Tohoku Math. J.
51 (1999), 489–537

A NOTE ON THE FACTORIZATION THEOREM OF TORIC BIRATIONAL MAPS AFTER MORELLI AND ITS TOROIDAL EXTENSION

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(Received April 23, 1998, revised June 29, 1999)

Abstract. Building upon a work of Morelli, we give a coherent presentation of Morelli's algorithm for the weak and strong factorization of toric birational maps. We also discuss its toroidal extension, which plays a crucial role in the recent solutions by Włodarczyk and Abramovich-Karu-Matsuki-Włodarczyk of the weak factorization conjecture of general birational maps.

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0. Introduction. This paper is a result of series of seminars held by the authors during the summer of 1997 and continued from then on, toward a thorough understanding of the following weak and strong factorization theorem of toric birational maps by Morelli [Morelli1] (cf. [Włodarczyk1]).

THEOREM 0.1 (Factorization Theorem for Toric Birational Maps). *Every proper and equivariant birational map $f : X_\Delta \dashrightarrow X_{\Delta'}$ ("proper" in the sense of $\lceil$Iitaka$\rceil$) between two nonsingular toric varieties can be factored into a sequence of blowups and blowdowns with smooth centers which are the closures of orbits.*

If we allow the sequence to consist of blowups and blowdowns in any order, then the factorization is called *weak.*

[1] The first author is partially supported by NSF grant DMS-9700520 and by an Alfred P. Sloan research fellowship.
[2] The second author is partially supported by NSA grant MDA904-96-1-0008.
[3] The third author is partially supported by the Purdue Research Foundation.

1991 *Mathematics Subject Classification.* Primary 14M25; Secondary 14E05.

If we insist on the sequence to consist only of blowups immediately followed by blowdowns, then the factorization is called *strong*.

Our purpose is two-fold. The first is to give a coherent presentation of the proof in [Morelli1] both for the weak factorization and the strong factorization, modifying some discrepancies found by King [King2] and by the authors in the due course of the seminars checking the original arguments. Most of these discrepancies are minor, except for a couple of essential points in the process of π-desingularization and in the process of showing that the weak factorization implies the strong factorization.[4] It is a mere attempt to see in a transparent way the beautiful and brilliant original ideas of [Morelli1,2] by sweeping dust off the surface.

The second is the generalization to the toroidal case, whose details are worked out as a part of the Ph. D. thesis of the third author. Though it may be said that the toroidal generalization is straightforward and even implicit in the original papers [Morelli1,2] (cf. [Włodarczyk1]), we would like to emphasize its importance in a more far-reaching problem formulated as below, with a view toward its application to the factorization problem of general birational maps.[5]

In a most naive way the "far-reaching" problem can be stated as follows: Let $f : X \to Y$ be a morphism (one may put the condition "with connected fibers" if one wishes) between nonsingular complete (or projective) varieties. By replacing X and Y with their modifications X' and Y', how "NICE" can one make the morphism $f' : X' \to Y'$?

$$\begin{tikzcd} X \arrow[d, "f"] & X' \arrow[l] \arrow[d, "f'"] \\ Y & Y' \arrow[l] \end{tikzcd}$$

Depending upon how we interpret the word "NICE" mathematically and what restrictions we put on the modifications, we get the corresponding interesting questions such as semistable reduction (when the morphism f' is "NICE" if every fiber is reduced with only simple normal crossings, $\dim Y = 1$ and the modifications for Y are restricted to finite morphisms while the modifications for X are restricted to smooth blowups after base change), resolution of hypersurface singularities (when the morphism f' is "NICE" if every fiber has only simple normal crossings, this time not necessarily reduced, $\dim Y = 1$ and no modification for Y and only smooth blowups are allowed for X). When f is birational and we require f' to

[4] As of Jan. 1998 we learned from Professor Fulton that Morelli himself offers correction in his homepage [Morelli2] to the discrepancies in the process of π-desingularization found by King. We still need some clarification, as is presented in this paper, to understand the correction. We thank Professor Morelli for guiding us toward a better understanding through private communication.

[5] Recently two independent proofs have appeared for the weak factorization conjecture of general birational maps, one by [Włodarczyk3] another by [Abramovich-Karu-Matsuki-Włodarczyk]. (Both proofs are based upon the theory of birational cobordism of [Włodarczyk2], which is inspired by the combinatorial cobordism of [Morelli1] discussed in Section 2 of this paper.) The former uses the algorithm for π-desingularization, while the latter uses the strong factorization of toroidal birational maps directly in their proofs. Thus the importance of the toroidal extension has only increased, as well as the need for a clear coherent presentation for the π-desingularization process. The toroidalization conjecture and the strong factorization conjecture remain open.

be an isomorphism in order for it to be "NICE", restricting the modifications of X and Y to be smooth blowups, we obtain the long standing (and perhaps notorious) strong factorization problem for general birational morphisms (cf. [Hironaka]).

Our interpretation is that we put "toroidal" for the word "NICE" and restrict the modifications of X and Y to be only smooth blowups.

CONJECTURE 0.2 (Toroidalization Conjecture). *Let $f : X \to Y$ be a morphism between nonsingular complete varieties. Then there exist sequences of blowups with smooth centers for X and Y so that the induced morphism $f' : X' \to Y'$ is toroidal:*

$$\begin{tikzcd} X \arrow[d, "f"'] & X' \arrow[l, "\text{smooth blowups}"'] \arrow[d, "f'"] & \textit{toroidal} \\ Y & Y' \arrow[l, "\text{smooth blowups}"'] & \end{tikzcd}$$

The conjecture is closely related to the recent work of [Abramovich-Karu], which introduces the notion of "toroidal" morphisms explicitly for the first time, though implicitly it can be recognized in [Kempf-Knudsen-Mumford-SaintDonat]. By only requiring "NICE" morphisms to be toroidal instead of being isomorphisms, we can start dealing not only with birational morphisms but also with fibering morphisms between varieties of different dimensions. This seems to give us more freedom to seek some inductional structure. Actually we expect that the powerful inductive method of [Bierstone-Milman] for the canonical resolution of singularities, proceeding from the hypersurface case with only one defining equation to the general case with several defining equations through the ingeneous use of invariants, should be modified to be applied to our toroidalization problem, proceeding similarly from the case $\dim Y = 1$ to the general case $\dim Y > 1$.

This intèrpretation not only generalizes the statement of the classical factorization problem but also gives the following approach to it:

EXPECTATION 0.3 (A Conjectural Approach to the Strong Factorization Problem via Toroidalization). *Given a birational morphism $f : X \to Y$ between nonsingular complete varieties,*

(I) *make it "toroidal" $f' : X' \to Y'$ modifying X and Y into X' and Y' by blowing up along smooth centers via some Bierstone-Milman type argument, and then*

(II) *factor the toroidal birational morphism $f' : X' \to Y'$ into (equivariant) smooth blowups and blowdowns by applying the toroidal version of the method of* ⌈*Morelli*1,2⌉ *(or* ⌈*Włodarczyk*1⌉*).*

The toroidalization conjecture and the strong factorization of toroidal birational morphisms would imply the strong factorization of general birational maps between nonsingular complete varieties.

This line of ideas came up in our conversation as a day-dreaming inspired by [De-Jong], only to find out later that an almost identical approach was already presented in [King1] and has been pursued by him in reality, who has (privately) announced the affirmative solution to the toroidalization conjecture in the case $\dim X = 3$. Actually our formulation above follows his presentation in [King1]. He has also read [Morelli1] carefully and his correspondence with Morelli himself was kindly communicated to us by Bierstone. We thank both professors for their generosity sharing their ideas with us and our indebtedness to them is both explicitly and implicitly clear as well as to the original papers [Morelli1,2] and [Włodarczyk1]. Another big inspiration for the factorization problem comes from the recent result of [Cutkosky1], which affirmatively solves the local factorization problem in dimension 3 using valuation theory. We thank Professor Cutkosky for kindly teaching us his method using valuation theory via preprints and private conversations. In response, we communicated to him our idea above for the global factorization, which turns out to be very similar to the idea of [Christensen] toward the local factorization problem:

(I) First "monomialize" the given local birational morphism via valuation theory ([Cutkosky2] uses the word "monomialization", which is nothing but "toroidalization" in the local case.), then

(II) factor the local monomial birational morphism (which is a toroidal birational morphism).

[Cutkosky2,3] achieves the local factorization in arbitrary dimension along this line of ideas, extending his method using valuation theory.[6]

We remark that [Reid3] gives factorization of toric birational maps into extremal divisorial contractions and flips by establishing the Minimal Model Program for toric varieties in arbitrary dimension. The Minimal Model Program in general, also known as the Mori Program, is only established in dimension 3 (cf. [Mori1,2, Kawamata1,2,3, Kollár, Reid1,2, Shokurov]). We also remark that recently a new algorithm called the Sarkisov Program has emerged (cf. [Sarkisov, Reid4]) to factor birational maps among uniruled varieties. Though it is only established in dimension 3 in general (cf. [Corti]), the toric case is rather straightforward in arbitrary dimension (cf. [Matsuki]). We do not know of a way to solve the classical factorization problem into smooth blowups and blowdowns using such factorizations as above.

Our organization, as being a note to [Morelli1,2], follows exactly the structure of the original paper [Morelli1,2] with one last section on the toroidal case added. The content of each section is outlined at the end of Section 1, where we explain the main ideas of Morelli.

Our hearty thanks go to Professor Oda for giving us invaluable suggestions at many critical points of the paper. We thank the referee for a very careful reading of the first draft of the paper and for providing us with meticulous and constructive comments.

1. Basic ideas. The purpose of this section is to present the basic ideas of the brilliant solution of [Morelli1,2] (see also [Włodarczyk1]) to the following conjecture of Miyake and

[6] After monomializing a birational map in (I), which is the most subtle and difficult part, [Cutkosky2] refers to the results of Morelli in (II). The first version of [Cutkosky3] factors the monomialized map in his own algorithm in (II) avoiding the use of results of Morelli, which were found to contain discrepancies at the time. The second version of [Cutkosky3], upon our communication, uses the strong factorization theorem of this paper by Morelli in (II) and hence provides the strong factorization theorem in the local case.

Oda (cf. [Oda1]). We follow the usual notation and terminology concerning the toric varieties X_Δ and their corresponding fans Δ, as presented in [Danilov, Fulton, Oda2].

CONJECTURE 1.1 (Weak and Strong Factorization of Toric Birational Maps by Miyake and Oda). *Every proper and equivariant birational map $f : X_\Delta \dashrightarrow X_{\Delta'}$ ("proper" in the sense of [Iitaka]) between two nonsingular toric varieties can be factored into a sequence of blowups and blowdowns with smooth centers which are the closures of orbits.*

If we allow the sequence to consist of blowups and blowdowns in any order, then the factorization is called weak.

If we insist on the sequence to consist only of blowups immediately followed by blowdowns, then the factorization is called strong.

In short, a toric birational map admits not only a weak factorization but also a strong factorization.

As the toric varieties X_Δ correspond to the fans Δ in $N_{\boldsymbol{Q}} = N \otimes \boldsymbol{Q}$, where N is the lattice of one-parameter subgroups of the torus, and blowups to the smooth star subdivisions of Δ, we can reformulate the above conjecture in the following purely combinatorial language:

CONJECTURE 1.2 (=Conjecture 1.1 in terms of Fans). *Let Δ and Δ' be two nonsingular fans in $N_{\boldsymbol{Q}}$ with the same support. Then there is a sequence of smooth star subdivisions and inverse operations called smooth star assemblings starting from Δ and ending with Δ'.*

If we allow the sequence to consist of smooth star subdivisions and smooth star assemblings in any order, then the factorization is called weak.

If we insist on the sequence to consist only of smooth star subdivisions immediately followed by smooth star assemblings, then the factorization is called strong.

In order to understand Morelli's strategy toward the solution of Conjecture 1.1, we look at the following simple example.

EXAMPLE 1.3. We take two fans Δ and Δ' to consist of the maximal cones in $N_{\boldsymbol{Q}} \cong \boldsymbol{Z}^3 \otimes \boldsymbol{Q}$

$$\Delta = \{\gamma_{123}, \gamma_{124}\}$$
$$\Delta' = \{\gamma_{134}, \gamma_{234}\}$$

where $\gamma_{ijk} = \langle v_i, v_j, v_k \rangle$ with

$$v_1 = (1,0,0)\,, \quad v_2 = (0,1,0)\,, \quad v_3 = (0,0,1)\,, \quad v_4 = (1,1,-1)\,.$$

Then we observe that by taking the common refinement $\tilde{\Delta}$ of Δ and Δ', subdivided by the vector

$$v_1 + v_2 = v_3 + v_4\,,$$

the transformation from Δ to Δ' can be factored into a smooth star subdivision immediately followed by a smooth star assembling

$$\Delta \leftarrow \tilde{\Delta} \rightarrow \Delta'$$

as asserted by Conjecture 1.2.

Morelli's great idea is to incorporate all the information of this factorization into a "cobordism" Σ, a fan in the vector space $N_{\boldsymbol{Q}}^{+} = N_{\boldsymbol{Q}} \oplus \boldsymbol{Q}$ of one dimension higher, with its lower face $\partial_{-}\Sigma$ being Δ and its upper face $\partial_{+}\Sigma$ being Δ'. Namely, we take

$$\Sigma = \{\sigma \text{ and its proper faces}\} \subset N_{\boldsymbol{Q}}^{+} = N_{\boldsymbol{Q}} \otimes \boldsymbol{Q}$$

where $\sigma = \langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ with

$$\rho_1 = (v_1, 0)\,, \quad \rho_2 = (v_2, 0)\,, \quad \rho_3 = (v_3, 1)\,, \quad \rho_4(v_4, 1)$$

and where the projection is denoted by

$$\pi : N_{\boldsymbol{Q}}^{+} = N_{\boldsymbol{Q}} \oplus \boldsymbol{Q} \to N_{\boldsymbol{Q}}\,.$$

The lower face

$$\partial_{-}\Sigma = \{\langle \rho_1, \rho_2, \rho_3 \rangle, \langle \rho_1, \rho_2, \rho_4 \rangle\}$$

maps isomorphically onto Δ by the projection π and so does the upper face

$$\partial_{+}\Sigma = \{\langle \rho_1, \rho_3, \rho_4 \rangle, \langle \rho_2, \rho_3, \rho_4 \rangle\}$$

isomorphically onto Δ'.

Moreover, since σ does not map isomorphically onto its image by π, i.e., since σ is π-dependent, we have the linear relation, unique up to scalar multiple, among the primitive vectors v_i of the projections of the generators ρ_i of σ

$$v_1 + v_2 - v_3 - v_4 = 0\,.$$

From this linear relation, we can read off the point

$$v_1 + v_2 = v_3 + v_4$$

by which we have to subdivide Δ and Δ' to reach the common refinement $\tilde{\Delta}$.

In short, we can realize the factorization from constructing the cobordism.

We can summarize Morelli's idea, demonstrated by the above example, in the following.

BASIC IDEA 1.4 (Morelli's Idea for Factorization). *Let Δ and Δ' be two nonsingular fans in $N_{\boldsymbol{Q}}$ with the same support. Then we can realize the* (*weak*) *factorization by constructing a cobordism Σ, a simplicial fan consisting of π-strongly convex cones* (*See Section* 3 *for the precise definition.*) *in $N_{\boldsymbol{Q}}^{+} = N_{\boldsymbol{Q}} \oplus \boldsymbol{Q}$ such that*

(1.4.1) *the lower face $\partial_{-}\Sigma$ and upepr face $\partial_{+}\Sigma$ of Σ map isomorphically onto Δ and Δ' by the projection π*

$$\pi : \partial_{-}\Sigma \xrightarrow{\sim} \Delta\,, \quad \pi : \partial_{+}\Sigma \xrightarrow{\sim} \Delta'\,,$$

(1.4.2) *Σ is π-nonsingular* (*See Section* 3 *for the precise definition*),

(1.4.3) *Σ is collapsible* (*See Section* 4 *for the precise definition.*).

In fact, let σ be a minimal simplex in Σ which is π-dependent. (We call such simplex σ a circuit.) If σ is generated by the extremal rays ρ_i

$$\sigma = \langle \rho_1, \rho_2, \dots, \rho_k \rangle\,,$$

then we have the linear relation among the primitive vectors v_i of the projections of the generators ρ_i

$$\sum_{i=1}^{k} r_i v_i = 0 .$$

Now by the π-nonsingularity of Σ and minimality of σ, it follows that we may assume that all the coefficients r_i are either $+1$ or -1 after rescaling (cf. Theorem 3.2). Thus after renumbering the v_i, we may assume that the linear relation is given by

$$v_1 + v_2 + \cdots + v_l - v_{l+1} - \cdots - v_k = 0 .$$

We then observe that by taking the common refinement subdivided by the vector

$$v_1 + v_2 + \cdots + v_l = v_{l+1} + \cdots + v_k$$

the transformation from $\partial_-\sigma$ to $\partial_+\sigma$ can be factored into a smooth star subdivision of $\partial_-\sigma$ immediately followed by a smooth star assembling into $\partial_+\sigma$.

Or more generally, we obtain the factorization between the lower face $\partial_-\overline{\mathrm{Star}(\sigma)}$ and upper face $\partial_+\overline{\mathrm{Star}(\sigma)}$ of the closed star $\overline{\mathrm{Star}(\sigma)}$ of σ, where

$$\overline{\mathrm{Star}(\sigma)} = \{\zeta \in \Sigma ; \zeta \subset \eta \supset \sigma \text{ for some cone } \eta \in \Sigma\} .$$

The π-nonsingularity also guarantees that the π-projections of all the lower and upper faces and the common refinement obtained through the star subdivisions are nonsingular and the star subdivisions are smooth.

This achieves the (weak) factorization for $\overline{\mathrm{Star}(\sigma)}$ for one circuit σ of Σ. In order to achieve the (weak) factorization for the entire $\Sigma = \bigcup \overline{\mathrm{Star}(\sigma)}$, where the union is taken over all the circuits σ in Σ, we have to coordinate the way we take the (weak) factorizations for all the circuits. This is done by requiring the collapsibility of the cobordism Σ.

In Section 2, we construct a cobordism Σ between two simplicial fans Δ and Δ' with the same support. The simplicial cobordism constructed in this section only satisfies the condition (1.4.1) above of Morelli's idea. The construction is done via a slick use of Sumihiro's equivariant completion theorem [Sumihiro1,2].

In Section 3, we discuss the (weak) factorization between the lower face $\partial_-\overline{\mathrm{Star}(\sigma)}$ and upper face $\partial_+\overline{\mathrm{Star}(\sigma)}$, which we call the bistellar operation, more in detail assuming the π-nonsingularity.

In Section 4, we achieve the condition (1.4.3), the collapsibility for the simplicial cobordism Σ. By star subdividing Σ further to obtain $\tilde{\Sigma}$, we can make $\tilde{\Sigma}$ projective via the use of toric version of Moishezon's theorem. Projectivity implies collapsibility, achieving a collapsible and simplicial cobordism $\tilde{\Sigma}$ between $\partial_-\tilde{\Sigma}$ and $\partial_+\tilde{\Sigma}$. We can explicitly construct a collapsible and simplicial cobordism Σ_1 (resp. Σ_2) between Δ and $\partial_-\tilde{\Sigma}$ (resp. between $\partial_+\tilde{\Sigma}$ and Δ'), as the latter is obtained through star subdivisions (resp. star assemblings) from the former. Now we only have to take the composite $\Sigma_1 \circ \tilde{\Sigma} \circ \Sigma_2$ to be the one providing a new collapsible and simplicial cobordism between Δ and Δ'.

Section 5 is the most sutble and difficult part of the proof, achieving the condition (1.4.2), i.e., the π-nonsingularity of the cobordism Σ. We introduce the invariant "π-multiplicity profile" of a simplicial cobordism, which measures how far Σ is from being π-nonsingular, and observe that it strictly drops after some appropriate star subdivisions. By the descending chain condition on the set of the π-multiplicity profiles, we acquire the π-nonsingularity after finitely many star subdivisions.

The arguments in Sections 2 through 5 put together provide the weak factorization, solving the weak form of Conjecture 1.1 affirmatively. The results are summarized in Section 6.

We should emphasize that the weak form of Conjecture 1.1 is also solved by [Włodarczyk1] along a similar line of ideas but in a more combinatorial language.

