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ON A LOCAL ERGODIC THEOREM FOR FINITE-DIMENSIONAL-HILBERT-SPACE-VALUED FUNCTIONS

Dedicated to Professor Satoru Igari on the occasion of his retirement from Tohoku University

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Abstract. We consider a Banach space of finite-dimensional-Hilbert-space-valued functions on a sigma-finite measure space. The norm of the function space is assumed to satisfy some suitable conditions. Then we prove a pointwise local ergodic theorem for a (C_0)-semigroup of linear contractions on the function space, under an additional norm condition for operators of the semigroup. Our result extends Baxter and Chacon's local ergodic theorem for scalar-valued functions.

§1. Introduction. Let $(H, \|\cdot\|)$ be a finite-dimensional Hilbert space and (Ω, Σ, μ) a σ-finite measure space. Let $(L, \|\cdot\|_L)$ be a Banach space of H-valued strongly measurable functions on (Ω, Σ, μ). In what follows, two functions f and g in L are not distinguished provided that $f(\omega) = g(\omega)$ for almost all $\omega \in \Omega$. We will assume throughout the paper that the norm $\|\cdot\|_L$ of the space L satisfies the following properties:

(I) If $f, g \in L$ and $\|f(\omega)\| \le \|g(\omega)\|$ for almost all $\omega \in \Omega$, then $\|f\|_L \le \|g\|_L$.

(II) If g is an H-valued strongly measurable function on Ω and $\|g(\omega)\| \le \|f(\omega)\|$ for almost all $\omega \in \Omega$ for some $f \in L$, then $g \in L$.

(III) If $E_n \in \Sigma$, $E_n \supset E_{n+1}$ for each $n \ge 1$ and $\bigcap_{n=1}^{\infty} E_n = \emptyset$, then for every $f \in L$

$$\lim_{n\to\infty} \|\chi_{E_n} \cdot f\|_L = 0\,,$$

where χ_{E_n} denotes the characteristic function of E_n.

(IV) If f and g are in L, $\|f(\omega)\| \le \|g(\omega)\|$ for almost all $\omega \in \Omega$ and $\|f\|_L = \|g\|_L$, then $\|f(\omega)\| = \|g(\omega)\|$ for almost all $\omega \in \Omega$.

It should be remarked that in addition to the usual H-valued L_p-spaces, with $1 \le p < \infty$, there are many interesting H-valued function spaces which satisfy Properties (I) to (IV) (e.g., H-valued Lorentz spaces and H-valued Orlicz spaces, etc.). By simple examples it follows that Properties (III) and (IV) are independent.

Let $\boldsymbol{T} = \{T(t) \mid t \ge 0\}$ be a (C_0)-semigroup of linear contractions on L. This means that for each $t \ge 0$, $T(t)$ is a linear operator on L with $\|T(t)\| \le 1$, where $\|T(t)\|$ denotes the operator norm of $T(t)$ determined by the norm $\|\cdot\|_L$ of L, and the following hold:

(i) $T(0) = I$ (the identity operator) and $T(t+s) = T(t)T(s)$ for $t, s \ge 0$.

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(ii) For every $f \in L$, $\lim_{t\to 0} \|T(t)f - f\|_L = 0$.

Since for an $f \in L$ the vector-valued function $t \mapsto T(t)f$ is continuous on $[0,\infty)$, it is Bochner integrable on every bounded interval $[a,b] \subset [0,\infty)$ with respect to the Lebesgue measure. In this paper we would like to study the a.e. convergence of the ergodic averages

$$\frac{1}{\alpha}\int_0^\alpha T(t)f\,dt \quad (\alpha > 0)$$

as α tends to zero. Since this does not make sense, when the ergodic averages are members of the space L and not actual functions and α ranges through all positive reals, we must restrict ourselves to consider the case where α ranges through a countable subset of the interval $(0,\infty)$. Thus, let $\boldsymbol{D}$ denote a countable dense subset of $(0,\infty)$. We then use the notation

$$\boldsymbol{D}\text{-}\lim_{\alpha\to 0} \quad \text{and} \quad \boldsymbol{D}\text{-}\limsup_{\alpha\to 0}$$

to mean that these limits are taken as α tends to zero through the set $\boldsymbol{D}$. However, the affirmative answer cannot be expected in general if the semigroup $\boldsymbol{T} = \{T(t)\}$ does not satisfy any additional hypothesis, as is found by an example of Akcoglu and Krengel (see [1]). Therefore we assume in Theorem 1 below the following additional hypothesis (cf. [2]–[5]):

$(*)$ There exists a constant $K \geq 1$ such that if $f \in L \cap L_\infty((\Omega,\Sigma,\mu);H)$, then for every $t \geq 0$ we have $T(t)f \in L_\infty((\Omega,\Sigma,\mu);H)$ and

$$\|T(t)f\|_\infty \leq K\|f\|_\infty\,, \tag{1}$$

where the norm $\|\cdot\|_\infty$ of $L_\infty((\Omega,\Sigma,\mu);H)$ is given as

$$\|f\|_\infty = \inf\{\alpha > 0 \mid \|f(\omega)\| \leq \alpha \text{ for almost all } \omega\} \quad (<\infty)\,.$$

Here we remark that, since $\lim_{t\to 0}\|T(t)f - f\|_L = 0$ for $f \in L$, if K satisfies (1) for all $t > 0$ and $f \in L \cap L_\infty((\Omega,\Sigma,\mu);H)$, then we must have $K \geq 1$, and that there exists an example of $\boldsymbol{T} = \{T(t)\}$ for which the hypothesis $(*)$ holds with some constant $K > 1$, but K cannot be replaced by 1. To see this, let w be a periodic continuous function on the real line such that the range of w coincides with the interval [1, 2], and let p be a real number suth that $1 \leq p < \infty$. Then define

$$L = L_p(w\,dm) = \left\{ f \;\middle|\; \int_{-\infty}^{\infty} |f|^p w\,dm < \infty \right\},$$

and

$$\|f\|_L = \left(\int_{-\infty}^{\infty} |f|^p w\,dm \right)^{1/p}$$

for $f \in L$, where f is a complex-valued Lebesgue measurable function and m denotes the Lebesgue measure on the real line. It is clear that $(L, \|\cdot\|_L)$ becomes a Banach space satisfying Properties (I) to (IV). If we define, for $t \geq 0$ and $f \in L$,

$$T(t)f(x) = f(t+x)\left(\frac{w(t+x)}{w(x)}\right)^{1/p},$$

then, by a straightforward calculation, we see that $\boldsymbol{T} = \{T(t) \mid t \geq 0\}$ becomes a (C_0)-semigroup of linear isometries on L. Further, since

$$\max\left\{\left(\frac{w(t+x)}{w(x)}\right)^{1/p} \,\middle|\, x \in (-\infty, \infty),\ t \geq 0\right\} = 2^{1/p},$$

the semigroup $\boldsymbol{T} = \{T(t)\}$ satisfies the hypothesis $(*)$ with $K = 2^{1/p}$, and $2^{1/p}$ is the best constant in the sense that if the hypothesis $(*)$ holds with some constant K', then we must have $K' \geq 2^{1/p}$.

