinitely many zeros. A short announcement of the results in this paper appeared in [KR3].

1991 *Mathematics Subject Classification*. Primary 53C50; Secondary 53A30, 58G30.
Key words. Warped product, conformal gradient field, pendulum equation, oscillator equation.

2. Conformal gradient fields and harmonic oscillator manifolds. From the viewpoint of infinitesimal transformations [Y1], a vector field V is said to preserve a certain geometric quantity if the Lie derivative $\mathscr{L}_V$ of this quantity vanishes. On a pseudo-Riemannian manifold (M, g) a vector field V is called *isometric* if it preserves the metric or if $\mathscr{L}_V g=0$. Recall that by definition $(\mathscr{L}_V g)(X, Y)=g(\nabla_X V, Y)+g(X, \nabla_Y V)$ for arbitrary tangent vectors X, Y where ∇ denotes the Levi-Civita connection. V is called *conformal* if it preserves the conformal class of the metric or if $\mathscr{L}_V g=2\cdot\varphi\cdot g$ for some function φ. Necessarily this function is $\varphi=(1/n)\operatorname{div} V$ in this case. V is called *homothetic* if φ is constant. In the particular case of a gradient field $V=\operatorname{grad} f$ we have $\mathscr{L}_V g==2\nabla^2 f$, hence grad f is conformal if and only if $\nabla^2 f=\varphi\cdot g$ where $n\cdot\varphi=\Delta f=\operatorname{div}(\operatorname{grad} f)$ is the Laplacian. As a short notation we will use the symbol $(\ )^0$ for the traceless part of a (0, 2)-tensor, e.g.

$$(\nabla^2 f)^0=\nabla^2 f-\frac{\Delta f}{n}\cdot g\,.$$

In particular, a gradient field grad f is conformal if and only if $(\nabla^2 f)^0=0$. The equation $(\nabla^2 f)^0=0$ has been extensively studied in many papers [Fi], [Y2], [T], [Bo], [Kb1], [Fe1], [Kü]. It arises in various contexts. Especially, it occurs in connection with the behavior of the Ricci tensor Ric_g in a conformal class of metrics and for conformal vector fields on Einstein spaces, see [Br], [Kan], [BK], [KR1], [KR4]. The local structure of all solutions of $(\nabla^2\varphi)^0=0$ for any function φ is well understood at least in the case where grad φ is not a null vector on an open set. We recall the following lemma:

2.1. Lemma. *Let (M, g) be a pseudo-Riemannian manifold admiting a non-constant solution φ of the equation $(\nabla^2\varphi)^0=0$. Then the following holds*:

(i) *In a neighborhood of any point with $\|\operatorname{grad}\varphi\|^2\neq 0$, g is a warped product $g=\eta dt^2+\varphi'^2(t)g_*$ ($\eta=\pm 1$ is the sign of $\|\operatorname{grad}\varphi\|^2$), φ is a function depending only on t, the trajectories of $\operatorname{grad}\varphi/\|\operatorname{grad}\varphi\|$ are geodesics, and φ satisfies $\varphi''=\eta\cdot\Delta\varphi/n$ along these trajectories.*

(ii) *The zeros of $\operatorname{grad}\varphi$ are isolated. In a neighborhood of such a zero the metric is a warped product in polar coordinates $g=\eta dt^2+(\varphi_\eta'^2(t)/\varphi_\eta''^2(0))g_\eta$ where g_η denotes the induced metric on the 'unit sphere' $\{x\,|\,\|x\|^2=\eta\}$ in the pseudo-Euclidean space of the same signature as g. In particular, near a critical point of φ the metric g is conformally flat.*

A poof of Lemma 2.1 has been given in our previous paper [KR2]. (i) is originally due to Fialkow [Fi]. In the Riemannian case (ii) has been observed by Tashiro [T].

2.2. Proposition (cf. [T], [Ob1]). *Assume that (M, g) is a complete Riemannian manifold admitting a non-constant solution φ of the differential equation $\nabla^2\varphi+c^2\varphi g=0$, where $c\neq 0$ is a real constant. Then (M, g) is isometric with the standard sphere of curvature c^2.*

The proof of Proposition 2.2 is an easy consequence of Lemma 2.1 by the following argument: along any trajectory of $\operatorname{grad}\varphi/\|\operatorname{grad}\varphi\|$, the function φ satisfies $\varphi''+c^2\varphi=0$, the equation of the *harmonic oscillator*, hence φ is periodic. From a critical point one determines the metric using Lemma 2.1(ii). The standard sphere may be called the 'harmonic oscillator manifold' because it is determined by the harmonic oscillator equation.