In Section 7, we finally show the strong factorization, based upon the weak factorization achieved in the previous sections. We obtain $\tilde{\Sigma}$ by further star subdividing the cobordism Σ corresponding to the weak factorization between Δ and Δ', without affecting the lower face of Σ but possibly smooth star subdividing the upper face of Σ, so that the bistellar operations of the circuits in $\tilde{\Sigma}$ only provide blowups starting from the lower face. We achieve the strong factorization

$$\Delta \cong \partial_{-}\Sigma = \partial_{-}\tilde{\Sigma} \leftarrow \partial_{+}\tilde{\Sigma} \rightarrow \partial_{+}\Sigma \cong \Delta',$$

the first left arrow representing a sequence of smooth star subdivisions and the second right arrow representing a sequence of smooth star assemblings immediately after.

Section 7 discusses the generalization to the toroidal case. All the arguments above for the toric case can be lifted immediately to the toroidal case, except for the existence of a cobordism and π-collapsibility, where we used the global results like Sumihiro's and Moishezon's theorems only valid in the toric case. We circumvent these difficulties by a trick embedding a toroidal conical complex into a usual toric fan after barycentric star subdivisions.

2. Cobordism. We follow the usual notation and terminology concerning the toric varieties X_Δ and their corresponding fans Δ, as presented in [Danilov, Fulton, Oda2].

We recall the notion of star subdivisions of a fan Δ, the key operation repeatedly used in this note.

DEFINITION 2.1. Let $\tau \in \Delta$ be a cone in a fan Δ. Let ρ be a ray passing in the relative interior of τ. (Note that such $\tau \in \Delta$ containing ρ in its relative interior is uniquely determined once the ray ρ is fixed.) Then we define the star subdivision $\rho \cdot \Delta$ of Δ with respect to ρ to be

$$\rho \cdot \Delta = (\Delta - \mathrm{Star}(\tau)) \cup \{\rho + \tau' + \nu;\ \tau' \text{ a proper face of } \tau,\ \nu \in \mathrm{link}_\Delta(\tau)\}$$

where

$$\begin{aligned}
\mathrm{Star}(\tau) &= \{\zeta \in \Delta;\ \zeta \supset \tau\} \\
\overline{\mathrm{Star}(\tau)} &= \{\zeta \in \Delta;\ \zeta \subset \eta \text{ for some } \eta \in \mathrm{Star}(\tau)\} \\
\mathrm{link}_\Delta(\tau) &= \{\zeta \in \overline{\mathrm{Star}(\tau)};\ \zeta \cap \tau = \emptyset\}
\end{aligned}$$

We call the inverse of a star subdivision a star assembling.

When $\tau = \langle \rho_1, \dots, \rho_l \rangle$ is generated by extremal rays ρ_i with the primitive vectors $v_i = n(\rho_i)$ and the ray ρ is generated by the vector $v_1 + \cdots + v_l$, the star subdivision is called the barycentric star subdivision with respect to τ.

When Δ is nonsingular, the barycentric star subdivision with respect to a face τ is called a smooth star subdivision and its inverse a smooth star assembling.

The notion of a cobordism as defined below sits in the center of Morelli's idea.

DEFINITION 2.2. Let Δ and Δ' be two fans in $N_{\boldsymbol{Q}} = N \otimes \boldsymbol{Q}$ with the same support, where N is the lattice of one-parameter subgroups of the torus. A cobordism Σ is a fan in $N_{\boldsymbol{Q}}^{+} = (N \oplus \boldsymbol{Z}) \otimes \boldsymbol{Q} = N_{\boldsymbol{Q}} \oplus \boldsymbol{Q}$ equipped with the natural projection

$$\pi : N_{\boldsymbol{Q}}^{+} = N_{\boldsymbol{Q}} \oplus \boldsymbol{Q} \to N_{\boldsymbol{Q}}$$

such that

(2.2.1) any cone $\tau \in \Sigma$ is π-strongly convex, i.e.,

$$x, y \in \tau, \pi(x) = -\pi(y) \Rightarrow x = y = 0,$$

(2.2.2) the projection π gives an isomorphism btween $\partial_{-}\Sigma$ and Δ (resp. $\partial_{+}\Sigma$ and Δ') as linear complexes, i.e., there is a one-to-one correspondence between the cones σ_{-} of $\partial_{-}\Sigma$ (resp. σ'_{+} of $\partial_{+}\Sigma$) and the cones σ of Δ (resp. σ' of Δ') such that $\pi : \sigma_{-} \to \sigma$ (resp. $\pi : \sigma'_{+} \to \sigma'$) is a linear isomorphism for each σ_{-} (resp. σ'_{+}) and its corresponding σ (resp. σ'). (Note that we do NOT require the map of lattices $\pi : (N \oplus \boldsymbol{Z}) \cap \sigma_{-} \to N \cap \sigma$ (resp. $\pi : (N \oplus \boldsymbol{Z}) \cap \sigma'_{+} \to N \cap \sigma'$) to be an isomorphism.) We denote this isomorphism by

$$\pi : \partial_{-}\Sigma \xrightarrow{\sim} \Delta \ (\text{resp. } \pi : \partial_{+}\Sigma \xrightarrow{\sim} \Delta')$$

where

$$\begin{aligned} \partial_{-}\Sigma = \{\tau \in \Sigma;\ & (x, y - \varepsilon) \notin \operatorname{Supp}(\Sigma) \\ & \text{for any } (x, y) \in \tau \text{ with } x \in N_{\boldsymbol{Q}},\ y \in \boldsymbol{Q} \text{ and any sufficiently small } \varepsilon > 0\} \\ (\text{resp. } \partial_{+}\Sigma = \{\tau \in \Sigma;\ & (x, y + \varepsilon) \notin \operatorname{Supp}(\Sigma) \\ & \text{for any } (x, y) \in \tau \text{ with } x \in N_{\boldsymbol{Q}},\ y \in \boldsymbol{Q} \text{ and any sufficiently small } \varepsilon > 0\}) \end{aligned}$$

(2.2.3) the support $\operatorname{Supp}(\Sigma)$ of Σ lies between the lower face $\partial_{-}\Sigma$ and the upper face $\partial_{+}\Sigma$, i.e.,

$$\begin{aligned} \operatorname{Supp}(\Sigma) = \{(x, y) \in N_{\boldsymbol{Q}}^{+};\ & x \in \operatorname{Supp}(\Delta) = \operatorname{Supp}(\Delta') \text{ and } y_{-}^{x} \le y \le y_{+}^{x} \\ & \text{where } (x, y_{-}^{x}) \in \operatorname{Supp}(\partial_{-}\Sigma) \text{ and } (x, y_{+}^{x}) \in \operatorname{Supp}(\partial_{+}\Sigma)\}. \end{aligned}$$

We remark that actually we only need the condition (2.2.2) for the definition of a cobordism and that the conditions (2.2.1) and (2.2.3) follow as the consequences of (2.2.2). We put all of these conditions as parts of the definition above to clarify its basic properties.

THEOREM 2.3. *Let Δ and Δ' be two simplicial fans in $N_{\boldsymbol{Q}} = N \otimes \boldsymbol{Q}$ with the same support. Then there exists a cobordism Σ between Δ and Δ'. We may also require Σ to be simplicial.*

PROOF. First we embed Δ "at the level -1" into $N_{\boldsymbol{Q}}^{+}$ so that the embedding Δ_{-} maps isomorphically back onto Δ by the projection π. Namely, we take the fan Δ_{-} in $N_{\boldsymbol{Q}}^{+}$ consisting of the cones σ_{-} of the form

$$\sigma_{-} = \langle (v_1, -1), \cdots, (v_k, -1) \rangle ,$$

where the corresponding cone $\sigma = \langle \rho_1, \ldots, \rho_k \rangle \in \Delta$ is generated by the extremal rays ρ_i with the primitive vectors $v_i = n(\rho_i)$. Similarly we embed Δ' "at the level $+1$" into $N_{\boldsymbol{Q}}^{+}$ so that the embedding Δ'_{+} maps isomorphically back onto Δ' by the projection π.

We take Γ to be the fan in $N_{\boldsymbol{Q}}^{+}$ consisting of the cones in Δ_{-} and Δ'_{+} and the cones ζ of the form

$$\zeta = \langle (v, -1), (v, +1) \rangle ,$$

where the v vary among all the primitive vectors for the extremal rays ρ_v such that ρ_v is a generator for some $\sigma \in \Delta$ and some $\sigma' \in \Delta'$ simultaneously.

Now by Sumihiro's equivariant completion theorem [Sumihiro1,2], there exists a fan Σ° with $\mathrm{Supp}(\Sigma^{\circ}) = N_{\boldsymbol{Q}}^{+}$ and containing Γ as a subfan.

We only have to take Σ to be

$$\Sigma = \{\tau \in \Sigma^{\circ}; \mathrm{Supp}(\tau) \subset S\}$$

where the set S is described as

$$\begin{aligned} S = \{(x, y) \in N_{\boldsymbol{Q}}^{+}; x \in \mathrm{Supp}(\Delta) = \mathrm{Supp}(\Delta'), y_{-}^{x} \le y \le y_{+}^{x} \\ \text{with } (x, y_{-}^{x}) \in \Delta_{-}, (x, y_{+}^{x}) \in \Delta'_{+}\} . \end{aligned}$$

The cobordism Σ constructed as above may not be simplicial. We take all the cones in Σ which are not simplicial, and give them the partial order according to the inclusion relation. We take a succession of barycentric star subdivisions with respect to these cones in the order compatible with the partial order, starting with the maximal ones. The resulting fan $\tilde{\Sigma}$ is simplicial with the property

$$\pi : \partial_{-}\tilde{\Sigma} = \partial_{-}\Sigma \xrightarrow{\sim} \Delta$$
$$\pi : \partial_{+}\tilde{\Sigma} = \partial_{+}\Sigma \xrightarrow{\sim} \Delta' ,$$

providing a simplicial cobordism between Δ and Δ'. (We also refer the reader to [Oda-Park, Park] for a more systematic treatment.)

3. Circuits and bistellar operations. In this section, we discuss how to read off the information on the factorization from the circuits of a π-nonsingular cobordism.

DEFINITION 3.1. Let Σ be a simplicial fan in $(N \oplus Z) \otimes \boldsymbol{Q} = N_{\boldsymbol{Q}}^{+}$ with the natural projection $\pi : N_{\boldsymbol{Q}}^{+} \to N_{\boldsymbol{Q}}$. Assume that all the cones in Σ are π-strictly convex.

A cone $\sigma \in \Sigma$ is π-indepenent if $\pi : \sigma \to \pi(\sigma)$ is an isomorphism. Otherwise σ is π-dependent.

A cone $\sigma \in \Sigma$ is called a circuit if it is minimal among the π-dependent cones, i.e., if σ is π-dependent and any proper face of σ is π-independent.

A cone $\sigma \in \Sigma$ is π-nonsingular if the projection $\pi(\tau)$ of each π-independent face $\tau \subset \sigma$ is nonsingular as a cone in $N_{\boldsymbol{Q}}$ with respect to the lattice N. We say that the fan Σ is π-nonsingular if all the cones in Σ are π-nonsingular.

The following theorem describes the transformation, which we call the bistellar operation, from the lower face $\partial_-\sigma$ to the upper face $\partial_+\sigma$ of a circuit σ of a simplicial and π-nonsingular cobordism Σ. (More generally the theorem describes the transformation from the lower face $\partial_-\overline{\mathrm{Star}(\sigma)}$ to the upper face $\partial_+\overline{\mathrm{Star}(\sigma)}$ of the closed star of a circuit σ.) It turns out that the bistellar operation corresponds to a smooth blowup immediately followed by a smooth blowdown.

THEOREM 3.2. *Let Σ be a simplicial and π-nonsingular coborbism in $N_{\boldsymbol{Q}}^+$. Let $\sigma = \langle \rho_1, \dots, \rho_k \rangle \in \Sigma$ be a circuit generated by the extremal rays ρ_i. From each extremal ray ρ_i we take the vector of the form $(v_i, w_i) \in N_{\boldsymbol{Q}}^+ = N_{\boldsymbol{Q}} \oplus \boldsymbol{Q}$ where $v_i = n(\pi(\rho_i))$ is the primitive vector of the projection $\pi(\rho_i)$.*

(3.2.1) *There is a unique linear relation among the v_i (up to renumbering) of the form*

$$\sum r_\alpha v_\alpha = v_1 + \cdots + v_l - v_{l+1} - \cdots - v_k = 0 \textit{ for some } 0 \le l \le k$$

with

$$\sum r_\alpha w_\alpha = w_1 + \cdots + w_l - w_{l+1} - \cdots - w_k > 0 .$$

(3.2.2) *All the maximal faces γ_i (resp. γ_j) of $\partial_-\sigma$ (resp. $\partial_+\sigma$) are of the form*

$$\gamma_i = \langle \rho_1, \dots, \check{\rho}_i, \dots, \rho_l, \rho_{l+1}, \dots, \rho_k \rangle \quad 1 \le i \le l$$

$$(\textit{resp. } \gamma_j = \langle \rho_1, \dots, \rho_l, \rho_{l+1}, \dots, \check{\rho}_j, \dots, \rho_k \rangle \quad l+1 \le j \le k) .$$

(3.2.3) *Let l_σ be the extremal ray in $N_{\boldsymbol{Q}}$ generated by the vector*

$$v_1 + \cdots + v_l = v_{l+1} + \cdots + v_k .$$

The smooth star subdivision of $\pi(\partial_-\sigma)$ with respect to l_σ coincides with the smooth star subdivision of $\pi(\partial_+\sigma)$ with respect to l_σ, whose maximal faces are of the form

$$\langle \pi(\gamma_{ij}), l_\sigma \rangle = \langle \pi(\rho_1), \dots, \check{\pi(\rho_i)}, \dots, \pi(\rho_l), \pi(\rho_{l+1}), \dots, \check{\pi(\rho_j)}, \dots, \pi(\rho_k), l_\sigma \rangle .$$

Thus the transformation from $\pi(\partial_-\sigma)$ to $\pi(\partial_+\sigma)$ is a smooth star subdivision followed immediately after by a smooth star assembling. We call the transformation a bistellar operation.

Similarly, the transformation from $\pi(\partial_-\overline{\mathrm{Star}(\sigma)})$ to $\pi(\partial_+\overline{\mathrm{Star}(\sigma)})$ is a smooth star subdivision followed immediately after by a smooth star assembling.

PROOF. (3.2.1) Since σ is a circuit, it is π-dependent and minimal by definition. Hence we have a linear relation

$$\sum r_i v_i = 0 \text{ with } r_i \neq 0 \text{ for all } i .$$

Since σ is simplicial, the ρ_i are linearly independent in $N_{\boldsymbol{Q}}^+$ and hence $\sum r_i w_i \neq 0$. We choose the signs of the r_i so that $\sum r_i w_i > 0$. We only have to prove $|r_1| = |r_2| = \cdots = |r_k|$.

Indeed, since σ is π-nonsingular, we have

$$1 = |\det(\check{v}_1, v_2, \dots, v_k)| = |\det(v_1, \dots, \check{v}_i, \dots, v_k)|$$
$$= \left|\det\left(\frac{1}{r_1}\left(\sum_{\alpha\neq 1} r_\alpha v_\alpha\right), v_2, \dots, \check{v}_i, \dots, v_k\right)\right| = \left|\frac{r_i}{r_1}\det(\check{v}_1, v_2, \dots, v_k)\right|,$$

which implies $|r_1| = |r_i|$ for all i.

(3.2.2) Note that since σ is a circuit, any maximal face γ of σ belongs either to $\partial_-\sigma$ or to $\partial_+\sigma$, exclusively.

Suppose $\gamma_i = \langle \rho_1, \dots, \check{\rho}_i, \dots, \rho_k\rangle \in \partial_-\sigma$. Then since σ is a circuit, for any point

$$p = \sum_{\alpha\neq i} c_\alpha(v_\alpha, w_\alpha) \in \mathrm{RelInt}(\gamma_i) \text{ with } c_\alpha > 0,$$

we have

$$p + (0, \varepsilon) \in \sigma \text{ for sufficiently small } \varepsilon > 0.$$

By setting $\varepsilon = t_\varepsilon \cdot \sum r_\alpha w_\alpha$ for $t_\varepsilon > 0$, we obtain

$$p + (0, \varepsilon) = \sum_{\alpha\neq i}(c_\alpha + t_\varepsilon r_\alpha)(v_\alpha, w_\alpha) + t_\varepsilon r_i(v_i, w_i) \in \sigma,$$

which implies $c_\alpha + t_\varepsilon r_\alpha > 0$ for $\alpha \neq i$ and $t_\varepsilon r_i > 0$. Therefore, we have $r_i > 0$. Similarly, if $\gamma_j = \langle \rho_1, \dots, \check{\rho}_j, \dots, \rho_k\rangle \in \partial_+\sigma$, then we have $r_j < 0$. This proves the assertion (3.2.2).

The assertion (3.2.3) follows immediately from (3.2.1) and (3.2.2).

The assertion about the transformation from $\pi(\partial_-\overline{\mathrm{Star}(\sigma)})$ to $\pi(\partial_+\overline{\mathrm{Star}(\sigma)})$ is an easy consequence of the description of the transformation from $\pi(\partial_-\sigma)$ to $\pi(\partial_+\sigma)$.

This completes the proof of Theorem 3.2.

4. Collapsibility.

Let Σ be a simplicial cobordism between simplicial fans Δ and Δ'. Noting that

$$\Sigma = \bigcup_\sigma \overline{\mathrm{Star}(\sigma)} \cup \partial_-\Sigma$$

where the union is taken over the circuits σ, we may try to factorize the transformation from Δ to Δ' into smooth star subdivisions and smooth star assemblings by replacing $\partial_-\overline{\mathrm{Star}(\sigma)}$ with $\partial_+\overline{\mathrm{Star}(\sigma)}$, if Σ is π-nonsingular. If we think of the cobordism built up out of "bubbles" $\overline{\mathrm{Star}(\sigma)}$, this process might be considered as a succession of "collapsing" these bubbles. The following simple example shows that this succession of collapsing, which should correspond to the factorization into smooth star subdivisions and smooth star assemblings, is not always possible, unless we can arrange the way we break these bubbles in a certain order. This possibility for the certain nice arrangement is what we call "collapsibility" in this section.

EXAMPLE 4.1. We take two sets of vectors in $N_Q = Z^2 \otimes Q$

$$\{v_1 = (1, 0), v_2 = (0, 1), v_3 = (-1, 0), v_4 = (0, -1)\}$$
$$\{v_1' = (1, 1), v_2' = (-1, 1), v_3' = (-1, -1), v_4' = (1, -1)\}$$

and fans Δ and Δ' whose maximal cones consist of

$$\Delta \ni \sigma_{12} = \langle v_1, v_2 \rangle\,, \quad \sigma_{23} = \langle v_2, v_3 \rangle\,, \quad \sigma_{34} = \langle v_3, v_4 \rangle\,, \quad \sigma_{41} = \langle v_4, v_1 \rangle$$
$$\Delta' \ni \sigma'_{12} = \langle v'_1, v'_2 \rangle\,, \quad \sigma'_{23} = \langle v'_2, v'_3 \rangle\,, \quad \sigma'_{34} = \langle v'_3, v'_4 \rangle\,, \quad \sigma'_{41} = \langle v'_4, v'_1 \rangle\,.$$

If we take the simplicial fan Σ in N_Q^+ whose maximal cones consist of

$$\begin{aligned}
\sigma_{124'} &= \langle (v_1, 0), (v_2, 0), (v'_4, 1) \rangle \\
\sigma_{231'} &= \langle (v_2, 0), (v_3, 0), (v'_1, 1) \rangle \\
\sigma_{342'} &= \langle (v_3, 0), (v_4, 0), (v'_2, 1) \rangle \\
\sigma_{413'} &= \langle (v_4, 0), (v_1, 0), (v'_3, 1) \rangle \\
\sigma_{4'1'2} &= \langle (v'_4, 1), (v'_1, 1), (v_2, 0) \rangle \\
\sigma_{1'2'3} &= \langle (v'_1, 1), (v'_2, 1), (v_3, 0) \rangle \\
\sigma_{2'3'4} &= \langle (v'_2, 1), (v'_3, 1), (v_4, 0) \rangle \\
\sigma_{3'4'1} &= \langle (v'_3, 1), (v'_4, 1), (v_1, 0) \rangle\,,
\end{aligned}$$

then Σ is a simplicial π-nonsingular cobordism between Δ and Δ'.