In this paper we will prove the following local ergodic theorem.

THEOREM 1. *If $\boldsymbol{T} = \{T(t) \mid t \geq 0\}$ is a (C_0)-semigroup of linear contractions on L and satisfies the additional hypothesis $(*)$, then for every $f \in L$*

$$\boldsymbol{D}\text{-}\lim_{\alpha \to 0} \frac{1}{\alpha} \int_0^\alpha T(t) f \, dt = f \quad \text{a.e. } on\ \Omega\,. \tag{2}$$

Here we note that Theorem 1 reduces to Baxter and Chacon's local ergodic theorem (cf. [2]) for scalar-valued functions, when $H = \boldsymbol{C}$ (the field of complex numbers), $L = L_p((\Omega, \Sigma, \mu); \boldsymbol{C})$ with $1 \leq p < \infty$, and $K = 1$. The purpose of this paper is to extend their theorem to finite-dimensional-Hilbert-space-valued function spaces including Lorentz spaces and Orlicz spaces, etc. To this end we have examined their arguments thoroughly and found out that their methods can be adapted to prove our Theorem 1. Incidentally, the hypothesis that $\boldsymbol{T}$ is a contraction semigroup cannot be weakened. This follows from Theorem 2 of [6]. The hypothesis that H is finite-dimensional is essential in the paper, because Lemma 6 below does not hold without it. It would be natural to ask whether or not Theorem 1 holds when H is an infinite-dimensional Hilbert space. This is an open problem.

As in [2], we also obtain a more general result. Namely, we have

THEOREM 2. *Let $\boldsymbol{T} = \{T(t) \mid t \geq 0\}$ be a (C_0)-semigroup of linear contractions on L. Suppose there exist a scalar-valued measurable function h on $[0, \infty) \times \Omega$ and a constant $K \geq 1$ such that*

(i) $h(t, \omega) > 0$ *for every* $(t, \omega) \in [0, \infty) \times \Omega$, *and*

(ii) $f \in L$ *and* $\|f(\omega)\| \leq h(t, \omega)$ *for almost all* $\omega \in \Omega$ *imply that* $\|T(s) f(\omega)\| \leq K h(t+s, \omega)$ *for almost all* $\omega \in \Omega$ *and for every* $t, s \geq 0$.

Then (2) *holds for all* $f \in L$.

As is easily seen, to prove these theorems it may be assumed without loss of generality that H is a *real* Hilbert space. Thus, in the following, H will denote a real Hilbert space.

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§2. Lemmas. In this section we prove some necessary lemmas. H and L will be the same as in Introduction.

LEMMA 1. *If (f_n) is a sequence of functions in L such that $\sum_{n=1}^\infty \|f_n\|_L < \infty$, then $\sum_{n=1}^\infty \|f_n(\omega)\| < \infty$ for almost all $\omega \in \Omega$, and the function $f(\omega) = \sum_{n=1}^\infty f_n(\omega)$ is in L and* *satisfies*

$$\lim_{k\to\infty}\left\| f - \sum_{n=1}^{k} f_n \right\|_L = 0\,.$$

LEMMA 2. *Let $f \in L$ and $f_n \in L$ for $n \geq 1$. If $\lim_{n\to\infty} \|f - f_n\|_L = 0$, then there exists a subsequence $(f_{n'})$ of (f_n) such that $\lim_{n'\to\infty} f_{n'}(\omega) = f(\omega)$ for almost all $\omega \in \Omega$.*

LEMMA 3. *Let (f_n) be a sequence of functions in L. If $\lim_{n\to\infty} f_n(\omega) = 0$ for almost all $\omega \in \Omega$ and if there exists an $f \in L$ such that $\|f_n(\omega)\| \leq \|f(\omega)\|$ for almost all $\omega \in \Omega$ and for every $n \geq 1$, then $\lim_{n\to\infty} \|f_n\|_L = 0$.*

These are elementary and hence proofs are omitted (see e.g. [7]).

An important consequence of Lemma 2 is that, to prove Theorems 1 and 2, it may be assumed without loss of generality that $\boldsymbol{D}$ is the set of all positive rationals. Hence, we will assume below that

$$\boldsymbol{D} = \{r > 0 \mid r \text{ is rational}\}\,.$$

Next, let $\Omega^{\sim} = [0, \infty) \times \Omega$. Let $\Sigma^{\sim}$ be the usual product σ-algebra of the Lebesgue measurable subsets of $[0, \infty)$ and Σ, and $\mu^{\sim}$ the product measure of the Lebesgue measure on $[0, \infty)$ and μ. Suppose $\boldsymbol{T} = \{T(t) \mid t \geq 0\}$ is a (C_0)-semigroup of linear contractions on L. For an $f \in L$ and $n \geq 1$, define the function $F_n : [0, \infty) \to L$ by

$$F_n(t) = T(i/n!)f \quad \text{if} \quad i/n! \leq t < (i+1)/n!\,. \tag{3}$$

(The factor $n!$ will be useful for the proof of Lemma 5 below.) Since $\boldsymbol{T}$ is a (C_0)-semigroup, it is easily seen that there exists a subsequence $(n(k))$ of (n) such that

$$\sum_{k=1}^{\infty} \|F_{n(k)}(t) - T(t)f\|_L < \infty \quad \text{for all } t \geq 0\,.$$

Thus, by Lemma 1,

$$\sum_{k=1}^{\infty} \|F_{n(k)}(t)(\omega) - T(t)f(\omega)\| < \infty$$

for almost all $\omega \in \Omega$, and hence we get

$$\lim_{k\to\infty} \|F_{n(k)}(t)(\omega) - T(t)f(\omega)\| = 0 \tag{4}$$

for almost all $\omega \in \Omega$. Taking this into account, let

$$F(t, \omega) = \begin{cases} \lim\limits_{k\to\infty} F_{n(k)}(t)(\omega) & \text{if the limit exists}\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{5}$$

Then we have the following

LEMMA 4. *The function $F : \Omega^{\sim} \to H$ is strongly measurable, and $F(t, \cdot)$ is a representative of the element $T(t)f \in L$ for each $t \geq 0$.*

PROOF. Obvious from the above construction of F.

In the following we will denote

$$T(t)(f,\omega) := F(t,\omega) \quad \text{for } (t,\omega) \in \Omega^{\sim}\,. \tag{6}$$

By Fubini's theorem the function $(t,\omega) \mapsto T(t)(f,\omega)$ is uniquely determined modulo sets of measure zero.

LEMMA 5. *Assume that the semigroup $\boldsymbol{T}$ satisfies the additional hypothesis $(*)$, and that $f \in L \cap L_\infty((\Omega,\Sigma,\mu);H)$. Then, for every $\beta > 0$, the function $f_\beta \in L$ defined by $f_\beta = \int_0^\beta T(t)f\,dt$ satisfies*

$$\boldsymbol{D}\text{-}\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha T(t)f_\beta dt = f_\beta \quad \textit{a.e. on } \Omega\,.$$

PROOF. We may assume without loss of generality that

$$\|T(t)(f,\omega)\| \le K\|f\|_\infty \quad \text{for all } (t,\omega)\in\Omega^{\sim}\,.$$

For each fixed $\omega\in\Omega$, the H-valued function $t\mapsto T(t)(f,\omega)$ is Bochner integrable on every bounded interval in $[0,\infty)$ with respect to the Lebesgue measure by Fubini's theorem, and the H-valued function F_β on $\Omega^{\sim}$ defined by

$$F_\beta(u,\omega) = \int_u^{u+\beta} T(t)(f,\omega)dt \quad \text{for } (u,\omega)\in\Omega^{\sim} \tag{7}$$

is strongly measurable with respect to $(\Omega^{\sim},\Sigma^{\sim},\mu^{\sim})$.