2.3. PROPOSITION (cf. [T], [Kb2]). *Assume that (M,g) is a complete pseudo-Riemannian manifold admitting a non-constant solution φ of the differential equation $\nabla^2\varphi-c^2\varphi g=0$, where $c\neq 0$ is a real constant. Assume furthermore that φ has a critical point. Then (M,g) is isometric with the (pseudo-)hyperbolic space of curvature $-c^2$: H^n, H^n_k or a covering of H^n_1.*

Proposition 2.3 follows by the same procedure as Proposition 2.2: from the critical point one determines the metric by the equation $\varphi''+\eta c^2\varphi=0$ where $\eta=1$ in spacelike directions and $\eta=-1$ in timelike directions. In the indefinite case we can replace g by $-g$. Then the equation becomes $\nabla^2\varphi+c^2\varphi g=0$, and the solution becomes the pseudo-sphere. If φ has no critical point on the manifold, then it is impossible to determine its metrical or even topological type: examples are warped products with—more or less—arbitrary fiber. A reformulation of Proposition 2.2 is the following:

2.4. COROLLARY. *Let (M,g) be a complete Riemannian manifold carrying a non-homothetic conformal vector field V. Assume that $\mathscr{L}_V(L+(c^2/2)\cdot g)=0$ holds for a certain real constant $c\neq 0$. Then (M,g) is isometric with the sphere of curvature c^2.*

PROOF. By assumption $\mathscr{L}_V g=2\varphi g$ for a non-constant function φ. On the other hand, from the well known equations (cf. [Y1; p. 160]) $\mathscr{L}_V \operatorname{Ric}=-(n-2)\nabla^2\varphi-\Delta\varphi\cdot g$ and $\mathscr{L}_V S=-(2/n)\Delta\varphi-2S\cdot\varphi$ one obtains $\mathscr{L}_V L=\nabla^2\varphi$ where $L:=(1/(n-2))((n/2)Sg-\operatorname{Ric})$ denotes the *Schouten tensor*. This implies $0=\mathscr{L}_V L+(c^2/2)\mathscr{L}_V g=\nabla^2\varphi+c^2\varphi g$. Now the assertion follows from Proposition 2.2. □

3. Oscillator and pendulum manifolds. The argument in the proof of Proposition 2.2 does not essentially depend on the particular geometry of the sphere. It only depends on the warped product structure of the metric and the periodicity of the solution φ, regarded as a function in one real variable. This allows a generalization to a large class of manifolds or equations, respectively. In this section we characterize certain metrics on complete manifolds by the equation of the *undamped oscillator* or the *pendulum*. The starting point is the characterization of standard spaces by the harmonic oscillator equations in Propositions 2.2 and 2.3.

3.1. THEOREM (undamped oscillator manifolds). *Let $h:\boldsymbol{R}\to\boldsymbol{R}$ be any given locally Lipschitzian function satisfying $xh(x)>0$ for all $x\neq 0$ and $\int_0^x h(\xi)d\xi\to\infty$ for $\to x\pm\infty$. Let (M,g) be a complete Riemannian manifold admitting a non-constant solution φ of the equation $\nabla^2\varphi+h(\varphi)\cdot g=0$. Then M is conformally diffeomorphic to the standard sphere.*

Moreover, the metric g is uniquely determined by h and a certain positive constant which corresponds to the energy of the undamped oscillator. In other words: The equation $\nabla^2\varphi + h(\varphi)\cdot g = 0$ determines a one-parameter family of Riemannian manifolds, each conformal to the sphere.

(Alternatively: assume that $\operatorname{grad}\varphi$ is a conformal vector field with $\operatorname{div}(\operatorname{grad}\varphi) = -nh(\varphi)$.) In the special case $h(\varphi) = c^2\varphi$ (the harmonic oscillator) this one-parameter family of metrics degenerates to the unique standard metric on S^n of curvature c^2. Other examples are given by $h(\varphi) := \varphi^{2k+1}$.

