Generalization of value distribution and uniqueness of certain types of difference polynomials

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Abstract

In this paper, we investigate the distribution of zeros as well as the uniqueness problems of certain type of differential polynomials sharing a small function with finite weight. The result obtained improves and generalizes the recent results.

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1 Introduction and main results

Let $f(z)$ be a meromorphic function of finite order, we define difference operator as,

$$\triangle_c f = f(z+c) - f(z), \text{ and}$$

$$\triangle_c^n f = \triangle_c^{n-1}(\triangle_c f),\ n \geq 2$$

where c is a non zero constant. In particular, if $c = 1$, we use the usual difference notation $\triangle_c f = \Delta f$. For a non-constant meromorphic function h, we denote by $T(r,h)$, the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}$.

Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. If the zeros of $f-a$ and $g-a$ coincide in locations and multiplicity, we say f and g share the value a CM (counting multiplicities). On the other hand, if the zeros of $f-a$ and $g-a$ coincide only in their locations then we say that f and g share the value a IM (ignoring multiplicities). By $S(r,f)$, we mean any quantity satisfying $S(r,f) = o\{T(r,f)\}$ as $r \to \infty$, possibly outside a set of finite logarithmic measure. We say that $\alpha(z)$ is a small function of f, if $T(r,\alpha(z)) = S(r,f)$. For a positive integer p, we denote by $N_p(r,a;f)$ the counting function of $a-$points of f, where an $a-$point of multiplicity m is counting m times if $m \leq p$ and p times if $m > p$.

Let k be a non-negative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$, we denote by $E_k(a;f)$ the set of all $a-$points of f where an $a-$point of multiplicity m is counted m times if $m \leq k$ and $k+1$ times if $m > k$. If $E_k(a;f) = E_k(a;g)$ then we say that f,g share the value a with weight k. We write f,g share (a,k) to mean that f,g share the value a with weight k. Clearly, if f,g share (a,k) then f,g share (a,p) for any integer p, $0 \leq p < k$. Also, we note that f,g share the value a IM or CM if and only if f,g share $(a,0)$ or (a,∞) respectively. If α is a small function of f and g, then f,g share α with weight k means that $f-\alpha$, $g-\alpha$ share the value 0 with weight k.

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In addition, we need following definitions:

Definition 1.[26] Let $a \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid= 1)$ the counting function of simple a points of f. For a positive integer k we denote by $N(r, a; f \mid\leq k)$ the counting function of those a-points of f (counted with proper multiplicities) whose multiplicities are not greater than k. By $\overline{N}(r, a; f \mid\leq k)$ we denote the corresponding reduced counting function. Analogously we can define $N(r, a; f \mid\geq k)$ and $\overline{N}(r, a; f \mid\geq k)$.

Definition 2.[10] Let k be a positive integer or infinity. We denote by $N_k(r, a; f)$ the counting function of a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and k times if $m > k$. Then

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

A handful of astonishing research works on entire and meromorphic functions whose differential polynomials share certain value or fixed point have been done by many researchers in the world (see [6], [17], [22], [23], [24], [25]). Recently, there has been an increasing interest in studying difference equations in the complex plane. In 2006, R.G.Halburd and R.J.Korhdnen [7] established a version of Nevanlinna theory based on difference operators. The difference logarithmic derivative lemma given in [4], [8], plays an important role in the difference analogue of Nevanlinna theory. With this development many researchers paid their attention to the distribution of zeros of different types of difference polynomials. In 2010, X. G. Qi, L. Z. Yang and K. Liu [14] proved the following uniqueness result.

Theorem A. Let f, g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to both f and g. Suppose that c is a non zero complex constant and $n \geq 7$ is an integer. If $f^n(z)(f(z)-1)f(z+c)$ and $g^n(z)(g(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f = g$.

In 2013, S. S. Bhoosnrmath and S. R. Kabbur [2] considered the zeros of difference polynomials of the form $f^n(z)(f^m(z) - 1)f(z + c)$, where n, m are positive integers and c is a non zero complex constant and obtained the following theorems.

Theorem B. Let f be an entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c a non zero complex constant and n, m are positive integers. If $n \geq 2$, then $f^n(z)(f^m(z) - 1)f(z + c) - \alpha(z)$ has infinitely many zeros.