On the other hand, since $\boldsymbol{T} = \{T(t)\}$ is strongly continuous on $[0,\infty)$, it follows that

$$\lim_{n\to\infty}\left\| \int_u^{u+\beta} T(t)f\,dt - \frac{1}{n!}\sum_{i=[n!\,u]}^{[n!\,(u+\beta)]} T(i/n!)f \right\|_L = 0 \tag{8}$$

for every $u \ge 0$, $[t]$ being the largest integer contained in $[0,t]$ for $t\in[0,\infty)$. Here we notice that if $i/n! \le t < (i+1)/n!$, i.e. $i = [n!t]$, then, by (3),

$$T(i/n!)f(\omega) = F_n(t)(\omega) = T(i/n!)(f,\omega) \quad \text{for } \omega\in\Omega\,.$$

Hence, by Fubini's theorem together with (4) and (5), to the subsequence $(n(k))$ in (4) there corresponds a set $N(f)\in\Sigma$, with $\mu(N(f)) = 0$, such that if $\omega\notin N(f)$, then

$$\lim_{k\to\infty} T(i(k)/n(k)!)(f,\omega) = T(t)(f,\omega)$$

for almost every $t\in[0,\infty)$, where $i(k) = [n(k)!t]$. Thus, by the Lebesgue convergence theorem,

$$\begin{aligned} &\lim_{k\to\infty}\frac{1}{n(k)!}\sum_{i=[n(k)!\,u]}^{[n(k)!\,(u+\beta)]} T(i/n(k)!)f(\omega) \\ &\quad = \int_u^{u+\beta} T(t)(f,\omega)dt = F_\beta(u,\omega) \quad \text{for } \omega\notin N(f)\,. \end{aligned} \tag{9}$$

Combining this with (8), we then see from Lemma 2 that for each fixed $u\ge 0$, the H-valued function $\omega\mapsto F_\beta(u,\omega)$ is a representative of the element $T(u)f_\beta = \int_u^{u+\beta} T(t)f\,dt \in L$.

Hence the function $F_\beta(u,\omega)$ on $\Omega^\sim$ can be regarded as the function $T(u)(f_\beta,\omega)$, defined in (6), for the element $f_\beta \in L \cap L_\infty((\Omega,\Sigma,\mu);H)$. Since for each $\omega\in\Omega$ the function $u \mapsto F_\beta(u,\omega)$ is continuous on $[0,\infty)$, it follows as above that for every $\alpha>0$ the function $\omega \mapsto \int_0^\alpha F_\beta(u,\omega)du$ on Ω is a representative of $\int_0^\alpha T(u)f_\beta du \in L$, and therefore

$$\begin{aligned}(10)\qquad \boldsymbol{D}\text{-}\lim_{\alpha\to 0}\left(\frac{1}{\alpha}\int_0^\alpha T(u)f_\beta du\right)(\omega) &= \boldsymbol{D}\text{-}\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha F_\beta(u,\omega)du \\ &= F_\beta(0,\omega) = T(0)f_\beta(\omega) = f_\beta(\omega)\end{aligned}$$

for almost all $\omega\in\Omega$, whence the proof is complete.

LEMMA 6. *Let $\delta>0$ and $E\in\Sigma$. Assume that $\{F_\alpha \mid \alpha\in\boldsymbol{D}\}$ is a family of H-valued strongly measurable functions on (Ω,Σ,μ) such that $\|F_\alpha(\omega)\|\le\delta$ on E for all $\alpha\in\boldsymbol{D}$, and such that to each $\alpha\in\boldsymbol{D}$ and $\omega\in E$ there corresponds $\beta\in\boldsymbol{D}$ with*

$$(11)\qquad \beta\le\alpha \quad \textit{and} \quad \|F_\beta(\omega)\| = \delta\,.$$

Then there exists an H-valued strongly measurable function F_0 on E, with $\|F_0(\omega)\|=\delta$ on E, such that to each $\omega\in E$ there corresponds a sequence (α_i) in $\boldsymbol{D}$ with

$$(12)\qquad \alpha_i\downarrow 0 \quad \textit{and} \quad \lim_{i\to\infty} F_{\alpha_i}(\omega) = F_0(\omega)\,.$$

PROOF. Since H is finite-dimensional by hypothesis, the set

$$H(\delta) = \{x\in H \mid \|x\|=\delta\}$$

is a compact subset of H. Thus there exists a sequence (Δ_n) of partitions of $H(\delta)$ such that the diameter of each member of Δ_n is less than $1/2^n$, and such that Δ_{n+1} is a refinement of Δ_n for each $n\ge 1$. We may write

$$\Delta_1 = \{A_1(1),\dots,A_1(l_1)\}\,,$$

and

$$\Delta_n = \{A_n(i_1,\dots,i_n) \mid 1\le i_1\le l_1,\dots,1\le i_n\le l_n\} \quad \text{for } n\ge 2\,,$$

where $\{A_n(i_1,\dots,i_{n-1},j) \mid 1\le j\le l_n\}$ becomes a partition of the set $A_{n-1}(i_1,\dots,i_{n-1})$.

Define

$$(i_1,\dots,i_n) < (j_1,\dots,j_n)$$

if there exists some k, with $1\le k\le n$, for which $i_k<j_k$ and $(i_1,\dots,i_{k-1}) = (j_1,\dots,j_{k-1})$ hold. Using this order defined on each set

$$\Gamma_n := \{1,\dots,l_1\}\times\dots\times\{1,\dots,l_n\}\,,$$

we will construct a sequence $(h_n(\omega))$ for $\omega\in E$, where $h_n(\omega)$ is an element of Γ_n for each $n\ge 1$. First, if $\omega\in E$, let

$$h_1(\omega) = \min\{1\le i\le l_1 \mid \{\beta\in\boldsymbol{D} \mid \beta\le\alpha \text{ and } F_\beta(\omega)\in A_1(i)\}\ne\emptyset \text{ for each } \alpha\in\boldsymbol{D}\}\,.$$

(Here we notice that, since $F_\beta(\omega)$ depends on ω, if $\alpha\in\boldsymbol{D}$ is fixed, then the number i for which $F_\beta(\omega)\in A_1(i)$ holds for some $\beta\le\alpha$, depends also on ω. This observation leads to the conclusion that $h_1(\omega)$ is a function of $\omega \in E$.) Next, if $h_1(\omega), \dots, h_n(\omega)$ have been defined, then let

$$h_{n+1}(\omega) = \min\left\{ (i_1, \dots, i_n, i_{n+1}) \,\middle|\, \begin{array}{l} h_n(\omega) = (i_1, \dots, i_n), \text{ and for each } \alpha \in \boldsymbol{D} \\ \{\beta \in \boldsymbol{D} \mid \beta \le \alpha \text{ and } F_\beta(\omega) \in A_{n+1}(i_1, \dots, i_n, i_{n+1})\} \neq \emptyset \end{array} \right\}.$$