Observe, however, that we cannot "collapse" any one of the maximal cones $\sigma_{ijk'}$ to replace $\partial_-\sigma_{ijk'}$ with $\partial_+\sigma_{ijk'}$. In fact, the circuit graph attached to Σ as defined below is a directed cycle consisting of eight vertices

$$\sigma_{124'} \to \sigma_{4'1'2} \to \sigma_{231'} \to \sigma_{1'2'3} \to \sigma_{342'} \to \sigma_{2'3'4} \to \sigma_{413'} \to \sigma_{3'4'1} \to \sigma_{124'}\,.$$

DEFINITION 4.2. Let Σ be a simplicial cobordism in N_Q^+. We define a directed graph, which we call the circuit graph of Σ as follows: The vertices of the circuit graph consist of the circuits σ of Σ. We draw an edge from σ to σ' if there is a point $p \in \partial_+\overline{\mathrm{Star}(\sigma)} \cap \partial_-\overline{\mathrm{Star}(\sigma')}$ such that

$$p - (0, \varepsilon) \in \overline{\mathrm{Star}(\sigma)}\,, \quad p + (0, \varepsilon) \in \overline{\mathrm{Star}(\sigma')} \text{ for sufficiently small } \varepsilon > 0\,.$$

We say Σ is collapsible if the circuit graph contains no directed cycle. When Σ is collapsible, the circuit graph determines a partial order among the circuits: $\sigma \leq \sigma'$ if there is an edge $\sigma \to \sigma'$.

THEOREM 4.3. *Let Δ and Δ' be two simplicial fans in N_Q with the same support. Then there exists a simplicial and collapsible cobordism Σ in N_Q^+ between Δ and Δ'.*

PROOF. The proof consists of several steps. The main idea of Morelli's is to reduce the collapsibility to the projectivity.

Step 1. Show that the projectivity induces the collapsibility.

PROPOSITION 4.4. *Let Σ be a simplicial cobordism in N_Q^+ and assume that Σ is a (part of a) projective fan. Then Σ is collapsible.*

PROOF. Since Σ is a part of a projective fan (i.e. a part of a fan Σ' whose corresponding toric variety $X_{\Sigma'}$ is projective), there exists a function $h : \mathrm{Supp}(\Sigma) \to \boldsymbol{Q}$ which is piecewise linear with respect to the fan Σ and which is strictly convex, i.e., we have

$$\frac{1}{2}\{h(v)+h(u)\} \geq h\left(\frac{1}{2}\{v+u\}\right)$$

whenever the line segment $\overline{vu}$ is in $\mathrm{Supp}(\Sigma)$ and the strict inequality holds whenever v and u are in two distinct maximal cones (cf. [Fulton, Oda2]).

Let σ and σ' be two circuits with a directed edge, i.e., there exists a point $p \in \partial_+\overline{\mathrm{Star}(\sigma)} \cap \partial_-\overline{\mathrm{Star}(\sigma')}$ such that

$$p-(0,\varepsilon) \in \overline{\mathrm{Star}(\sigma)}, \quad p+(0,\varepsilon) \in \overline{\mathrm{Star}(\sigma')} \text{ for sufficiently small } \varepsilon > 0 .$$

Take a maximal π-dependent cone $p \in \eta \supset \sigma$ (resp. $p \in \eta' \supset \sigma'$) of $\overline{\mathrm{Star}(\sigma)}$ (resp. $\overline{\mathrm{Star}(\sigma')}$) such that $p-(0,\varepsilon) \in \eta$ (resp. $p+(0,\varepsilon) \in \eta'$).

Take also linear functions h_η, $h_{\eta'}$, h_σ, $h_{\sigma'}$ which coincide with $h\big|_\eta$, $h\big|_{\eta'}$, $h\big|_\sigma$, $h\big|_{\sigma'}$, respectively.

Then by the strict convexity of the function h, setting the coordinates of $p=(x,y)$ we have $\{h(x,y+\varepsilon)+h(x,y-\varepsilon)\}/2 > h(x,y)$ or equivalently $h_{\eta'}(0,1) > h_\eta(0,1)$, and hence $h_{\sigma'}(0,1) > h_\sigma(0,1)$. (Note that $(0,1) \in \mathrm{span}_Q(\sigma)$ for any π-dependent cone σ.)

If $\sigma_1, \ldots, \sigma_l$ are circuits determining a directed path in the circuit graph of Σ, then the above observation shows $h_{\sigma_1}(0,1) < \cdots < h_{\sigma_l}(0,1)$. Thus the path cannot be a cycle. Therefore, Σ is collapsible.

Step 2. Show the toric version of Moishezon's theorem.

THEOREM 4.5. *Let Σ be a fan in N_Q^+. Then there exists a fan $\tilde{\Sigma}$ obtained from Σ by a sequence of star subdivisions such that $\tilde{\Sigma}$ is a (part of a) projective fan.*

PROOF. We may assume that $\mathrm{Supp}(\Sigma) = N_Q^+$ and that Σ is simplicial and nonsingular by applying some appropriate sequence of star subdivisions to the original Σ.

By the toric version of Chow's Lemma (see, e.g., [Oda2], § 2.3), we have a projective fan Σ' mapping to Σ, i.e., we have a projective toric variety $X_{\Sigma'}$ with an equivariant proper birational morphism onto X_Σ

$$g : X_{\Sigma'} \to X_\Sigma .$$

By the toric version of Hironaka's elimination of points of indeterminacy (cf. [DeConcini-Procesi].) we can take a fan $\tilde{\Sigma}$ obtained from Σ by a sequence of smooth star subdivisions such that there exists an equivariant proper birational morphism

$$f : X_{\tilde{\Sigma}} \to X_{\Sigma'} .$$

Since $g \circ f$ is projective as it is a sequence of smooth blowups and since g is separated, f is also projective. Now since Σ' is a projective fan, so is $\tilde{\Sigma}$.

Step 3. Composition of (collapsible) cobordisms.

Starting from a simplicial cobordism Σ between Δ and Δ' constructed as in Theorem 2.3 and then applying Step 2, we obtain a simplicial cobordism $\tilde{\Sigma}$ between $\partial_-\tilde{\Sigma}$ and $\partial_+\tilde{\Sigma}$, where $\tilde{\Sigma}$ is a (part of a) projective fan and hence collapsible and where $\partial_-\tilde{\Sigma}$ (resp. $\partial_+\tilde{\Sigma}$) is obtained from Δ (resp. Δ') by a sequence of star subdivisions. (Or equivalently Δ (resp.

Δ') is obtained from $\partial_-\tilde{\Sigma}$ (resp. $\partial_+\tilde{\Sigma}$) by a sequence of star assemblings.) We only have to construct a collapsible and simplicial cobordism Σ_Δ between Δ and $\partial_-\tilde{\Sigma}$ and another $\Sigma_{\Delta'}$ between $\partial_+\tilde{\Sigma}$ and Δ' so that we compose them together $\Sigma_\Delta \circ \tilde{\Sigma} \circ \Sigma_{\Delta'}$ to obtain a collpasible and simplicial cobordism between Δ and Δ'.

PROPOSITION-DEFINITION 4.6. *Let Σ_1 and Σ_2 be cobordisms in N_Q^+ such that*

(4.6.1) *$\Sigma_1 \cup \Sigma_2$ is again a fan in N_Q^+,*

(4.6.2) *$\Sigma_1 \cap \Sigma_2 = \partial_+\Sigma_1 \cap \partial_-\Sigma_2$,*

(4.6.3) *for any cone $\sigma \in \partial_+\Sigma_2$*

$$\pi(\sigma) \not\subset \partial\{\pi(\partial_+\Sigma_1 \cup \partial_+\Sigma_2)\} \quad \text{and} \quad \pi(\sigma) \subset \partial(\pi(\partial_+\Sigma_2)) \Rightarrow \sigma \in \partial_+\Sigma_1\,,$$

and for any cone $\sigma \in \partial_+\Sigma_1$

$$\pi(\sigma) \not\subset \partial\{\pi(\partial_-\Sigma_1 \cup \partial_-\Sigma_2)\} \quad \text{and} \quad \pi(\sigma) \subset \partial(\pi(\partial_-\Sigma_1)) \Rightarrow \sigma \in \partial_-\Sigma_2\,.$$

Then the union $\Sigma_1 \cup \Sigma_2$, which we call the composite of Σ_1 with Σ_2 and denote by $\Sigma_1 \circ \Sigma_2$, is a cobordism.

Moreover, if both Σ_1 and Σ_2 are simplicial and collapsible, then so is the composite $\Sigma_1 \circ \Sigma_2$.

PROOF. By the condition (4.6.1) the composite $\Sigma_1 \circ \Sigma_2$ is a fan. The conditions (4.6.2) and (4.6.3) guarantee $\pi : \partial_-(\Sigma_1 \circ \Sigma_2) \to N_Q$ and $\pi : \partial_+(\Sigma_1 \circ \Sigma_2) \to N_Q$ are isomorphisms of linear complexes onto their images. Thus $\Sigma_1 \circ \Sigma_2$ is a cobordism. The "Moreover" part of the assertion is also clear.

We note that in case $\partial_+\Sigma_1 = \partial_-\Sigma_2$ the condition (4.6.3) is automatically satisfied.

PROPOSITION 4.7. *Let $\tilde{\Delta}$ be a simplicial fan in N_Q obtained from another simplicial fan Δ in N_Q by a sequence of star subdivisions and star assemblings. Suppose Δ is embedded in N_Q*

$$s : \Delta \hookrightarrow N_Q^+$$

so that $\pi \circ s$ is the identity of the fan.

Then there exists a simplicial and collapsible cobordism Σ between Δ and $\tilde{\Delta}$ (resp. between $\tilde{\Delta}$ and Δ) such that $\partial_-\Sigma = s(\Delta)$ (resp. $\partial_+\Sigma = s(\Delta)$).

PROOF. We only have to prove the assertion when the sequence consists of a single star subdivision or a star assembling.

Suppose $\tilde{\Delta}$ is obtained from Δ by a star subdivision with respect to a ray ρ passing through the relative interior of a face $\tau \in \Delta$. Say that the ray ρ is generated by a primitive vector v_ρ. Then we only have to take by fixing some sufficiently large $y_\rho > 0$

$$\Sigma = s(\Delta) \cup \{\langle s(\zeta),\ (v_\rho, y_\rho)\rangle;\ \zeta \in \Delta,\ \zeta \subset \sigma \text{ for some } \sigma \in \Delta \text{ with } \sigma \ni \rho\}\,.$$

Suppose $\tilde{\Delta}$ is obtained from Δ by a star assembling, which is the inverse of a star subdivision with respect to a ray ρ passing through the relative interior of a face $\tau \in \tilde{\Delta}$. Let $\tau = \langle \rho_1, \dots, \rho_k \rangle$ be generated by extremal rays ρ_i with the primitive vectors $v_{\rho_i} = n(\rho_i)$. We construct $\Sigma_1, \dots, \Sigma_k$ with $s_i : \Delta \xrightarrow{\sim} \partial_+\Sigma_i$ and Σ as required inductively.

Fixing some sufficiently large $y_{\rho_1} > 0$, we take

$$\Sigma_1 = s(\Delta) \cup \{\langle s(\zeta),\ (v_{\rho_1}, y_{\rho_1})\rangle;\ \zeta \in \Delta,\ \zeta \subset \sigma \text{ for some } \sigma \in \Delta \text{ with } \sigma \ni \rho_1\}.$$

Obviously $\partial_+\Sigma_1$ is isomorphic to Δ via the projection π, and we set the inverse $s_1 : \Delta \xrightarrow{\sim} \partial_+\Sigma_1$.

Suppose we have already constructed $\Sigma_1, \dots, \Sigma_{i-1}$ with a sequence of positive numbers $0 < y_{\rho_1} < \cdots < y_{\rho_{i-1}}$ where each positive number is sufficiently larger than the previous one, and with the isomorphisms $s_1, \dots, s_{i-1}$ from Δ to $\partial_+\Sigma_1, \dots, \partial_+\Sigma_{i-1}$, respectively. Then by fixing some positive number y_{ρ_i} which is sufficiently larger than $y_{\rho_{i-1}}$, we take

$$\Sigma_i = \Sigma_{i-1} \cup \{\langle s_{i-1}(\zeta),\ (v_{\rho_i}, y_{\rho_i})\rangle;\ \zeta \in \Delta,\ \zeta \subset \sigma \text{ for some } \sigma \in \Delta \text{ with } \sigma \ni \rho_i\}.$$

Again clearly $\partial_+\Sigma_i$ is isomorphic to Δ via the projection π, and we set the inverse $s_i : \Delta \xrightarrow{\sim} \partial_+\Sigma_i$.

Thus we have constructed $\Sigma_1, \dots, \Sigma_k$. We only have to set

$$\tilde{\Sigma} = \Sigma_k \cup \langle s_k(\rho_1), \dots, s_k(\rho_k)\rangle \cup \langle s_k(\rho_1), \dots, s_k(\rho_k), s_k(\rho)\rangle.$$

This completes the proof of Proposition 4.7.

Thus we complete Step 3 and hence the proof of Theorem 4.5.

In Section 5, starting from a collapsible and simplicial cobordism between two nonsingular fans Δ and Δ' (which we constructed in this section), we try to construct another cobordism which is not only collapsible and simplicial but also π-nonsingular, by further star subdividing the original cobordism. It is worthwhile to note that the collapsibility is preserved under star subdivisions.

LEMMA 4.8. *Let Σ be a simplicial cobordism in N_Q^+, which is collapsible. Then any simplicial cobordism $\tilde{\Sigma}$ obtained from Σ by a star subdivision, with respect to a ray ρ, is again collapsible.*

PROOF. Note first that if Σ consists of the closed star of a single circuit, then $\rho \cdot \Sigma = \rho \cdot \overline{\mathrm{Star}(\sigma)}$ is easily seen to be collapsible.

In general, number the circuits $\sigma_1, \sigma_2, \dots, \sigma_m$ of Σ so that σ_i is minimal among σ_i, $\sigma_{i+1}, \dots, \sigma_m$ according to the order given by the circuit graph. Then setting

$$\Sigma = \bigcup_{i=1}^{m} \overline{\mathrm{Star}(\sigma_i)} \cup \partial_+\Sigma,$$

we have

$$\rho \cdot \Sigma = \bigcup_{i=1}^{m} \rho \cdot \overline{\mathrm{Star}(\sigma_i)} \cup \rho \cdot \partial_+\Sigma$$

and

$$\rho \cdot \Sigma = \{\rho \cdot \overline{\mathrm{Star}(\sigma_1)}\} \circ \cdots \circ \{\rho \cdot \overline{\mathrm{Star}(\sigma_m)}\} \circ \{\rho \cdot \partial_+\Sigma\}$$

is collapsible by the first observation and by Proposition-Definition 4.6.

5. π-Desingularization. The purpose of this section, which is technically the most subtle, is to show the following theorem of "π-desingularization".

THEOREM 5.1. *Let Σ be a simplicial cobordism in N_Q^+. Then there exists a simplicial cobordism $\tilde{\Sigma}$ obtained from Σ by a sequence of star subdivisions such that $\tilde{\Sigma}$ is π-nonsingular. Moreover, the sequence can be taken so that any π-independent and already π-nonsingular face of Σ remains unaffected during the process.*

Naively, just LIKE the case of the usual desingularization of toric fans, we would like to subdivide any π-independent face with π-multiplicity bigger than 1 so that its π-multiplicity drops. However, UNLIKE the case of the usual desingularization, we might introduce a new π-independent face of uncontrollably high π-multiplicity if we subdivide blindly, though we may succeed in decreasing the π-multiplicity of the π-independent face that we picked originally. This is where the difficulty lies! We outline Morelli's ingeneous strategy to subdivide carefully to avoid introducing new π-independent faces with high π-multiplicity and achieve π-desingularization. It consists of the following four steps:

Step 1: Introduce the invariant "π-multiplicity profile" π-m.p.(Σ) of a simplicial cobordism Σ, which measures how far Σ is from being π-nonsingular.

Step 2: Observe that the star subdivision $\eta' = \mathrm{Mid}(\tau, l_q) \cdot \eta$ of a simplex η by an interior point of a face τ does not increase the π-multiplicity profile, i.e.,

$$\pi\text{-m.p.}(\eta') \leq \pi\text{-m.p.}(\eta)$$

if

(i) τ is "codefinite" with respect to η, and

(ii) the interior point corresponds to the midray $\mathrm{Mid}(\tau, l_q)$, where the ray l_q is generated by a lattice point $q \in \mathrm{par}(\pi(\tau))$.

Moreover, if τ is contained in a maximal π-independent face γ of η with the maximum π-multiplicity h_η, i.e., if

$$\tau \subset \gamma \quad \text{and} \quad \pi\text{-mult}(\gamma) = h_\eta = \max\{\pi\text{-mult}(\zeta); \zeta \subset \eta\},$$

then the π-multiplicity profile strictly drops

$$\pi\text{-m.p.}(\eta') < \pi\text{-m.p.}(\eta).$$

Step 3: Let τ be a π-independent face in the closed star $\overline{\mathrm{Star}(\sigma)}$ of a circuit σ in Σ. Introduce the notion of the star subdivision by the negative or positive center point of σ. We can find Σ° such that

(i) Σ° is obtained by a succession of appropriate star subdivisions by negative or positive center points of circuits inside of σ,

(ii) the π-multiplicity profile does not increase, i.e.,

$$\pi\text{-m.p.}(\Sigma^\circ) \leq \pi\text{-m.p.}(\Sigma),$$

(iii) τ is a face of Σ° such that τ is codefinite with respect to every cone $\eta \in \Sigma^\circ$ containing τ.

Step 4: Combine Step 2 and Step 3 to find $\tilde{\Sigma}$ obtained from Σ by a succession of star subdivisions such that the π-multiplicity profile strictly drops

$$\pi\text{-m.p.}(\tilde{\Sigma}) < \pi\text{-m.p.}(\Sigma)\,.$$

As the set of the π-multiplicity profiles satisfies the descending chain condition, we reach a π-nonsingular cobordism after finitely many star subdivisions as required.

In fact, by Step 3 we can find a π-independent face τ of a maximal cone $\eta' \subset \Sigma^\circ$ such that

(i) π-m.p.(η') is maximum among the π-multiplicity profiles of all the maximal cones of Σ°,

(ii) τ is contained in a maximal π-independent face γ of η' with the maximum π-multiplicity π-mult$(\gamma) = h_{\eta'}$,

(iii) τ is codefinite with respect to η' and with respect to all the other maximal cones containing τ,

(iv) we can find a lattice point $q \in \operatorname{par}(\pi(\tau))$.

We only have to set $\tilde{\Sigma} = \operatorname{Mid}(\tau, l_q) \cdot \Sigma^\circ$ to observe by Step 2 that π-m.p.$(\tilde{\Sigma}) <$ π-m.p.(Σ).

This completes the process of π-desingularization.

Now we discuss the details of each step.

Step 1.

DEFINITION 5.2. Let γ be a simplicial cone in N_Q^+. If γ is π-independent, then we define the π-multiplicity of γ to be

$$\pi\text{-mult}(\gamma) = |\det(v_1, \dots, v_k)|\,,$$

where the $v_i = n(\pi(\rho_i))$ are the primitive vectors of the projections of the extremal rays ρ_i generating $\gamma = \langle \rho_1, \dots, \rho_k \rangle$. If γ is π-dependent, then we set π-mult$(\gamma) = 0$ by definition.

Let η be a simplicial and π-strictly convex cone in N_Q^+ with

$$\begin{aligned}
h_\eta &= \max\{\pi\text{-mult}(\gamma);\ \gamma \text{ is a } \pi\text{-independent face of } \eta\}\,,\\
k_\eta &= \dim\sigma \text{ where } \sigma \text{ is the unique circuit contained in } \eta\,,\\
r_\eta &= \text{the number of the maximal } \pi\text{-independent faces of } \eta\\
&\quad\ \text{having the maximum } \pi\text{-multiplicity } h_\eta\,.
\end{aligned}$$

We define the π-multiplicity profile π-m.p.(η) of η to be the ordered quadruple of numbers

$$\pi\text{-m.p.}(\eta) = (a_\eta, b_\eta, c_\eta, d_\eta)$$

where

$$a_\eta = h_\eta\,,$$
$$b_\eta = \begin{cases} 0 & \text{if } r_\eta \le 1 \\ 1 & \text{if } r_\eta > 1\,, \end{cases}$$
$$c_\eta = \begin{cases} 0 & \text{if } b_\eta = 0 \\ k_\eta & \text{if } b_\eta = 1\,, \end{cases}$$
$$d_\eta = \begin{cases} 0 & \text{if } c_\eta = 0 \\ r_\eta & \text{if } c_\eta > 0\,. \end{cases}$$

We order the set of the π-multiplicity profiles of all the simplicial and π-strictly convex cones in N_Q^+ lexicographically.