PROOF. By Lemma 2.1 along any trajectory of $\operatorname{grad}\varphi/\|\operatorname{grad}\varphi\|$, the function $\varphi = \varphi(t)$ satisfies the equation of the undamped oscillator

$$\varphi'' + h(\varphi) = 0 .$$

The metric g is a warped product

$$g = dt^2 + \varphi'^2(t)\cdot g_* .$$

It is well known (see [BN]) that under the given assumption every solution of the undamped oscillator is periodic. Therefore every solution has a critical point. Then by Lemma 2.1(ii) g_* is the standard metric on the sphere of certain radius.

In more detail the solutions are the following: Integration of the equation of the undamped oscillator leads to

$$\frac{1}{2}\varphi'^2 + \int_0^{\varphi} h(x)dx = C \quad (=\text{constant}) .$$

This constant C is nothing but the total energy function if one regards $\varphi = \varphi(t)$ as the motion of a point depending on the time t. By assumption it is clear that $C > 0$. Let $H(y) := \int_0^y h(x)dx$. Then a second step of integration leads to the following expression for the inverse function $t = t(\varphi)$:

$$t(\varphi) = \frac{\pm\sqrt{2}}{2}\int_{\varphi_0}^{\varphi}\frac{dy}{\sqrt{C - H(y)}} ,$$

where φ_0 must be chosen so that $H(\varphi_0) < C$. The periodicity of φ can be expressed by the fact that this formula for $t(\varphi)$ describes a branch between two zeros of the denominator, i.e. between two points φ_1, φ_2 with $H(\varphi_1) = H(\varphi_2) = C$.

After a shift of the parameter t we can assume that $\varphi'(0) = 0$ and $\varphi(0) > 0$. Then $\varphi(0)$ is uniquely determined by $H(\varphi) = C$ because H is monotone in $[0, \infty)$. The equation $\varphi'' + h(\varphi) = 0$ itself determines $\varphi''(0) < 0$. Therefore around the corresponding critical point of φ on M the metric g looks like

$$g = dt^2 + \frac{\varphi'^2(t)}{\varphi''^2(0)}\, g_1 ,$$

where g_1 is the metric of the unit sphere (cf. Proposition 2.3(ii)). This expression for g holds for any interval $[0, t]$ as long as $\varphi'(\tilde{t}) \neq 0$ for $0 < \tilde{t} < t$. If τ denotes the smallest positive zero of φ', then

$$dt^2 + \frac{\varphi'^2(t)}{\varphi''^2(0)} g_1$$

produces a singularity at $t = \tau$ unless the factor of g_1 is appropriate. On the other hand, by assumption g has no singularity. Therefore this factor must be appropriate, i.e. $(\varphi'')^2(\tau) = (\varphi'')^2(0)$. Therefore the warped product $(0, \tau) \times_{\varphi'} S^{n-1}$ together with the two critical points at $t = 0$ and $t = \tau$ is conformal to the standard sphere. The metric is uniquely determined by h and C.

If σ denotes the smallest zero of φ' which is greater than τ, then in the interval $[\tau, \sigma]$ the same happens. The warped product metric is exactly the same, just parametrized beckwards. Here we use the facts that $h(\sigma) = h(0)$ and that φ'' depends only on φ. If in addition h is an odd function, then the metric has an additional symmetry: there is an isometric reflection interchanging the two exceptional points. Furthermore it is fairly clear that the metric g depends continuously on the choice of C because the expression for the inverse function φ^{-1} does. In the special case when h is linear this choice of C has no influence on the metric: (M, g) is a round sphere of a radius depending only on h. □

If one tries to find a generalization of Theorem 3.1 in the case of an indefinite metric, one has the same kind of relationship as between Propositions 2.2 and 2.3. First of all, one has to assume that the solution has a critical point. Secondly, there is the problem that the equation $\varphi'' + h(\varphi) = 0$ along spacelike directions corresponds to the equation $\varphi'' - h(\varphi) = 0$ along timelike directions. The solutions of the latter equation are not necessarily defined over $\boldsymbol{R}$ which would exclude complete metrics of this type. Therefore, we have to restrict ourselves to a special subclass of equations.