By this process we get an infinite sequence $h_1(\omega), h_2(\omega), \dots$. Then, by putting

$$\{F_0(\omega)\} = \bigcap_{n=1}^{\infty} \overline{A_n(h_n(\omega))}, \tag{13}$$

where $\overline{A_n(h_n(\omega))}$ denotes the closure of the set $A_n(h_n(\omega))$, we define an H-valued strongly measurable function F_0 on E such that $\|F_0(\omega)\| = \delta$ for all $\omega \in E$. Then, from the definition of $F_0(\omega)$, there exists a sequence (α_i) in $\boldsymbol{D}$, with $\alpha_i \downarrow 0$, such that

$$\lim_{i\to\infty} \|F_{\alpha_i}(\omega) - F_0(\omega)\| = 0. \tag{14}$$

This completes the proof.

Lemma 6 is a key lemma of the paper. As is observed in a simple example, it does not hold if H is not finite-dimensional. The next lemma is Lemma 1 of [2], which is proved easily by induction and hence we omit the proof.

LEMMA 7. *Let T be a linear operator on L and $f \in L$. If $h_k, g_k \in L$ for $0 \le k \le n$ and $d_k \in L$ for $1 \le k \le n$ satisfy $f = h_0 + g_0$ and $Tg_k = d_{k+1} + g_{k+1}$, $h_{k+1} = d_{k+1} + h_k$ for $0 \le k \le n-1$, then*

$$T^n f = T^n h_0 + \sum_{i=0}^{n-1} T^i d_{n-1} + g_n, \quad \textit{and} \tag{15}$$

$$\sum_{i=0}^{n} T^i f = \sum_{i=0}^{n} T^i h_{n-i} + \sum_{i=0}^{n} g_i. \tag{16}$$

We extend Baxter and Chacon's truncation operation for complex numbers to vectors of H as follows. For $\gamma > 0$, let

$$S(\gamma) = \{x \in H \mid \|x\| \le \gamma\},$$

and for $(x, y) \in S(\gamma) \times H$, define

$$C_\gamma(x, y) = x + \lambda \cdot (y - x) \tag{17}$$

with $\lambda = \max\{t \mid 0 \le t \le 1, \|x + t \cdot (y - x)\| \le \gamma\}$. It follows easily that C_γ is continuous on $S(\gamma) \times H$, and hence if f and g are H-valued strongly measurable functions on Ω and $\|f(\omega)\| \le \gamma$ for $\omega \in \Omega$, then the H-valued function

$$C_\gamma(f, g)(\omega) = C_\gamma(f(\omega), g(\omega)) \tag{18}$$

becomes strongly measurable on Ω.

LEMMA 8. *Suppose T is a linear contraction on L. Assume that there exists a constant $K \geq 1$ such that if $f \in L \cap L_\infty((\Omega, \Sigma, \mu); H)$, then $\|T^n f(\omega)\| \leq K\|f\|_\infty$ for almost all $\omega \in \Omega$ and for every $n \geq 0$. Let $f \in L$, $A \in \Sigma$ and $\beta > 0$ be such that $\beta > \|f(\omega)\|$ for $\omega \in A$. Assume that R is an H-valued strongly measurable function on A and $N \geq 1$ is an integer such that to each $\omega \in A$ there corresponds j, with $0 \leq j \leq N$, for which*

$$R(\omega) = \frac{1}{j+1} \sum_{i=0}^{j} T^i f(\omega) \tag{19}$$

holds. Then $\|R(\omega)\| \geq 3K\beta$ for $\omega \in A$ implies that there exist functions $d_1, \ldots, d_N$, g in L such that

(i) $d_k(\omega) = 0$ *for* $\omega \in \Omega \setminus A$ *and* $\|d_1(\omega) + \cdots + d_k(\omega)\| \leq 2\beta$ *for* $\omega \in A$, $1 \leq k \leq N$,
(ii) $T^N f = T^N C_\beta(0, f) + (d_N + T d_{N-1} + \cdots + T^{N-1} d_1) + g$,
(iii) $\|g\|_L \leq \|f - C_\beta(0, f)\|_L$,
(iv) $C_\beta(f, R)(\omega) = f(\omega) + d_1(\omega) + \cdots + d_N(\omega)$ *for almost every* $\omega \in A$.

PROOF. Let $h_0 = C_\beta(0, f)$ and $g_0 = f - h_0$. If h_i and g_i for $0 \leq i \leq k$ and d_i for $1 \leq i \leq k$ have been defined in L, then using the function $a(\omega)$ on Ω defined by

$$a(\omega) = \begin{cases} \mathrm{sgn}[C_\beta(f, R)(\omega) - h_0(\omega)] & \text{for } \omega \in A\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\mathrm{sgn}\, x = x/\|x\|$ if $x \in H$ with $x \neq 0$, and $\mathrm{sgn}\, x = 0$ if $x = 0 \in H$, we set

$$U_{k+1}(\omega) = \begin{cases} \langle T g_k(\omega), a(\omega)\rangle \cdot a(\omega) & \text{if } \langle T g_k(\omega), a(\omega)\rangle \text{ is positive}\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product of H (it is here that we use the fact that H is a real Hilbert space),

$$h_{k+1} = C_\beta(h_k, h_k + U_{k+1})\,, \tag{20}$$

$$d_{k+1} = h_{k+1} - h_k\,, \tag{21}$$

$$g_{k+1} = T g_k - d_{k+1}\,. \tag{22}$$

By (20) and (21)

$$d_{k+1}(\omega) = \lambda_{k+1}(\omega) U_{k+1}(\omega) \quad \text{with } 0 \leq \lambda_{k+1}(\omega) \leq 1\,,$$

and hence $\|g_{k+1}(\omega)\| \leq \|T g_k(\omega)\|$ for $\omega \in \Omega$. It follows from Property (I) and the hypothesis $\|T\| \leq 1$ that

$$\|g_{k+1}\|_L \leq \|T g_k\|_L \leq \|g_k\|_L\,.$$

Then, letting $g = g_N$, we obtain $\|g\|_L = \|g_N\|_L \leq \|g_0\|_L = \|f - C_\beta(0, f)\|_L$, whence (iii) follows. (ii) is a consequence of Lemma 7.

Since $U_{k+1}(\omega) = 0$ on $\Omega \setminus A$, $0 \leq k \leq N-1$, by definition, it follows from (20) and (21) that $d_k(\omega) = 0$ on $\Omega \setminus A$, $1 \leq k \leq N$. On the other hand, since

$$h_k - h_0 = d_k + d_{k-1} + \cdots + d_1 \quad \text{on } \Omega \qquad \text{(by (21))}\,, \tag{23}$$

it follows that $\|d_1(\omega) + \cdots + d_k(\omega)\| \le \|h_k(\omega)\| + \|h_0(\omega)\| \le 2\beta$ for $\omega \in A$. Thus (i) follows.

