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BOUNDARY VALUE PROBLEMS OF NONSINGULAR TYPE ON THE SEMI-INFINITE INTERVAL

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Abstract. Existence of a positive solution is established for second order boundary value problems on the semi-infinite interval.

1. Introduction. In this paper we discuss boundary value problems on the semi-infinite interval. In particular we examine

$$\begin{cases} y'' + \phi(t) f(t, y, y') = 0, \quad 0 < t < \infty \\ y(0) = 0, \quad y \text{ bounded on } [0, \infty)\,, \end{cases} \tag{1.1}$$

$$\begin{cases} y'' + \phi(t) f(t, y, y') = 0, \quad 0 < t < \infty \\ y(0) = 0, \quad \lim\limits_{t \to \infty} y(t) \text{ exists}\,, \end{cases} \tag{1.2}$$

and

$$\begin{cases} y'' + \phi(t) f(t, y, y') = 0, \quad 0 < t < \infty \\ y(0) = 0, \quad \lim\limits_{t \to \infty} y'(t) = 0\,, \end{cases} \tag{1.3}$$

in Section 2; here $f : [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty)$. By putting physically reasonable assumptions on f, we will show (1.1), (1.2) and (1.3) have solutions $y \in C^1[0, \infty) \cap C^2(0, \infty)$ with $y > 0$ on $(0, \infty)$ even if $y \equiv 0$ is also a solution. Problems of the above type have been discussed by many authors in the literature; we refer the reader to [2–11] and their references. The technique we use to establish existence is based on (i) establishing new results (see [1] also) on the finite interval $[0, n]$ for each $n \in N^+ = \{1, 2, \dots\}$ and (ii) a diagonalization argument. Consequently, the results of this paper are new and they extend and complement previously known results. We remark that the diagonalization argument applied in this paper has been used by many authors; see [2, 4, 6, 7].

To conclude this section we recall the following well-known existence principle [10] for

$$\begin{cases} y'' + \phi(t) F(t, y, y') = 0, \quad 0 < t < n \\ y(0) = a \geq 0 \\ y'(n) = b \geq 0; \quad \text{here } n \in \{1, 2, \dots\} \text{ is fixed}\,. \end{cases} \tag{1.4}$$

THEOREM 1.1. *Suppose*

$$\phi \in C(0, n) \ \textit{with}\ \phi > 0 \ \textit{on}\ (0, n) \ \textit{and}\ \phi \in L^1[0, n] \tag{1.5}$$

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and

$$F : [0, n] \times \boldsymbol{R}^2 \to \boldsymbol{R} \ \text{ is continuous} \tag{1.6}$$

are satisfied. In addition, suppose there is a constant $M > a + bn$, independent of λ, with

$$|y|_1 = \max\{|y|_0, |y'|_0\} \neq M$$

for any solution $y \in C^1[0, n] \cap C^2(0, n)$ to

$$(1.7)_\lambda \qquad \begin{cases} y'' + \lambda\phi(t)F(t, y, y') = 0, & 0 < t < n \\ y(0) = a, \quad y'(n) = b \end{cases}$$

for each $\lambda \in (0, 1)$; here $|u|_0 = \sup_{[0,n]} |u(t)|$. Then (1.4) *has a solution $y \in C^1[0, n] \cap C^2(0, n)$ with $|y|_1 \leq M$.*

2. Semi-infinite problem. In this section we discuss (1.1), (1.2) and (1.3). Throughout this section we will assume the following conditions hold:

$$\phi \in C(0, \infty) \ \text{ with } \ \phi > 0 \ \text{ on } (0, \infty)\,, \tag{2.1}$$

$$\begin{cases} f : [0, \infty) \times [0, \infty) \times [0, \infty) \to [0, \infty) \ \text{ is continuous with} \\ f(t, u, p) > 0 \ \text{ for } (t, u, p) \in [0, \infty) \times (0, \infty) \times (0, \infty)\,, \end{cases} \tag{2.2}$$

$$Q_\infty = \int_0^\infty \phi(s)ds < \infty\,, \tag{2.3}$$

$$R_\infty = \int_0^\infty s\phi(s)ds < \infty\,, \tag{2.4}$$

$$\begin{cases} f(t, u, p) \leq w(\max\{u, p\}) \ \text{ on } \ [0, \infty) \times (0, \infty) \times (0, \infty) \ \text{ with} \\ w \geq 0 \ \text{ continuous and nondecreasing on } \ [0, \infty)\,, \end{cases} \tag{2.5}$$

