Tôhoku Math. J.
49 (1997), 347–353

# ON THE INTEGRALS OF RIEMANN-LEBESGUE TYPE

CHANG-PAO CHEN* AND YU-YING CHEN

(Received January 30, 1996, revised April 15, 1996)

**Abstract.** In this paper, a theorem on the integrals of Riemann-Lebesgue type is established. It not only generalizes Theorem II.7.1 in Widder's book, but also gives a unified version of Young's test, Dirichlet-Jordan test, de la Vallée-Poussin test, and Dini's test.

**1. Introduction.** Let $f: [0, \delta] \mapsto C$ and $g: [0, \infty) \mapsto C$ be measurable. The integral in the question is of the form:

$$I(f, g; \lambda) \equiv \int_0^\delta f(t)g(\lambda t)dt \qquad (\lambda > 0) .$$

The famous Riemann-Lebesgue theorem says that if $f \in L^1(0, \delta)$ and $g(t) = \sin t$, then $I(f, g; \lambda) \to 0$ as $\lambda \to \infty$. In this paper, we try to find conditions on $f$ and $g$ such that the limit of $I(f, g; \lambda)$ exists as $\lambda \to \infty$. As indicated below, such kind of results are closely connected to the pointwise convergence problem of Fourier series. The purpose of this paper is to establish the following theorem.

THEOREM 1.1. *Set $f(t) = f_1(t) + f_2(t)$. Assume that the following conditions are satisfied for some $p \geq 0$ and some $\delta > 0$:*

(i) $tf_1(t) \to a$ *as* $t \to 0^+$,

(ii) $t^{p+1}f_1(t)$ *is of bounded variation on* $[0, \delta]$ *and*

$$V(t^{p+1}f_1(t); 0, h) = O(h^p) \quad as \quad h \to 0^+ ,$$

(iii)

$$\lim_{\lambda \to \infty} \int_0^\delta f_2(t)g(\lambda t)dt = 0 ,$$

(iv) $t^{-1}g(t)$ *is locally integrable on* $[0, \infty)$ *and the improper integral* $\int_0^\infty t^{-1} g(t)dt$ *converges,*

(v)

$$\sup_{t \geq 0} \left| \int_0^t g(\tau)d\tau \right| < \infty .$$

---

1991 *Mathematics Subject Classification.* Primary 42A32; Secondary 42B05, 42A20.

* This research is supported by National Science Council, Taipei, R.O.C. under Grant #NSC 85-2121-M-007-021.

*Then*

$$\lim_{\lambda\to\infty}\int_0^\delta f(t)g(\lambda t)dt=a\left(\int_0^\infty \frac{g(t)}{t}\,dt\right).$$

Here $V(\theta;\, a, b)$ denotes the variation of $\theta$ on $[a, b]$. Obviously, the function $g(t)=\sin t$ satisfies (iv) and (v). The Riemann-Lebesgue theorem implies (iii) in the case $f_2\in L^1(0,\delta)$ and $g(t)=\sin t$. Theorem 1.1 generalizes [W, Theorem II.7.1]. The latter corresponds to the case $p=0$, $f_1(t)=t^{-1}\alpha(t)$, $f_2(t)=0$, and $g(t)=\sin t$.

Let $s_n(\phi;\, x)$ denote the $n$th partial sum of the Fourier series of $\phi\in L^1(T)$, where $T\equiv[-\pi,\pi]$. As indicated in [B, §37 of Chapter I], we have $s_n(\phi;\, x)\to\check{\phi}(x)$ (as $n\to\infty$) if and only if for some $\delta\in(0,\pi)$,

$$\int_0^\delta \frac{\phi_x(t)}{t}\sin nt\,dt=o(1)\qquad(\text{as } n\to\infty),$$

where

$$\phi_x(t)\equiv\frac{\phi(x+t)+\phi(x-t)-2\check{\phi}(x)}{2}.$$

Set $f_1(t)=t^{-1}\phi_x(t)$, $f_2(t)=0$, and $g(t)=\sin t$ in Theorem 1.1. Then we get:

THEOREM 1.2 (generalized Young's test). *Let $\phi\in L^1(T)$. Assume that the following are satisfied for some $p\ge 0$:*

(i) $\phi_x(t)\to 0$ *as* $t\to 0^+$,

(ii) $V(t^p\phi_x(t);\, 0, h)=O(h^p)$ *as* $h\to 0^+$.