To prove (iv), we use (23) and the fact that $f(\omega) = C_\beta(0, f)(\omega) = h_0(\omega)$ for $\omega \in A$; it suffices to show that the following holds:

$$C_\beta(f, R)(\omega) = h_N(\omega) \quad \text{for almost all } \omega \in A\,. \tag{24}$$

If (24) did not hold, then there would exist $E \in \Sigma$, with $E \subset A$ and $\mu(E) > 0$, such that

$$C_\beta(f, R)(\omega) \ne h_N(\omega) \quad \text{for } \omega \in E\,. \tag{25}$$

Then for $\omega \in E$ and $0 \le k \le N - 1$ we have, from the definition of $h_N(\omega)$, that

$$d_{k+1}(\omega) = U_{k+1}(\omega) \quad \text{and} \quad h_k(\omega) \ne C_\beta(f, R)(\omega) \tag{26}$$

and that

$$\langle g_{k+1}(\omega), C_\beta(f, R)(\omega) - f(\omega)\rangle \le 0\,. \tag{27}$$

On the other hand, by hypothesis, there exists j with $0 \le j \le N$ such that

$$\begin{aligned}(j+1)R(\omega) &= \sum_{i=0}^{j} T^i f(\omega) \\ &= h_j(\omega) + Th_{j-1}(\omega) + \cdots + T^j h_0(\omega) + \sum_{i=0}^{j} g_i(\omega)\,,\end{aligned}$$

by Lemma 7. Thus, from (27) and the fact that $\operatorname{sgn}[R(\omega) - f(\omega)] = a(\omega)$ $(\ne 0)$, we find that

$$\langle (j+1)R(\omega), a(\omega)\rangle \le \left\langle \sum_{i=0}^{j} T^i h_{j-i}(\omega), a(\omega)\right\rangle. \tag{28}$$

However,

$$\begin{aligned}\langle R(\omega), a(\omega)\rangle &= \langle R(\omega) - f(\omega), a(\omega)\rangle + \langle f(\omega), a(\omega)\rangle \\ &> 2K\beta - \beta \ge K\beta\,,\end{aligned}$$

because $\|R(\omega) - f(\omega)\| > 2K\beta$ and $\|f(\omega)\| < \beta \le K\beta$ for $\omega \in E$. Moreover,

$$\left\langle \sum_{i=0}^{j} T^i h_{j-i}(\omega), a(\omega)\right\rangle \le (j+1)K\beta\,,$$

because $\|T^i h_{j-i}(\omega)\| \le K\|h_{j-i}\|_\infty \le K\beta$. Hence, from (28), we deduce that $(j+1)K\beta < (j+1)K\beta$, a contradiction. This completes the proof.

LEMMA 9. *Suppose the semigroup $\boldsymbol{T} = \{T(t)\}$ satisfies the additional hypothesis $(*)$. Then for every $f \in L$,*

$$\boldsymbol{D}\text{-}\limsup_{\alpha \to 0} \left\| \frac{1}{\alpha}\left(\int_0^\alpha T(t) f\,dt\right)(\omega)\right\| \le 3K\|f(\omega)\| \tag{29}$$

for almost all $\omega \in \Omega$.

PROOF. Let F^* be the nonnegative measurable function on Ω defined by

$$F^*(\omega) = \boldsymbol{D}\text{-}\limsup_{\alpha\to 0} \left\| \frac{1}{\alpha}\left(\int_0^\alpha T(t) f\,dt\right)(\omega)\right\| \quad (\omega \in \Omega)\,. \tag{30}$$

It suffices to show that if $\gamma > 0$ and $F^*(\omega) > 3K\gamma$ for almost all $\omega \in E$, where $E \in \Sigma$ and $0 < \mu(E) < \infty$, then $\|f(\omega)\| \geq \gamma$ for almost all $\omega \in E$.

Assume the contrary. Namely, there exist $\gamma > 0$ and $E \in \Sigma$ with $0 < \mu(E) < \infty$ such that

$$F^*(\omega) > 3K\gamma \quad \text{and} \quad \|f(\omega)\| < \gamma \quad \text{for } \omega \in E\,. \tag{31}$$

Here, using Property (II), we may assume without loss of generality that there exists $e \in L$ with

$$\chi_E(\omega) = \|e(\omega)\| \quad \text{for } \omega \in \Omega\,. \tag{32}$$

For an $\alpha \in \boldsymbol{D}$, let F_α be the function in L defined by

$$F_\alpha(\omega) = \begin{cases} C_{3K\gamma}\left(f, \dfrac{1}{\alpha}\displaystyle\int_0^\alpha T(t) f\,dt\right)(\omega) & \text{if } \omega \in E\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{33}$$

Since $F^*(\omega) > 3K\gamma$ for $\omega \in E$ by (31), the family $\{F_\alpha \mid \alpha \in \boldsymbol{D}\}$ satisfies the hypothesis of Lemma 6 with $\delta = 3K\gamma$. Thus there exists a function $F_0 \in L$ such that

$$\|F_0(\omega)\| = 3K\gamma \cdot \chi_E(\omega) \quad \text{for } \omega \in \Omega\,, \tag{34}$$

and also such that to each $\omega \in E$ there corresponds a sequence (α_i) in $\boldsymbol{D}$, with $\alpha_i \downarrow 0$, for which

$$\left\| \frac{1}{\alpha_i}\left(\int_0^{\alpha_i} T(t) f\,dt\right)(\omega)\right\| > 3K\gamma \quad \text{for } i \geq 1\,, \tag{35}$$

and

$$\lim_{i\to\infty} F_{\alpha_i}(\omega) = F_0(\omega) \tag{36}$$

hold. Let h be the function in L defined by

$$h(\omega) = \begin{cases} C_\gamma(f, F_0)(\omega) - f(\omega) & \text{if } \omega \in E\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{37}$$

Since $h(\omega) \neq 0$ for $\omega \in E$ and $f(\omega) - C_\gamma(0, f)(\omega) = 0$ for $\omega \in E$, by (31) and (34), it follows from Property (IV) that

$$\|f - C_\gamma(0, f) - h\|_L > \|f - C_\gamma(0, f)\|_L\,.$$

Take an $\varepsilon > 0$ so that

$$5\varepsilon < \|f - C_\gamma(0, f) - h\|_L - \|f - C_\gamma(0, f)\|_L\,. \tag{38}$$