We define the π-multiplicity profile π-m.p.(Σ) of a simplicial cobordism Σ in N_Q^+ to be

$$\pi\text{-m.p.}(\Sigma) = [g_\Sigma; s]$$

where

$$g_\Sigma = \max\{\pi\text{-m.p.}(\eta);\ \eta \text{ is a maximal simplicial cone of } \Sigma\}$$

and where s is the number of the maximal simplicial cones of Σ having the maximum π-multiplicity profile g_Σ.

When a simplicial cobordism Σ consists of only one maximal simplicial and π-strictly convex cone η (and its faces), we understand as a convention

$$\pi\text{-m.p.}(\Sigma) = [\pi\text{-m.p.}(\eta); 1] = \pi\text{-m.p.}(\eta)\,.$$

The definition of the invariant π-multiplicity profile may look heuristic at this point. At the end of the section, we discuss how Morelli reached this definition after a couple of false trials in [Morelli1,2]. The behavior of the π-multiplicity profile under several kinds of star subdivisions will be the key in Step 3.

Step 2.

DEFINITION 5.3. Let η be a simplicial, π-dependent and π-strictly convex cone in N_Q^+. A π-independent face τ of η is said to be codefinite with respect to η if the set of generators of τ does not contain both positive and negative extremal rays ρ_i of η. That is to say, if $\sum r_i v_i = 0$ is the nontrivial linear relation for η among the primitive vectors $v_i = n(\pi(\rho_i))$, then the generators for τ contain only those extremal rays in the set $\{\rho_i; r_i < 0\}$ or in the set $\{\rho_i; r_i > 0\}$, exclusively.

NOTATION 5.4. Let τ be a cone in a simplicial cobordism Σ in N_Q^+ and l a ray in $\pi(\tau)$. Then we define the "midray" $\mathrm{Mid}(\tau, l)$ to be the ray generated by the middle point of the line segment $\tau \cap \pi^{-1}(n(l))$. (If $\tau \cap \pi^{-1}(n(l))$ consists of a point, then $\mathrm{Mid}(\tau, l)$ is the ray generated by that point.)

Let $\gamma = \langle \rho_1, \dots , \rho_k \rangle$ be a π-independent cone in N_Q^+ generated by the extremal rays ρ_i with the corresponding primitive generators $v_i = n(\pi(\rho_i)) \in N$. Then we define the set

$$\mathrm{par}(\pi(\gamma)) = \left\{ m \in N; m = \sum_i a_i v_i, 0 < a_i < 1 \right\} .$$

PROPOSITION 5.5. *Let τ be a π-independent face of a simplicial, π-dependent and π-strictly convex cone η in N_Q^+. Assume τ is codefinite with respect to η. Let $\eta' = \mathrm{Mid}(\tau, l_q) \cdot \eta$ be the star subdivision of η by the midray $\mathrm{Mid}(\tau, l_q)$ where the ray l_q is generated by a lattice point $q \in \mathrm{par}(\pi(\tau))$. Then the π-multiplicity profile does not increase under the star subdivision, i.e.,*

$$\pi\text{-m.p.}(\eta') \leq \pi\text{-m.p.}(\eta) .$$

Moreover, if τ is contained in a maximal codimension one face γ of η with

$$\pi\text{-mult}(\gamma) = h_\eta = \max\{\pi\text{-mult}(\zeta); \zeta \subset \eta\} ,$$

then the π-multiplicity strictly decreases, i.e.,

$$\pi\text{-m.p.}(\eta') < \pi\text{-m.p.}(\eta) .$$

PROOF. We claim first that all the NEW maximal π-independent faces γ' of η' have π-multiplicities strictly smaller than h_η, i.e.,

$$\pi\text{-mult}(\gamma') < h_\eta .$$

Let $\tau = \langle \rho_1, \dots , \rho_n \rangle$ be generated by the extremal rays ρ_i with the corresponding primitive vectors $v_i = n(\pi(\rho_i)), i = 1, \dots , n$. We can write $0 \neq q = \sum_i a_i v_i$ with $0 < a_i < 1$ for all i, as $q \in \mathrm{par}(\pi(\tau))$.

Any new maximal π-independent face γ' in η' has the form

$$\gamma' = \rho' + \tau' + \nu$$

where $\rho' = \mathrm{Mid}(\tau, l_q)$, τ' is a proper face of τ with $\rho' \not\subset \tau'$ and where $\nu \in \mathrm{link}_\eta(\tau)$.

Observe that in general a maximal π-independent face of a simplicial cone in N_Q^+ has codimension at most one and hence we may assume that in the above expression τ' has codimension at most two in τ.

Case: τ' has codimension one in τ.

The face τ' omits, say, ρ_j among the extremal rays of τ. Then

$$\pi\text{-mult}(\rho' + \tau' + \nu') \leq a_j \cdot \pi\text{-mult}(\tau + \nu) \leq a_j \cdot h_\eta < h_\eta .$$

Case: τ' has codimension two in τ.

The face τ' omits, say, ρ_j and ρ_k among the extremal rays of τ. Observe that in this case $\tau + \nu$ is necessarily π-dependent. Indeed, if $\tau + \nu$ is π-independent, then there exists a codimension one face $\tau''(\supset \tau')$ of τ such that we have π-independent faces

$$\tau + \nu \supset \rho' + \tau'' + \nu \underset{\neq}{\supset} \rho' + \tau' + \nu ,$$

contradicting the maximality of $\rho' + \tau' + \nu$.

Let $\nu = \langle \rho_{n+1}, \dots, \rho_m \rangle$ be generated by the extremal rays ρ_i with the corresponding primitive vectors $v_i = n(\pi(\rho_i))$, $i = n+1, \dots, m$ as before. Then, since $\tau + \nu$ is π-dependent, we have a nontrivial linear dependence relation $\sum_{i=1}^m r_i v_i = 0$. In order to compute the π-multiplicity, choose a basis of $\{\mathrm{span}_Q \pi(\tau + \nu)\} \cap N$. Then

$$
\begin{aligned}
&\pi\text{-mult}(\rho' + \tau' + \nu) \\
&\qquad \le |\det(q, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m)| \\
(\diamond) \qquad &\qquad = \left| \sum_i a_i \cdot \det(v_i, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m) \right| \\
&\qquad = |a_j \cdot \det(v_j, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m) \\
&\qquad\quad + a_k \cdot \det(v_k, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m)| .
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
0 &= \left| \sum_i r_i \cdot \det(v_i, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m) \right| \\
&= |r_j \cdot \det(v_j, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m) \\
&\quad + r_k \cdot \det(v_k, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m)| .
\end{aligned}
$$

Since τ is codefinite with respect to η, either r_j and r_k have the same sign or one of them is 0. (If $r_j = r_k = 0$, then $\tau' + \nu$ would be π-dependent since $\sum_{i \neq j,k} r_i v_i = \sum_i r_i v_i = 0$. But $\rho' + \tau' + \nu$, containing $\tau' + \nu$, is π-independent, a contradiction!) In the former case, $\det(v_j, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m)$ and $\det(v_k, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m)$ have opposite signs and hence continuing the formula $(\diamond)$ we have

$$
\begin{aligned}
&\le \max\{a_j \cdot |\det(v_j, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m)|, \\
&\qquad a_k \cdot |\det(v_k, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m)|\} \\
&\le \max\{a_j \cdot h_\eta, a_k \cdot h_\eta\} < h_\eta .
\end{aligned}
$$

In the latter case (say, $r_j = 0$ while $r_k \neq 0$), we have

$$\det(v_k, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m) = 0$$

and hence continuing the formula $(\diamond)$ we obtain

$$= |a_j \cdot \det(v_j, v_1, \dots, \check{v}_j, \dots, \check{v}_k, \dots, v_m)| \le a_j \cdot h_\eta < h_\eta .$$

This completes the proof of the claim.

Observe that $\eta = \langle \rho_1, \dots, \rho_n, \rho_{n+1}, \dots, \rho_m \rangle$ and that a maximal cone ζ' of η' has the form

$$\zeta' = \langle \rho', \rho_1, \dots, \check{\rho}_j, \dots, \rho_m \rangle \quad \text{for some } j = 1, \dots, m .$$

The only possible and old maximal π-independent face of ζ' is $\langle \rho_1, \dots, \overset{\vee}{\rho_j}, \dots, \rho_m \rangle$ and hence the above claim implies π-m.p.$(\zeta') \leq (h_\eta, 0, 0, 0)$. Note that π-m.p.$(\eta) \geq (h_\eta, 0, 0, 0)$ and that if the equality holds then there is only one maximal π-independent face $\gamma \subset \eta$ with π-mult$(\gamma) = h_\eta$ and hence we have possibly only one maximal cone ζ' of η', namely the one containing γ, having the π-multiplicity profile equal to $(h_\eta, 0, 0, 0)$. Therefore, we have either

$$\pi\text{-m.p.}(\eta) = (h_\eta, 1, *, *) = [(h_\eta, 1, *, *); 1] > [(h_\eta, 0, 0, 0), s] \geq \pi\text{-m.p.}(\eta')$$

or

$$\pi\text{-m.p.}(\eta) = (h_\eta, 0, 0, 0) = [(h_\eta, 0, 0, 0); 1] \geq \pi\text{-m.p.}(\eta').$$

If τ is contained in a maximal codimension one face γ of η with π-mult$(\gamma) = h_\eta$, then in the latter case we have the strict inequality.

This completes the proof of Proposition 5.5.

As shown above, the star subdivision by a π-independent face behaves well (choosing an appropriate division point in the interior) if it is codefinite with respect to a π-dependent cone containing it, i.e., if it is codefinite with respect to a circuit in its closed star. In the following, we study how to make a given π-independent face codefinite with respect to all the circuits in its closed star, after some specific star subdivisions.

Step 3. Let $\sigma = \langle \rho_1, \dots, \rho_k \rangle$ be a simplicial and π-strictly convex cone which is a circuit of dimension k in $N^+_{\mathbf{Q}}$, where the extremal rays ρ_i of σ are generated by $(v_i, w_i) \in N^+_{\mathbf{Q}} = N_{\mathbf{Q}} \oplus \mathbf{Q}$ with $v_i = n(\pi(\rho_i))$, $i = 1, \dots, k$, being the primitive vectors in N. Let τ be a codimension one face of σ with the maximum π-multiplicity h_σ among all the π-independent faces of σ. Say,

$$\tau = \tau_\alpha = \langle \rho_1, \dots, \overset{\vee}{\rho_\alpha}, \dots, \rho_k \rangle.$$

We have the unique linear dependence relation

$$(\natural) \quad \sum_{i=1}^{k} r_i v_i = 0 \text{ with the conditions } |r_\alpha| = 1 \text{ and } r_1 w_1 + \cdots + r_k w_k > 0.$$

We note that $0 < |r_i| \leq 1$ for $i = 1, \dots, k$ where $|r_i| = 1$ if and only if π-mult$(\tau_i) = h_\sigma$ for $\tau_i = \langle \rho_1, \dots, \overset{\vee}{\rho_i}, \dots, \rho_k \rangle$.

The first inequality $0 < |r_i|$ comes from the fact that σ is a circuit and the second inequality and the assertion about the equality comes from the easy observation

$$\begin{aligned}
\pi\text{-mult}(\tau_i) &= \pi\text{-mult}(\langle \rho_1, \dots, \overset{\vee}{\rho_i}, \dots, \rho_k \rangle) \\
&= \left| \det \left(v_1, \dots, \overset{\vee}{v_i}, \dots, -r_\alpha v_\alpha = \sum_{j \neq \alpha} r_j v_j, \dots, v_k \right) \right| \\
&= |r_i| \cdot |\det(v_1, \dots, v_i, \dots, \overset{\vee}{v_\alpha}, \dots, v_k)| \\
&= |r_i| \cdot \pi\text{-mult}(\tau) \leq \pi\text{-mult}(\tau).
\end{aligned}$$

Thus we conclude that the relation ($\natural$) is independent of the choice of a codimension one π-independent face τ of σ as long as τ has the maximum π-multiplicity h_σ.

DEFINITION 5.6. Let σ be a circuit of N_Q^+ as above. Then the negative (resp. positive) center point $\mathrm{Ctr}_-(\sigma)$ (resp. $\mathrm{Ctr}_+(\sigma)$) of σ is defined to be

$$\mathrm{Ctr}_-(\sigma) = \sum_{r_i<0} v_i \quad \left(\text{resp. } \mathrm{Ctr}_+(\sigma) = \sum_{r_i>0} v_i\right).$$

LEMMA 5.7. *Let σ be a circuit of N_Q^+. Then*

$$\mathrm{Ctr}_-(\sigma), \mathrm{Ctr}_+(\sigma) \in \mathrm{RelInt}(\pi(\sigma)).$$

PROOF. We observe

$$\begin{aligned}
\mathrm{Ctr}_-(\sigma) &= \sum_{r_i<0} v_i = \sum_{r_i<0} v_i + \sum_{i=1}^{k} r_i v_i = \sum_{r_i>0} r_i v_i + \sum_{r_i<0}(1+r_i)v_i \\
&= (1-\varepsilon)\left\{\sum_{r_i<0} v_i\right\} + \varepsilon\left\{\sum_{r_i>0} r_i v_i + \sum_{r_i<0}(1+r_i)v_i\right\} \quad \text{for } 0<\varepsilon<1 \\
&= \sum_{i=1}^{k} c_i v_i,
\end{aligned}$$

where

$$c_i = \begin{cases} = \varepsilon r_i \text{ when } r_i > 0 \\ = 1-\varepsilon+\varepsilon(1+r_i) \text{ when } r_i < 0 \end{cases}.$$

Since $r_i \neq 0$ and $r_i \geq -1$ for all i, we see

$$c_i > 0 \quad \text{for all } i = 1, \ldots, k.$$

Thus we conclude

$$\mathrm{Ctr}_-(\sigma) \in \mathrm{RelInt}(\pi(\sigma)).$$

The argument for the statement $\mathrm{Ctr}_+(\sigma) \in \mathrm{RelInt}(\pi(\sigma))$ is identical.

LEMMA 5.8. *Let σ be a circuit in N_Q^+ as above with the negative center point* $\mathrm{Ctr}_-(\sigma)$ (*resp. the positive center point* $\mathrm{Ctr}_+(\sigma)$). *Let l_- (resp. l_+) be the ray generated by* $\mathrm{Ctr}_-(\sigma)$ (*resp.* $\mathrm{Ctr}_+(\sigma)$) *and* $\sigma' = \mathrm{Mid}(\sigma, l_-)\cdot\sigma$ (*resp.* $\sigma' = \mathrm{Mid}(\sigma, l_+)\cdot\sigma$) *be the subdivision of σ by the midray* $\mathrm{Mid}(\sigma, l_-)$ (*resp.* $\mathrm{Mid}(\sigma, l_+)$). *Then every codimension one face ζ of σ with the maximum π-multiplicity h_σ (which stays unchanged through the subdivision and hence can be considered a face $\zeta \in \sigma'$) is codefinite with respect to the (unique) maximal cone in the closed star of ζ in σ'.*

PROOF. We use the same notation as above. We only prove the statement for the negative center as the proof is identical for the positive center.

Observe fist that we have

$$\mathrm{Mid}(\sigma, l_-) \in \mathrm{RelInt}(\sigma),$$

since $\mathrm{Ctr}_-(\sigma) \in \mathrm{RelInt}(\pi(\sigma))$ by Lemma 5.7. Therefore, the star subdivision with respect to $\mathrm{Mid}(\sigma, l_-)$ does not affect ζ, i.e., $\zeta \in \sigma'$.

Observe secondly (say, $\zeta = \zeta_j = \langle \rho_1, \dots, \overset{\vee}{\rho_j}, \dots, \rho_k \rangle$) that

$$\zeta_j \text{ has maximal } \pi\text{-multiplicity} \Longleftrightarrow |r_j| = 1 .$$

Case: $r_j = 1$.

In this case, since $\mathrm{Ctr}_-(\sigma) = \sum_{r_i<0} v_i$ and since the maximal cone σ_j containing ζ_j in σ' is of the form $\sigma_j = \langle \mathrm{Mid}(\sigma, l_-), \rho_1, \dots, \overset{\vee}{\rho_j}, \dots, \rho_k \rangle$, the linear relation for σ_j is given by

$$\mathrm{Ctr}_-(\sigma) - \sum_{r_i<0} v_i = 0 .$$

As ζ_j contains only the extremal rays corresponding to the v_i, which have the same sign (or 0) in the linear relation, ζ_j is codefinite with respect to the (unique) maximal cone σ_j in the closed star of ζ_j in σ'.

Case: $r_j = -1$.

In this case, since $\mathrm{Ctr}_-(\sigma) = \sum_{r_i>0} r_i v_i + \sum_{-1<r_i<0}(1 + r_i)v_i$ and since the maximal cone σ_j containing ζ_j is of the form $\sigma_j = \langle \mathrm{Mid}(\sigma, l_-), \rho_1, \dots, \overset{\vee}{\rho_j}, \dots, \rho_k \rangle$, the linear relation for σ_j is given by

$$\mathrm{Ctr}_-(\sigma) - \sum_{r_i>0} r_i v_i - \sum_{-1<r_i<0} (1 + r_i)v_i = 0 .$$

As ζ_j contains only the extremal rays corresponding to the v_i, which have the same sign (or 0) in the linear relation, ζ_j is codefinite with respect to the (unique) maximal cone in the closed star σ_j of ζ_j in σ'.

This completes the proof of Lemma 5.8.

This lemma suggests that we should use the star subdivision by the negative or positive center of a circuit to achieve codefiniteness of a face τ in order to bring the situation of Proposition 5.4 in Step 2. But the lemma only achieves the codefiniteness for a face τ which is contained in a maximal π-independent face with the maximum π-multiplicity but does not analyze the behavior of the π-multiplicity profile. In our process of π-desingularization, we need to achieve codefiniteness for a face τ which is not contained in a maximal π-independent face with the maximum π-multiplicity and the analysis of the π-multiplicity profile is crucial. Both of these needs are fulfilled by the following proposition, which is at the technical heart of this section.

PROPOSITION 5.9. *Let σ be a circuit of* $\dim \sigma > 2$ *in N_Q^+. Then, by choosing σ' to be either the star subdivision of σ coresponding to the negative center point or the one by the positive center point, i.e.,*

$$\sigma' = \mathrm{Mid}(\sigma, l_-) \cdot \sigma \quad or \quad \mathrm{Mid}(\sigma, l_+) \cdot \sigma ,$$

where l_- (resp. l_+) is the ray generated by the negative (resp. positive) center point $\mathrm{Ctr}_-(\sigma)$ *(resp.* $\mathrm{Ctr}_+(\sigma)$*), we see σ' satisfies one of the following:*

A: *Every maximal cone δ' of σ' has the π-multiplicity profile strictly smaller than that of σ, i.e.,*

$$\pi\text{-m.p.}(\delta') < \pi\text{-m.p.}(\sigma)\,.$$

In particular, we have

$$\pi\text{-m.p.}(\sigma') < \pi\text{-m.p.}(\sigma)\,.$$

B: *Every maximal cone δ' of σ', except for one maximal cone κ', has the π-multiplicity profile strictly smaller than that of σ, i.e.,*

$$\pi\text{-m.p.}(\delta') < \pi\text{-m.p.}(\sigma)$$

and the exceptional maximal cone κ' has the same π-multiplicity profile as that of σ, i.e.,

$$\pi\text{-m.p.}(\kappa') = \pi\text{-m.p.}(\sigma)\,.$$

In particular, we have

$$\pi\text{-m.p.}(\sigma') = \pi\text{-m.p.}(\sigma)\,.$$

Moreover, there exists a maximal π-independent face γ' of κ' such that

(B-o) *γ' is also a face of σ (i.e., γ' remains untouched under the subdivision),*

(B-ii) *γ' has the maximum π-multiplicity, i.e., π-mult$(\gamma') = h_{\sigma'} = h_\sigma$,*

(B-iii) *γ' is codefinite with respect to κ'.*

PROOF. Let $\sigma = \langle \rho_1, \dots, \rho_k\rangle$, where the extremal rays ρ_i are generated by $(v_i, w_i) \in N_{\boldsymbol{Q}}^+$ with $v_i = n(\pi(\rho_i))$, $i = 1, \dots, k$, being the primitive vectors for the projections.