*Then* $s_n(\phi;\, x)\to\check{\phi}(x)$ *as* $n\to\infty$.

The case $p=1$ of Theorem 1.2 is the classical Young test (see [B, §4 of Chapter III]). As for $p=0$, choose $\check{\phi}(x)=\{\phi(x^+)+\phi(x^-)\}/2$. Then Theorem 1.2 gives the following Dirichlet-Jordan test.

COROLLARY 1.3 (Dirichlet-Jordan test). *Let $\phi$ be of bounded variation in some neighborhood of $x$. Then* $s_n(\phi;\, x)\to\{\phi(x^+)+\phi(x^-)\}/2$ *as* $n\to\infty$.

Set $f_1(t)=t^{-2}\Phi(t)$, $f_2(t)=(tf_1(t))'$, and $g(t)=\sin t$, where

$$\Phi(t)\equiv\int_0^t \phi_x(u)du.$$

Then $t^{-1}\phi_x(t)=f_1(t)+f_2(t)$ for almost all $t$. With the help of [WZ, Corollary 7.23], we get the following consequence of Theorem 1.1.

COROLLARY 1.4 (de la Vallée-Poussin test). *Let $\phi\in L^1(T)$. Assume that the following are satisfied:*

(i) $t^{-1}\Phi(t)\to 0$ *as* $t\to 0^+$,

(ii) $V(t^{-1}\Phi(t); 0, h) = O(1)$ *as* $h \to 0^+$.
*Then* $s_n(\phi; x) \to \check{\phi}(x)$ *as* $n \to \infty$.

Consider the case $p=0$, $f_1(t)=0$, $f_2(t)=t^{-1}\phi_x(t)$, and $g(t)=\sin t$. Then Dini's test follows directly from Theorem 1.1.

COROLLARY 1.5 (Dini's test). *If* $\phi \in L^1(T)$ *and* $t^{-1}\phi_x(t) \in L^1(0, \delta)$ *for some* $\delta > 0$, *then* $s_n(\phi; x) \to \check{\phi}(x)$ *as* $n \to \infty$.

**2. Proof of Theorem 1.1.** To prove Theorem 1.1, we need the following form of integration by parts. The case $f(x) = 1$ reduces to the classical one.

THEOREM (generalized integration by parts). *Assume that* $f$ *is continuous on* $[a, b]$ *and* $\alpha$, $\beta$ *are of bounded variation on* $[a, b]$. *If at least one of* $\alpha$ *and* $\beta$ *is continuous on* $[a, b]$, *then*

$$\int_a^b f(x)d(\alpha(x)\beta(x)) = \int_a^b f(x)\alpha(x)d\beta(x) + \int_a^b f(x)\beta(x)d\alpha(x).$$

PROOF. Without loss of generality, we may assume that both $\alpha$ and $\beta$ are positive and increasing on $[a, b]$. Therefore, $\alpha\beta$ is positive and increasing on $[a, b]$. Let $P$: $a = x_0 < x_1 < \cdots < x_n = b$ be any partition of $[a, b]$. By integration by parts, we find

$$\int_{x_{j-1}}^{x_j} d(\alpha(x)\beta(x)) = \int_{x_{j-1}}^{x_j} \alpha(x)d\beta(x) + \int_{x_{j-1}}^{x_j} \beta(x)d\alpha(x).$$