Then choose an integer $l \geq 1$ so that

$$\frac{4\gamma\|e\|_L}{l} < \varepsilon\,. \tag{39}$$

Next choose $\delta \in \boldsymbol{D}$ with $0 < \delta < 1$ so that $0 \le t < (l+1)\delta$ implies

$$\|(I - T(t))f\|_L < \varepsilon\,, \quad \|(I - T(t))C_\gamma(0, f)\|_L < \varepsilon\,, \tag{40}$$

and

$$\|(I - T(t))h\|_L < \varepsilon\,. \tag{41}$$

Finally, let $\eta > 0$ be fixed arbitrarily. By (35) and (36) we can choose an integer $n \ge 1$ and a set $E_1 \in \Sigma$, with $E_1 \subset E$ and $\mu(E \setminus E_1) < \eta$, so that if $\omega \in E_1$, then there exists an integer k, with $1 \le k \le n$, for which

$$\frac{1}{k\delta/n}\left\|\left(\int_0^{k\delta/n} T(t) f\,dt\right)(\omega)\right\| > 3K\gamma \tag{42}$$

and

$$\|C_\gamma(f, F_0)(\omega) - C_\gamma(f, F_{k\delta/n})(\omega)\| < \eta \tag{43}$$

hold. Thus, by (33) and the fact that $\gamma < 3K\gamma$, we deduce that

$$C_\gamma\left(f, \frac{n}{k\delta}\int_0^{k\delta/n} T(t) f\,dt\right)(\omega) = C_\gamma(f, F_{k\delta/n})(\omega)\,,$$

and that

$$\left\|C_\gamma(f, F_0)(\omega) - C_\gamma\left(f, \frac{n}{k\delta}\int_0^{k\delta/n} T(t) f\,dt\right)(\omega)\right\| < \eta\,. \tag{44}$$

We then apply Lemma 2 and Cantor's diagonal method to infer that there exists a strictly increasing sequence $(n(k))$ of positive integers such that

$$\left(\int_0^{\alpha} T(t) f\,dt\right)(\omega) = \lim_{k\to\infty} \frac{\delta}{n(k)!} \sum_{i=0}^{(\alpha/\delta)n(k)!-1} T\left(\frac{i\delta}{n(k)!}\right) f(\omega)$$

for almost all $\omega \in \Omega$ and for every $\alpha \in \boldsymbol{D}$. Then, using (42) and (44), we can choose a sufficiently large integer $N \ge 1$ and a set $A \in \Sigma$, with $A \subset E_1 \subset E$ and $\mu(E \setminus A) < \eta$, such that N/n is a positive integer and if $\omega \in A$, then there exists an integer k, with $1 \le k \le n$, for which

$$\left\|\frac{n}{kN} \sum_{i=0}^{(k/n)N-1} T\left(\frac{i\delta}{N}\right) f(\omega)\right\| > 3K\gamma \tag{45}$$

and

$$\left\|C_\gamma(f, F_0)(\omega) - C_\gamma\left(f,\ \frac{n}{kN} \sum_{i=0}^{(k/n)N-1} T\left(\frac{i\delta}{N}\right) f\right)(\omega)\right\| < \eta \tag{46}$$

hold. Denoting by $k(\omega)$ the smallest positive integer k satisfying (45) and (46), we then define a function R in L by

$$R(\omega) = \begin{cases} \dfrac{n}{k(\omega)N} \displaystyle\sum_{i=0}^{(k(\omega)/n)N-1} T\left(\frac{i\delta}{N}\right) f(\omega) & \text{if } \omega \in A\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{47}$$

Let $T = T(\delta/N)$. Then for $\omega \in A$, we have $\|R(\omega)\| > 3K\gamma$, and further

$$R(\omega) = \frac{1}{j+1} \sum_{i=0}^{j} T^i f(\omega) \quad \text{with} \quad j = \frac{k(\omega)}{n} N - 1\,.$$

Thus by Lemma 8 there exist functions $d_1, \dots, d_N$, g in L such that

(i) $d_k(\omega) = 0$ for $\omega \in \Omega \setminus A$ and $\|\sum_{i=1}^{k} d_i(\omega)\| \le 2\gamma$ for $\omega \in A$, $1 \le k \le N$,
(ii) $T^N f = T^N C_\gamma(0, f) + (d_N + T d_{N-1} + \cdots + T^{N-1} d_1) + g$,
(iii) $\|g\|_L \le \|f - C_\gamma(0, f)\|_L$,
(iv) $C_\gamma(f, R)(\omega) = f(\omega) + \sum_{i=1}^{N} d_i(\omega)$ for almost all $\omega \in A$.

Let

$$W = \frac{1}{lN} \sum_{i=0}^{lN-1} T^i\,.$$

Then, since

$$\begin{aligned} &W\left(\sum_{i=1}^{N} T^{N-i} d_i - \sum_{i=1}^{N} d_i\right) \\ &\quad = \frac{1}{lN} \sum_{k=1}^{N-1} [-T^{k-1}(d_1 + \cdots + d_{N-k}) + T^{lN+k-1}(d_1 + \cdots + d_{N-k})]\,, \end{aligned}$$

we apply (i) and (39) to obtain that

$$\begin{aligned} \left\| W\left(\sum_{i=1}^{N} T^{N-i} d_i - \sum_{i=1}^{N} d_i\right)\right\|_L &\le \frac{2}{lN} \sum_{k=1}^{N-1} \|d_1 + \cdots + d_{N-k}\|_L \\ &\le \frac{2(N-1)}{lN} 2\gamma \cdot \|e\|_L < \varepsilon\,. \end{aligned}$$

Hence, by (ii), we have

$$\left\| W(T^N f - T^N C_\gamma(0, f)) - W\left(\sum_{i=1}^{N} d_i + g\right)\right\|_L < \varepsilon\,. \tag{48}$$

On the other hand, since

$$W(T^N f - T^N C_\gamma(0, f)) = \frac{1}{lN} \sum_{i=0}^{lN-1} T\left(\frac{i+N}{N}\delta\right)(f - C_\gamma(0, f))\,,$$

it follows from (40) that

$$\|W(T^N f - T^N C_\gamma(0, f)) - (f - C_\gamma(0, f))\|_L < 2\varepsilon \,. \tag{49}$$

Combining this with (48) yields

$$\left\|(f - C_\gamma(0, f)) - W\left(\sum_{i=1}^{N} d_i + g\right)\right\|_L < 3\varepsilon \,. \tag{50}$$

Next, let

$$h_1(\omega) = \begin{cases} C_\gamma(f, R)(\omega) - f(\omega) & \text{if } \omega \in A \,, \\ 0 & \text{otherwise} \,. \end{cases} \tag{51}$$

By (37), (46), (47) and (31) we then deduce that

$$\|h(\omega) - h_1(\omega)\| \le \eta \cdot \chi_A(\omega) + 2\gamma \cdot \chi_{E\setminus A}(\omega) \quad \text{for } \omega \in \Omega \,. \tag{52}$$

Hence, if $\eta > 0$ is taken to be sufficiently small, then, since $\mu(E \setminus A) < \eta$, it follows from Property (III) that

$$\|h - h_1\|_L \le \eta \|e\|_L + 2\gamma \|e_{E\setminus A}\|_L < \varepsilon \,, \tag{53}$$

where $e_{E\setminus A}(\omega) = e(\omega)$ if $\omega \in E \setminus A$, and $= 0$ if $\omega \notin E \setminus A$. We also deduce by (iv), (i) and (51) that

$$W(d_1 + \cdots + d_N) - h = Wh_1 - h = W(h_1 - h) + Wh - h \,,$$

and by (41),

$$\|Wh - h\|_L \le \frac{1}{lN} \sum_{i=0}^{lN-1} \left\|T\left(\frac{i\delta}{N}\right)h - h\right\|_L < \varepsilon \,.$$

Thus $\|W(d_1 + \cdots + d_N) - h\|_L < 2\varepsilon$, and consequently we get

$$\|f - C_\gamma(0, f) - h - Wg\|_L < 5\varepsilon \,. \tag{54}$$

But this is impossible, because

$$\begin{aligned} \|f - C_\gamma(0, f) - h - Wg\|_L &\ge \|f - C_\gamma(0, f) - h\|_L - \|Wg\|_L \\ &\ge \|f - C_\gamma(0, f) - h\|_L - \|g\|_L > 5\varepsilon \qquad \text{(by (iii) and (38))}\,, \end{aligned}$$

and hence the proof is complete.