Theorem C. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant and n, m are positive integers such that $n \geq m+6$. If $f^n(z)(f^m(z)-1)f(z+c)$ and $g^n(z)(g^m(z)-1)g(z+c)$ share $\alpha(z)$ CM, then $f = tg$, where $t^m = 1$.

Theorem D. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant and n, m are positive integers such that $n \geq 4m + 12$. If $f^n(z)(f^m(z) - 1)f(z + c)$ and $g^n(z)(g^m(z) - 1)g(z + c)$

share $\alpha(z)$ IM, then $f = tg$, where $t^m = 1$.

Recently, P. Sahoo and B. Saha [21] studied the zeros and uniqueness of certain type of difference polynomial sharing a small function with finite weight and obtained the following results.

Theorem E. Let f be a transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c is a non zero complex constant, $n(\geq 1), m(\geq 1)$ and $k(\geq 0)$ are integers. If $n \geq k+2$, then $[f^n(z)(f^m(z)-1)f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem F. Let f be a transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that c is a non zero complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers. If $n \geq k+2$, when $m \leq k+1$ and $n \geq 2k-m+3$ when $m > k+1$, then $[f^n(z)(f(z)-1)^m f(z+c)]^{(k)} - \alpha(z)$ has infinitely many zeros.

Theorem G. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$. If $[f^n(z)(f^m(z)-1)f(z+c)]^{(k)}$ and $[g^n(z)(g^m(z)-1)g(z+c)]^{(k)}$ share $(\alpha, 2)$ then $f = tg$, where $t^m = 1$.

Theorem H. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 2k+m+6$ when $m \leq k+1$ and $n \geq 4k-m+10$ when $m > k+1$. If $[f^n(z)(f(z)-1)^m f(z+c)]^{(k)}$ and $[g^n(z)(g(z)-1)^m g(z+c)]^{(k)}$ share $(\alpha, 2)$ then either $f = g$ or f and g satisfy the algebraic equation $R(f, g) = 0$ where $R(f, g)$ is given by

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m \omega_1(z+c) - \omega_2^n(\omega_2 - 1)^m \omega_2(z+c).$$

Theorem I. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 5k+4m+12$. If $[f^n(z)(f^m(z)-1)f(z+c)]^{(k)}$ and $[g^n(z)(g^m(z)-1)g(z+c)]^{(k)}$ share $\alpha(z)$ IM, then $f = tg$, where $t^m = 1$.

Theorem J. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that c is a non zero complex constant, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 5k+4m+12$ when $m \leq k+1$ and $n \geq 10k-m+19$ when $m > k+1$. If $[f^n(z)(f(z)-1)^m f(z+c)]^{(k)}$ and $[g^n(z)(g(z)-1)^m g(z+c)]^{(k)}$ share $\alpha(z)$, then conclusion of theorem H hold.

Regarding Theorems E-J, the following question is inevitable which is motivation of the present paper.

Question. What would happen if one replaces the difference polynomials $[f^n(z)(f^m(z)-1)f(z+c)]^{(k)}$ by $f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}$ in Theorems E-J, where k is any positive integer?

In this paper, we study the zero and uniqueness of difference polynomial of the form $f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{v_j}$ and $f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}$ where $c_j(j=1,2,\cdots,d)$ are complex constants, $v_j(j=1,2,\cdots,d)$ are non-negative integers and $\sigma=v_1+v_2+\cdots+v_d=\sum_{j=1}^{d}v_j$ and hence obtain the following results.

Theorem 1. Let f be a transcendental entire function of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that $c_j(j=1,2,\cdots,d)$ are non zero complex constants, $v_j(j=1,2,\cdots,d)$ are non-negative integers, $n,m\geq 1$ and $k(\geq 0)$ are integers. If $n\geq k+2$, then $\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k)}-\alpha(z)$ has infinitely many zeros.

Theorem 2. Let f be a transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f. Suppose that $c_j(j=1,2,\cdots,d)$ is a non zero complex constants, $v_j(j=1,2,\cdots,d)$ are non-negative integers, $n,m\geq 1$ and $k(\geq 0)$ are integers. If $n\geq k+2$ when $m\leq k+1$ and $n\geq 2k-m+3$ when $m>k+1$, then $\left[f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k)}-\alpha(z)$ has infinitely many zeros.