Applying the intermediate value theorem, we find $c_j$, $c_j^*$, $c_j^{**} \in [x_{j-1}, x_j]$ such that

$$\begin{aligned} \int_{x_{j-1}}^{x_j} f(x)d(\alpha(x)\beta(x)) &= f(c_j)\int_{x_{j-1}}^{x_j} d(\alpha(x)\beta(x)) \\ &= f(c_j)\left\{\int_{x_{j-1}}^{x_j} \alpha(x)d\beta(x) + \int_{x_{j-1}}^{x_j} \beta(x)d\alpha(x)\right\}, \end{aligned}$$

$$\int_{x_{j-1}}^{x_j} f(x)\alpha(x)d\beta(x) = f(c_j^*)\int_{x_{j-1}}^{x_j} \alpha(x)d\beta(x),$$

$$\int_{x_{j-1}}^{x_j} f(x)\beta(x)d\alpha(x) = f(c_j^{**})\int_{x_{j-1}}^{x_j} \beta(x)d\alpha(x),$$

which imply

$$\begin{aligned} (2.1)\quad & \left|\int_{x_{j-1}}^{x_j} f(x)d(\alpha(x)\beta(x)) - \int_{x_{j-1}}^{x_j} f(x)\alpha(x)d\beta(x) - \int_{x_{j-1}}^{x_j} f(x)\beta(x)d\alpha(x)\right| \\ &\leq \left|f(c_j) - f(c_j^*)\right| \int_{x_{j-1}}^{x_j} \alpha(x)d\beta(x) + \left|f(c_j) - f(c_j^{**})\right| \int_{x_{j-1}}^{x_j} \beta(x)d\alpha(x) \end{aligned}$$

$$\le M_j(P)\left\{\int_{x_{j-1}}^{x_j}\alpha(x)d\beta(x)+\int_{x_{j-1}}^{x_j}\beta(x)d\alpha(x)\right\}$$

$$=M_j(P)\int_{x_{j-1}}^{x_j}d(\alpha(x)\beta(x))\,,$$

where

$$M_j(P)\equiv\max_{x,y\in[x_{j-1},x_j]}|f(x)-f(y)|\,.$$

Summing up both sides of (2.1) with respect to $j$ yields

$$\left|\int_a^b f(x)d(\alpha(x)\beta(x))-\int_a^b f(x)\alpha(x)d\beta(x)-\int_a^b f(x)\beta(x)d\alpha(x)\right|$$

$$\le\sum_{j=1}^n M_j(P)\int_{x_{j-1}}^{x_j}d(\alpha(x)\beta(x))\le\left(\max_j M_j(P)\right)\int_a^b d(\alpha(x)\beta(x))\,.$$

Taking infimum with respect to $P$ gives

$$\left|\int_a^b f(x)d(\alpha(x)\beta(x))-\int_a^b f(x)\alpha(x)d\beta(x)-\int_a^b f(x)\beta(x)d\alpha(x)\right|$$

$$\le\left\{\inf_P\left(\max_j M_j(P)\right)\right\}\int_a^b d(\alpha(x)\beta(x))=0\,.$$

The last equality is based on the fact that $f$ is uniformly continuous on $[a, b]$. The proof is completed.

PROOF OF THEOREM 1.1. It suffices to prove this theorem in the case $f_2(t)=0$. In this case, $f(t)=f_1(t)$. Without loss of generality, we assume that $f$ is real-valued. We first consider the case $a=0$ and $p>0$. Set $\varphi(t)=t^{p+1}f(t)$. Then (ii) says that $\varphi$ is of bounded variation on $[0,\delta]$ and $V(\varphi;0,h)=O(h^p)$ as $h\to 0^+$. By (i), $|\varphi(t)|\le|tf(t)|\to 0$ as $t\to 0^+$, and $|\varphi(t)-\varphi(0)|\le V(\varphi;0,t)=O(t^p)$ as $t\to 0^+$. Hence, $\varphi(0)=0$ and there exists $M>0$ such that $|\varphi(t)|\le Mt^p$ for all $t\in[0,\delta]$. Write $\varphi=\varphi_1-\varphi_2$, where $\varphi_1(t)=V(\varphi;0,t)$. Then $\varphi_1(0)=\varphi_2(0)=0$ and $|\varphi_j(t)|=|\varphi_j(t)-\varphi_j(0)|\le V(\varphi_j;0,t)=O(t^p)$ as $t\to 0^+$. Hence, we can rearrange $M$ so that

$$\max(|\varphi(t)|,|\varphi_1(t)|,|\varphi_2(t)|)\le Mt^p\qquad\text{for all}\quad t\in[0,\delta]\,.\tag{2.2}$$