§3. Proof of Theorem 1. Let $f \in L$. Since the set

$$M = \left\{ g \in L \,\middle|\, \boldsymbol{D}\text{-}\lim_{\alpha \to 0} \left(\frac{1}{\alpha}\int_0^\alpha T(t) g\, dt\right)(\omega) = g(\omega) \text{ for almost all } \omega \in \Omega \right\}$$

is dense in L by Lemma 5 together with Lemma 3, we can choose a sequence (f_n) of functions in M such that $\lim_{n\to\infty} \|f - f_n\|_L = 0$ and also such that

$$\boldsymbol{D}\text{-}\lim_{\alpha \to 0} \frac{1}{\alpha}\left(\int_0^\alpha T(t) f_n\, dt\right)(\omega) = f_n(\omega) \quad \text{for almost all } \omega \in \Omega \tag{55}$$

for each $n \geq 1$. Here, by Lemma 2, we may assume that

$$\lim_{n\to\infty} f_n(\omega) = f(\omega) \quad \text{for almost all } \omega \in \Omega\,. \tag{56}$$

Then, since

$$\frac{1}{\alpha}\int_0^\alpha T(t) f\,dt - f = \frac{1}{\alpha}\int_0^\alpha T(t)(f - f_n)dt + \left(\frac{1}{\alpha}\int_0^\alpha T(t) f_n dt - f_n\right) + (f_n - f)\,,$$

we have

$$\begin{aligned} &\boldsymbol{D}\text{-}\limsup_{\alpha\to 0}\left\|\frac{1}{\alpha}\left(\int_0^\alpha T(t) f\,dt\right)(\omega) - f(\omega)\right\| \\ &\quad \leq \boldsymbol{D}\text{-}\limsup_{\alpha\to 0}\left\|\frac{1}{\alpha}\left(\int_0^\alpha T(t)(f - f_n)dt\right)(\omega)\right\| + \|f_n(\omega) - f(\omega)\| \\ &\quad \leq 4K\|f_n(\omega) - f(\omega)\| \quad \text{for almost all } \omega \in \Omega\,, \end{aligned}$$

by Lemma 9. Hence, by (56), we find

$$\boldsymbol{D}\text{-}\limsup_{\alpha\to 0}\left\|\frac{1}{\alpha}\left(\int_0^\alpha T(t) f\,dt\right)(\omega) - f(\omega)\right\| = 0$$

for almost all $\omega \in \Omega$, and this completes the proof.

§4. Proof of Theorem 2. Let $f^\sim$ be an H-valued strongly measurable function on $\Omega^\sim$. Assume that $f^\sim(t,\cdot) \in L$ for almost all $t \geq 0$. Then, since there exists a sequence $(f_n^\sim)$ of H-valued stongly measurable simple functions on $\Omega^\sim$ such that for every $\omega^\sim = (t,\omega) \in \Omega^\sim$

$$\|f_n^\sim(\omega^\sim)\| \leq \|f_{n+1}^\sim(\omega^\sim)\| \quad \text{for } n \geq 1, \tag{57}$$

$$\|f_n^\sim(\omega^\sim) - f^\sim(\omega^\sim)\| \leq 2\|f^\sim(\omega^\sim)\| \quad \text{for } n \geq 1\,, \quad \text{and} \tag{58}$$

$$\lim_{n\to\infty}\|f_n^\sim(\omega^\sim) - f^\sim(\omega^\sim)\| = 0\,, \tag{59}$$

it follows from Lemma 3 that for almost all $t \geq 0$, $\lim_{n\to\infty}\|f^\sim(t,\cdot) - f_n^\sim(t,\cdot)\|_L = 0$, and thus

$$\|f^\sim(t,\cdot)\|_L = \lim_{n\to\infty}\|f_n^\sim(t,\cdot)\|_L\,.$$

Since the functions $t \mapsto \|f_n^\sim(t,\cdot)\|_L$ are Lebesgue measurable on the interval $[0,\infty)$, which can be seen from Lemma 3 together with a standard approximation argument, it follows that the function $t \mapsto \|f^\sim(t,\cdot)\|_L$ is Lebesgue measurable on the interval $[0,\infty)$, and thus we can define

$$\|f^\sim\|_{\Omega^\sim} = \int_0^\infty \|f^\sim(t,\cdot)\|_L dt\,. \tag{60}$$

Let $L^\sim$ be the set of all $f^\sim$ such that $\|f^\sim\|_{\Omega^\sim} < \infty$, and put

$$\|f^\sim\|_{L^\sim} = \|f^\sim\|_{\Omega^\sim} \ (< \infty) \quad \text{for } f^\sim \in L^\sim\,.$$

It is easily checked that $(L^\sim, \|\cdot\|_{L^\sim})$ is a Banach space satisfying Properties (I) to (IV) replaced $(L, \|\cdot\|_L)$ with $(L^\sim, \|\cdot\|_{L^\sim})$.

Let T be a bounded linear operator on L. If $f^{\sim}$ is a strongly measurable simple function in $L^{\sim}$ of the form

$$f^{\sim}(\omega^{\sim}) = \sum_{i=1}^{n} \chi_{E_i^{\sim}}(\omega^{\sim}) \cdot x_i \,,$$

where $x_i \in H$ and $E_i^{\sim}$ $(\in \Sigma^{\sim})$ has the form $E_i^{\sim} = B_i \times E_i$ for some Lebesgue measurable subset B_i of $[0, \infty)$ and $E_i \in \Sigma$, then define

$$(T^{\sim} f^{\sim})(t, \omega) = (T f^{\sim}(t, \cdot))(\omega) \quad \text{for } (t, \omega) \in \Omega^{\sim} \,.$$

It follows that $T^{\sim} f^{\sim}$ is in $L^{\sim}$, and that

$$\|T^{\sim} f^{\sim}\|_{L^{\sim}} = \int_0^{\infty} \|T f^{\sim}(t, \cdot)\|_L dt \le \|T\| \int_0^{\infty} \|f^{\sim}(t, \cdot)\|_L dt = \|T\| \|f^{\sim}\|_{L^{\sim}} \,.$$

