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A SINGULAR NONLINEAR SECOND-ORDER PERIODIC BOUNDARY VALUE PROBLEM

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Abstract. A singular nonlinear second-order periodic boundary value problem is studied and the uniqueness and existence of solutions are obtained by employing a priori estimates, perturbation techniques and comparison principles.

1. Introduction and main results. During the last two decades, singular nonlinear two-point boundary value problems, not including periodic ones, have been studied extensively. For details, see, for instance, papers [1]–[20] and the references therein. However, the works on singular nonlinear periodic boundary value problems are quite rarely seen. It is well known that periodic boundary value problems have always been attended to. So we study in this paper a singular nonlinear periodic boundary value problem of the form

$$(1)\qquad \begin{cases} -u''(t)+\rho^2u(t)=Q(t,u(t))\,, & 0\le t\le 2\pi\,,\\ u(0)=u(2\pi)\,, \quad u'(0)=u'(2\pi)\,, \end{cases}$$

where ρ is a positive constant and the nonlinear function $Q(t,u)=f(t,u)+h(t,u)$ is assumed to be defined on $[0,2\pi]\times(0,+\infty)$ and satisfy the following assumptions:

(A_1) For each fixed $u\in(0,+\infty)$, $f(t,u)$ is nonnegative integrable on $[0,2\pi]$.

(A_2) For almost all $t\in[0,2\pi]$, $f(t,u)$ is nonincreasing in $u>0$ and

$$\int_0^{2\pi} f(s,0+)ds>0\,,\quad f(t,0+)=\lim_{u\to 0+} f(t,u)\,.$$

(A_3) $h(t,u)$ is a Caratheodory function defined on $[0,2\pi]\times(0,\infty)$, i.e., for each fixed $u\in(0,\infty)$, the function $h(t,u)$ is nonnegative integrable on $[0,2\pi]$ and for almost all $t\in[0,2\pi]$, the function $h(t,u)$ is continuous in $u>0$.

(A_4) There exists a nonnegative integrable function $k(t)$ defined on $[0,2\pi]$ and a nonnegative continuous increasing function $H(u)$ defined on $(0,+\infty)$ such that

$$h(t,u)\le k(t)H(u) \quad \text{for almost all} \quad (t,u)\in[0,2\pi]\times(0,\infty)\,,$$

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where $H(u)$ and $k(t)$ respectively satisfy

$$\overline{\lim_{u\to+\infty}}\frac{H(u)}{u}=B \tag{2}$$

and

$$G(0)B\int_0^{2\pi}k(s)ds<1\,. \tag{3}$$

Hence $G(0)$ is the positive maximum of the Green function

$$G(|t-s|):=G(t,s):=\begin{cases}\dfrac{e^{\rho(t-s)}+e^{\rho(2\pi-t+s)}}{2\rho(e^{2\rho\pi}-1)}\,, & 0\le s\le t\le 2\pi\,,\\[2ex] \dfrac{e^{\rho(s-t)}+e^{\rho(2\pi-s+t)}}{2\rho(e^{2\rho\pi}-1)}\,, & 0\le t\le s\le 2\pi\,.\end{cases}$$

A direct calculation shows that

$$\frac{e^{\rho\pi}}{\rho(e^{2\rho\pi}-1)}=G(\pi)\le G(t,s)\le G(0)=\frac{e^{2\rho\pi}+1}{2\rho(e^{2\rho\pi}-1)}\quad\text{on}\ [0,2\pi]\times[0,2\pi]\,.$$

The assumptions (A_1) and (A_2) allow but not require $f(t,u)$ to have singularity at $u=0$ and to be discontinuous with respect to u. For example, the function

$$f(t,u)=|\sin 8\pi t|^{-\alpha}\left(\sum_{j=1}^{\infty}\eta\left(\frac{1}{j}-u\right)+u^{-\beta}\right)$$

satisfies (A_1) and (A_2). Here $\alpha\in(0,1)$, $\beta>0$, and $\eta(s)$ is the Heaviside function

$$\eta(s):=\begin{cases}1\,, & s\ge 0\,,\\ 0\,, & s<0\,.\end{cases}$$

A function $u(t)$ is said to be a solution to the problem (1), if

(i) $u\in C^1[0,2\pi]$, $u(0)=u(2\pi)$, $u'(0)=u'(2\pi)$,

(ii) $u''(t)$ exists almost everywhere and integrable on $[0,2\pi]$, and

$$-u''(t)+\rho^2u(t)=Q(t,u(t))\quad\text{a.e. on}\quad[0,2\pi]\,.$$

Furthermore, if $u(t)>0$ on $[0,2\pi]$, then it is called a positive solution.