Let $k\pi\lambda^{-1}<\delta$. The precise values of $k$ and $\lambda$ will be determined later. We have

$$\left|\int_0^{k\pi/\lambda}f(t)g(\lambda t)dt\right|\le\left(\sup_{0<t\le k\pi/\lambda}|tf(t)|\right)\left(\int_0^{k\pi/\lambda}\left|\frac{g(\lambda t)}{t}\right|dt\right)\tag{2.3}$$

$$=\left(\sup_{0<t\leq k\pi/\lambda}|tf(t)|\right)\left(\int_0^{k\pi}\left|\frac{g(t)}{t}\right|dt\right).$$

On the other hand, set

$$G(t)\equiv\int_0^t g(\tau)d\tau\,.$$

Then $G$ is bounded on $[0, \infty)$ and absolutely continuous on every interval $[0, t]$ with $t>0$. Integration by parts gives

$$\text{(2.4)}\qquad \left|\int_{k\pi/\lambda}^{\delta} f(t)g(\lambda t)dt\right|=\left|\int_{k\pi/\lambda}^{\delta} f(t)d\left(\frac{G(\lambda t)}{\lambda}\right)\right|$$
$$\leq\left|\frac{f(\delta)G(\lambda\delta)}{\lambda}-\frac{f(k\pi/\lambda)G(k\pi)}{\lambda}\right|+\left|\int_{k\pi/\lambda}^{\delta}\frac{G(\lambda t)}{\lambda}\,df(t)\right|$$
$$=I_1+I_2\,,\quad\text{say}\,.$$

We have $f(t)=t^{-p-1}\varphi(t)$. Applying the generalized integration by parts, we obtain

$$\text{(2.5)}\qquad \lambda I_2\leq\left|\int_{k\pi/\lambda}^{\delta}\frac{G(\lambda t)}{t^{p+1}}\,d\varphi(t)\right|+\int_{k\pi/\lambda}^{\delta}\frac{(p+1)|G(\lambda t)\varphi(t)|}{t^{p+2}}\,dt$$
$$=I_{21}+I_{22}\,,\quad\text{say}\,.$$

Since $\varphi_1$ and $\varphi_2$ are positive and increasing on $[0, \delta]$, by (2.2),

$$\text{(2.6)}\qquad I_{21}\leq\left(\sup_{t>0}|G(t)|\right)\left(\int_{k\pi/\lambda}^{\delta}\frac{d\varphi_1(t)}{t^{p+1}}+\int_{k\pi/\lambda}^{\delta}\frac{d\varphi_2(t)}{t^{p+1}}\right)$$
$$\leq\left(\sup_{t>0}|G(t)|\right)\left(\frac{\varphi_1(\delta)+\varphi_2(\delta)}{\delta^{p+1}}+\int_{k\pi/\lambda}^{\delta}\frac{(p+1)(\varphi_1(t)+\varphi_2(t))}{t^{p+2}}\,dt\right)$$
$$\leq\left(\sup_{t>0}|G(t)|\right)\left(\frac{\varphi_1(\delta)+\varphi_2(\delta)}{\delta^{p+1}}+\int_{k\pi/\lambda}^{\delta}\frac{2M(p+1)}{t^2}\,dt\right)$$
$$\leq\left(\sup_{t>0}|G(t)|\right)\left(\frac{\varphi_1(\delta)+\varphi_2(\delta)}{\delta^{p+1}}+\frac{2M(p+1)\lambda}{k\pi}\right).$$