$$\sup_{c \in (0,\infty)} \frac{c}{w(c)\max\{Q_\infty, R_\infty\}} > 1 \tag{2.6}$$

and

$$\begin{cases} \text{for a constant } \ H > 0 \ \text{ there exists a function } \ \psi_H \ \text{ continuous} \\ \text{on } \ [0, \infty) \ \text{ and positive on } \ (0, \infty), \ \text{ and a constant } \ \gamma, 0 \leq \gamma < 1 \\ \text{with } \ f(t, u, p) \geq \psi_H(t)p^\gamma \ \text{ on } \ [0, \infty) \times [0, H]^2\,. \end{cases} \tag{2.7}$$

THEOREM 2.1. *Suppose* (2.1)–(2.7) *hold. Then* (1.1), (1.2) *and* (1.3) *have solutions $y \in C^1[0, \infty) \cap C^2(0, \infty)$ with $y > 0$ on $(0, \infty)$.*

PROOF. First *fix* $n \in N^+ = \{1, 2, \dots\}$ and consider the family of problems

$$\begin{cases} y'' + \phi(t)f(t, y, y') = 0, & 0 < t < n \\ y(0) = y'(n) = 0\,. \end{cases} \tag{2.8}$$

Choose $M > 0$ with

$$\frac{M}{w(M)\max\{Q_\infty, R_\infty\}} > 1. \tag{2.9}$$

Next choose $\varepsilon > 0$ with $\varepsilon < M/(n+1)$ and

$$\frac{M}{w(M)\max\{Q_\infty, R_\infty\} + 2\varepsilon} > 1. \tag{2.10}$$

Let $n_0 \in N^+$ be chosen so that $n/n_0 < \varepsilon$ and let $N_0 = \{n_0, n_0+1, \dots\}$.

We first show that

$$(2.11)^m \qquad \begin{cases} y'' + \phi(t) f^\star(t, y, y') = 0, & 0 < t < n \\ y(0) = y'(n) = 1/m \end{cases}$$

has a solution for each $m \in N_0$; here

$$f^\star(t,u,p) = \begin{cases} f(t,u,p), & u \geq 1/m, \quad p \geq 1/m \\ f(t,u,1/m), & u \geq 1/m, \quad p < 1/m \\ f(t,1/m,p), & u < 1/m, \quad p \geq 1/m \\ f(t,1/m,1/m), & u < 1/m, \quad p < 1/m. \end{cases}$$

To show that $(2.11)^m$ has a solution, we consider the family of problems

$$(2.12)^m_\lambda \qquad \begin{cases} y'' + \lambda\phi(t) f^\star(t, y, y') = 0, & 0 < t < n \\ y(0) = y'(n) = 1/m, & m \in N_0 \end{cases}$$

for $0 < \lambda < 1$. Let $y \in C^1[0,n] \cap C^2(0,n)$ be any solution of $(2.12)^m_\lambda$. Then $y' \geq 1/m$ and $y \geq 1/m$ on $[0,n]$. Also from (2.5) we have

$$-y''(t) \leq \phi(t) w(|y|_1) \quad \text{for } t \in (0,n);$$

here $|y|_1 = \max\{|y|_0, |y'|_0\}$ and $|u|_0 = \sup_{[0,n]} |u(t)|$. Integrate from t to n to obtain

$$y'(t) \leq w(|y|_1) \int_t^n \phi(x)dx + \frac{1}{m} \quad \text{for } t \in [0,n]. \tag{2.13}$$

In particular

$$y'(0) \leq w(|y|_1) Q_\infty + \varepsilon. \tag{2.14}$$

Also, by using (2.13) and the equality $\int_0^n s\phi(s)ds = \int_0^n \int_t^n \phi(x)dxdt$,

$$y(n) \leq \frac{n}{m} + \frac{1}{m} + w(|y|_1) \int_0^n s\phi(s)ds$$

and so

$$y(n) \leq 2\varepsilon + w(|y|_1) R_\infty. \tag{2.15}$$

Combine (2.14) and (2.15) to obtain

$$\frac{|y|_1}{w(|y|_1)\max\{Q_\infty, R_\infty\} + 2\varepsilon} \leq 1. \tag{2.16}$$

Now (2.10) together with (2.16) implies $|y|_1 \neq M$.