Let $\sum r_i v_i = 0$ be the nontrivial linear relation so that $\sum r_i w_i = 0$ and

$$|r_i| = 1 \Longleftrightarrow \pi\text{-mult}(\tau_i) = h_\sigma \quad \text{for } \tau_i = \langle \rho_1, \dots, \overset{\vee}{\rho_i}, \dots, \rho_k\rangle\,.$$

Note that the maximal cones σ_i' of σ' are of the form

$$\sigma_i' = \langle \rho_0, \rho_1, \dots, \overset{\vee}{\rho_i}, \dots, \rho_k\rangle$$

where ρ_0 is the midray $\mathrm{Mid}(\sigma, l_-)$ or $\mathrm{Mid}(\sigma, l_+)$ depending on the choice of the negative or positive center point.

We compute the π-multiplicity of the maximal faces τ_{ij}' of σ_i'

$$\tau_{ij}' = \langle \rho_0, \rho_1, \dots, \overset{\vee}{\rho_i}, \dots, \overset{\vee}{\rho_j}, \dots, \rho_k\rangle$$

as follows:

Case of the negative center point: $\rho_0 = \mathrm{Mid}(\sigma, l_-)$.

We let $e_- \in \boldsymbol{N}$ be the integer such that $\sum_{r_\alpha < 0} v_\alpha = e_- \cdot n(\pi(\rho_0))$ with $n(\pi(\rho_0))$ being the primitive vector.

Subcase $r_i > 0$:

$$\pi\text{-mult}(\tau'_{i0}) = \pi\text{-mult}(\tau_i)\,,$$
$$\pi\text{-mult}(\tau'_{ij}) = \frac{1}{e_-}\left|\det\left(\sum_{r_\alpha<0} v_\alpha, v_1, \dots, \overset{\vee}{v_i}, \dots, \overset{\vee}{v_j}, \dots, v_k\right)\right|$$
$$= \begin{cases} 0 & \text{when } r_j > 0\,, \\ \dfrac{1}{e_-}\pi\text{-mult}(\tau_i) & \text{when } r_j < 0\,. \end{cases}$$

Subcase $r_i < 0$:

$$\pi\text{-mult}(\tau'_{i0}) = \pi\text{-mult}(\tau_i)\,,$$
$$\pi\text{-mult}(\tau'_{ij}) = \frac{1}{e_-}\left|\det\left(\sum_{r_\alpha<0} v_\alpha, v_1, \dots, \overset{\vee}{v_i}, \dots, \overset{\vee}{v_j}, \dots, v_k\right)\right|$$
$$= \begin{cases} \dfrac{1}{e_-}\pi\text{-mult}(\tau_j) & \text{when } r_j > 0\,, \\ \dfrac{1}{e_-}|\pi\text{-mult}(\tau_j) - \pi\text{-mult}(\tau_i)| & \text{when } r_j < 0\,. \end{cases}$$

Note that $\det(v_i, v_1, \dots, \overset{\vee}{v_i}, \dots, \overset{\vee}{v_j}, \dots, v_k)$ and $\det(v_j, v_1, \dots, \overset{\vee}{v_i}, \dots, \overset{\vee}{v_j}, \dots, v_k)$ have opposite signs, since

$$\begin{aligned} 0 &= \det\left(\sum r_\alpha v_\alpha, v_1, \dots, \overset{\vee}{v_i}, \dots, \overset{\vee}{v_j}, \dots, v_k\right) \\ &= r_i(v_i, v_1, \dots, \overset{\vee}{v_i}, \dots, \overset{\vee}{v_j}, \dots, v_k) + r_j(v_j, v_1, \dots, \overset{\vee}{v_i}, \dots, \overset{\vee}{v_j}, \dots, v_k) = 0\,. \end{aligned}$$

Symmetrically we compute the other case.

Case of the positive center point: $\rho_0 = \mathrm{Mid}(\sigma, l_+)$.

We let $e_+ \in \boldsymbol{N}$ be the integer such that $\sum_{r_\alpha>0} v_\alpha = e_+ \cdot n(\pi(\rho_0))$ with $n(\pi(\rho_0))$ being the primitive vector.

Subcase $r_i < 0$:

$$\pi\text{-mult}(\tau'_{i0}) = \pi\text{-mult}(\tau_i)\,,$$
$$\pi\text{-mult}(\tau'_{ij}) = \frac{1}{e_+}\left|\det\left(\sum_{r_\alpha<0} v_\alpha, v_1, \dots, \overset{\vee}{v_i}, \dots, \overset{\vee}{v_j}, \dots, v_k\right)\right|$$
$$= \begin{cases} 0 & \text{when } r_j < 0\,, \\ \dfrac{1}{e_+}\pi\text{-mult}(\tau_i) & \text{when } r_j > 0\,. \end{cases}$$

Subcase $r_i > 0$:

$$\pi\text{-mult}(\tau'_{i0}) = \pi\text{-mult}(\tau_i)\,,$$

$$\pi\text{-mult}(\tau'_{ij}) = \frac{1}{e_+}\left|\det\left(\sum_{r_\alpha>0} v_\alpha, v_1, \dots, \overset{\vee}{v_i}, \dots, \overset{\vee}{v_j}, \dots, v_k\right)\right|$$

$$= \begin{cases} \dfrac{1}{e_+}\pi\text{-mult}(\tau_j) & \text{when } r_j < 0\,, \\ \dfrac{1}{e_+}|\pi\text{-mult}(\tau_j) - \pi\text{-mult}(\tau_i)| & \text{when } r_j > 0\,. \end{cases}$$

Using this computation, we can now easily derive the conclusion of the proposition dividing it into the cases according to the cardinalities of the following sets:

$$I^+ = \{i; r_i > 0\}\,, \quad I^- = \{i; r_i < 0\}\,, \quad I_1^+ = \{i; r_i = 1\}\,, \quad I_1^- = \{i; r_i = -1\}\,.$$

Case: $2 \le \sharp I_1^- \le \sharp I_1^+$

In this case, we choose the negative center point and let $\rho_0 = \mathrm{Mid}(\sigma, l_-)$.

When $0 < r_i < 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$.

When $0 < r_i = 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$. If $e_- > 1$, then $r_{\sigma'_i} = 1$ and hence $b_{\sigma'_i} = 0 < b_\sigma = 1$. If $e_- = 1$, then $r_{\sigma'_i} \ge \sharp I_1^- \ge 2$ and hence $b_{\sigma'_i} = 1 = b_\sigma$. But $c_{\sigma'_i} = k_{\sigma'_i} < k_\sigma = c_\sigma$, since $\pi\text{-mult}(\tau'_{ij}) = 0$ for $j \in I^+ \supset I_1^+$.

When $-1 < r_i < 0$ and $e_- > 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$.

When $-1 = r_i < 0$ and $e_- > 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$. But $r_{\sigma'_i} = 1$ and hence $b_{\sigma'_i} = 0 < b_\sigma = 1$.

When $-1 \le r_i < 0$ and $e_- = 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$, $r_{\sigma'_i} \ge \sharp I_1^+ \ge 2$ and hence $b_{\sigma'_i} = 1 = b_\sigma$. We also have $c_{\sigma'_i} = k_{\sigma'_i} \le k_\sigma = c_\sigma$, since σ is a circuit. But $d_{\sigma'_i} = r_{\sigma'_i} = r_\sigma - \sharp I_1^- + 1 < r_\sigma = d_\sigma$, since $\tau'_{ij} = |\pi\text{-mult}(\tau_j) - \pi\text{-mult}(\tau_i)| < h_\sigma$ for $j \in I_1^-$, $j \ne i$.

Thus we have

$$\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma)$$

for all the maximal cones σ'_i of σ'.

Therefore, in this case with the choice of the negative center we conclude we are in Case A and

$$\pi\text{-m.p.}(\sigma') < \pi\text{-m.p.}(\sigma)\,.$$

Case: $1 = \sharp I_1^- < \sharp I_1^+$

In this case, we choose the negative center point and let $\rho_0 = \mathrm{Mid}(\sigma, l_-)$.

When $0 < r_i < 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} < h_\sigma = a_\sigma$ and hence $\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma)$.

When $0 < r_i = 1$, we have $a_{\sigma'_i} = h_{\sigma'_i} = h_\sigma = a_\sigma$. If $\sharp I^- = \sharp I_1^- = 1$ or $e_- > 1$, then $r_{\sigma'_i} = 1$ and hence $b_{\sigma'_i} = 0 < b_\sigma = 1$. If $\sharp I^- > 1$ and $e_- = 1$, then $r_{\sigma'_i} > 1$ and hence $b_{\sigma'_i} = 1 = b_\sigma$. But we have $c_{\sigma'_i} = k_{\sigma'_i} < k_\sigma = c_\sigma$, since $\pi\text{-mult}(\tau'_{ij}) = 0$ for $j \in I^+ \supset I_1^+$. Thus we have $\pi\text{-m.p.}(\sigma'_i) < \pi\text{-m.p.}(\sigma)$.

When $-1 < r_i < 0$ and $e_- > 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} < h_\sigma = a_\sigma$ and hence π-m.p.$(\sigma_i') <$ π-m.p.(σ).

When $-1 = r_i < 0$ and $e_- > 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} = h_\sigma = a_\sigma$. But $r_{\sigma'} = 1$ and hence $b_{\sigma'} = 0 < b_\sigma = 1$. Thus we have π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $-1 \leq r_i < 0$ and $e_- = 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} = h_\sigma = a_\sigma$, $r_{\sigma_i'} \geq \sharp I_1^+ \geq 2$ and hence $b_{\sigma_i'} = 1 = b_\sigma$. We also have $c_{\sigma_i'} = k_{\sigma_i'} \leq k_\sigma = c_\sigma$, since σ is a circuit. But $d_{\sigma_i'} = r_{\sigma_i'} = r_\sigma - \sharp I_1^- + 1 \leq r_\sigma = d_\sigma$, since $\tau_{ij}' = |\pi\text{-mult}(\tau_j) - \pi\text{-mult}(\tau_i)| < h_\sigma$ for $j \in I_1^-$, $j \neq i$. Thus we have π-m.p.$(\sigma_i') \leq \pi$-m.p.(σ). The equality holds only when $r_i = -1$ with i being the sole member of I_1^-, in which case the face $\tau_{i0}' = \tau_i$ has the maximum π-multiplicity h_σ and it is codefinite with respect to σ_i' by Lemma 5.8.

Therefore, in this case with the choice of the negative center we conclude that we are in Case A and

$$\pi\text{-m.p.}(\sigma') < \pi\text{-m.p.}(\sigma) \quad \text{if } e_- > 1$$

and that we are in Case B and

$$\pi\text{-m.p.}(\sigma') = \pi\text{-m.p.}(\sigma) \quad \text{if } e_- = 1\,.$$

Case: $1 = \sharp I_1^- = \sharp I_1^+ < \sharp I^+$.

In this case, we choose the negative center point and let $\rho_0 = \mathrm{Mid}(\sigma, l_-)$.

When $0 < r_i < 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} < h_\sigma = a_\sigma$ and hence π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $0 < r_i = 1$ and $e_- > 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} = h_\sigma = a_\sigma$. But $r_{\sigma'} = 1$ and hence $b_{\sigma_i'} = 0 < b_\sigma = 1$. Thus we have π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $0 < r_i = 1$ and $e_- = 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} = h_\sigma = a_\sigma$. If $\sharp I^- = \sharp I_1^- = 1$, then $r_{\sigma_i'} = 1$ and hence $b_{\sigma_i'} = 0 < b_\sigma = 1$. If $\sharp I^- > 1$, then $r_{\sigma_i'} > 1$ and hence $b_{\sigma_i'} = 1 = b_\sigma$. But we have $c_{\sigma_i'} = k_{\sigma_i'} < k_\sigma = c_\sigma$, since π-mult$(\tau_{ij}') = 0$ for $j \in I^+ \supset I_1^+$, $j \neq i$. Thus we have π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $-1 < r_i < 0$ and $e_- > 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} < h_\sigma = a_\sigma$ and hence π-m.p.$(\sigma_i') <$ π-m.p.(σ).

When $-1 < r_i < 0$ and $e_- = 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} = h_\sigma = a_\sigma$. But $r_{\sigma_i'} = 1$ and hence $b_{\sigma_i'} = 0 < b_\sigma = 1$. Thus we have π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $r_i = -1$ and $e_- > 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} = h_\sigma = a_\sigma$. But $r_{\sigma_i'} = 1$ and hence $b_{\sigma_i'} = 0 < b_\sigma = 1$. Thus we have π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $r_i = -1$ and $e_- = 1$, i is the sole member of I_1^- and we have $a_{\sigma_i'} = h_{\sigma_i'} = h_\sigma = a_\sigma$, $r_{\sigma_i'} = 2$ and hence $b_{\sigma_i'} = 1 = b_\sigma$. Moreover, we have $c_{\sigma_i'} = k_{\sigma_i'} = k_\sigma = c_\sigma$ and $d_{\sigma_i'} = r_{\sigma_i'} = 2 = r_\sigma = d_{\sigma_i}$. Thus we have π-m.p.$(\sigma_i') = \pi$-m.p.(σ). The face $\tau_{i0}' = \tau_i$ has the maximum π-multiplicity h_σ and it is codefinite with respect to σ_i' by Lemma 5.8.

Therefore, in this case with the choice of the negative center we conclude that we are in Case A and

$$\pi\text{-m.p.}(\sigma') < \pi\text{-m.p.}(\sigma) \quad \text{if } e_- > 1$$

and that we are in Case B and

$$\pi\text{-m.p.}(\sigma') = \pi\text{-m.p.}(\sigma) \quad \text{if } e_- = 1\,.$$

Case: $0 = \sharp I_1^- < 2 \leq \sharp I_1^+$.

In this case, we choose the positive center point and let $\rho_0 = \mathrm{Mid}(\sigma, l_+)$.

When $-1 < r_i < 0$, we have $a_{\sigma_i'} = h_{\sigma_i'} < h_\sigma = a_\sigma$ and hence π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $0 < r_i < 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} < h_\sigma = a_\sigma$ and hence π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $r_i = 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} = h_\sigma = a_\sigma$. But $r_{\sigma_i'} = 1$ and hence $b_{\sigma_i'} = 0 < b_\sigma = 1$. Thus we have π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

Therefore, in this case with the choice of the positive center we conclude that we are in Case A and

$$\pi\text{-m.p.}(\sigma') < \pi\text{-m.p.}(\sigma)\,.$$

Case: $0 = \sharp I_1^- < 1 = \sharp I_1^+$

In this case, we choose the positive center point and let $\rho_0 = \mathrm{Mid}(\sigma, l_+)$.

When $-1 < r_i < 0$, we have $a_{\sigma_i'} = h_{\sigma_i'} < h_\sigma = a_\sigma$ and hence π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $0 < r_i < 1$, we have $a_{\sigma_i'} = h_{\sigma_i'} < h_\sigma = a_\sigma$ and hence π-m.p.$(\sigma_i') < \pi$-m.p.(σ).

When $r_i = 1$, i.e., i is the sole member of I_1^+, we have $a_{\sigma_i'} = h_{\sigma_i'} = h_\sigma = a_\sigma$, $r_{\sigma_i'} = 1$ and hence $b_{\sigma_i'} = 0 = b_\sigma$. Moreover, we have $c_{\sigma_i'} = k_{\sigma_i'} = k_\sigma = c_\sigma$ and $d_{\sigma_i'} = r_{\sigma_i'} = 1 = r_\sigma = d_{\sigma_i}$. Thus we have π-m.p.$(\sigma_i') = \pi$-m.p.(σ). The face $\tau_{i0}' = \tau_i$ has the maximum π-multiplicity h_σ and it is codefinite with respect to σ_i' by Lemma 5.8.

Therefore, in this case with the choice of the positive center we conclude we are in Case B and

$$\pi\text{-m.p.}(\sigma') = \pi\text{-m.p.}(\sigma)\,.$$

Symmetrically, we also conclude:

Case: $2 \leq \sharp I_1^+ \leq \sharp I_1^-$

With the choice of the positive center point, we are in Case A.

Case: $1 = \sharp I_1^+ < \sharp I_1^-$

With the choice of the positive center point, we are in Case A if $e_+ > 1$ and in Case B if $e_+ = 1$.

Case: $1 = \sharp I_1^+ = \sharp I_1^- < \sharp I^-$

With the choice of the positive center point, we are in Case A if $e_+ > 1$ and in Case B if $e_+ = 1$.

Case: $0 = \sharp I_1^+ < 2 \leq \sharp I_1^-$

With the choice of the negative center point, we are in Case A.

Case: $0 = \sharp I_1^+ < 1 = \sharp I_1^-$

With the choice of the negative center point, we are in Case B.

Since the above cases exhaust all the possibilities, we complete the proof for Proposition 5.9.

The next lemma shows that the π-multiplicity of a cone can be computed easily from that of the unique circuit contained in it.

LEMMA 5.10. *Let σ be a circuit in a simplicial cobordism Σ in N_Q^+ and η be a maximal cone in $\overline{\mathrm{Star}(\sigma)}$. Then any maximal π-independent face γ of η is of the form*

$$\gamma = \tau + \nu$$

where $\tau = \gamma \cap \sigma$ is a maximal π-independent face of σ and ν is the unique maximal cone of $\mathrm{link}_\eta(\sigma)$.

Moreover, there exists $e \in N$ such that for any γ as above (once η is fixed) we have the formula

$$\pi\text{-mult}(\gamma) = \pi\text{-mult}(\tau) \cdot e\,.$$

In particular, we have

$$\pi\text{-m.p.}(\eta) = (a_\eta, b_\eta, c_\eta, d_\eta) = (e \cdot a_\sigma, b_\sigma, c_\sigma, d_\sigma)\,.$$

PROOF. Let $\sigma = \langle \rho_1, \dots, \rho_k \rangle$ and $\eta = \langle \rho_1, \dots, \rho_k, \rho_{k+1}, \dots, \rho_l \rangle$ be generated by the extremal rays ρ_i with the corresponding primitive vectors of the projections $v_i = n(\pi(\rho_i)) \in N$. Then a maximal π-independent face γ of η is of the form

$$\gamma = \langle \rho_1, \dots, \overset{\vee}{\rho_j}, \dots, \rho_k, \rho_{k+1}, \dots, \rho_l \rangle = \tau + \nu$$

where $\tau = \langle \rho_1, \dots, \overset{\vee}{\rho_j}, \dots, \rho_k \rangle = \gamma \cap \sigma$ and $\nu = \langle \rho_{k+1}, \dots, \rho_l \rangle$ is the unique maximal cone of $\mathrm{link}_\eta(\sigma)$. This proves the first assertion.

For "Moreover" part, we have the exact sequence

$$0 \to L \to N_\eta \to Q \to 0$$

where $L = \mathrm{span}_{\mathbf{Q}}(\pi(\sigma)) \cap N$, $N_\eta = \mathrm{span}_{\mathbf{Q}}(\pi(\eta)) \cap N$ and Q is the cokernel, which is torsion free and hence a free $\mathbf{Z}$-module. Take a $\mathbf{Z}$-basis $\{u_1, \dots, u_{k-1}, u_{k+1}, \dots, u_l\}$ of N_η so that $\{u_1, \dots, u_{k-1}\}$ is a $\mathbf{Z}$-basis of L and $\{u_{k+1}, \dots, u_l\}$ maps to a $\mathbf{Z}$-basis of Q. With respect to this basis of N_η, the π-multiplicity of γ can be computed

$$\pi\text{-mult}(\gamma) = \det \begin{pmatrix} A & B \\ 0 & E \end{pmatrix} = \det A \cdot \det E = \pi\text{-mult}(\tau) \cdot e\,,$$

where

$$(v_1, \dots, \overset{\vee}{v_j}, \dots, v_k) = \begin{pmatrix} A \\ 0 \end{pmatrix} \quad \text{and} \quad (u_{k+1}, \dots, u_l) = \begin{pmatrix} B \\ E \end{pmatrix}.$$

This completes the proof of the lemma.

Now it is easy to see the following main consequence of Step 3.