Similarly, (2.2) implies

$$\text{(2.7)}\qquad I_{22}\leq M(p+1)\left(\sup_{t>0}|G(t)|\right)\int_{k\pi/\lambda}^{\delta}t^{-2}dt$$
$$\leq\frac{M(p+1)\lambda}{k\pi}\left(\sup_{t>0}|G(t)|\right).$$

By (2.5)–(2.7), we get

$$I_2 \leq \left( \sup_{t>0} |G(t)| \right) \left( \frac{\varphi_1(\delta)+\varphi_2(\delta)}{\lambda \delta^{p+1}} + \frac{3M(p+1)}{k\pi} \right). \tag{2.8}$$

Combining (2.3)–(2.4) with (2.8) yields

$$\left| \int_0^\delta f(t) g(\lambda t) dt \right| \leq \left( \sup_{0<t\leq k\pi/\lambda} |tf(t)| \right) \left( \int_0^{k\pi} \left| \frac{g(t)}{t} \right| dt \right) + \left| \frac{f(\delta)G(\lambda\delta)}{\lambda} - \frac{f(k\pi/\lambda)G(k\pi)}{\lambda} \right|$$
$$+ \left( \sup_{t>0} |G(t)| \right) \left( \frac{\varphi_1(\delta)+\varphi_2(\delta)}{\lambda \delta^{p+1}} + \frac{3M(p+1)}{k\pi} \right).$$

Choose large $k$ first and then let $\lambda \to \infty$. By (i), (iv), and (v), we infer that

$$\int_0^\delta f(t) g(\lambda t) dt = o(1) \quad \text{as} \quad \lambda \to \infty . \tag{2.9}$$

Next, we consider the case $a = p = 0$. By (ii), we can still find $M > 0$ so that (2.2) holds. From the argument given in (2.3)–(2.8), we see that (2.9) still holds in this case. This finishes the proof in the case $a = 0$.

For general $a$, set $f^*(t) = f(t) - at^{-1}$. Then $tf^*(t) = tf(t) - a = o(1)$ as $t \to 0^+$, $t^{p+1} f^*(t)$ is of bounded variation on $[0, \delta]$, and

$$V(t^{p+1} f^*(t); 0, h) \leq V(t^{p+1} f(t); 0, h) + |a| V(t^p; 0, h)$$
$$= O(h^p) \quad \text{as} \quad h \to 0^+ .$$

Hence, the preceding result in the case $a = 0$ leads us to

$$\lim_{\lambda\to\infty} \int_0^\delta f(t) g(\lambda t) dt = \lim_{\lambda\to\infty} \int_0^\delta (f^*(t) + at^{-1}) g(\lambda t) dt$$
$$= a \left\{ \lim_{\lambda\to\infty} \int_0^\delta \frac{g(\lambda t)}{t} dt \right\} = a \left( \int_0^\infty \frac{g(t)}{t} dt \right).$$

This completes the proof.

ACKNOWLEDGMENT. The authors are grateful to the referee and Professor Satoru Igari for valuable suggestions which improved the presentation of this paper.

## REFERENCES


[B] N. K. BARY, A Treatise on Trigonometric Series, Vol. 1, Pergamon Press, Oxford, 1964.

[W] D. V. WIDDER, The Laplace Transform, Princeton Univ. Press, Princeton, 1946.

[WZ] R. L. WHEEDEN AND A. ZYGMUND, Measure and Integral, An Introduction to Real Analysis, Marcel Dekker, New York/Basel, 1977.

[Z] A. ZYGMUND, Trigonometric series, Cambridge Univ. Press, London/New York, 1959.

DEPARTMENT OF MATHEMATICS
NATIONAL TSING HUA UNIVERSITY
HSINCHU, TAIWAN 30043
REPUBLIC OF CHINA

*E-mail address*: cpchen@math.nthu.edu.tw