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Associated primes of h-wheels

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We study the associated primes of the powers of the cover ideal of h-wheels. The main result generalizes a theorem of Kesting, Pozzi, and Striuli (2011).

Several pieces of information about an ideal I in a commutative noetherian ring R are enclosed in its primary decomposition: Given an ideal I we can write $I = \bigcap_{i=1}^{\ell} Q_i$, where the radical ideal of each ideal Q_i is given by a prime ideal P_i of the ring R. The prime ideals P_i for $i = 1, \dots, \ell$ are called associated primes of the ideal I. The finiteness conditions imposed by a noetherian ring not only allow the decomposition of an ideal into primary components, but also have stronger repercussions, as shown in the following statement proved by Brodmann [1979] in which the set $\operatorname{Ass}(R/I)$ denotes the set of all the associated primes of I:

- *Let I be an ideal in a commutative noetherian ring*; *then the set*

$$\bigcup_{i=1}^{\infty} \operatorname{Ass}(R/I^i)$$

is finite. Moreover, *there exists an integer m such that for all $k \geq m$ the equality* $\operatorname{Ass}(R/I^m) = \operatorname{Ass}(R/I^k)$ *holds.*

The positive integer m identified by Brodmann's theorem is called the index of stability for the associated primes of I, denoted by $\operatorname{astab}(I)$. Despite the simplicity of the statement, the value of $\operatorname{astab}(I)$ remains generally unknown.

Much work has been done recently for graded ideals in polynomial rings. While a large upper bound for $\operatorname{astab}(I)$ for monomial ideals was given in [Hoa 2006] in terms of properties of the ideal itself, a lot of recent work supports the conjecture that in a polynomial ring $\mathsf{k}[x_1, \dots, x_d]$ the uniform bound $\operatorname{astab}(I) \leq d$ for every graded ideal $I \subseteq \mathsf{k}[x_1, \dots, x_d]$ holds; see for example [Herzog and Asloob Qureshi 2015, Theorem 4.1] for polymatroid ideals.

More cases for which the conjecture holds true come from ideals that arise from graphs. In this paper, a graph G is given by a set of vertices $V_G = \{x_1, \dots, x_d\}$ and a set of edges E_G; elements of E_G are subsets of V_G of cardinality 2. In particular,

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if $\{x_i, x_j\}$ is an edge then we say that x_i and x_j are adjacent vertices. Given such a graph G, the *edge ideal* of G is an ideal of the polynomial ring $\mathsf{k}[x_1, \dots, x_d]$ generated by the monomials $x_i x_j$ such that $\{x_i, x_j\} \in E_G$.

The conjecture is verified for edge ideals. It follows from [Simis et al. 1994, Theorem 5.9] that $\operatorname{astab}(I)$ is equal to 1 for edge ideals of bipartite graphs. In [Chen et al. 2002, Proposition 4.3], the authors show the conjecture, and in fact a stronger statement, holds for edge ideals of nonbipartite graphs.

The authors of [Francisco et al. 2011] look at cover ideals of graphs (in fact the paper deals with the more general notion of a hypergraph). We define the cover ideal later, but in Corollary 4.9 of the paper above, the authors prove that if J is the cover ideal of a simple graph then $\operatorname{astab}(J) \leq \chi(G) - 1$, where $\chi(G)$ is the coloring number of the graph (which is bounded above by the number of vertices of a graph). Further, they fully characterize prime ideals that appear as associated primes of the second power of the cover ideal.

In line with this work, in [Kesting et al. 2011] the authors study which prime ideals appear as associated primes of the third power of the cover ideal. They prove that the *wheel* corresponds to an element of $\operatorname{Ass}(R/J^3)$.

In this paper we generalize the work of [Kesting et al. 2011]. Given an integer h, we define the h-wheel and prove the following:

0.1. Theorem. *Let G be graph with vertex set $V_G = \{x_1, \dots, x_d\}$ that is an h-wheel. Denote by $J_G \subseteq \mathsf{k}[x_1, \dots, x_d]$ the cover ideal of G. Then the prime ideal $(x_1, \dots, x_d)$ belongs to $\operatorname{Ass}(R/J^n)$ if and only if $n \geq h + 2$.*

As a corollary, for every integer $d \geq 6$ we deliver an ideal I_d in a polynomial ring with d variables such that $\operatorname{astab}(I_d) \geq d - 3$.

1. Definitions

We now introduce the notation and give the definitions used in the paper.