3.2. Theorem (undamped oscillator manifolds with indefinite metric). *Let $h: \boldsymbol{R} \to \boldsymbol{R}$ be a globally Lipschitzian function satisfying $xh(x) > 0$ for all $x \neq 0$ and $\int_0^x h(\xi)d\xi \to \infty$ for $x \to \pm\infty$. Let (M, g) be a complete pseudo-Riemannian manifold with an indefinite metric admitting a non-constant solution φ of the equation $\nabla^2\varphi + h(\varphi) \cdot g = 0$ with at least one critical point. Then M is conformally diffeomorphic to S^n_k or to a covering of S^n_{n-1}. Moreover, the metric g is uniquely determined by h and a certain positive constant. Conversely, there is a complete metric g of this type.*

In the special case $h(\varphi) = c^2\varphi$ the metric is uniquely determined, see Proposition 2.3 for $-g$ instead of g. The assumption concerning a critical point in Theorem 3.2 can be replaced by the assumption that $\|\operatorname{grad}\varphi\|^2 > 0$ somewhere.

Proof. The proof consists of a combination of the proofs of Proposition 2.3 and Theorem 3.1. Along radial geodesics emanating from the critical point of φ we have

$$\varphi'' \pm h(\varphi) = 0\,,$$

where the sign is the sign of $\|\operatorname{grad}\varphi\|^2$. In spacelike directions this leads to the same solution as in Theorem 3.1. They all are periodic by the assumptions on h. Therefore φ has at least two critical points, just as in the case of an indefinite metric in Proposition 2.3. In timelike directions the equation $\varphi'' - h(\varphi) = 0$ has a complete solution $\varphi\colon \boldsymbol{R} \to \boldsymbol{R}$ by the assumption of the global Lipschitz condition. In some sense h is 'almost linear' leading to an 'almost sinh-like' solution φ. The global conformal equivalence with the pseudo-sphere or its covering follows from [KR2; Sect. 6]. □

Furthermore, we want to discuss the analogous problem for the equation of the undamped pendulum $\varphi'' + \omega^2 \sin\varphi = 0$ for a real constant $\omega > 0$. In this case the function $h(\varphi) = \omega^2 \sin\varphi$ does not satisfy the assumption of Theorem 3.1. In fact, it is well known that the solutions are periodic only for small energy and that they are non-periodic beyond a critical level. Thus we expect the topology of the manifold to change exactly at this critical energy level.

3.3. Remark (pendulum equation). Solutions of the *pendulum equation*

$$f''(t) + \omega^2 \sin f(t) = 0$$

or, equivalently,

$$f'^2(t) - 2\omega^2 \cos f(t) + 2\omega^2 = C$$

with positive total energy C can be expressed using Jacobi's *elliptic functions* $u \mapsto \operatorname{sn}(u, k)$, $u \mapsto \operatorname{cn}(u, k)$, $u \mapsto \operatorname{dn}(u, k)$ (cf. [Law; 5.1]). Here $k \in (0, 1)$ denotes the *modulus*, $k' \in (0, 1)$ the *complementary modulus* satisfying $k^2 + k'^2 = 1$.

There are analytic functions $K = K(k)$, $K' = K'(k)$ which allow to express the periods of Jacobi's elliptic functions in terms of the modulus.

If the total energy C satisfies $C < 4\omega^2$ then the solution

$$f(t) = 2\arcsin(k\operatorname{sn}(\omega t, k))$$

with $C = 4\omega^2 k^2$ and $f(0) = 0$ is periodic with period $T = 4K/\omega$. In this case the amplitude is $f(K/\omega) = 2\arcsin(k) \in (0, \pi)$.

The elliptic function

$$G(z) = 2\omega k \operatorname{cn}(\omega t, k)$$

has periods $4K/\omega$ and $(4K'/\omega)i$. The zeros occur at $z = (K/\omega)(2m+1) + (2K'/\omega)in$; m, $n \in \boldsymbol{Z}$ and the poles at $z = (2K/\omega)m + (K'/\omega)(2n+1)i$; $m, n \in \boldsymbol{Z}$. All poles and zeros are simple. Then the solution f satisfies $f'(t) = G(t)$ for all $t \in \boldsymbol{R}$.

For later use we define

$$f_+(t)=f\left(t+\frac{K}{\omega}\right)=2\arcsin(k\,\mathrm{sn}(\omega t+K,k))$$

and

$$f_-(t)=\pi-2\arcsin(k'\,\mathrm{sn}(\omega t+K',k')).$$

Then it follows that $f_\pm$ are solutions of $f_\pm''\pm\omega^2\sin f_\pm=0$ or of $(f_\pm')^2\mp\omega^2\cos(f_\pm)+2\omega^2=4\omega^2k^2$, respectively.