Thus Theorem 1.1 implies that $(2.11)^m$ has a solution $y_{m,n}$ with $|y_{m,n}|_1 \leq M$. In fact

$$\frac{1}{m} \leq y_{m,n}(t) \leq M \quad \text{and} \quad \frac{1}{m} \leq y'_{m,n}(t) \leq M \quad \text{for } t \in [0,n] \tag{2.17}$$

and $y_{m,n}$ satisfies

$$\begin{cases} y'' + \phi(t) f(t, y, y') = 0, & 0 < t < n \\ y(0) = y'(n) = 1/m. \end{cases}$$

Now (2.7) guarantees the existence of a function $\psi_M(t)$ continuous on $[0, \infty)$ and positive on $(0, \infty)$, and a constant γ, $0 \leq \gamma < 1$, with $f(t, y_{m,n}(t), y'_{m,n}(t)) \geq \psi_M(t)[y'_{m,n}(t)]^\gamma$ for $(t, y_{m,n}(t), y'_{m,n}(t)) \in [0, n] \times [0, M]^2$. Of course, we have immediately that

$$y'_{m,n}(t) \geq \left((1-\gamma) \int_t^n \psi_M(s)\phi(s)ds\right)^{1/(1-\gamma)} \quad \text{for } t \in [0,n] \tag{2.18}$$

and so

$$y_{m,n}(t) \geq \int_0^t \left((1-\gamma) \int_x^n \psi_M(s)\phi(s)ds\right)^{1/(1-\gamma)} dx \quad \text{for } t \in [0,n]. \tag{2.19}$$

It is also immediate that

$$\begin{cases} \{y^{(j)}_{m,n}\}_{m \in N_0} \text{ is a bounded, equicontinuous} \\ \text{family on } [0,n] \text{ for each } j = 0, 1. \end{cases} \tag{2.20}$$

The Arzelà-Ascoli Theorem guarantees the existence of a subsequence N of N_0 and a function $y_n \in C^1[0, n]$ with $y^{(j)}_{m,n}$ converging uniformly on $[0, n]$ to $y^{(j)}_n$ as $m \to \infty$ through N; here $j = 0, 1$. Also $y_n(0) = 0 = y'_n(n)$ and

$$y_n(t) \geq \int_0^t \left((1-\gamma) \int_x^n \psi_M(s)\phi(s)ds\right)^{1/(1-\gamma)} dx \quad \text{for } t \in [0,n], \tag{2.21}$$

in particular, $y_n > 0$ on $(0, n]$. Now $y_{m,n}$, $m \in N$, satisfies

$$\begin{aligned} y_{m,n}(t) = \frac{1}{m} + \frac{1}{m}t &+ \int_0^t s\phi(s) f(s, y_{m,n}(s), y'_{m,n}(s))ds \\ &+ t\int_t^n \phi(s) f(s, y_{m,n}(s), y'_{m,n}(s))ds. \end{aligned} \tag{2.22}$$

Fix $t \in [0, n]$ and let $m \to \infty$ through N in (2.22) to obtain

$$y_n(t) = \int_0^t s\phi(s) f(s, y_n(s), y'_n(s))ds + t\int_t^n \phi(s) f(s, y_n(s), y'_n(s))ds.$$

Consequently, $y \in C^2(0, n]$ with $y''_n + \phi(t) f(t, y_n, y'_n) = 0$ for $0 < t < n$. Also from (2.17) we have

$$0 \leq y_n(t) \leq M \quad \text{and} \quad 0 \leq y'_n(t) \leq M \quad \text{for } t \in [0,n], \tag{2.23}$$

and the differential equation yields

$$0 \leq -y''_n(t) \leq \phi(t) H_\infty \quad \text{for } t \in (0,n]; \tag{2.24}$$

here $H_\infty = \sup\{f(t,u,p); (t,u,p) \in [0,\infty) \times [0,M]^2\}$. In addition we have

$$y_n'(t) \le H_\infty \int_t^n \phi(x)dx \le H_\infty \int_t^\infty \phi(x)dx \quad \text{for } t \in [0,n]. \tag{2.25}$$