1.1. Given a graph G with vertex set $V_G = \{x_1, \dots, x_d\}$, we consider the polynomial ring $\mathsf{k}[x_1, \dots, x_d]$, which we often denote by $\mathsf{k}[V_G]$. If S is a subset of V_G, then the prime monomial ideal P_S is the ideal generated by the variables $x \in S$. If $S = V_G$, then we denote P_S by $\mathfrak{m}_G$, the maximal homogeneous ideal in $\mathsf{k}[V_G]$. It is worth noting that a prime monomial ideal is always generated by a subset of the variables. In this setting, given a monomial $\boldsymbol{m} \in \mathsf{k}[x_1, \dots, x_d]$ we can write $\boldsymbol{m} = \prod_{i=1}^{d} x_i^{\alpha_i}$, where $\alpha_i \geq 0$. The support of $\boldsymbol{m}$ is the set of variables $\{x_i \mid \alpha_i > 0\}$ and it is denoted as $\operatorname{supp}(\boldsymbol{m})$. We denote by $\operatorname{ver}(\boldsymbol{m})$ the subset of V_G of vertices labeled by the variables appearing in $\operatorname{supp}(\boldsymbol{m})$.

1.2. Definition. Given a graph G with vertex set $V_G = \{x_1, \dots, x_d\}$ and edge set E_G, a *cover* of G is a subset S of V_G such that each edge in E_G has a nonempty intersection with S.

The cover ideal $J_G \subset \mathsf{k}[x_1, \dots, x_d]$ is the monomial ideal generated by monomials $\boldsymbol{m}$ such that $\mathrm{ver}(\boldsymbol{m})$ is a cover of G.

The following definition is a particular case of the definition of associated prime given in [Eisenbud 1995, page 89].

1.3. Definition. Let I be a monomial ideal of the polynomial ring $\mathsf{k}[x_1, \dots, x_d]$ and let $P = (x_{i_1}, \dots, x_{i_\ell})$ be a monomial prime ideal containing I. We say that P is an associated prime of I, and we write $P \in \mathrm{Ass}(R/I)$, if there exists a monomial $\boldsymbol{w} \in \mathsf{k}[x_1, \dots, x_d]$ such that $\boldsymbol{w} \notin I$, $x_i \boldsymbol{w} \in I$ for $i = i_1, \dots, i_\ell$, but $x_i \boldsymbol{w} \notin I$ for $i \neq i_1, \dots, i_\ell$.

The monomial $\boldsymbol{w}$ is called a witness of P for the ideal I.

As shown in [Eisenbud 1995, Theorem 3.10], the associated primes of a monomial ideal I defined in the previous definition are exactly the prime ideals that are radical ideals in a minimal primary decomposition of I.

Let G be a connected graph with vertex set $\{x_1, \dots, x_d\}$. The edge ideal and the cover ideal of G are dual to each other with respect to the Alexander duality; see for a proof [Bruns and Herzog 1993, Chapter 5] or consult [Van Tuyl 2013] for a quicker introduction to the subject. This fact implies that a prime ideal P is an associated prime of the cover ideal if and only if $P = (x_i, x_j)$, where $\{x_i, x_j\}$ is in E_G.

The following theorem extends the knowledge of associated primes to second powers of the cover ideal [Francisco et al. 2010, Corollary 3.4].

1.4. Let G be a connected graph, let S be a subset of the vertex set V_G, and let $R = \mathsf{k}[V_G]$. A prime ideal $P_S \subset \mathsf{k}[V_G]$ belongs to $\mathrm{Ass}(R/J_G^2)$ if and only if the induced subgraph generated by S is an odd cycle in G or S is an edge.

We concentrate our attention on a family of graphs called h-wheels, whose definition is given below. First we need the following notion:

1.5. Let G be a graph with vertex set V_G. Given a vertex $x \in V_G$ and a subset $S \subseteq V_G$ of vertices of G, we denote by $N_S(x)$ the subset of S consisting of adjacent vertices to x. If S is the set of all vertices in G then we use $N(x)$ to denote the set of all vertices adjacent to x.

1.6. Definition. A graph G with vertex set V_G is an h-wheel if V_G can be written as the union of two disjoint sets, the set of rim vertices R^G and the set of center vertices C^G, such that the following conditions hold:

(1) The subgraph induced by C^G is the complete graph on h vertices.

(2) The subgraph induced by R^G is an odd cycle.

(3) There exist $x_1, \dots, x_k \in R^G$ with $k \geq 3$ such that $N_{R^G}(y) = \{x_1, \dots, x_k\}$ for all $y \in C^G$.