Since $f_+'(t)=G(t+K/\omega)$ and $f_-'(t)=iG(i(t+K/\omega))$, it follows that for all $m\geq 0$

$$f_+^{(2m)}(0)=(-1)^m f_-^{(2m)}(0).$$

$k^2=k'^2=1/2$ implies $f_-(t)=\pi-f_+(t)$. In the limit case $C=4\omega^2$ (i.e., $k\to 1$) we obtain

$$f(t)=2\arcsin(\tanh(\omega t)).$$

Hence f is monotonic and approaches for $t\to\pm\infty$ the unstable equilibrium given by the bottom up position of the pendulum.

If $C>4\omega^2$, then the solution f is strictly monotonic and satisfies

$$\sin\frac{f(t)}{2}=\mathrm{sn}\left(\frac{\omega t}{k},k\right)$$

with $C=4\omega^2/k^2$, $k\in(0,1)$. The derivative

$$f'(t)=\frac{2\omega}{k}\,\mathrm{dn}\left(\frac{\omega t}{k},k\right)$$

is periodic with period $2kK/\omega$.

3.4. Theorem (pendulum manifolds). *Let (M,g) be a complete Riemannian manifold admitting a non-constant solution φ of the equation of the undamped pendulum*

$$\nabla^2\varphi+\omega^2\cdot\sin\varphi\cdot g=0.$$

Depending on the choice of a certain positive constant C (the total energy of the pendulum), the following holds:

(i) *if $C<4\omega^2$, then (M,g) is conformally diffeomorphic to S^n; the metric is uniquely determined by C,*

(ii) *if $C>4\omega^2$, then (M,g) is a warped product $\boldsymbol{R}\times_{\varphi'}M_*$ where φ' is periodic,*

(iii) *if $C=4\omega^2$, then (M,g) is a warped product $\boldsymbol{R}\times_{\varphi'}M_*$ where $\lim_{t\to\pm\infty}\varphi'=0$.*

Proof. The proof follows the pattern of the proof of Theorem 3.1. Along the trajectories of $\mathrm{grad}\,\varphi/\|\mathrm{grad}\,\varphi\|$ the function φ satisfies the pendulum equation $\varphi''+\omega^2\cdot\sin\varphi=0$. Locally g is a warped product $g=dt^2+\varphi'^2(t)g_*$. The total energy C is given by $C=\varphi'^2(t)-2\omega^2\cos\varphi+2\omega^2$. Therefore, by Remark 3.3 there are the following three cases to be considered.

If $C < C_0 := 4\omega^2$ then the solution $\varphi(t) = 2\arcsin(k\,\mathrm{sn}(\omega t + K, k))$ is periodic. Therefore φ' has zeros leading to a conformal sphere as in Theorem 3.1 (= Case (i)).

If $C = C_0$ then the solution is $\varphi(t) = 2\arcsin(\tanh(\omega t))$, hence $\varphi'(t) = 2\,\omega\,\mathrm{sech}(\omega t)$, i.e. φ' is positive everywhere leading to the warped product $\boldsymbol{R} \times_{\varphi'} M_*$. This case corresponds to the critical energy $C_0 = 4\omega^2$ of an unstable equilibrium of the pendulum ([BN; pp. 177–178). More precisely, the ideal boundary points $t = \pm\infty$ represent this unstable bottom up position of the pendulum.

If $C > C_0$ then the solution φ is given by $\sin(\varphi(t)/2) = \mathrm{sn}(\omega t/k, k)$, hence $\varphi'(t) = (2\omega/k)\,\mathrm{dn}(\omega t/k, k)$ is positive everywhere and periodic leading again to the warped product $\boldsymbol{R} \times_{\varphi'} M_*$.

It is particularly interesting to study the same pendulum equation $\nabla^2\varphi + \omega^2 \sin(\varphi) g = 0$ on pseudo-Riemannian manifolds with an indefinite metric because a new type of examples comes in. In fact, this leads to a conformal type of non-compact spaces carrying conformal gradient fields with infinitely many zeros. This manifold $M(\boldsymbol{Z})$ and one particular analytic metric has been constructed in [KR2]. For the reader's convenience we briefly recall the construction. It is denoted by $M(\boldsymbol{Z})$ because the set of zeros is in natural bijection with $\boldsymbol{Z}$ including a $\boldsymbol{Z}$-action which is half transitive on the set of zeros.