COROLLARY 5.11. *Let τ be a π-independent face contained in the closed star $\overline{\mathrm{Star}(\sigma)}$ of a circuit σ. Then there exists $\{\overline{\mathrm{Star}(\sigma)}\}^\circ$ obtained by a succession of star subdivisions by the negative or positive center points of the circuits (of the intermediate subdivisions) inside of σ such that*

(5.11.1) *the π-multiplicity profile does not increase, i.e.,*

$$\pi\text{-m.p.}(\{\overline{\mathrm{Star}(\sigma)}\}^\circ) \leq \pi\text{-m.p.}(\overline{\mathrm{Star}(\sigma)})\,,$$

(5.11.2) *τ is a face of $\{\overline{\mathrm{Star}(\sigma)}\}^\circ$ and τ is codefinite with respect to every cone $\nu \in \{\overline{\mathrm{Star}(\sigma)}\}^\circ$ containing τ.*

PROOF. We prove the assertion by induction on the π-multiplicity profile of σ.

If π-m.p.$(\sigma) = (1, *, *, *)$, then by taking $\{\overline{\mathrm{Star}(\sigma)}\}^\circ$ to be the star subdivision corresponding to the negative or positive point of σ, we see easily that the condition (5.11.1) is satisfied, while the condition (5.11.2) is a consequence of Lemma 5.8.

We assume that the assertion holds for the case with the π-multiplicity profile smaller than π-m.p.(σ). If $\dim\sigma = 2$, then τ is already codefinite with respect to σ and there is nothing more to prove. So we may assume $\dim\sigma > 2$. We take the star subdivision by the negative or positive center of σ, according to Proposition 5.9, so that either the case A or the case B holds and hence the π-multiplicity does not increase.

If the case A holds, noting that the circuits of all the maximal cones of the star subdivision are contained in σ we see the assertion holds immediately by the induction hypothesis, since all the maximal cones have the π-multiplicity profile strictly smaller than π-m.p.(σ).

Suppose the case B holds. If $\tau \cap \sigma$ is contained in κ', then $\tau \cap \sigma$ is necessarily contained in γ' and hence codefinite with respect to κ'. The other maximal cones have the π-multiplicity profile strictly smaller than π-m.p.(σ) and the assertion again holds by the induction hypothesis.

This completes the proof of Corollary 5.11 and Step 3.

Now we discuss Step 4.

Step 4. We start from a simplicial cobordism Σ.

If Σ is π-nonsingular, then we are done.

So we may assume Σ is not π-nonsingular and hence π-m.p.$(\Sigma) = (g_\Sigma; s)$ with $g_\Sigma > (1, *, *, *)$. We only have to construct $\tilde{\Sigma}$ obtained from Σ by a succession of star subdivisions such that π-m.p.$(\tilde{\Sigma}) < \pi$-m.p.(Σ).

Let η be a maximal cone of Σ such that π-m.p.$(\eta) = g_\Sigma$ with σ being the unique circuit contained in η.

If $\dim\sigma \le 2$, then we let γ be a maximal π-independent face of η with π-mult$(\gamma) = h_\eta$. We let τ be a minimal π-singular (i.e. not π-nonsingular) face of γ so that we can pick a point $q \in \mathrm{par}(\pi(\tau))$.

If $\dim\sigma > 2$, then we take the star subdivision Σ' of Σ with respect to the negative or positive center point of σ so that either the case A or the case B occurs according to Proposition 5.9.

If the case A occurs, then π-m.p.$(\Sigma') < \pi$-m.p.(Σ) and we simply have to set $\Sigma^\circ = \Sigma'$.

If the case B occurs, then we take the exceptional cone κ' of σ' with π-m.p.$(\kappa') = \pi$-m.p.(σ) as described in Proposition 5.9 and take the maximal π-independent face γ of η such that $\gamma \cap \sigma' = \gamma'$, where γ' is a face of κ' satisfying the conditions (B-o), (B-i) and (B-ii) in Proposition 5.9. Observe that by Lemma 5.10 there is a maximal cone η' of Σ' such that $\eta' \cap \sigma' = \kappa'$, π-m.p.$(\eta') = g_{\Sigma'} = g_\Sigma$, γ is a face of η' as well as that of η, π-mult$(\gamma) = h_{\eta'} = h_\eta$ and that γ is codefinite with respect to η'.

We also take τ to be a minimal π-singular (i.e. not π-nonsingular) face of γ so that we can pick a point $q \in \operatorname{par}(\pi(\tau))$.

Now we consider the situation where $\dim\sigma \leq 2$ and the situation where $\dim\sigma > 2$ with the case B together.

Take all the circuits θ' (except for the one contained in κ') of Σ' such that $\tau \subset \overline{\operatorname{Star}(\theta')}$. By Corollary 5.11 of Step 3 for each θ' we can find $\{\overline{\operatorname{Star}(\theta')}\}^\circ$ obtained by a succession of star subdivisions by the negative or positive center points of the circuits (of the intermediate subdivisions) inside of θ' such that the π-multiplicity profile does not increase, i.e.,

$$\pi\text{-m.p.}(\{\overline{\operatorname{Star}(\theta')}\}^\circ) \leq \pi\text{-m.p.}(\overline{\operatorname{Star}(\theta')}),$$

and that τ is a face of $\{\overline{\operatorname{Star}(\theta')}\}^\circ$ and τ is codefinite with respect to every cone $\nu \in \{\overline{\operatorname{Star}(\theta)}\}^\circ$ containing τ.

Note that these star subdivisions can be carried out simultaneously without affecting each other and that hence we obtain a simplicial cobordism Σ° obtained from Σ by a successive star subdivisions such that

(o) the π-multiplicity profile does not increase, i.e.,

$$\pi\text{-m.p.}(\Sigma^\circ) \leq \pi\text{-m.p.}(\Sigma),$$

(i) η' ($\eta' = \eta$ in the case $\dim\sigma = 2$) is a maximal cone in Σ° with

$$\pi\text{-m.p.}(\eta') = g_{\Sigma^\circ} = g_{\Sigma'} = g_\Sigma,$$

(ii) τ is contained in a maximal π-independent face γ of η' with the maximum π-multiplicity $\pi\text{mult}(\gamma) = h_{\eta'}$,

(iii) τ is codefinite with respect to η' and with respect to all the other maximal cones containing τ,

(iv) we can find a lattice point $q \in \operatorname{par}(\pi(\tau))$.

We only have to set $\tilde{\Sigma} = \operatorname{Mid}(\tau, l_q) \cdot \Sigma^\circ$ to observe by Proposition 5.5 in Step 2 that

$$\pi\text{-m.p.}(\tilde{\Sigma}) < \pi\text{-m.p.}(\Sigma).$$

By the descending chain condition of the set of the π-multiplicity profiles, this completes the process of π-desingularization. Remark that by construction the process leaves any π-independent and already π-nonsingular face of Σ unaffected.

REMARK 5.12. We discuss the comparison of our arguments with the original papers [Morelli1,2].

(5.12.1) (Definition of the negative or positive center point.)

The definition of the negative or positive center point $\operatorname{Ctr}_-(\sigma)$, $\operatorname{Ctr}_+(\sigma)$ as presented here and in [Morelli2] is different from the original definition of the center point $\operatorname{Ctr}(\sigma,\tau)$ in [Morelli1]. In spite of the assertions in [Morelli1], $\operatorname{Ctr}(\sigma,\tau)$ is not always in $\operatorname{RelInt}(\pi(\tau))$, as one can see in some easy examples. This causes a problem in the original argument in [Morelli1], as the subdivision corresponding to the center point may affect not only the cones in the closed star $\overline{\operatorname{Star}(\sigma)}$ but also possibly some other cones, which we do not have any control over. This is the first problematic point in the argument of [Morelli1] noticed by [King2].

(5.12.2) (Definition of the π-multiplicity profile.)

In [Morelli1], the π-multiplicity profile π-m.p.(η) of a simplicial cone η was defined to be

$$\pi\text{-m.p.}(\eta) = (\pi\text{-mult}(\gamma_1), \dots, \pi\text{-mult}(\gamma_l))$$

where $\gamma_1, \dots, \gamma_l$ are the maximal π-independent faces of η with

$$\pi\text{-mult}(\gamma_1) \geq \pi\text{-mult}(\gamma_2) \geq \cdots \geq \pi\text{-mult}(\gamma_l).$$

Proposition 5.5 holds with this definition, while Proposition 5.9 fails to hold, as [King2] noticed.

(With the slightly coarser definition of the π-multiplicity profile

$$\pi\text{-m.p.}(\eta) = (h_\eta, r_\eta),$$

Proposition 5.5 holds, while Proposition 5.9 fails to hold in a similar way.)

In [Morelli2], the π-multiplicity profile π-m.p.(η) of a simplicial cone η was changed and defined to be

$$\pi\text{-m.p.}(\eta) = (h_\eta, k_\eta, r_\eta).$$

Proposition 5.9 holds with this definition, while now in turn Proposition 5.5 fails to hold.

The current and correct definition of the π-multiplicity profile, as presented here, was suggested to us by Morelli after we discussed the dilemma as above through e-mail.

(5.12.3) (How to choose τ with $q \in \operatorname{par}(\pi(\tau))$ and make it codefinite.)

[Morelli1] could be read (by a naive reader like us) in such a way that it suggests that for a maximal π-independent face γ with the maximum π-multiplicity π-mult$(\gamma) = h > 1$ we could take $q \in \operatorname{par}(\pi(\gamma))$, which is clearly false in the case $\dim N_Q \geq 3$. The subdivision with respect to $q \in \operatorname{par}(\pi(\gamma))$ would only affect the cones in the star $\operatorname{Star}(\gamma)$ and we would only have to analyze those circuits σ such that $\gamma \subset \overline{\operatorname{Star}(\sigma)}$. Then the face $\zeta = \gamma \cap \sigma$ has the maximum π-multiplicity h_σ and only Lemma 5.8 would suffice to achieve codefiniteness after the subdivision by the negative or positive center point.

But in general it is only a subface $\tau \subset \gamma$ which contains a point $q \in \operatorname{par}(\pi(\tau))$. Now we have to analyze those circuits σ such that $\tau \subset \overline{\operatorname{Star}(\sigma)}$ but maybe $\gamma \not\subset \overline{\operatorname{Star}(\sigma)}$. Lemma 5.8 is not sufficient any more to achieve the codefiniteness for τ. This is another problematic point in the argument of [Morelli1] noticed by [King2].

[Morelli2] tries to fix this problem via the use of Proposition 5.9 and what Morelli calls the trivial subdivision of a circuit σ.

Our argument here to achieve Corollary 5.11 solves the problem by induction on π-multiplicity profile based upon Proposition 5.9 and does not use the trivial subdivision.

6. The weak factorization theorem. In this section, we harvest the fruit "Weak Factorization Theorem" grown upon the tree of the results of the previous sections.

PROPOSITION 6.1. *We have the weak factorization of a proper equivariant birational map between two nonsingular toric varieties X_Δ and $X_{\Delta'}$ if and only if there exists a simplicial, collapsible and π-nonsingular cobordism Σ between the fans Δ and Δ'.*

PROOF. Suppose we have the weak factorization of a proper equivariant birational map between two nonsingular toric varieties X_Δ and $X_{\Delta'}$. Then the fan Δ' is obtained from Δ by a sequence of smooth star subdivisions and smooth star assemblings (in arbitrary order). By Proposition 4.7 there exists a simplicial and collapsible cobordism Σ between Δ and Δ', which is also π-nonsingular by construction (cf. the proof of Proposition 4.7).

Conversely, suppose there exists a simplicial, collapsible and π-nonsingular cobordism Σ between the fans Δ and Δ'. Write

$$\Sigma = \bigcup_\sigma \overline{\mathrm{Star}(\sigma)} \cup \partial_- \Sigma$$

where the union is taken over all the circuits σ. By the collapsibility of Σ, we can order the circuits $\sigma_1, \ldots, \sigma_m$ so that each σ_i is minimal among the circuits $\sigma_i, \sigma_{i+1}, \ldots, \sigma_m$ with respect to the partial order given by the circuit graph of Σ. Accordingly, we have a sequence of fans

$$\begin{aligned}
\Delta = \Delta_0 = \pi(\partial_-\Sigma) &= \pi\left(\partial_-\left\{\bigcup_{i=1}^{k} \overline{\mathrm{Star}(\sigma_i)} \cup \partial_+\Sigma\right\}\right)\\
\Delta_1 &= \pi\left(\partial_-\left\{\bigcup_{i=2}^{k} \overline{\mathrm{Star}(\sigma_i)} \cup \partial_+\Sigma\right\}\right)\\
&\ldots\\
\Delta_j &= \pi\left(\partial_-\left\{\bigcup_{i=j+1}^{k} \overline{\mathrm{Star}(\sigma_i)} \cup \partial_+\Sigma\right\}\right)\\
&\ldots\\
\Delta_k &= \partial_+\Sigma = \Delta'.
\end{aligned}$$

Note that the fan Δ_{j+1} is obtained from Δ_j by replacing $\partial_-\overline{\mathrm{Star}(\sigma_j)}$ with $\partial_+\overline{\mathrm{Star}(\sigma_j)}$, which is the bistellar operation analyzed in Section 3 and corresponds to a smooth star subdivision followed by a smooth star assembling. Therefore, we conclude $X_{\Delta'}$ is obtained from X_Δ by a sequence of equivariant smooth blowups and smooth blowdowns.

THEOREM 6.2 (The Weak Factorization Theorem). *We have the weak factorization for every proper and equivariant birational map between two nonsingular toric varieties X_Δ and $X_{\Delta'}$, i.e., Conjecture* 1.1 *holds in the weak from.*

PROOF. Let Δ and Δ' be the corresponding nonsingular fans in N_Q with the same support. Then by Theorem 4.3 there exists a simplicial and collapsible cobordism Σ in N_Q^+ between Δ and Δ'. Theorem 5.1 implies there is a simplicial fan $\tilde{\Sigma}$ obtained from Σ by a sequence of star subdivisions such that $\tilde{\Sigma}$ is π-nonsingular and that the process leaves all the π-independent and π-nonsingular cones of Σ unaffected. By Lemma 4.8 we see that $\tilde{\Sigma}$ is also collapsible as well as simplicial and π-nonsingular and that the lower face and upper face of $\tilde{\Sigma}$ are unaffected and hence isomorphic to Δ and Δ', respectively. Thus $\tilde{\Sigma}$ is a simplicial, collapsible and π-nonsingular cobordism between Δ and Δ'. By Proposition 6.1, we have the weak factorization between X_Δ and $X_{\Delta'}$. This completes the proof of Theorem 6.2.

7. The strong factorization theorem. The purpose of this section is to show the strong factorization therem, i.e., a proper and equivariant birational map $X_\Delta \dashrightarrow X_{\Delta'}$ between smooth toric varieties can be factored into a sequence of smooth equivariant blowups $X_\Delta \leftarrow X_{\Delta''}$ followed immediately by smooth equivariant blowdowns $X_{\Delta''} \rightarrow X_{\Delta'}$, based upon the weak factorization theorem (of Section 6 or [Włodarczyk1]). The main difference between the weak and strong factorization theorems is that the former allows the sequence to consist of blowups and blowdowns in any order for the factorization, while the latter allows the sequence to consist only of blowups first and immediately followed by blowdowns. We should emphasize that this section uses only the statement of the weak factorization theorem and hence is independent of the methods of the previous sections and that the reader, if he wishes, can use [Włodarczyk1]'s result as the starting point for this section (though we continue to phrase the statements in Morelli's terminology that we have been using up to Section 6).

Our strategy goes as follows. We start with a simplicial, collapsible and π-nonsingular cobordism Σ between Δ and Δ', whose existence is guaranteed by Theorem 6.2. We construct a new cobordism $\tilde{\Sigma}$ from Σ applying an appropriate sequence of star subdivisions such that $\partial_- \Sigma = \partial_- \tilde{\Sigma}$ is unaffected through the process of the star subdivisions and that the cobordism $\tilde{\Sigma}$ represents, via the bistellar operations (cf. Theorem 3.2), a sequence consisting only of smooth star subdivisions starting from $\Delta = \pi(\partial_- \Sigma) = \pi(\partial_- \tilde{\Sigma})$ and ending with $\pi(\partial_+ \tilde{\Sigma})$. Observing that $\pi(\partial_+ \tilde{\Sigma})$ is obtained from $\pi(\partial_+ \Sigma) = \Delta'$ by a sequence consisting only of smooth star subdivisions, or equivalently $\Delta' = \pi(\partial_+ \Sigma)$ is obtained from $\pi(\partial_+ \tilde{\Sigma})$ by a sequence consisting only of smooth star assemblings, we achieve the strong factorization

$$\Delta = \pi(\partial_- \Sigma) = \pi(\partial_- \tilde{\Sigma}) \leftarrow \pi(\partial_+ \tilde{\Sigma}) \rightarrow \pi(\partial_+ \Sigma) = \Delta' .$$

First we identify the condition for the bistellar operation to consist of a single smooth star subdivision.

DEFINITION 7.1. A π-nonsingular simplicial circuit

$$\sigma = \langle (v_1, w_1), (v_2, w_2), \cdots, (v_k, w_k) \rangle \subset N_{\boldsymbol{Q}} \oplus \boldsymbol{Q} = N_{\boldsymbol{Q}}^+$$

is called pointing up (resp. pointing down) if it has exactly one positive (resp. negative) extremal ray, i.e., we have the linear relation among the primitive vectors $v_i = n(\pi(\rho_i))$ of the projections of the extremal rays ρ_i for σ (after re-numbering)

$$\begin{aligned} v_1 - v_2 - \cdots - v_k = 0 \quad &\text{with } w_1 - w_2 - \cdots - w_k > 0 \\ (\text{resp. } -v_1 + v_2 + \cdots + v_k = 0 \quad &\text{with } -w_1 + w_2 + \cdots + w_k > 0) . \end{aligned}$$

LEMMA 7.2. *Let Σ be a simplicial and π-nonsingular cobordism in $N_{\boldsymbol{Q}}^+$ and $\sigma \in \Sigma$ a circuit which is pointing up. Let*

$$\sigma = \langle (v_1, w_1), (v_2, w_2), \cdots, (v_k, w_k) \rangle \subset N_{\boldsymbol{Q}} \oplus \boldsymbol{Q} = N_{\boldsymbol{Q}}^+$$

with the linear relation among the primitive vectors $v_i = n(\pi(\rho_i))$ of the projections of the extremal rays ρ_i for σ

$$v_1 - v_2 - \cdots - v_k = 0 \quad \textit{with } w_1 - w_2 - \cdots - w_k > 0 .$$

Then the bistellar operation going from $\pi(\partial_-\overline{\mathrm{Star}(\sigma)})$ to $\pi(\partial_+\overline{\mathrm{Star}(\sigma)})$ is a smooth star subdivision with respect to the ray generated by

$$v_1 = v_2 + \cdots + v_k .$$

If σ is pointing down with the linear relation

$$-v_1 + v_2 + \cdots + v_k = 0 \quad \textit{with } -w_1 + w_2 + \cdots + w_k > 0 ,$$

then the bistellar operation going from $\pi(\partial_-\overline{\mathrm{Star}(\sigma)})$ to $\pi(\partial_+\overline{\mathrm{Star}(\sigma)})$ is a smooth star assembling, the inverse of a smooth star subdivision going from $\pi(\partial_+\overline{\mathrm{Star}(\sigma)})$ to $\pi(\partial_-\overline{\mathrm{Star}(\sigma)})$ with respect to the ray generated by

$$v_1 = v_2 + \cdots + v_k .$$

The proof is immediate from Theorem 3.2.

LEMMA 7.3. *Let Σ be a simplicial and π-nonsingular cobordism. Let*

$$\tau = \langle (v_1, w_1), \cdots , (v_l, w_l) \rangle$$

be a π-independent cone of Σ with the $v_i = n(\pi(\rho_i))$ being the primitive vectors of the projections of the extremal rays ρ_i for τ. Let ρ_τ be the midray $\mathrm{Mid}(\tau, l_{r(\tau)})$, where $r(\tau) \in N$ is the vector $r(\tau) = v_1 + \cdots + v_l$, called the "$\pi$-barycenter" of τ. If τ is codefinie with respect to all the circuits $\sigma \in \Sigma$ with $\tau \in \overline{\mathrm{Star}(\sigma)}$, then $\rho_\tau \cdot \Sigma$ stays π-nonsingular.

PROOF. Note that though in the statement of Proposition 5.5 the point q was assumed to be taken from $\mathrm{par}(\pi(\tau))$, we only need the description

$$q = \sum a_i v_i \quad \text{with } 0 \le a_i \le 1$$

(allowing the equality $a_i = 1$) to conclude that the maximum of the π-multiplicities of the π-independent cones does not increase. Thus we can apply the argument in the proof of Proposition 5.5 with

$$q = r(\tau) = v_1 + \cdots + v_l$$

to conclude that the maximum of the π-multiplicities of the π-independent cones does not increase and in particular $\rho_\tau \cdot \Sigma = \mathrm{Mid}(\tau, l_\tau) \cdot \Sigma$ stays π-nonsingular.