The main results of this paper are as follows.

THEOREM 1. *Let* (A_1)–(A_4) *hold. Then there exist two positive numbers* δ *and* N *such that for all solutions* $u(t)$ *to the problem* (1)

$$\delta\le u(t)\le N\quad on\quad[0,2\pi]\,.$$

THEOREM 2. *Assume that* (A_1)–(A_4) *are satisfied and* $f(t,u)$ *is continuous in* $u>0$. *Then the problem* (1) *has at least one positive solution.*

THEOREM 3. *Assume that* $h(t, u) \equiv 0$ *and* $f(t, u)$ *satisfies* (A_1)–(A_2). *Then the problem* (1) *has a unique positive solution.*

Our arguments for establishing the existence and uniqueness of solutions to the problem (1) involve only the positivity of the Green function, a priori estimates, perturbation techniques, and comparison principles.

2. Proof of Theorem 1. It is readily verified that the problem (1) is equivalent to the integral equation

$$u(t) = \int_0^{2\pi} G(t, s) Q(s, u(s)) ds, \qquad 0 \leq t \leq 2\pi. \tag{4}$$

Let $u(t)$ be a solution to the problem (1) and let

$$m := \min\{u(t) : 0 \leq t \leq 2\pi\}, \quad M := \max\{u(t) : 0 \leq t \leq 2\pi\}.$$

Then, by (4), we obtain

$$G(\pi) \int_0^{2\pi} Q(s, u(s)) ds \leq m \leq M \leq G(0) \int_0^{2\pi} Q(s, u(s)) ds,$$

where $G(\pi)$ is the positive minimum of $G(t, s)$. Consequently, we have

$$M \leq m G(0)/G(\pi). \tag{5}$$

We now prove that there exists a $\delta > 0$ such that for all solutions $u(t)$ to the problem (1)

$$u(t) \geq \delta \quad \text{on} \quad [0, 2\pi]. \tag{6}$$

If this is not the case, then there exists a sequence of solutions to the problem (1), $\{u_j(t)\}$, such that

$$m_j := \min\{u_j(t) : 0 \leq t \leq 2\pi\} \to 0 \qquad \text{as} \quad j \to \infty.$$

From (A_1), (A_2), (A_3), (5) and (6), we have

$$\begin{aligned} m_j &\geq G(\pi) \int_0^{2\pi} f(s, u(s)) ds \\ &\geq G(\pi) \int_0^{2\pi} f(s, M_j) ds \\ &\geq G(\pi) \int_0^{2\pi} f(s, m_j G(0)/G(\pi)) ds, \end{aligned}$$

where $M_j := \max\{u_j(t) : 0 \leq t \leq 2\pi\}$. Letting $j \to \infty$ in the above, we lead to

$$0 \geq G(\pi) \int_0^{2\pi} f(s, 0+) ds > 0 ,$$

a contradiction. This shows that (6) holds. Therefore, the condition (A_2) excludes the case when $u(t) \equiv 0$ is a solution to the problem (1).

Next, we prove that there exists an $N > 0$ such that for all solution $u(t)$ to the problem (1)

$$u(t) \leq N \quad \text{on} \quad [0, 2\pi] . \tag{7}$$

From (3), we choose an $\varepsilon > 0$ such that

$$G(0)(B+\varepsilon) \int_0^{2\pi} k(s) ds < 1 .$$

It follows from (2) that there exists an $N^* > \delta$ such that

$$H(u) \leq (B+\varepsilon)u \quad \text{for all} \quad u \geq N^* .$$

Let

$$N := \frac{N^* + G(0) \int_0^{2\pi} f(s, \delta) ds + G(0) H(N^*) \int_0^{2\pi} k(s) ds}{1 - G(0)(B+\varepsilon) \int_0^{2\pi} k(s) ds}$$

and let

$$M := \max\{u(t) : 0 \leq t \leq 2\pi\} ,$$

where $u(t)$ is an arbitrary solution to the problem (1). Without loss of generality, we may assume that $M \geq N^*$. From (4), we have

$$\begin{aligned} M &\leq G(0) \int_0^{2\pi} [f(s, u(s)) + h(s, u(s))] ds \\ &\leq G(0) \int_0^{2\pi} [f(s, \delta) + k(s) H(u(s))] ds \\ &< N^* + G(0) \int_0^{2\pi} f(s, \delta) ds + G(0) \int_0^{2\pi} k(s) ds H(N^*) \\ &\quad + G(0)(B+\varepsilon) M \int_0^{2\pi} k(s) ds \end{aligned}$$

i.e., $M \leq N$. This shows that (7) holds. Theorem 1 is thus proved.