For fixed $k \in (0, 1)$ and $\omega > 0$ we define a *building block* B as the following open subspace of $\boldsymbol{R}^n_\kappa$:

$$B := \left\{ y \in \boldsymbol{R}^n_\kappa \,\middle|\, -\frac{K'^2}{\omega^2} < \|y\|^2_\kappa < \frac{K^2}{\omega^2} \right\}.$$

We define the manifold

$$M(\boldsymbol{Z}) = \cdots \cup B_{-1} \cup B_0 \cup B_1 \cup \cdots$$

by gluing copies B_j of B along the components

$$\partial B_j^+ = \left\{ y \in \boldsymbol{R}^n_\kappa \,\middle|\, \|y\|^2_\kappa = \frac{K^2}{\omega^2} \right\}, \qquad \partial B_j^- = \left\{ y \in \boldsymbol{R}^n_\kappa \,\middle|\, \|y\|^2_\kappa = -\frac{K'^2}{\omega^2} \right\}$$

of the boundary ∂B as follows: Identify ∂B_1^+ with ∂B_2^+ and ∂B_1^- with ∂B_0^- and then ∂B_2^- with ∂B_3^-. By proceeding this way we obtain $M(\boldsymbol{Z})$ as a differentiable manifold. On each building block B we have (outside the light cone of 0) polar coordinates

$$(t, x) \in \boldsymbol{R} \times (S^{n-1}_\kappa \cup H^{n-1}_{\kappa+1})$$

(cf. [KR2; Ch. 3]). The induced metric of $\boldsymbol{R}^n_\kappa$ on each block together with the inversion

$$(t, x) \longmapsto \left(\frac{1}{t}, x \right),$$

understood as a map from B_j to B_{j+1} for $\|x\|^2 > 0$ and as a map from B_j to B_{j-1} for

$\|x\|^2<0$, defines a conformal structure on $M(\boldsymbol{Z})$. For our purpose here, we describe a metric induced by the pendulum equation and its solution as follows:
Let

$$\varphi_+(t)=2\arcsin(k\,\mathrm{sn}(\omega t+K,k))$$

and

$$\varphi_-(t)=\pi-2\arcsin(k'\,\mathrm{sn}(\omega t+K',k'))\,.$$

Then

$$\varphi''_\pm \pm\omega^2\sin\varphi=0\,,$$

see Remark 3.3. We define a function by $\varphi(t,x)=\varphi_\eta(t)$ $\eta=\|x\|^2$ outside the light cone of 0. The equations in Remark 3.3 imply that this function extends to an analytic function on B, as was shown in Proposition 3.5 of [KR2]. Consequently,

$$g(t,x)=\eta dt^2+\frac{\varphi_\eta'^2(t)}{\varphi_\eta''^2(0)}\,g_\eta$$

extends to an analytic metric on B. It also follows that the function φ on B satisfies

$$\nabla^2\varphi+\omega^2\sin\varphi\cdot g=0\,.$$

The only critical point on B is the center 0 of B. The boundary ∂B is totally geodesic since $\varphi''_+(K/\omega)=\varphi_+(K/\omega)=0$ resp. $\varphi''_-(K'/\omega)=\varphi_-(K'/\omega)=0$. Hence the metric extends to an analytic metric $g_{k,\omega}$ on $M(\boldsymbol{Z})$ and the function extends to a function ψ on $M(\boldsymbol{Z})$ solving the pendulum equation. The critical points are precisely the central points of the building blocks. Using the pendulum equation and the differential equations for geodesics in a warped product [ON; p. 208], one can also show that $g_{k,\omega}$ is geodesically complete. Moreover, $g_{k,\omega}$ admits an isometric $\boldsymbol{Z}$-action permuting the set of critical points of φ. Every geodesic through a critical point is closed. It describes the periodic motion of an ordinary pendulum.

3.5. Theorem (pendulum manifolds with indefinite metric). *Let (M,g) be a complete pseudo-Riemannian manifold with a metric of signature (κ,n) for $2\le\kappa\le n-2$. Assume that for a constant $\omega\neq 0$ there exists a non-constant solution φ of the pendulum equation*

$$\nabla^2\varphi+\omega^2\cdot\sin\varphi\cdot g=0\,.$$

Depending on the choice of a certain positive constant C (the total energy of the pendulum) the following holds:

(i) *If $C<4\omega^2$, then (M,g) is isometric to one of the manifolds $(M(\boldsymbol{Z}),g_{k,\omega})$ constructed above where $k=C/4\omega^2$.*

(ii) *If $C>4\omega^2$, then (M,g) is a warped product $\boldsymbol{R}\times_{\varphi'}M_*$ where φ' is periodic.*

Conversely, for any choice of $C \neq 4\omega^2$ *and any signature* (*including* $(1, n)$ *and* $(n-1, n)$) *there exists such a complete pseudo-Riemannian manifold* (M, g).