Theorem 3. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that $c_j(j=1,2,\cdots,d)$ are non zero complex constants, $v_j(j=1,2,\cdots,d)$ are non-negative integers, $n,m\geq 1$ and $k(\geq 0)$ are integers satisfying $n\geq 2k+m+\sigma+5$. If $\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k)}$ and $\left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\right]^{(k)}$ share $(\alpha,2)$, then $f=tg$ where $t^m=1$.

Theorem 4. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that $c_j(j=1,2,\cdots,d)$ are non zero complex constants, $v_j(j=1,2,\cdots,d)$ are non-negative integers, $n,m\geq 1$ and $k(\geq 0)$ are integers satisfying $n\geq 2k+m+\sigma+5$ when $m\leq k+1$ and $n\geq 4k-m+\sigma+9$ when $m>k+1$. If $\left[f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k)}$ and $\left[g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{v_j}\right]^{(k)}$ share $(\alpha,2)$, then either $f=g$ or f and g satisfy the algebraic equation $R(f,g)=0$ where $R(f,g)$ is given by

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m\prod_{j=1}^{d}\omega_1(z+c_j)^{v_j}-\omega_2^n(\omega_2-1)^m\prod_{j=1}^{d}\omega_2(z+c_j)^{v_j}.$$

Theorem 5. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that $c_j(j=1,2,\cdots,d)$ are non-negative complex constants, $v_j(j=1,2,\cdots,d)$ are non-negative integers, $n,m\geq 1$ and $k(\geq 0)$ are integers satisfying $n\geq 5k+4m+4\sigma+8$. If $\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\right]^{(k)}$ and $\left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\right]^{(k)}$ share $\alpha(z)$ IM, then $f=tg$ where $t^m=1$.

Theorem 6. Let f and g be two transcendental entire functions of finite order and $\alpha(z)(\not\equiv 0)$ be a small function with respect to f and g. Suppose that $c_j(j=1,2,\cdots,d)$ are non zero complex constants, $v_j (j = 1, 2, \cdots, d)$ are non-negative integers, $n, m \geq 1$ and $k(\geq 0)$ are integers satisfying $n \geq 5k + 4m + 4\sigma + 8$ when $m \leq k + 1$ and $n \geq 10k - m + 4\sigma + 15$ when $m > k + 1$. If $\left[f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ and $\left[g^n(z)(g(z)-1)^m \prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$ share $\alpha(z)$ IM, then the conclusion of theorem 4 hold.

Remark. For $\sigma = 1$ in Theorems 1 to 6, we get Theorems E to J. Hence Theorems 1 to 6 generalizes Theorems E to J.

2 Preliminary Lemmas

Let F and G be two non-constant meromorphic functions defined in the complex plane $\mathbb{C}$. We denote by H the following functions. $H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right)$

Lemma 2.1 (see [13]). Let f be a meromorphic function of finite order ρ and let $c(\neq 0)$ be a fixed non zero complex constant. Then

$$\overline{N}(r, \infty; f(z+c)) \leq \overline{N}(r, \infty; f) + S(r, f)$$

outside a possible exceptional set of finite logarithmic measure.

Lemma 2.2 (see [3]). Let f be an entire function of finite order and $F = f^n(z)(f^m(z)-1)f(z+c)$. Then $T(r, F) = (n + m + 1)T(r, f) + S(r, f)$.

Arguing in a similar manner as in Lemma 2.6[3] we obtain the following Lemma.

Lemma 2.3. Let f be an entire function of finite order and $F = f^n(z)(f(z)-1)^m \prod_{j=1}^{d} f(z+c_j)$. Then $T(r, F) = (n + m + \sigma)T(r, f) + S(r, f)$.

Lemma 2.4(see [20]). Let f be a non-constant meromorphic functions and p, k be two positive integers. Then

$$N_p(r, 0; f^{(k)}) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}(r, 0; f) + S(r, f) \tag{2.1}$$

$$N_p(r, 0; f^{(k)}) \leq k\overline{N}(r, \infty, f) + N_{p+k}(r, 0; f) + S(r, f) \tag{2.2}$$

Lemma 2.5 (see [10]). Let f and g be two non-constant meromorphic functions sharing (1,2). Then one of the following cases holds.

(i) $T(r) \leq N_2(r, 0; f) + N_2(r, 0; g) + N_2(r, \infty; f) + N_2(r, \infty; g) + S(r)$,

(ii) $f = g$,

(iii) $fg = 1$, when $T(r) = max\{T(r, f), T(r, g)\}$ and $S(r) = o\{T(r)\}$.