To show that (1.1), (1.2) and (1.3) have a solution, we will apply a diagonalization argument. Let

$$u_n(t) = \begin{cases} y_n(t), & t \in [0,n] \\ y_n(n), & t \in [n,\infty). \end{cases}$$

Notice that $u_n \in C^1[0,\infty)$ with

$$0 \le u_n(t) \le M \quad \text{and} \quad 0 \le u_n'(t) \le M \quad \text{for } t \in [0,\infty), \tag{2.26}$$

and for $t, s \in [0,\infty)$ it is easy to see that

$$|u_n'(t) - u_n'(s)| \le H_\infty \left| \int_s^t \phi(x)dx \right|. \tag{2.27}$$

In addition

$$u_n'(t) \le H_\infty \int_t^\infty \phi(x)dx \quad \text{for } t \in [0,\infty), \tag{2.28}$$

and

$$u_n(t) \ge \int_0^t \left((1-\gamma) \int_x^n \psi_M(s)\phi(s)ds \right)^{1/(1-\gamma)} dx \quad \text{for } t \in [0,n]. \tag{2.29}$$

Also notice for $n \in N^+$ that

$$u_n(t) \ge \int_0^t \left((1-\gamma) \int_x^1 \psi_M(s)\phi(s)ds \right)^{1/(1-\gamma)} dx \equiv a_1(t) \quad \text{for } t \in [0,1]. \tag{2.30}$$

The Arzelà-Ascoli Theorem guarantees the existence of a subsequence N_1 of N^+ and a function $z_1 \in C^1[0,1]$ with $u_n^{(j)}$ converging uniformly on $[0,1]$ to $z_1^{(j)}$ as $n \to \infty$ through N_1; here $j = 0, 1$. Also from (2.30), $z_1(t) \ge a_1(t)$ for $t \in [0,1]$ (in particular, $z_1 > 0$ on $(0,1]$).

Let $N_1^+ = N_1 \setminus \{1\}$. Notice from (2.29) that

$$u_n(t) \ge \int_0^t \left((1-\gamma) \int_x^2 \psi_M(s)\phi(s)ds \right)^{1/(1-\gamma)} dx \equiv a_2(t) \quad \text{for } t \in [0,2]. \tag{2.31}$$

The Arzelà-Ascoli Theorem guarantees the existence of a subsequence N_2 of N_1^+ and a function $z_2 \in C^1[0,2]$ with $u_n^{(j)}$ converging uniformly on $[0,2]$ to $z_2^{(j)}$ as $n \to \infty$ through N_2; here $j = 0, 1$. Also from (2.31), $z_2(t) \ge a_2(t)$ for $t \in [0,2]$ (in particular, $z_2 > 0$ on $(0,2]$). Note that $z_2 = z_1$ on $[0,1]$, since $N_2 \subseteq N_1^+$. Let $N_2^+ = N_2 \setminus \{2\}$. Proceed inductively to obtain for $k = 1, 2, \ldots$ a subsequence N_k of N_{k-1}^+ and a function $z_k \in C^1[0,k]$ with $u_n^{(j)}$ converging uniformly on $[0,k]$ to $z_k^{(j)}$ as $n \to \infty$ through N_k; here $j = 0, 1$. Also

$$z_k(t) \ge a_k(t) = \int_0^t \left((1-\gamma) \int_x^k \psi_M(s)\phi(s)ds \right)^{1/(1-\gamma)} dx \quad \text{for } t \in [0,k],$$

(so in particular, $z_k > 0$ on $(0, k]$). Note that $z_k = z_{k-1}$ on $[0, k-1]$. Let $N_k^+ = N_k \setminus \{k\}$.

Define a function y as follows. Fix $t \in (0, \infty)$ and let $k \in N^+$ with $t \leq k$. Define $y(t) = z_k(t)$. Note that y is well-defined and $y(t) = z_k(t) > 0$. We can do this for each $t \in (0, \infty)$ and so $y \in C^1[0, \infty)$ with $y > 0$ on $(0, \infty)$. In addition, $0 \leq y(t) \leq M$, $0 \leq y'(t) \leq M$, and $y'(t) \leq H_\infty \int_t^\infty \phi(x)dx$ for $t \in [0, \infty)$.