For $C = 4\omega^2$ there is such a warped product as in Theorem 3.4 but it is not null-complete (compare [ON; p. 209]). In the case of signature $(1, n)$ there are additional global conformal types of manifolds, due to the fact that the 'unit sphere' has three or four connected components.

PROOF. First of all, the 'improper' case $\|\operatorname{grad}\varphi\|^2 = 0$ everywhere does not occur because otherwise $\nabla^2\varphi = 0$ and consequently φ is constant. Along any trajectory of $\operatorname{grad}\varphi$, parametrized by are length, the equation is

$$\varphi'' \pm \omega^2 \sin\varphi = 0\,,$$

where the sign is the sign of $\|\operatorname{grad}\varphi\|^2$. It is equivalent to

$$\varphi'^2 \mp 2\omega^2 \cos\varphi + 2\omega^2 = C\,,$$

where $C > 0$ denotes the total energy (see Remark 3.3).

(i) If $C < 4\omega^2$, then any solution φ is periodic and has therefore a critical point. In polar coordinates (t, x) around a critical point, φ can be written as $\varphi(t, x) = \varphi_\eta(t)$, $\eta = \|x\|^2$, and g takes the form

$$g = \eta dt^2 + \frac{\varphi_\eta'^2(t)}{\varphi_\eta''^2(0)}\, g_\eta\,,$$

where $\varphi_\eta'(0) = 0$. This is nothing but the metric $g_{k,\omega}$ on the building block B (see above). By periodicity and completeness g coincides globally with $g_{k,\omega}$ on $M(\boldsymbol{Z})$.

(ii) If $C > 4\omega^2$, then any solution φ is strictly monotonic with a periodic derivative φ'. By Lemma 2.1 g contains a warped product $\eta dt^2 + \varphi_\eta'^2(t) g_*$. This is complete, and therefore it coincides with g. □

3.6. REMARKS. The manifold $M(\boldsymbol{Z})$ carries a conformal vector field which is essential and complete with a set of zeros in natural bijection with $\boldsymbol{Z}$. For an arbitrary proper subset $A \subset \boldsymbol{Z}$ we can restrict the vector field to $M(\boldsymbol{Z}) \setminus A$ and still obtain an essential and complete conformal field. In this way one obtains uncountably many conformally distinct examples. In the Riemannian case an essential and complete conformal vector field exists only on S^n and E^n (see [A1], [Ob2], [La2], [Fe2], [Yo]). Such an analogous classification theorem does not seem to be known in the indefinite case. In fact, at least all the examples $M(\boldsymbol{Z}) \setminus A$ would have to occur in such a classification. Furthermore, the essential homothetic vector fields studied in [A2] would have to occur also.

REFERENCES

[A1] D. V. ALEKSEEVSKII, Groups of conformal transformations of Riemannian spaces, (Russian), Mat. Sb. (N.S.) 89 (131) 1972, 280–296 = (Engl. transl.) Math. USSR Sb. 18 (1972), 285–301.

[A2] D. V. ALEKSEEVSKII, Selfsimilar Lorentzian manifolds, Ann. Global Anal. Geom. 3 (1985), 59–84.

[BK] C. BARBANCE AND Y. KERBRAT, Sur les transformations conformes des variétés d'Einstein, C. R. Acad. Sci. Paris 286 (1978), 391–394.

[Bo] J. P. BOURGUIGNON, Transformations infinitésimales conformes fermées des variétés riemanniennes connexes complètes, C. R. Acad. Sci. Paris 270 (1970), 1593–1596.

[BN] F. BRAUER AND J. A. NOHEL, The qualitative theory of ordinary differential equations, W. A. Benjamin, New York, 1969.

[Br] H. W. BRINKMANN, Einstein spaces which are mapped conformally on each other, Math. Ann. 94 (1925), 119–145.

[Fe1] J. FERRAND, Concircular transformations of Riemannian manifolds, Ann. Acad. Sci. Fenn. Ser. A. I. 10 (1985), 163–171.