Lemma 2.6 (see [1]). Let F and G be two non-constant meromorphic functions sharing the value 1 IM and $H \not\equiv 0$. Then

$$\begin{aligned} T(r, F) \leq & N_2(r, 0; F) + N_2(r, 0; G) + N_2(r, \infty; F) + N_2(r, \infty; G) + 2\overline{N}(r, 0; F) + \overline{N}(r, 0; G) \\ & + 2\overline{N}(r, \infty; F) + \overline{N}(r, \infty; G) + S(r, F) + S(r, G) \end{aligned}$$

and the same inequality holds for $T(r,G)$.

Lemma 2.7. Let f and g be two entire functions, suppose that $c_j (j=1,2,\cdots,d)$ are non zero complex constants, $v_j (j=1,2,\cdots,d)$ are non-negative integers, $n,m \geq 1$ and $k(\geq 0)$ are integers and let $F=\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ and $G=\left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$. If there exists non zero constants c_1 and c_2 such that $\overline{N}(r,c_1;F)=\overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G)=\overline{N}(r,0;F)$, then $n \leq 2k+m+\sigma+2$.

Proof. We put $F_1=f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$ and $G_1=g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}$, by the second fundamental theorem of Nevanlinna, we have

$$\begin{aligned} T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}(r,c_1;F)+S(r,F) \\ &\leq \overline{N}(r,0;F)+\overline{N}(r,0;G)+S(r,F) \end{aligned} \tag{2.3}$$

Using equation (2.3), in Lemmas 2.2 and 2.4, we obtain

$$\begin{aligned} (n+m+\sigma)T(r,f) &\leq T(r,F)-\overline{N}(r,0;F)+N_{k+1}(r,0;F_1)+S(r,f) \\ &\leq \overline{N}(r,0;G)+N_{k+1}(r,0;F_1)+S(r,f) \\ &\leq N_{k+1}(r,0;F_1)+N_{k+1}(r,0;G_1)+S(r,f)+S(r,g) \\ &\leq (k+m+\sigma+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \end{aligned} \tag{2.4}$$

Similarly,

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+1)(T(r,f)+T(r,g))+S(r,f)+S(r,g). \tag{2.5}$$

Combining (2.4) and (2.5), we obtain

$$(n-2k-m-\sigma-2)(T(r,f)+T(r,g)) \leq S(r,f)+S(r,g).$$

Which gives $n \leq 2k+m+\sigma+2$.
This proves the lemma.

Lemma 2.8. Let f and g be two entire functions, suppose that $c_j (j=1,2,\cdots,d)$ are non zero complex constants, $v_j (j=1,2,\cdots,d)$ are non-negative integers, $n,m \geq 1$ and $k(\geq 0)$ are integers and let $F=\left[f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)}$ and $G=\left[g^n(z)(g(z)-1)^m\prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$. If there exists non zero constants c_1 and c_2 such that $\overline{N}(r,c_1;F)=\overline{N}(r,0;G)$ and $\overline{N}(r,c_2;G)=\overline{N}(r,0;F)$, then $n \leq 2k+m+\sigma+2$ when $m \leq k+1$ and $n \leq 4k-m+\sigma+4$ when $m > k+1$.

Proof. By the same reasoning as in proof of Lemma 2.7, we can easily deduce the result. Hence we omit the details.

Arguing in a similar manner as in lemma 5([2]), we obtain the following lemma.

Lemma 2.9. Suppose that f and g are two transcendental entire function of finite order. Suppose that $c_j (j=1,2,\cdots,d)$ are non zero complex constants, $v_j (j=1,2,\cdots,d)$ are non-negative integers, $n,m \geq 1$ and $k(\geq 0)$ are integers. If $n \geq m+5$ and $\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)} = \left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)}$ then $f=tg$ where $t^m=1$.