Thus $T^{\sim}$ can be uniquely extended to a bounded linear operator on $L^{\sim}$. We will use the same symbol $T^{\sim}$ to denote the extended operator. By applying Lemmas 1 and 2 to both $(L, \|\cdot\|_L)$ and $(L^{\sim}, \|\cdot\|_{L^{\sim}})$ and using an approximation argument, we see without difficulty that if $f^{\sim} \in L^{\sim}$, then there exists a representative $(T^{\sim} f^{\sim})(t, \omega)$ of the element $T^{\sim} f^{\sim} \in L^{\sim}$ such that for almost every $t \ge 0$,

$$(T^{\sim} f^{\sim})(t, \omega) = (T f^{\sim}(t, \cdot))(\omega) \quad \text{for almost all } \omega \in \Omega \,. \tag{61}$$

Let S be another bounded linear operator on L. By using (61), we deduce immediately that $(TS)^{\sim} = T^{\sim} S^{\sim}$ on $L^{\sim}$. Thus, if $\boldsymbol{T}^{\sim} = \{T(t)^{\sim} \mid t \ge 0\}$ denotes the family of linear contractions on $L^{\sim}$ induced from the semigroup $\boldsymbol{T} = \{T(t) \mid t \ge 0\}$ on L by the above method, then $\boldsymbol{T}^{\sim}$ becomes a semigroup on $L^{\sim}$, and an approximation argument implies that

$$\lim_{t \to 0} \|T(t)^{\sim} f^{\sim} - f^{\sim}\|_{L^{\sim}} = 0 \tag{62}$$

for $f^{\sim} \in L^{\sim}$. That is, $\boldsymbol{T}^{\sim}$ is a (C_0)-semigroup of linear contractions on $L^{\sim}$.

For $t \in [0, \infty)$ and $f^{\sim} \in L^{\sim}$, let

$$(A(t) f^{\sim})(u, \omega) = \begin{cases} 0 & \text{if } 0 \le u < t \,, \\ f^{\sim}(u - t, \omega) & \text{if } u \ge t \,. \end{cases}$$

Clearly, $\boldsymbol{A} = \{A(t) \mid t \ge 0\}$ is a (C_0)-semigroup of linear isometries on $L^{\sim}$ such that $A(t)T(s)^{\sim} = T(s)^{\sim} A(t)$ on $L^{\sim}$ for all $t, s \ge 0$. Define

$$V(t) = A(t) T(t)^{\sim} \quad (t \ge 0) \,. \tag{63}$$

It then follows that $\boldsymbol{V} = \{V(t) \mid t \ge 0\}$ becomes a (C_0)-semigroup of linear contractions on $L^{\sim}$.

Let h be the function appearing in Theorem 2. Then define

$$L^{\sim}(h) = \{f^{\sim}/h \mid f^{\sim} \in L^{\sim}\} \quad \text{and} \quad \|f^{\sim}/h\|_{L^{\sim}(h)} = \|f^{\sim}\|_{L^{\sim}} \,.$$

Obviously, $(L^{\sim}(h), \|\cdot\|_{L^{\sim}(h)})$ is a Banach space satisfying Properties (I) to (IV) replaced $(L, \|\cdot\|_L)$ with $(L^{\sim}(h), \|\cdot\|_{L^{\sim}(h)})$, and the mapping $\vartheta : L^{\sim} \to L^{\sim}(h)$ defined by

$$\vartheta f^{\sim} = f^{\sim}/h$$

is an invertible linear isometry with the property that, for every family $\{f^\sim_\alpha \mid \alpha \in \boldsymbol{D}\} \subset L^\sim$, $\boldsymbol{D}\text{-}\lim_{\alpha\to 0} f^\sim_\alpha$ exists a.e. on $\Omega^\sim$ if and only if $\boldsymbol{D}\text{-}\lim_{\alpha\to 0} \vartheta f^\sim_\alpha$ exists a.e. on $\Omega^\sim$.

Let $\boldsymbol{U} = \{U(t) \mid t \geq 0\}$ be a (C_0)-semigroup of linear contractions on $L^\sim(h)$ defined by

$$U(t) = \vartheta V(t)\vartheta^{-1} \quad \text{for } t \geq 0\,. \tag{64}$$

Then for $f^\sim/h \in L^\sim(h) \cap L_\infty((\Omega^\sim, \Sigma^\sim, \mu^\sim); H)$ we have

$$U(t)(f^\sim/h) = \frac{1}{h}V(t)f^\sim = \frac{1}{h}A(t)T(t)^\sim f^\sim = \frac{1}{h}T(t)^\sim A(t)f^\sim\,,$$

and without loss of generality we may assume that

$$\|f^\sim(u,\omega)\| \leq \|f^\sim/h\|_\infty h(u,\omega)$$

for all $(u,\omega) \in \Omega^\sim$. Hence we see that

(a) for every u with $0 \leq u < t$, $(U(t)(f^\sim/h))(u,\omega) = 0$ for $\omega \in \Omega$, and

(b) for almost every u with $u \geq t$ (with respect to the Lebesgue measure),

$$\begin{aligned}\|(U(t)(f^\sim/h))(u,\omega)\| &= \frac{1}{h(u,\omega)}\|(T(t)f^\sim(u-t,\cdot))(\omega)\| \\ &\leq K\|f^\sim/h\|_\infty\end{aligned}$$

for almost all $\omega \in \Omega$, by (61) and (ii) of Theorem 2.

Therefore we can apply Theorem 1 to the semigroup $\boldsymbol{U} = \{U(t) \mid t \geq 0\}$ to infer that

$$\boldsymbol{D}\text{-}\lim_{\alpha\to 0}\frac{1}{\alpha}\int_0^\alpha V(t)f^\sim dt = f^\sim \quad \text{a.e. on } \Omega^\sim \tag{65}$$

for each $f^\sim \in L^\sim$.

To complete the proof, fix $f \in L$ and $b > 0$. Define a function $f^\sim$ in $L^\sim$ by

$$f^\sim(u,\omega) = \begin{cases} f(\omega) & \text{if } 0 \leq u < b\,, \\ 0 & \text{if } u \geq b\,. \end{cases}$$

It follows that

$$\begin{aligned}(V(t)f^\sim)(u,\omega) &= (A(t)T(t)^\sim f^\sim)(u,\omega) \\ &= \begin{cases} (T(t)f)(\omega) & \text{if } u \in [t, b+t)\,, \\ 0 & \text{if } u \in [0,\infty)\setminus[t,b+t)\,. \end{cases}\end{aligned}$$

In particular, if $0 \leq t \leq b/2 \leq u < b$, then

$$(V(t)f^\sim)(u,\omega) = (T(t)f)(\omega) \quad \text{for } \omega \in \Omega\,, \tag{66}$$

so that for any α, with $0 < \alpha < b/2$, and any u, with $b/2 \leq u < b$, we find

$$\frac{1}{\alpha}\left(\int_0^\alpha V(t)f^\sim dt\right)(u,\omega) = \frac{1}{\alpha}\left(\int_0^\alpha T(t)f\,dt\right)(\omega) \quad \text{for } \omega \in \Omega\,.$$

Hence, by (65),

$$\boldsymbol{D}\text{-}\lim_{\alpha\to 0}\frac{1}{\alpha}\left(\int_0^\alpha T(t)f\,dt\right)(\omega) = f(\omega)$$

for almost all $\omega \in \Omega$. This completes the proof.

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