[Fe2] J. FERRAND, Sur une lemme d'Alekseevskii relatif aux transformations conformes, C. R. Acad. Sci. Paris, Sér. A 284 (1977), 121–123.

[Fi] A. FIALKOW, Conformal geodesics, Trans. Amer. Math. Soc. 45 (1939), 443–473.

[Kan] M. KANAI, On a differential equation characterizing a Riemannian structure of a manifold, Tokyo J. Math. 6 (1983), 143–151.

[Kb1] Y. KERBRAT, Existence de certains champs de vecteurs sur les variétés riemanniennes complètes, C. R. Acad. Sci. Paris 270 (1970), 1430–1433.

[Kb2] Y. KERBRAT, Transformations conformes des variétés pseudo-Riemanniennes, J. Differential Geom. 11 (1976), 547–571.

[Ke1] M. G. KERCKHOVE, Conformal transformations of pseudo-Riemannian Einstein manifolds, Thesis, Brown Univ., Providence, RI, 1988.

[Ke2] M. G. KERCKHOVE, The structure of Einstein spaces admitting conformal motions, Classical Quantum Gravity 8 (1991), 819–825.

[Kü] W. KÜHNEL, Conformal transformations between Einstein spaces, Conformal Geometry (R. S. Kulkarni and U. Pinkall, eds.), 105–146, Aspects of Math. E 12, Vieweg-Verlag, Braunschweig–Wiesbaden, 1988.

[KR1] W. KÜHNEL AND H.-B. RADEMACHER, Conformal diffeomorphisms preserving the Ricci tensor, Proc. Amer. Math. Soc. 123 (1995), 2841–2848.

[KR2] W. KÜHNEL AND H.-B. RADEMACHER, Essential conformal fields in pseudo-Riemannian geometry, J. Math. Pures Appl. (9) 74 (1995), 453–481.

[KR3] W. KÜHNEL AND H.-B. RADEMACHER, Oscillator and pendulum equation on pseudo-Riemannian manifolds, and conformal vector fields, Geometry and Topology of Submanifolds VII (F. Dillen et al., eds.), 159–163, World Scientific, Singapore, 1995.

[KR4] W. KÜHNEL AND H.-B. RADEMACHER, Conformal vector fields on pseudo-Riemannian spaces, Differential Geom. Appl. (to appear).

[La1] J. LAFONTAINE, Conformal geometry from the Riemannian view-point, Conformal Geometry (R. S. Kulkarni and U. Pinkall (eds.), 65–92, Aspects of Math. E 12, Vieweg-Verlag, Braunschweig–Wiesbaden, 1988.

[La2] J. LAFONTAINE, The theorem of Lelong-Ferrand and Obata, Conformal Geometry (R. S. Kulkarni and U. Pinkall, eds.), 93–103, Aspects of Math. E 12, Vieweg-Verlag, Braunschweig–Wiesbaden, 1988.

[Law] D. F. LAWDEN, Elliptic Functions and Applications, Springer, Berlin–Heidelberg–New York, 1989.

[Ob1] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, J. Math. Soc. Japan 14 (1962), 333–340.

[Ob2] M. Obata, The conjectures about conformal transformations, J. Differential Geom. 6 (1971), 247–258.

[ON] B. O'Neill, Semi-Riemannian geometry, Academic Press, New York–London, 1983.

[T] Y. Tashiro, Complete Riemannian manifolds and some vector fields, Trans. Amer. Math. Soc. 117 (1965), 251–275.

[Y1] K. Yano, The theory of Lie derivatives and its applications, North-Holland, Amsterdam, 1957.

[Y2] K. Yano, Concircular geometry I–V, Proc. Imp. Acad. Tokyo 16 (1940), 195–200, 354–360, 442–448, 505–511; ibid. 18 (1942), 446–451.

[YN] K. Yano and T. Nagano, Einstein spaces admitting a one-parameter group of conformal transformations, Ann. of Math. (2) 69 (1959), 451–461.

[Yo] Y. Yoshimatsu, On a theorem of Alekseevskii concerning conformal transformations, J. Math. Soc. Japan 28 (1976), 278–289.

Mathematisches Institut B
Universität Stuttgart
70550 Stuttgart
Germany

E-mail address:
kuehnel@mathematik.uni-stuttgart.de

Mathematisches Institut
Universität Leipzig, Augustusplatz 10/11
04109 Leipzig
Germany

E-mail address:
rademacher@mathematik.uni-leipzig.de