3 Proof of the Theorem

Proof of Theorem 1. Let $F_1 = f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$. Then F_1 is a transcendental entire function.
If possible, we assume $F_1^{(k)} - \alpha(z)$ has only finitely many zeros. Then, we have

$$N(r, \alpha, F_1^{(k)}) = o\{\log r\} = S(r, f). \tag{3.1}$$

Using (2.1), (3.1) and Nevanlinna's three small function theorem, we obtain

$$\begin{aligned} T(r, F_1^{(k)}) &\leq \overline{N}(r, 0, F_1^{(k)}) + \overline{N}(r, \alpha; F_1^{(k)}) + S(r, f) \\ &\leq T(r, F_1^{(k)}) - T(r, F_1) + N_{k+1}(r, 0; F_1) + S(r, f) \end{aligned} \tag{3.2}$$

Applying lemma 2.2, we obtain from (3.2),

$$\begin{aligned} (n+m+\sigma)T(r, f) &\leq N_{k+1}(r, 0; F_1) + S(r, f) \\ &\leq (k+m+\sigma+1)T(r, f) + S(r, f) \end{aligned}$$

This gives

$$(n-k-1)T(r, f) \leq S(r, f),$$

a contradiction with the assumption that $n \geq k+2$. This proves the theorem.

Proof of Theorem 2. Let $F_2 = f^n(z)(f(z)-1)^m\prod_{j=1}^{d} f(z+c_j)^{v_j}$. Then F_2 is a transcendental entire function.
If possible, suppose that $F_2^{(k)} - \alpha(z)$ has only finitely many zeros. Then, we have

$$N(r, \alpha, F_2^{(k)}) = o\{\log r\} = S(r, f) \tag{3.3}$$

Now, using (2.1), (3.3) and Nevanlinna's three small function theorem, we obtain

$$\begin{aligned} T(r, F_2^{(k)}) &\leq \overline{N}(r, 0, F_2^{(k)}) + \overline{N}(r, \alpha; F_2^{(k)}) + S(r, f) \\ &\leq T(r, F_2^{(k)}) - T(r, F_2) + N_{k+1}(r, 0; F_2) + S(r, f) \end{aligned} \tag{3.4}$$

Applying lemma 2.3, we obtain from (3.4)

$$\begin{aligned} (n+m+\sigma)T(r, f) &\leq N_{k+1}(r, 0; F_2) + S(r, f) \\ &\leq (k+m+\sigma+1)T(r, f) + S(r, f) \end{aligned} \tag{3.5}$$

If $m \leq k+1$, we deduce from (3.5) that

$$(n-k-1)T(r, f) \leq S(r, f),$$

a contradiction to the assumption that $n \geq k+2$.
If $m < k+1$, by (3.5) we get,

$$(n+m-2k-2)T(r, f) \leq S(r, f)$$

a contradiction with the assumption that $n \geq 2k - m + 3$. This proves the theorem.

Proof of Theorem 3. Let $F_1 = f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}$ and $G_1 = g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}$
$F = \dfrac{F_1^{(k)}}{\alpha(z)}$ and $G = \dfrac{G_1^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share (1,2) except the zeros and poles of $\alpha(z)$. Using (2.1) and lemma 2.2, we get

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\leq T(r,F_1^{(k)}) - (n+m+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f) \\ &\leq T(r,F) - (n+m+\sigma)T(r,f) + N_{k+2}(r,0;F_1) + S(r,f) \end{aligned}$$

From this we get,

$$(n+m+\sigma)T(r,f) \leq T(r,F) + N_{k+2}(r,0;F_1) - N_2(r,0;F) + S(r,f) \tag{3.6}$$

Again by (2.2), we have

$$\begin{aligned} N_2(r,0;F) &\leq N_2(r,0;F_1^{(k)}) + S(r,f) \\ &\leq N_{k+2}(r,0;F_1) + S(r,f) \end{aligned} \tag{3.7}$$

Suppose, if possible that (i) of Lemma 2.5 holds, Then, using (3.7), we obtain from (3.6)

$$\begin{aligned} (n+m+\sigma)T(r,f) &\leq N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + N_{k+2}(r,0;F_1) + S(r,f) + S(r,g) \\ &\leq N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + S(r,f) + S(r,g) \\ &\leq (k+2+m+\sigma)\{T(r,f) + T(r,g)\} + S(r,f) + S(r,g) \end{aligned} \tag{3.8}$$

In a similar manner we obtain,

$$(n+m+\sigma)T(r,g) \leq (k+2+m+\sigma)\{T(r,f)+T(r,g)\} + S(r,f) + S(r,g). \tag{3.9}$$

From (3.8) and (3.9) together gives,

$$(n-2k-m-\sigma-4)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g),$$

contradicting with the fact that $n \geq 2k+m+\sigma+5$. Therefore, by Lemma 2.5 we have either $FG = 1$ or $F = G$