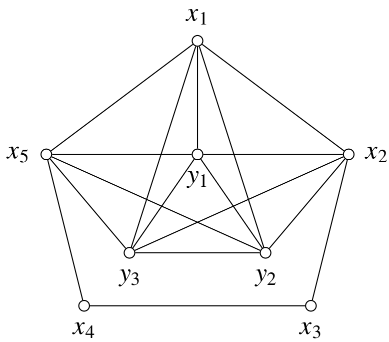

Figure 1. A 3-wheel.

(4) For every $y \in C^G$, the vertex y belongs to at least two odd cycles in the subgraph induced by y and $N_{R^G}(y)$.

We call k the radial number for G. For each $i = 1, \dots, k-1$, set ℓ_i as the length of the path along the subgraph induced by R^G from x_i to x_{i+1}, and set ℓ_k as the length from x_k to x_1. The positive integers $\ell_1, \dots, \ell_k$ are called the radial lengths.

In [Kesting et al. 2011], the authors studied the 1-wheel, which we call a wheel for simplicity. Notice that given an h-wheel G and a vertex $y \in C^G$, the subgraph induced by y and R^G is a wheel.

1.7. Example. Figure 1 is a representation of a 3-wheel G. We have

$$C^G = \{y_1, y_2, y_3\}, \quad R^G = \{x_1, x_2, x_3, x_4, x_5\},$$
$$N_{R^G}(y_1) = N_{R^G}(y_2) = N_{R^G}(y_3) = \{x_1, x_2, x_3\}.$$

In the rest of the paper we rely on the following constructions.

1.8. Definition. Given a graph G and a vertex $x \in V_G$, the *contraction* of G via x is a new graph obtained from G by deleting x and connecting all the vertices in $N(x)$ to each other.

1.9. Definition. Given a graph G, let x_1 and x_2 be two adjacent vertices in G. A *subdivision* of G via the edge $\{x_1, x_2\}$ is a graph obtained from G by deleting the edge $\{x_1, x_2\}$, adding a new vertex y, and adding two new edges $\{x_1, y\}$ and $\{x_2, y\}$.

2. Preliminary lemmas

We now prove several lemmas that are used to prove our main result.

The first lemma describes necessary conditions for a monomial to be a witness for a power of the cover ideal of a graph G.

2.1. Lemma. *Let G be a graph with vertex set V_G, and let J_G be the cover ideal of G in the ring $R = \mathsf{k}[V_G]$. Let $S \subseteq V_G$, and assume that $P_S \in \operatorname{Ass}(R/J_G^n)$. Let $\boldsymbol{w}$ be a witness for P_S. Then x^n does not divide $\boldsymbol{w}$ for any $x \in S$.*

Proof. By the definition of witness, $\boldsymbol{w} \notin J_G^n$.

Suppose toward contradiction that there exists $x \in S$ such that x^n divides $\boldsymbol{w}$. Since the monomial $x\boldsymbol{w}$ is in J_G^n, there exist $\boldsymbol{m}_1, \dots, \boldsymbol{m}_n \in J_G$ such that $x\boldsymbol{w} = \boldsymbol{m}_1 \cdots \boldsymbol{m}_n$. Moreover, since $x^n \mid \boldsymbol{w}$, by the pigeonhole principle we know that there exists an integer s such that $1 \le s \le n$ and x^2 divides $\boldsymbol{m}_s$. Let $\boldsymbol{m}'_s$ be the monomial $\boldsymbol{m}_s/x$. Since $\boldsymbol{m}_s \in J_G$, it follows that $\operatorname{ver}(\boldsymbol{m}_s)$ is a cover for G. Since $\operatorname{supp}(\boldsymbol{m}_s) = \operatorname{supp}(\boldsymbol{m}'_s)$, we know $\operatorname{ver}(\boldsymbol{m}_{s'})$ is a cover for G, and it follows that $\boldsymbol{m}'_s \in J_G$. In particular $\boldsymbol{w}$ can be written as the product of the n monomials $\boldsymbol{m}_1 \cdots \boldsymbol{m}_{s-1}\boldsymbol{m}'_s \cdots \boldsymbol{m}_n$, which shows that $\boldsymbol{w} \in J_G^n$. □

In the rest of the paper, if $\boldsymbol{m} = \prod_{i=1}^d x_i^{\alpha_i}$ is a monomial in the ring $\mathsf{k}[x_1, \dots, x_d]$, then $\deg_{\boldsymbol{m}} x_i = \alpha_i$, while the total degree of $\boldsymbol{m}$ is given by $\sum_{i=1}^d \alpha_i$ and is denoted by $\operatorname{tot\,deg} \boldsymbol{m}$.