DEFINITION 7.4. Let I be a subset, consisting only of π-independent cones, of a simplicial cobordism Σ. Assume I is join closed, i.e.,

$$\tau, \tau' \in I \Longrightarrow \tau + \tau' \in I \quad (\text{provided } \tau + \tau' \in \Sigma) .$$

We denote

$$I \cdot \Sigma = \rho_{\tau_n} \cdots \rho_{\tau_1} \cdot \Sigma$$

where ρ_{τ_i} is the midray $\mathrm{Mid}(\tau_i, l_{r(\tau_i)})$ with $r(\tau_i)$ being the π-barycenter of τ_i, as described in Lemma 7.3, and where the τ_i are cones in I so ordered that $\dim \tau_i \ge \dim \tau_{i+1}$ for all

i. (Observe that, as I is join closed, $I \cdot \Sigma$ is independent of the choice of the order and is well-defined.)

The following simple observation of Morelli is the basis of our method in this section.

LEMMA 7.5. *Let σ be a circuit in a simplicial and π-nonsingular cobordism Σ. Let*

$$\sigma = \langle (v_1, w_1), \cdots, (v_m, w_m), (v_{m+1}, w_{m+1}), \cdots, (v_k w_k) \rangle ,$$

where $v_1, \ldots, v_m, v_{m+1}, \ldots, v_k$ are the primitive vectors in N of the projections of the extremal rays for σ, having the unique linear relation

$$v_1 + \cdots + v_m - v_{m+1} - \cdots - v_k = 0 \quad \textit{with } w_1 + \cdots + w_m - w_{m+1} - \cdots - w_k > 0 .$$

Let

$$\sigma_+ = \langle (v_1, w_1), \cdots, (v_m, w_m) \rangle \quad \textit{and} \quad \sigma_- = \langle (v_{m+1}, w_{m+1}), \cdots, (v_k, w_k) \rangle .$$

(7.5.1) *The fan $\rho_{\sigma_+} \cdot \overline{\mathrm{Star}(\sigma)}$, where ρ_{σ_+} is the midray* $\mathrm{Mid}(\sigma_+, l_{r(\sigma_+)})$ *with $r(\sigma_+)$ being the π-barycenter of σ_+, is π-nonsingular and the closed star of a π-nonsingular pointing up circuit σ'.*

(7.5.2) *If σ is pointing up and I is a join closed subset of σ_-, then $I \cdot \overline{\mathrm{Star}(\sigma)}$ is π-nonsingular and the closed star of a π-nonsingular pointing up circuit.*

PROOF. (7.5.1) First note that, since σ is π-strongly convex and hence does not contain a nonzero vector $0 \neq (0, w) \in N_{\mathbf{Q}}^+ = N_{\mathbf{Q}} \oplus \mathbf{Q}$, it is impossible to have all the coefficients in the linear relation to be $+1$ or all to be -1.

Let $\eta \in \mathrm{Star}(\sigma)$ be a simplicial cone of the form

$$\eta = \langle (u_1, w_1'), \cdots, (u_l, w_l'), (v_1, w_1), \cdots, (v_k, w_k) \rangle .$$

Then the maximal cones of $\rho_{\sigma_+} \cdot \eta$ are of the form

$$\Big\langle (u_1, w_1'), \cdots, (u_l, w_l'), (u_1, w_1), \cdots, (\overset{\vee}{v_i, w_i}), \cdots, (v_m, w_m), \\ (v_{m+1}, w_{m+1}), \cdots, (v_k, w_k), \left(r(\sigma_+), \sum_{i=1}^m w_i\right)\Big\rangle$$

omitting one of (v_i, w_i), $1 \leq i \leq m$, from the generators of σ_+. Therefore,

$$\sigma' = \left\langle \left(r(\sigma_+), \sum_{i=1}^m w_i\right), (v_{m+1}, w_{m+1}), \cdots, (v_k, w_k) \right\rangle$$

is the unique circuit in $\rho_{\sigma_+} \cdot \overline{\mathrm{Star}(\sigma)}$ and

$$\rho_{\sigma_+} \cdot \overline{\mathrm{Star}(\sigma)} = \overline{\mathrm{Star}(\sigma')} .$$

As ρ_{σ_+} is generated by the vector $(r(\sigma_+), \sum_{i=1}^m w_i) = (\sum_{i=1}^m v_i, \sum_{i=1}^m w_i)$, the unique linear relation for σ' is

$$n(\pi(\rho_{\sigma_+})) - v_{m+1} - \cdots - v_k = 0 ,$$

where $n(\pi(\rho_{\sigma_+})) = \sum_{i=1}^{m} v_i$ with $(\sum_{i=1}^{m} w_i) - w_{m+1} - \cdots - w_k > 0$. Therefore, the circuit σ' is pointing up. We note that π-nonsingularity is preserved as σ_+ is obviously codefinite with respect to the circuit σ.

(7.5.2) We use the same notation as in (7.5.1) with $\sigma_+ = \langle (v_1, w_1) \rangle$ being the only positive extremal ray of the pointing up circuit σ. Let ζ be the maximal cone in I. Then the maximal cones η' of $\rho_\zeta \cdot \eta$, where ρ_ζ is the midray $\mathrm{Mid}(\zeta, l_{r(\zeta)})$, are of the form

$$\Big\langle (u_1, w_1'), \cdots, (u_l, w_l'), (v_1, w_1),$$
$$(v_{1+1}, w_{1+1}), \cdots, \overset{\vee}{(v_j, w_j)}, \cdots, (v_k, w_k), \Big(r(\zeta), \sum_{(v_i, w_i) \in \zeta} w_i\Big)\Big\rangle, \quad (v_j, w_j) \in \zeta\,.$$

Therefore,

$$\sigma_\zeta = \Big\langle (v_1, w_1), \text{ all the } (v_i, w_i) \notin \zeta, \Big(r(\zeta), \sum_{(v_i, w_i) \in \zeta} w_i\Big)\Big\rangle$$

is the unique circuit in $\rho_\zeta \cdot \overline{\mathrm{Star}(\sigma)}$, which is pointing up with the unique linear relation

$$v_1 - \sum_{(v_i, w_i) \notin \zeta} v_i - n(\pi(\rho_\zeta)) = 0\,,$$

where $n(\pi(\rho_\zeta)) = \sum_{(v_i, w_i) \in \zeta} v_i$ with $w_1 - \sum_{(v_i, w_i) \notin \zeta} w_i - (\sum_{(v_i, w_i) \in \zeta} w_i) > 0$. With $\eta \in \mathrm{Star}(\sigma)$ being arbitrary, we also have

$$\rho_\zeta \cdot \overline{\mathrm{Star}(\sigma)} = \overline{\mathrm{Star}(\sigma_\zeta)}\,.$$

Moreover, every cone in the complement I' of ζ in I (i.e., I' consists of the proper subfaces of ζ) is disjoint from σ_ζ. Therefore, σ_ζ is still the unique circuit, which is pointing up, in

$$I \cdot \overline{\mathrm{Star}(\sigma)} = I' \cdot \rho_\zeta \cdot \overline{\mathrm{Star}(\sigma)}$$

and

$$I \cdot \overline{\mathrm{Star}(\sigma)} = \overline{\mathrm{Star}(\sigma_\zeta)}\,.$$

This completes the proof of Lemma 7.5.

The following is an easy consequence of Lemma 7.5.

LEMMA 7.6. *Let Σ be a simplicial, collapsible and π-nonsingular cobordism whose circuits are all pointing up and let $I \subset \partial_- \Sigma$ be a join closed subset. Assume the condition $(\star)$:*

$$(\star) \quad I \cap \overline{\mathrm{Star}(\sigma)} \subset \{\tau \in \Sigma; \tau \subset \sigma_-\} = \partial_- \sigma \quad \textit{for any circuit } \sigma \in \Sigma\,.$$

Then $\Sigma' = I \cdot \Sigma$ is again a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits.

PROOF. By Lemma 4.8 and Lemma 7.3 the cobordism Σ' is gain simplicial, collapsible and π-nonsingular. We only have to check that $I \cdot \overline{\mathrm{Star}(\sigma)} = (I \cap \overline{\mathrm{Star}(\sigma)}) \cdot \overline{\mathrm{Star}(\sigma)}$ contains only pointing up circuits for any circuit $\sigma \in \Sigma$, which follows immediately from the condition $(\star)$ and (7.5.2) in Lemma 7.5.

REMARK 7.7. Lemma 7.6 is a modification of Lemma 9.7 in [Morelli1] (together with the notion of "neatly founded"), which unfortunately has a counter-example as below. We observe that the notion of "neatly founded" is used only in the form of the condition ($\star$) in the argument of [Morelli1] and we carry out our argument here all through with the condition ($\star$) instead of the notion of "neatly founded".

Below we recall the definition of "neatly founded" and Lemma 9.7 in [Morelli1] and then present a counter-example.

[Morelli1] defines that Σ is "neatly founded" if for each down definite face $\tau \in \Sigma$ (A face $\tau \in \Sigma$ is down definite if $\tau \in \partial_-\Sigma$ but $\tau \notin \partial_+\Sigma$.), there is a circuit $\sigma \in \Sigma$ such that $\tau = \sigma_-$.

LEMMA 9.7 in [Morelli1]. *Let Σ be a neatly founded, simplicial, collapsible and π-nonsingular cobordism whose circuits are all pointing up, and let $I \subset \partial_-\Sigma$ be join closed. Then $\Sigma' = I \cdot \Sigma$ is again a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits.*

A counter-example to Lemma 9.7 in [Morelli1]:

We take

$$\begin{aligned}
\rho_1 &= (v_1, 0)\\
\rho_2 &= (v_2, 0)\\
\rho_3 &= (v_3, 0)\\
\rho_4 &= (v_1 + v_2 + v_3, 1)\\
\rho_5 &= (v_1 + v_2 + 2v_3, 2)
\end{aligned}$$

in $N^+_{\boldsymbol{Q}} = (N \oplus \boldsymbol{Z}) \otimes \boldsymbol{Q} = N_{\boldsymbol{Q}} \oplus \boldsymbol{Q}$ with $\dim N_{\boldsymbol{Q}} = 3$ where v_1, v_2, v_3 form a $\boldsymbol{Z}$-basis for N. We set Σ to be

$$\Sigma = \left\{ \begin{array}{l} \langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle \text{ and its faces,} \\ \langle \rho_2, \rho_3, \rho_4, \rho_5 \rangle \text{ and its faces,} \\ \langle \rho_1, \rho_3, \rho_4, \rho_5 \rangle \text{ and its faces} \end{array} \right\}.$$

The fan Σ is by construction a simplicial, collapsible and π-nonsingular cobordism between $\Delta = \partial_-\Sigma$ and $\Delta' = \partial_+\Sigma$.

The cobordism Σ is neatly founded as $\langle \rho_1, \rho_2, \rho_3 \rangle$ is the only down definite face and there is a circuit $\langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ such that

$$\langle \rho_1, \rho_2, \rho_3 \rangle = \langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle_- .$$

All circuits $\langle \rho_1, \rho_2, \rho_3, \rho_4 \rangle$ and $\langle \rho_3, \rho_4, \rho_5 \rangle$ are pointing up.

Take

$$I = \{\langle \rho_2, \rho_3 \rangle \text{ and its faces}\}.$$

Now Σ and I satisfy all the conditions of Lemma 9.7. On the other hand, $\Sigma' = I \cdot \Sigma$ contains a circuit

$$\langle \rho_2, M, \rho_4, \rho_5 \rangle \quad \text{where } M = (v_2 + v_3, 0),$$

which is NOT pointing up!

We resume our proof of the implication the "weak" factorization⇒the "strong" factorization.

PROPOSITION 7.8. *Let Σ be a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Then there is a simplicial, collapsible and π-nonsingular cobordism Σ' such that*

(7.8.1) *Σ' contains only pointing up circuits,*

(7.8.2) *Σ' satisfies the condition $(\star)$ for any join closed subset $I \subset \partial_-\Sigma'$,*

(7.8.3) *Σ' is obtained from Σ by a sequence of star subdivisions, none of which involve $\partial_-\Sigma$, of the π-independent faces which are codefinite with respect to all the circuits.*

PROOF. Express the collapsible Σ as

$$\Sigma = \overline{\mathrm{Star}(\sigma_m)} \circ \overline{\mathrm{Star}(\sigma_{m-1})} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+\Sigma$$

for the circuits $\sigma_m, \sigma_{m-1}, \dots, \sigma_1 \in \Sigma$ so that σ_i is minimal among $\sigma_i, \sigma_{i-1}, \dots, \sigma_1$ according to the partial order given by the circuit graph. We prove the lemma by induction on m.

Case $m = 1$: This case is the building block of the construction in the induction step and we state it in the form of a lemma as below.

LEMMA 7.9. *Let Σ be a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Let $\overline{\mathrm{Star}(\sigma)}$ be the closed star of a circuit $\sigma \in \Sigma$. Let*

$$J = \{\langle\sigma_+, \nu\rangle; \nu \in \mathrm{link}_\Sigma(\sigma)\}.$$

Then

(7.9.1) *$J \cdot \overline{\mathrm{Star}(\sigma)}$ contains only pointing up circuits,*

(7.9.2) *$J \cdot \overline{\mathrm{Star}(\sigma)}$ satisfies the condition $(\star)$ for any join closed subset $J \subset \partial_-\{J \cdot \overline{\mathrm{Star}(\sigma)}\}$, and*

(7.9.3) *$J \cdot \overline{\mathrm{Star}(\sigma)}$ is obtained from $\overline{\mathrm{Star}(\sigma)}$ by a sequence of star subdivisions, none of which involve $\partial_-\overline{\mathrm{Star}(\sigma)}$, of the π-independent faces which are codefinite with respect to all the circuits.*

PROOF. Let

$$\sigma = \langle (v_1, w_1), (v_2, w_2), \dots, (v_k, w_k)\rangle$$

where $v_1, v_2, \dots, v_k$ are primitive vectors in N satisfying the unique linear relation

$$v_1 - v_2 - \cdots - v_k = 0 \quad \text{with } w_1 - w_2 - \cdots - w_k > 0.$$

Let $\eta \in \mathrm{Star}(\sigma)$ be a simplicial cone of the form

$$\eta = \langle (u_1, w_1'), \dots, (u_l, w_l'), (v_1, w_1), \dots, (v_k, w_k)\rangle.$$

Then the circuits of $J \cdot \eta = \{J \cap \eta\} \cdot \eta$ are the cones of the form

$$\sigma_\nu = \left\langle \left(r(\langle\sigma_+, \nu\rangle), w_1 + \sum_{(u_j, w_j') \in \nu} w_j' \right), (v_2, w_2), \dots, (v_k, w_k), \text{ all the } (u_j, w_j') \in \nu \right\rangle$$

for $\nu \in \mathrm{link}_\eta(\sigma)$ (including $\sigma = \sigma_\emptyset = \langle (r(\sigma_+), w_1) = (v_1, w_1), (v_2, w_2), \ldots, (v_k, w_k)\rangle$ for $\nu = \emptyset$) satisfying the unique linear relation

$$\left(v_1 + \sum_{(u_j, w'_j) \in \nu} u_j\right) - v_2 - \cdots - v_k - \sum_{(u_j, w'_j) \in \nu} u_j = 0$$

$$\text{with } \left(w_1 + \sum_{(u_j, w'_j) \in \nu} w'_j\right) - w_2 - \cdots - w_k - \sum_{(u_j, w'_j) \in \nu} w'_j > 0\,.$$

Thus $J \cdot \eta$ contains only pointing up circuits. Since $\eta \in \mathrm{Star}(\sigma)$ is arbitrary, we conclude $J \cdot \overline{\mathrm{Star}(\sigma)}$ contains only pointing up circuits, proving (7.9.1).

We also observe that the maximal cones of $\overline{\mathrm{Star}(\sigma_\nu)}$ are of the form

$$\langle \sigma_\nu, \mathrm{Mid}(\langle v_1, \nu, (u_{p(1)}, w'_{p(1)}), \ldots, (u_{p(s)}, w'_{p(s)})\rangle, l_{r(\mathrm{Mid}(\langle v_1, \nu, (u_{p(1)}, w'_{p(1)}), \ldots, (u_{p(s)}, w'_{p(s)})\rangle))}),$$
$$s = 1, \ldots, l' = l - \sharp\{(u_j, w'_j) \in \nu\}\rangle$$

where

$$(u_{p(1)}, w'_{p(1)}), (u_{p(2)}, w'_{p(2)}), \ldots, (u_{p(l')}, w'_{p(l')})$$

are the (u_j, w'_j)'s NOT belonging to ν, ordered in the specified way by a permutation p. Therefore, any cone in the lower face $\partial_- \overline{\mathrm{Star}(\sigma_\nu)}$, if not included in σ_ν, is also in the upper face but not in the lower face of the closed star of some other circuit of $J \cdot \overline{\mathrm{Star}(\sigma)}$. Therefore, we conclude that for any join closed subset $I \subset \partial_-\{J \cdot \overline{\mathrm{Star}(\sigma)}\}$ we have

$$\begin{aligned} I \cap \overline{\mathrm{Star}(\sigma_\nu)} &= \partial_-\{J \cdot \overline{\mathrm{Star}(\sigma)}\} \cap \{\tau \in J \cdot \overline{\mathrm{Star}(\sigma)};\, \tau \subset \sigma_\nu\} \\ &\subset \{\tau \in J \cdot \overline{\mathrm{Star}(\sigma)};\, \tau \subset (\sigma_\nu)_-\} = \partial_- \sigma_\nu\,. \end{aligned}$$

Since $\eta \in \mathrm{Star}(\sigma)$ is arbitrary, this proves (7.9.2).

The condition (7.9.3) is obvious from the construction.

This completes the proof of Lemma 7.9.

We go back to the proof of Proposition 7.8 resuming the induction.

Suppose $m > 1$. Set

$$\Sigma_{m-1} = \overline{\mathrm{Star}(\sigma_{m-1})} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+ \Sigma$$

and apply the induction hypothesis to Σ_{m-1} to obtain Σ'_{m-1} satisfying the conditions (7.8.1), (7.8.2) and (7.8.3). Then $\overline{\mathrm{Star}(\sigma_m)} \circ \Sigma'_{m-1}$ is the result of a sequence of star subdivisions, none of which involve $\partial_-\Sigma$, of the π-independent faces which are codefinite with respect to all the circuits. Let

$$J = \{\langle (\sigma_m)_+, \nu\rangle;\, \nu \in \mathrm{link}_\Sigma(\sigma_m)\}\,.$$

We show that $J \cdot (\overline{\mathrm{Star}(\sigma_m)} \circ \Sigma'_{m-1})$ satisfies the conditions (7.8.1), (7.8.2) and (7.8.3).

Since Σ'_{m-1} satisfies the condition (7.8.1) and $J \subset \partial_- \Sigma'_{m-1}$ is join closed, the condition ($\star$) for J with Lemma 7.6 implies that $J \cdot \Sigma'_{m-1}$ is a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Lemma 7.9 implies that $J \cdot \overline{\mathrm{Star}(\sigma_m)}$ is also a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Therefore,

$$\Sigma' = J \cdot (\overline{\mathrm{Star}(\sigma_m)} \circ \Sigma'_{m-1}) = (J \cdot \overline{\mathrm{Star}(\sigma_m)}) \circ (J \cdot \Sigma'_{m-1})$$

is a simplicial, collapsible and π-nonsingular cobordism satisfying the condition (7.8.1).

Observe that

$$\partial_-\Sigma' = \partial_-\overline{\mathrm{Star}(\sigma_m)} \cup (\partial_-\Sigma'_{m-1} - \mathrm{RelInt}(J))\,.$$

Thus by construction we have the condition (7.8.2).