Fix $x \in [0, \infty)$ and choose $k \geq x$, $k \in N^+$. Then, for $n \in N_k^+$, we have

$$y_n(x) = y_n'(k)x + \int_0^x s\phi(s)f(s, y_n(s), y_n'(s))ds + x\int_x^k \phi(s)f(s, y_n(s), y_n'(s))ds\,.$$

Let $n \to \infty$ through N_k^+ to obtain

$$z_k(x) = z_k'(k)x + \int_0^x s\phi(s)f(s, z_k(s), z_k'(s))ds + x\int_x^k \phi(s)f(s, z_k(s), z_k'(s))ds\,.$$

Thus

$$y(x) = y'(k)x + \int_0^x s\phi(s)f(s, y(s), y'(s))ds + x\int_x^k \phi(s)f(s, y(s), y'(s))ds\,.$$

Consequently, $y \in C^2(0, \infty)$ with $y'' + \phi(t)f(t, y, y') = 0$ for $0 < t < \infty$. Thus y is a solution of (1.1) with $y > 0$ on $(0, \infty)$. In addition, y is a solution of (1.2), since $y' \geq 0$ on $[0, \infty)$ and $0 \leq y \leq M$ on $[0, \infty)$. Finally, since $y'(t) \leq H_\infty \int_t^\infty \phi(x)dx$ for $t \in [0, \infty)$, we have that y is a solution of (1.3). □

EXAMPLE 2.1. The boundary value problem

$$\begin{cases} y'' + (y')^\beta e^{-t} = 0, & 0 < t < \infty \\ y(0) = 0, & \lim_{t\to\infty} y'(t) = 0\,, \end{cases} \tag{2.32}$$

with $0 \leq \beta < 1$, has a solution $y \in C^1[0, \infty) \cap C^2(0, \infty)$ with $y > 0$ on $(0, \infty)$.

REMARK 2.1. Notice that $y \equiv 0$ is also a solution of (2.32) if $\beta \neq 0$. Of course, one could construct explicitly a solution to (2.32).

We will apply Theorem 2.1 with $\phi(t) = e^{-t}$ and $w(x) = x^\beta$. Clearly (2.1)–(2.5) and (2.7) (with $\psi_H = 1$ and $\gamma = \beta$) hold. Also

$$\sup_{c\in(0,\infty)} \frac{c}{w(c)\max\{Q_\infty, R_\infty\}} = \sup_{c\in(0,\infty)} \frac{c}{c^\beta} = \infty\,,$$

so (2.6) is satisfied. Theorem 2.1 now guarantees that (2.32) has a solution $y \in C^1[0, \infty) \cap C^2(0, \infty)$ with $y > 0$ on $(0, \infty)$.

EXAMPLE 2.2. Consider the boundary value problem

$$\begin{cases} y'' + \mu(y^\alpha + \eta_0)((y')^\beta + \eta_1)e^{-t} = 0, & 0 < t < \infty \\ y(0) = 0, & \lim_{t\to\infty} y'(t) = 0\,, \end{cases} \tag{2.33}$$

with $\alpha \geq 0$, $0 \leq \beta < 1$, $\eta_0 > 0$, $\eta_1 \geq 0$, and $\mu > 0$. If

$$\mu < \sup_{c\in(0,\infty)} \frac{c}{(c^\alpha + \eta_0)(c^\beta + \eta_1)}\,, \tag{2.34}$$

then (2.33) has a solution $y \in C^1[0,\infty) \cap C^2(0,\infty)$ with $y > 0$ on $(0,\infty)$.

REMARK 2.2. Notice that $y \equiv 0$ is also a solution of (2.33), if $\eta_1 = 0$ and $\beta \neq 0$.

REMARK 2.3. If $\alpha + \beta < 1$, then (2.34) is satisfied for all $\mu > 0$.

We will apply Theorem 2.1 with $\phi(t) = \mu e^{-t}$ and $w(x) = (x^\alpha + \eta_0)(x^\beta + \eta_1)$. Clearly, (2.1)–(2.5) and (2.7) (with $\psi_H = \eta_0$ and $\gamma = \beta$) hold. Also

$$\sup_{c\in(0,\infty)} \frac{c}{w(c)\max\{Q_\infty, R_\infty\}} = \frac{1}{\mu} \sup_{c\in(0,\infty)} \frac{c}{(c^\alpha + \eta_0)(c^\beta + \eta_1)},$$

so (2.34) guarantees that (2.6) is true. Theorem 2.1 now establishes the result.

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