Let $FG = 1$. Then,

$$\left[f^n(z)(f^m(z)-1)\prod_{j=1}^{d} f(z+c_j)^{v_j}\right]^{(k)} \cdot \left[g^n(z)(g^m(z)-1)\prod_{j=1}^{d} g(z+c_j)^{v_j}\right]^{(k)} = \alpha^2$$

$\Big[f^n(z)(f(z)-1)(f^{m-1}(z)+f^{m-2}(z)+\cdots+1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\Big]^{(k)}\cdot\Big[g^n(z)(g(z)-1)(g^{m-1}(z)+g^{m-2}(z)+\cdots+1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\Big]^{(k)}=\alpha^2$

It can be easily viewed from above that

$$N(r,0;f)=S(r,f)\text{ and }N(r,1;f)=S(r,f)$$

Thus,

$$\delta(0,f)+\delta(1,f)+\delta(\infty,f)=3,$$

Which is not possible. Therefore, we must have $F=G$, and then

$$\Big[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\Big]^{(k)}=\Big[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\Big]^{(k)}$$

Integrating above, we get,

$$\Big[f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}\Big]^{(k-1)}=\Big[g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}\Big]^{(k-1)}+C_{k-1}$$

Where C_{k-1} is a constant. If $C_{k-1}\neq 0$, using Lemma 2.7, it follows that $n\leq 2k+m+\sigma$ a contradiction. Hence $C_{k-1}=0$, repeating k times, we deduce that,

$$f^n(z)(f^m(z)-1)\prod_{j=1}^{d}f(z+c_j)^{v_j}=g^n(z)(g^m(z)-1)\prod_{j=1}^{d}g(z+c_j)^{v_j}$$

which by Lemma 2.9, gives $f=tg$ where t is a constant satisfying $t^m=1$. This proves Theorem 3.

Proof of Theorem 4. Let $F_1=f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{v_j}$ and $G_1=g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{v_j}$
$F=\dfrac{F_1^{(k)}}{\alpha(z)}$ and $G=\dfrac{G_1^{(k)}}{\alpha(z)}$. Then F and G are transcendental meromorphic functions that share (1,2) except possibly the zeros and poles of $\alpha(z)$.

Arguing in a manner similar to the proof of Theorem 3, we obtain either $FG=1$ or $F=G$. If $F=G$, then applying the same techniques as in the proof of Theorem 3 and using Lemma 2.8, we obtain.

$$f^n(z)(f(z)-1)^m\prod_{j=1}^{d}f(z+c_j)^{v_j}=g^n(z)(g(z)-1)^m\prod_{j=1}^{d}g(z+c_j)^{v_j}\tag{3.10}$$

Set $h=\dfrac{f}{g}$. If h is a constant, then substituting $f=gh$ in equation (3.10), we duduce that

$$g^n\prod_{j=1}^{d}g(z+c_j)^{v_j}\Big[g^m(h^{n+m+\sigma}-1)-{}^{m}c_1g^{m-1}(h^{n+m+\sigma-1}-1)+\cdots+(-1)^m(h^{n+\sigma}-1)\Big]=0$$

Since g is a transcendental entire function, we have $g^n \prod_{j=1}^{d} g(z+c_j)^{v_j} \neq 0$. So from above we obtain,

$$g^m(h^{n+m+\sigma}-1) - {}^{m}c_1 g^{m-1}(h^{n+m+\sigma-1}-1) + \cdots + (-1)^m(h^{n+\sigma}-1) = 0$$

which implies $h = 1$
Hence $f = g$. If h is not a constant, then it follows from equation (3.10) that f and g satisfy the algebraic equation $R(f,g) = 0$ where $R(f,g)$ is given by

$$R(\omega_1,\omega_2) = \omega_1^n(\omega_1-1)^m \prod_{j=1}^{d} \omega_1(z+c_j)^{v_j} - \omega_2^n(\omega_2-1)^m \prod_{j=1}^{d} \omega_2(z+c_j)^{v_j}$$

If $FG = 1$, proceeding in a like manner as in the proof of Theorem 3 we arrive at a contradiction. This completes the proof of Theorem 4.