3. Proof of Theorem 2. Let δ and N be two positive numbers determined by Theorem 1. Consider the modified problem

$$(8) \qquad \begin{cases} -u''(t)+\rho^2 u(t)=Q^*(t,u(t)), & 0\le t\le 2\pi, \\ u(0)=u(2\pi), \quad u'(0)=u'(2\pi), \end{cases}$$

where $Q^*(t,u):=f^*(t,u)+h^*(t,u)$ and

$$f^*(t,u):=\begin{cases} f(t,\delta) & \text{if } u<\delta, \\ f(t,u) & \text{if } u\ge\delta, \end{cases}$$

$$h^*(t,u):=\begin{cases} h(t,u) & \text{if } u\le N, \\ h(t,N) & \text{if } u>N. \end{cases}$$

It is clear that both $f^*(t,u)$ and $h^*(t,u)$ satisfy (A_1)–(A_4) and hence Theorem 1 holds for the problem (8).

We define a mapping $\Phi: D\to D$ by

$$(\Phi u)(t):=\int_0^{2\pi} G(t,s)Q^*(s,u(s))ds, \qquad \forall u\in D,$$

where $D:=\{u\in C[0,2\pi];\ u(t)\ge 0 \text{ on } [0,2\pi]\}$.

By the definition of Φ, it is readily verified that Φ is a compactly continuous mapping from D into D. The Schauder fixed point theorem tells us that Φ has at least one fixed point in D. Let $u(t)$ be a fixed point in D. Then it is easy to check that $u(t)$ is a solution to the problem (8). Since $\delta\le u(t)\le N$ on $[0,2\pi]$, $u(t)$ is also a positive solution to the problem (1).

4. Proof of Theorem 3. When $h(t,u)\equiv 0$, the following comparison principle holds.

LEMMA 1. *Let $f_j(t,u)$, $j=1,2$, satisfy (A_1) and (A_2) and let $u_j(t)$, $j=1,2$, be a solution to the problem (1) with $f=f_j$. If $f_1(t,u)\ge f_2(t,u)$ a.e. on $[0,2\pi]\times(0,+\infty)$, then $u_1(t)\ge u_2(t)$ for all $t\in[0,2\pi]$.*

PROOF OF LEMMA 1. Let $w(t):=u_1(t)-u_2(t)$. If the lemma is not true, then $w(t)<0$ for some $t\in[0,2\pi]$. Without loss of generality, we can consider only the following three cases.

Case I. $w(t)<0$ on $[0,2\pi]$. In this case, we have

$$(9) \qquad -w''(t)+\rho^2 w(t)=f_1(t,u_1(t))-f_2(t,u_2(t))\ge 0$$

for almost all $t\in[0,2\pi]$, i.e.,

$$w''(t)\le\rho^2 w(t)<0$$

for almost all $t\in[0,2\pi]$, which implies that $w'(0)>w'(2\pi)$. This contradicts the fact that $w'(0)=w'(2\pi)$.

Case II. $w(0)=w(2\pi)\ge 0$ and $w(t)<0$ for some $t\in(0,2\pi)$. In this case, there exists a subinterval (a,b) of $[0,2\pi]$ such that

$w(t)<0$ in (a, b), $w(a)=w(b)=0$, and hence $w'(a)\leq 0\leq w'(b)$.

By (9), we have

$$w''(t)\leq \rho^2 w(t)<0 \quad \text{for almost all} \quad t\in(a, b)\,,$$

which implies that $w'(a)>w'(b)$. This is a contradiction.

Case III. $w(0)=w(2\pi)<0$ and $w(t)\geq 0$ for some $t\in(0, 2\pi)$. In this case, there are two points $a, b\in(0, 2\pi)$, $a\leq b$, such that

$w(t)<0$ in $[0, a)\cup(b, 2\pi]$, $w(a)=w(b)=0$, and hence $w'(a)\geq 0\geq w'(b)$.