Let I be any join closed subset of $\partial_-\Sigma'$. Let $\sigma' \in \Sigma'$ be a circuit. If $\sigma' \in J \cdot \overline{\mathrm{Star}(\sigma_m)}$, then by Lemma 7.9 we have

$$I \cap \overline{\mathrm{Star}(\sigma')} = (I \cap J \cdot \overline{\mathrm{Star}(\sigma_m)}) \cap \overline{\mathrm{Star}(\sigma')} \subset \partial_-\sigma'\,.$$

If $\sigma' \in J \cdot \Sigma'_{m-1}$ and $\sigma' \notin \Sigma'_{m-1}$, then there exists a circuit $\sigma = \langle (v_1, w_1), \dots, (v_k, w_k)\rangle \in \Sigma'_{m-1}$ such that

$$\sigma' = \sigma_\zeta = \left\langle (v_1, w_1),\ \text{all the } (v_i, w_i)\ i \notin \zeta,\ \left(r(\zeta),\ \sum_{(v_i,w_i)\in\zeta} w_i\right)\right\rangle$$

where ζ is the maximal cone in $J \cap \{\tau \in \Sigma'_{m-1};\, \tau \subset \sigma\}$, using the same notation as in Lemma 7.5. Observe that for any maximal cone $\eta'' \in \overline{\mathrm{Star}(\sigma')}$ if a face $\tau \subset \eta''$ contains a new ray used for the subdividing operation "$J\cdot$" as one of the generators then $\tau \notin I$. Therefore, by looking at the description of η' in Lemma 7.5 and η'' obtained from η' by the star subdivision of some faces of ζ, we conclude

$$I \cap \{\tau \subset \Sigma';\, \tau \subset \eta''\} = (I \cap \{\tau \in \Sigma';\, \tau \subset \sigma'\}) \cap \partial_-\sigma' \subset \partial_-\sigma'\,.$$

If $\sigma' \in J \cdot \Sigma'_{m-1}$ and also $\sigma' \in \Sigma'_{m-1}$, then the condition ($\star$) for Σ'_{m-1} implies

$$I \cap \overline{\mathrm{Star}(\sigma')} \subset \partial_-\sigma'\,.$$

Thus we have the condition ($\star$) for Σ' proving the condition (7.8.2).

This completes the proof of Proposition 7.8.

THEOREM 7.10. *Any simplicial, collapsible and π-nonsingular cobordism Σ between Δ and Δ' can be made into a simplicial, collapsible and π-nonsingular cobordism Σ' between Δ and Δ'' by a sequence of star subdivisions such that Σ' contains only pointing up circuits and that Δ'' is obtained from Δ' by a sequence of smooth star subdivisions.*

PROOF. Express

$$\Sigma = \overline{\mathrm{Star}(\sigma_m)} \circ \overline{\mathrm{Star}(\sigma_{m-1})} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+\Sigma$$

for the circuits $\sigma_m, \sigma_{m-1}, \dots, \sigma_1 \in \Sigma$ so that σ_i is minimal among $\sigma_i, \sigma_{i-1}, \dots, \sigma_1$ according to the partial order given by the circuit graph.

Define a sequence of cobordisms $\tilde{\Sigma}_k$, $\tilde{\Sigma}'_k$ inductively as follows: Let

$$\tilde{\Sigma}_0 = \tilde{\Sigma}'_0 \doteq \partial_+ \Sigma$$
$$\tilde{\Sigma}_k = \rho_{\sigma_k^+} \cdot (\overline{\mathrm{Star}(\sigma_k)} \circ \tilde{\Sigma}'_{k-1})$$

where $\tilde{\Sigma}'_{k-1}$ for $k \geq 2$ is obtained from $\tilde{\Sigma}_{k-1}$ by the procedure described in Proposition 7.8 to satisfy the conditions (7.8.1), (7.8.2) and (7.8.3). We remark that

$$\partial_- \tilde{\Sigma}_k = \partial_- \tilde{\Sigma}'_k = \partial_-(\overline{\mathrm{Star}(\sigma_k)} \circ \overline{\mathrm{Star}(\sigma_{k-1})} \circ \cdots \circ \overline{\mathrm{Star}(\sigma_1)} \circ \partial_+ \Sigma)\,.$$

Note then that inductively by Lemma 7.5, Lemma 7.6 and Proposition 7.8 $\tilde{\Sigma}_k$ is a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits. Finally $\tilde{\Sigma} = \tilde{\Sigma}_m$ is a simplicial, collapsible and π-nonsingular cobordism containing only pointing up circuits between $\Delta = \pi(\partial_- \Sigma) = \pi(\partial_- \tilde{\Sigma})$ and $\Delta'' = \pi(\partial_+ \tilde{\Sigma})$, which is obtained from Δ' by a sequence of smooth star subdivisions.

This completes the proof of Theorem 7.10.

COROLLARY 7.11 (The Strong Factorization Theorem). *We have the strong factorization for every proper and equivariant birational map between two nonsingular toric varieties X_Δ and $X_{\Delta'}$, i.e., Conjecture* 1.1 *holds in the strong form. In particular, if both X_Δ and $X_{\Delta'}$ are projective, then the factorization can be chosen so that all the intermediate toric varieties are also projective.*

PROOF. Let Δ and Δ' be the corresponding two nonsingular fans in N_Q with the same support. Then by Proposition 6.1 and Theorem 6.2 there exists a simplicial, collapsible and π-nonsingular cobordism Σ between Δ and Δ'. By Theorem 7.10 we can make Σ into a simplicial, collapsible and π-nonsingular cobordism $\tilde{\Sigma}$ with only pointing up circuits between Δ and a fan Δ'' such that Δ'' is obtained from Δ' by a sequence of smooth star subdivisions. By Lemma 7.2 $\Delta'' = \pi(\partial_+ \tilde{\Sigma})$ is also obtained from $\pi(\partial_- \tilde{\Sigma}) = \Delta$ by a sequence of smooth star subdivisions. Thus we have the factorization

$$\Delta = \pi(\partial_- \Sigma) = \pi(\partial_- \tilde{\Sigma}) \leftarrow \pi(\partial_+ \tilde{\Sigma}) \rightarrow \pi(\partial_+ \Sigma) = \Delta'\,,$$

which corresponds to the strong factorization

$$X_\Delta \leftarrow X_{\Delta''} \rightarrow X_{\Delta'}\,.$$

8. The toroidal case. The purpose of this section is to generalize the main theorem of the previous sections, namely the strong factorization of a proper and equivariant birational map between two nonsingular toric varieties, to the one in the toroidal case.

First we recall several definitions about the toroidal embeddings (cf. [Kempf-Knudsen-Mumford-SaintDonat]) and the notion of a "toroidal" morphism as in [Abramovich-Karu].

DEFINITION 8.1 (Toroidal Embeddings). Given a normal variety X and an open subset $U_X \subset X$, the embedding $U_X \subset X$ is called toroidal if for every closed point $x \in X$ there exist an affine toric variety X_σ, a closed point $s \in X_\sigma$ and an isomorphism of complete local algebras

$$\hat{\mathcal{O}}_{X,x} \cong \hat{\mathcal{O}}_{X_\sigma,s}$$

so that the ideal in $\hat{\mathcal{O}}_{X,x}$ generated by the ideal of $X - U_X$ corresponds under this isomorphism to the ideal in $\hat{\mathcal{O}}_{X_\sigma,s}$ generated by the ideal of $X_\sigma - T$, where T is the torus. The affine toric variety X_σ is called a local model of X at x.

We will always assume that the irreducible components of $\bigcup_{i\in I} E_i = X - U_X$ are normal, i.e., $U_X \subset X$ is a toroidal embedding without self-intersection. (In fact, in most of the cases X is nonsingular and $\bigcup_{i\in I} E_i \subset X$ is a divisor with normal crossings whose irreducible components are all nonsingular.)

The irreducible components of $\bigcap_{i\in J} E_i$ for $J \subset I$, together with U_X, define a stratification of X. (These components and X are the closures of the strata. The closures of the strata formally correspond to the closures of the orbits in local models.)

Let S be a stratum in X, which is by definition an open set in an irreducible component of $\bigcap_{i\in J} E_i$ for some $J \subset I$. The star Star(S) is the union of those strata containing S in their closure (each of them corresponds to some $K \subset J \subset I$). To the stratum S one associates the following data:

M^S: the group of Cartier divisors in Star(S) supported in Star(S) $- U_X$
$N^S := \mathrm{Hom}(M^S, \boldsymbol{Z})$
$M^S_+ \subset M^S$: effective Cartier divisors
$\sigma^S \subset N^S_{\boldsymbol{R}}$: the dual of M^+_S.
If (X_σ, s) is a local model at $x \in X$ in the stratum S, then

$$M^S \cong M_\sigma/\sigma^\perp\,, \quad N^S \cong N_\sigma \cap \mathrm{span}(\sigma) \text{ and } \sigma^S \cong \sigma\,.$$

The cones glue together to form a conical complex

$$\Delta_X = (|\Delta_X|, \{\sigma^S\}, \{N^S\})\,,$$

where $|\Delta_X| = \bigcup_S \sigma^S$ is the support of Δ_X and the lattices N^S form an integral structure on Δ_X with $\sigma^S \hookrightarrow N^S_{\boldsymbol{R}}$.

DEFINITION 8.2 (Toroidal Morphisms). A dominant morphism

$$f : (U_X \subset X) \to (U_Y \subset Y)$$

of toroidal embeddings is called toroidal if for every closed point $x \in X$ there exist local models (X_σ, s) at x and $X_{\tau,t}$ at $y = f(x)$ and a toric morphism $g : X_\sigma \to X_\tau$ such that the following diagram commutes

$$\begin{array}{ccc} \hat{\mathcal{O}}_{X,x} & \cong & \hat{\mathcal{O}}_{X_\sigma,s} \\ \uparrow{\scriptstyle \hat{f}^*} & & \uparrow{\scriptstyle \hat{g}^*} \\ \hat{\mathcal{O}}_{Y,y} & \cong & \hat{\mathcal{O}}_{X_\tau,t}\,. \end{array}$$

Now we can state our main result of this section.

THEOREM 8.3. *Let*

$$f : (U_X \subset X) \to (U_Y \subset Y)$$

be a proper birational and toroidal morphism between toroidal embeddings, where X and Y are nonsingular and $\bigcup_{i\in I} E_i = X - U_X$ and $\bigcup_{j\in J} F_j = Y - U_Y$ are divisors with normal crossings whose irreducible components are all nonsingular. Then there exist a toroidal embedding (U_V, V) and sequences of blowups, with centers being smooth closed strata, which factor f

$$(U_X, X) \leftarrow (U_V, V) \rightarrow (U_Y, Y).$$

LEMMA 8.4. *Let*

$$f : (U_X \subset X) \rightarrow (U_Y \subset Y)$$

be a toroidal morphism between two toroidal embeddings.

(8.4.1) *f induces a morphism $f_\Delta : \Delta_X \rightarrow \Delta_Y$ of complexes such that each $\sigma^S \in \Delta_X$ maps to some $\sigma^{S'} \in \Delta_Y$ linearly $f_\Delta : \sigma^S \hookrightarrow \sigma^{S'}$ with the map of lattices of the integral structures $N_{\sigma^S} \rightarrow N_{\sigma^{S'}}$.*

(8.4.2) *If f is proper and birational, then each $\sigma^S \in \Delta_X$ maps injectively into some $\sigma^{S'} \in \Delta_Y$ linearly $f_\Delta : \sigma^S \hookrightarrow \sigma^{S'}$ and the lattice N_{σ^S} is a saturated sublattice of $N_{\sigma^{S'}}$. In short, Δ_X is a refinement of Δ_Y with $|\Delta_X| = |\Delta_Y|$ preserving the integral structure. Moreover, once we fix the toroidal embedding $(U_Y \subset Y)$, there is a one-to-one correspondence between the set of refinements $f_\Delta : \Delta_X \rightarrow \Delta_Y$ preserving the integral structures and the set of toroidal embeddings mapping proper birationally onto $(U_Y \subset Y)$ by toroidal morphisms $f : (U_X \subset X) \rightarrow (U_Y \subset Y)$.*

PROOF. For a proof, we refer the reader to [Kempf-Knudsen-Mumford-SaintDonat] and [Abramovich-Karu]. We only note that a proper birational toroidal morphism between toroidal embeddings without self-intersection is always allowable in the sense of [Kempf-Knudsen-Mumford-SaintDonat].

We can reformulate via the lemma our main theorem of this section in terms of the conical complexes (which are always assumed to be finite in this section).

THEOREM 8.5. *Let $f_\Delta : \Delta' \rightarrow \Delta$ be a map between two nonsingular conical complexes, which represents a refinement preserving the integral structure. Then there exist a nonsingular conical complex Δ'' obtained both from Δ' and from Δ by some sequences of smooth star subdivisions which factor f_Δ*

$$\Delta \leftarrow \Delta'' \rightarrow \Delta.$$

Given a conical complex Δ, we consider the space $N^S \oplus \mathbf{Z}$, for each $N^S = N_{\sigma^S}$ associated to the cone $\sigma^S \in \Delta$, which can be glued together naturally via the glueing of N^S to form the integral structure. We denote this space $N_\Delta \oplus \mathbf{Z}$. By considering the space $(N^S \oplus \mathbf{Z}) \otimes \boldsymbol{Q}$ and glueing them together, we obtain the space

$$(N_\Delta)^+_{\boldsymbol{Q}} = (N_\Delta \oplus \mathbf{Z}) \otimes \boldsymbol{Q} = (N_\Delta)_{\boldsymbol{Q}} \oplus \boldsymbol{Q}$$

with the lattices $N^S \oplus \mathbf{Z}$ also glued together to form the integral structure $N_\Delta \oplus \mathbf{Z}$.

If $f_\Delta : \Delta' \rightarrow \Delta$ is a refinement of Δ preserving the integral structures, then we can identify $(N_{\Delta'})^+_{\boldsymbol{Q}}$ with $(N_\Delta)^+_{\boldsymbol{Q}}$ having the same integral structure $N_{\Delta'} \oplus \mathbf{Z} = N_\Delta \oplus \mathbf{Z}$.

Observe that as in the case of toric fans we can define a cobordism Σ in the space $(N_\Delta)^+_{\boldsymbol{Q}}$ between Δ' and Δ as well as the notions of collapsibility, π-nonsingularity, pointing up, etc.

Once this is understood, we can carry out the same strategy as the one presented in Section 1 through 7 by Morelli to factor a proper birational toroidal morphism and we only have to prove:

THEOREM 8.6. *Let $f_\Delta : \Delta' \to \Delta$ be a map between two nonsingular conical complexes, which represents a refinement preserving the integral structure. Then there exists a simplicial, collapsible and π-nonsingular cobordism Σ in $(N_\Delta)^+_{\boldsymbol{Q}}$ between conical complexes Δ'' and Δ such that Δ'' is obtained from Δ' by a sequence of smooth star subdivisions and that Σ consists only of pointing up circuits and hence Δ'' is also obtained from Δ by a sequence of smooth star subdivisions.*

PROOF. We follow exactly the line of argument developed in the previous sections.

First we claim that there exists a simplicial and collapsible cobordism Σ between Δ and Δ'. Recall that in order to construct a cobordism and make it collapsible in the argument for the toric case we have utilized such global theorems as Sumihiro's and Moishezon's, which are no longer applicable in the toroidal case. This calamity can be avoided by using the following simple lemma.

LEMMA 8.7. *Let Δ be a simplicial conical complex. Then we can embed the barycentric star subdivision Δ_B (cf. Definition 2.1) into a toric fan Δ_B^T in some vector space $N_{\boldsymbol{Q}}$, i.e., there is a bijective map $i : |\Delta_B| \to |\Delta_B^T|$ such that it restricts to a linear isomorphism to each cone $i : \sigma \to \sigma^T$. (Note that we do NOT require i to preserve the integral structure.)*

PROOF. We prove by induction on the dimension d of Δ and the number of the cones of the maximal dimension d.

When $d = 1$, i.e., Δ is a finite number of lines, the assertion is obvious.

Suppose the assertion is proved already for a simplicial conical complex of either dimension $< d$ or dimension d with $k-1$ number of the cones of the maximal dimension d. Take a simplicial conical complex Δ of dimension d with k number of the cones of the maximal dimension d. Choose one cone σ of dimension d and let $\Delta_\sigma = \Delta - \{\sigma\}$. By the induction hypothesis, we can embed the barycentric star subdivision $(\Delta_\sigma)_B$ into a toric fan $(\Delta_\sigma)_B^T$ in some vector space $N'_{\boldsymbol{Q}}$

$$i' : |(\Delta_\sigma)_B| \xrightarrow{\sim} |(\Delta_\sigma)_B^T|.$$

We take $N_{\boldsymbol{Q}} = N'_{\boldsymbol{Q}} \oplus \boldsymbol{Q}$ and regard $N'_{\boldsymbol{Q}} = N'_{\boldsymbol{Q}} \oplus \{0\} \subset N_{\boldsymbol{Q}}$. We only have to take the embedding $i : \Delta_B \to \Delta_B^T$ to be the one such that

$$i\big|_{(\Delta_\sigma)_B} = i' : |(\Delta_\sigma)_B| \to |(\Delta_\sigma)_B^T| \subset N'_{\boldsymbol{Q}} \subset N_{\boldsymbol{Q}} \quad \text{and} \quad i(r(\sigma)) = (0,1) \in N'_{\boldsymbol{Q}} \oplus \boldsymbol{Q},$$

where $r(\sigma)$ is the barycenter of σ (in the sense of Definition 2.1 and hence corresponding to the sum of the primitive vectors of the extremal rays for σ) and the map i on the cones in Δ_B containing $r(\sigma)$ is defined in the obvious way.

We resume the proof of Theorem 8.6.

Take the barycentric star subdivisions Δ'_B and Δ_B of the conical complexes Δ' and Δ, respectively, and let $\tilde{\Delta}_B$ be a simplicial common refinement of Δ'_B and Δ_B. By Lemma 8.7 we can embed Δ'_B into a toric fan Δ'^T_B in some vector space N'_Q. As $\tilde{\Delta}_B$ is a refinement of Δ'_B, it can also be embedded as a toric fan $\tilde{\Delta}^T_B$ in the same space N'_Q by the extension of the same map. We can take $\Delta^{\circ T}_B$, obtained by a sequence of star subdivisions from Δ'^T_B such that it is a refinement of $\tilde{\Delta}^T_B$ (cf. [DeConcini-Procesi]). By replacing the original $\tilde{\Delta}_B$ with the pull-back of $\Delta^{\circ T}_B$, we may assume that $\tilde{\Delta}_B$ is a refinement of Δ_B and Δ'_B and that $\tilde{\Delta}_B$ is obtained from Δ'_B by a sequence of star subdivisions.

By Lemma 8.7 we can embed Δ_B into a toric fan Δ^T_B in some vector space N_Q. As $\tilde{\Delta}_B$ is a refinement of Δ_B, it can also be embedded as a toric fan $\tilde{\Delta}^T_B$ in the same space by the extension of the same map. Now we can apply the arguments in Section 3 and Section 4 to conclude there is a simplicial and collapsible cobordism in N^+_Q between $\widehat{(\Delta^T_B)}$ and $\widehat{(\tilde{\Delta}^T_B)}$, where $\widehat{(\Delta^T_B)}$ is obtained from Δ^T_B by a sequence of star subdivisions and $\widehat{(\tilde{\Delta}^T_B)}$ is obtained from $\tilde{\Delta}^T_B$ by another sequence of star subdivisions. We can pull back this cobordism to obtain a simplicial and collapsible cobordism $\tilde{\Sigma}$ in $(N_\Delta)^+_Q$ between $\widehat{(\Delta_B)}$ and $\widehat{(\tilde{\Delta}_B)}$, where $\widehat{(\Delta_B)}$ is obtained from Δ by a sequence of star subdivisions (via the barycentric star subdivision Δ_B) and $\widehat{(\tilde{\Delta}_B)}$ is obtained from Δ' by a sequence of star subdivisions (via the barycentric star subdivision Δ'_B and $\tilde{\Delta}_B$). Now we apply Proposition 4.8, which is also valid in the toroidal case, to the lower face $\partial_-\tilde{\Sigma}$ and to the upper face $\partial_+\tilde{\Sigma}$ to extend it to a simplicial and collapsible cobordism Σ between Δ and Δ'.

Now apply the process of π-desingularization described in Section 5, which is word for word valid also in the toroidal case to make Σ a simplicial, collapsible and π-nonsingular cobordism between Δ and Δ'.

Finally apply the process described in Section 7, which is again word for word valid in the toroidal case, to the cobordism above to obtain the desired simplicial, collapsible and π-nonsingular cobordism Σ between Δ'' and Δ such that Δ'' is obtained from Δ' by a sequence of smooth star subdivisions and that Σ consists only of pointing up circuits and hence Δ'' is also obtained from Δ by a sequence of smooth star subdivisions.

This completes the proof of Theorem 8.6 and the verification of the strong factorization theorem for proper birational toroidal morphisms.

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