Proof of Theorem 5. Let F, G, F_1 and G_1 be defined as in the proof of Theorem 3. Then, F and G are transcendental meromorphic functions that share the value 1 IM except the zeros and poles of $\alpha(z)$. We assume, if possible, that $H \not\equiv 0$. Using Lemma 2.6 and (3.7), we obtain from (3.6).

$$\begin{aligned}(n+m+\sigma)T(r,f) \leq\ & N_2(r,0;G) + N_2(r,\infty;F) + N_2(r,\infty;G) + 2\overline{N}(r,0;F) + \overline{N}(r,0;G)\\ & + N_{k+2}(r,0;F_1) + 2\overline{N}(r,\infty;F) + \overline{N}(r,\infty;G) + S(r,f) + S(r,g)\\ \leq\ & N_{k+2}(r,0;F_1) + N_{k+2}(r,0;G_1) + 2N_{k+1}(r,0;F_1) + 2N_{k+1}(r,0;G_1) + S(r,f) + S(r,g)\\ \leq\ & (3k+4+3m+3\sigma)T(r,f) + (2k+3+2m+2\sigma)T(r,g) + S(r,f) + S(r,g)\\ \leq\ & (5k+5m+5\sigma+7)T(r) + S(r)\end{aligned} \tag{3.11}$$

Similarly,

$$(n+m+\sigma)T(r,f) \leq (5k+5m+12)T(r) + S(r). \tag{3.12}$$

From equations (3.11) and (3.12), together yields

$$(n-4m-4\sigma-5k-7)T(r) \leq S(r),$$

which is a contradiction with the assumption that $n \geq 5k+4m+4\sigma+8$. We now assume that $H \equiv 0$. Then

$$\left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right) = 0$$

Integrating both sides of above equality twice, we get

$$\frac{1}{F-1} = \frac{A}{G-1} + B. \tag{3.13}$$

Where $A(\neq 0)$ and B are constants.
From (3.13) it is obvious that F, G share value 1 CM and hence they share (1,2). Therefore $n \geq 2k+m+\sigma+5$.

We now discuss the following three cases separately.
case i. Suppose that $B \neq 0$ and $A = B$, then from (3.13) we obtain.

$$\frac{1}{F-1} = \frac{BG}{G-1}. \tag{3.14}$$

If $B = -1$, then from (3.14), we obtain $FG = 1$, which is a contradiction as in the proof of Theorem 3.

If $B \neq -1$, from (3.14), we have,

$$\frac{1}{F} = \frac{BG}{(1+B)G-1}$$

and so $\overline{N}\left(r, \frac{1}{1+B}; G\right) = \overline{N}(r, 0; F)$.
Using (2.1), (2.2) and Second Fundamental Theorem of Nevanlinna, we deduce that

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G) + \overline{N}\left(r, \frac{1}{1+B}; G\right) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq \overline{N}(r,0;F) + \overline{N}(r,0;G) + \overline{N}(r,\infty;G) + S(r,G) \\ &\leq N_{k+1}(r,0;F_1) + T(r,G) + N_{k+1}(r,0;G_1) - (n+m+\sigma)T(r,g) + S(r,g) \end{aligned}$$

This gives,

$$(n+m+\sigma)T(r,g) \leq (k+m+\sigma+1)\{T(r,g)+T(r,g)\} + S(r,g)$$

Thus we obtain

$$(n-2k-m-\sigma-2)\{T(r,g)+T(r,g)\} \leq S(r,f) + S(r,g),$$

which is a contradiction as $n \geq 2k+m+\sigma+5$.
case ii. Let $B \neq 0$ and $A \neq B$. Then From (3.13), we get

$$F = \frac{(B+1)G-(B-A+1)}{BG+(A-B)},$$

and so $\overline{N}\left(r, \frac{B-A+1}{B+1}; G\right) = \overline{N}(r,0;F)$,
Proceeding in a manner similar to case i we can arrive at a contradiction.
case iii. Let $B = 0$ and $A \neq 0$. Then from (3.13) we get

$$F = \frac{G+A-1}{A} \text{ and } G = AF - (A-1)$$

If $A \neq 1$, it follows that $\overline{N}\left(r, \frac{A-1}{A}; F\right) = \overline{N}(r,0;G)$ and $\overline{N}\left(r, 1-A; G\right) = \overline{N}(r,0;F)$ Now applying Lemma 2.7, it can be shown that $n \leq 2k+m+\sigma+2$, which is a contradiction.

Thus, $A = 1$ and then $F = G$. Now the result follows from the proof of Theorem 3.
This completes the proof of Theorem 5.

Proof of Theorem 6. Proceeding as in the proof of Theorem 5, the conclusion of Theorem 6 follows. Here we omit the details.

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