By (9), we have

$$w''(t)\leq \rho^2 w(t)<0 \quad \text{for almost all} \quad t\in[0, a)\cup(b, 2\pi]\,,$$

which implies that $w'(0)>w'(a)\geq w'(b)>w'(2\pi)$. This is also a contradiction.

The Lemma is thus proved.

In very much the same way, we can prove the following statement.

LEMMA 2. *If $h(t, u)\equiv 0$, then the problem* (1) *has at most one solution.*

When $f(t, u)$ is not continuous in $u>0$ (for almost all $t\in[0, 2\pi]$), we employ a perturbation technique. We define

$$f(t, u, v):=\frac{1}{v}\int_u^{u+v} f^*(t, s)ds\,, \quad F(t, u, v):=\frac{1}{v}\int_{u-v}^{u} f^*(t, s)ds\,,$$

where $v>0$, $f^*(t, u)$ is defined as in the proof of Theorem 2, and

$$f_j(t, u):=f\left(t, u, \frac{1}{j}\right), \quad F_j(t, u):=F\left(t, u, \frac{1}{j}\right), \qquad j=1, 2, \ldots.$$

Then $f_j(t, u)$ and $F_j(t, u)$ satisfy (A_1), (A_2),

$$f_j(t, u)\leq f_{j+1}(t, u)\leq F_{j+1}(t, u)\leq F_j(t, u) \quad \text{on} \quad [0, 2\pi]\times[0, +\infty)\,,$$

and

$$\lim_{j\to\infty} f_j(t, u)=\lim_{j\to\infty} F_j(t, u)=f^*(t, u) \quad \text{pointwise on} \quad [0, 2\pi]\times[0, +\infty)\,,$$

since

$$\frac{\partial f(t, u, v)}{\partial u}\leq 0\,, \quad \frac{\partial F(t, u, v)}{\partial u}\geq 0\,, \quad \frac{\partial f(t, u, v)}{\partial v}\leq 0\,, \quad \frac{\partial F(t, u, v)}{\partial v}\geq 0\,,$$

$$\text{a.e. on} \quad [0, 2\pi]\times[0, +\infty)\times(0, +\infty)\,.$$

From Theorems 1 and 2, Lemmas 1 and 2, we conclude that the problem (1) with $f=f_j$ ($f=F_j$) has a unique solution $\delta\leq u_j(t)\leq N$ ($\delta\leq v_j(t)\leq N$) and

$$u_j(t)\leq u_{j+1}(t)\leq v_{j+1}(t)\leq v_j(t) \quad \text{on} \quad [0, 2\pi]\,.$$

Consequently, we have

$$u_j(t) \leq \lim_{j\to\infty} u_j(t) := u^*(t) \leq \lim_{j\to\infty} v_j(t) := v^*(t) \leq v_j(t) \quad \text{on} \quad [0, 2\pi]. \tag{10}$$

It follows by (4) that

$$\begin{aligned} u^*(t) \geq u_j(t) &= \int_0^{2\pi} G(t, s) f_j(s, u_j(s)) ds \\ &\geq \int_0^{2\pi} G(t, s) f_j(s, u^*(s)) ds \quad \text{on} \quad [0, 2\pi]. \end{aligned}$$

Letting $j \to \infty$ in the above yields

$$u^*(t) \geq \int_0^{2\pi} G(t, s) f(s, u^*(s)) ds \quad \text{on} \quad [0, 2\pi].$$

(Here we have used the fact that $\delta \leq u^*(t) \leq N$.) Similarly, we obtain

$$\begin{aligned} v^*(t) &\leq \int_0^{2\pi} G(t, s) f(s, v^*(s)) ds \\ &\leq \int_0^{2\pi} G(t, s) f(s, u^*(s)) ds \\ &\leq u^*(t) \quad \text{on} \quad [0, 2\pi]. \end{aligned} \tag{11}$$

From (10) and (11), it follows that

$$u^*(t) = v^*(t) = \int_0^{2\pi} G(t, s) f(s, u^*(s)) ds \quad \text{on} \quad [0, 2\pi].$$

This shows that $u^*(t)$ is a unique solution to problem (1).

This completes the proof of Theorem 3.

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