The following corollary is an immediate consequence of the previous lemma.

2.2. Corollary. *Let G be a graph with vertex set V_G of cardinality larger than 2. Let J_G be the cover ideal of G in the polynomial ring $\mathsf{k}[V_G]$. Assume that $\{x_1, x_2\}$ is an edge of G and assume that $\mathfrak{m}_G \in \operatorname{Ass}(R/J_G^n)$. If $\boldsymbol{w}$ is a witness of $\mathfrak{m}_G$, then $x_1, x_2 \in \operatorname{supp} \boldsymbol{w}$. Moreover, $\deg_{\boldsymbol{w}} x_1 + \deg_{\boldsymbol{w}} x_2 \ge n$.*

Proof. Assume for the sake of contradiction that x_2 does not divide $\boldsymbol{w}$. Let $x \in V_G \setminus \{x_1, x_2\}$. The monomial $x\boldsymbol{w}$ can be written as the product of n monomials $\boldsymbol{m}_1 \cdots \boldsymbol{m}_n$ such that $\boldsymbol{m}_i \in J_G$ for all $i = 1, \dots, n$. By Lemma 2.1 $\deg_{\boldsymbol{w}} x_1 \le n-1$, and therefore we can conclude that there exists an $i \in \{1, \dots, n\}$ such that x_1 does not divide $\boldsymbol{m}_i$. Since x_2 does not divide $\boldsymbol{w}$, it follows that x_2 does not divide $\boldsymbol{m}_i$. In particular, $\operatorname{ver}(\boldsymbol{m}_i)$ cannot be a cover of G, as neither x_1 nor x_2 are in $\operatorname{supp}(\boldsymbol{m}_i)$, while $\{x_1, x_2\}$ forms an edge.

Notice that either x_1 or x_2 divides $\boldsymbol{m}_i$, as $\boldsymbol{m}_i \in J_G$ for all $i = 1, \dots, n$, verifying the final statement. □

In the following K_h denotes the complete graph in h vertices. Notice that every cover of K_h contains at least $h-1$ vertices.

2.3. Lemma. *Let G be a graph with vertex set V_G. Let J_G be the cover ideal in the polynomial ring $R = \mathsf{k}[V_G]$. If G contains the complete graph K_h as an induced subgraph but $G \ne K_h$, then $\mathfrak{m}_G \notin \operatorname{Ass}(R/J_G^n)$ for all integers n such that $n \le h-1$.*

Proof. Suppose G contains K_h as an induced subgraph. Without loss of generality we may label the vertices of K_h with the variables $\{x_1, \dots, x_h\}$. Towards contradiction, assume that $\mathfrak{m}_G \in \operatorname{Ass}(R/J_G^n)$ with $n \le h-1$, and let $\boldsymbol{w}$ be a witness. For every monomial $\boldsymbol{c} \in J_G$, we have that $\boldsymbol{c} \in J_{K_h}$. This implies that at least

$h-1$ variables among $x_1, \dots, x_h$ belong to $\operatorname{supp} \boldsymbol{c}$. Therefore, if $\boldsymbol{c} \in J_G^n$ then $\sum_{i=1}^h \deg_{\boldsymbol{c}} x_i \geq n(h-1) = nh - n$.

However, we know from Lemma 2.1 that for each variable x_i the inequality $\deg_{\boldsymbol{w}} x_i \leq n-1$ holds, so that $\sum_{i=1}^h \deg_{\boldsymbol{w}} x_i \leq h(n-1) = hn - h$.

If $x \in V_G$ and $x \neq x_i$ for $i = 1, \dots, h$, then $x\boldsymbol{w} \in J_G^n$, as $\boldsymbol{w}$ is a witness of $\mathfrak{m}_G$, which yields

$$n(h-1) \leq \sum_{i=1}^{h} \deg_{x_j \boldsymbol{w}} x_i = \sum_{i=1}^{h} \deg_{\boldsymbol{w}} x_i \leq h(n-1).$$

This gives us the desired contradiction $h \leq n$. □

In the following lemma, under proper assumptions, we can be more specific about the degree formula presented in Corollary 2.2.

2.4. A monomial $\boldsymbol{n} \in \mathsf{k}[x_1, \dots, x_d]$ is said square-free if for all $i = 1, \dots, d$ the monomial x_i^2 does not divide $\boldsymbol{n}$. For a graph G with cover ideal J_G, given a monomial $\boldsymbol{m} \in J_G$, one can always find a square-free monomial $\boldsymbol{n} \in J_G$ such that $\boldsymbol{n}$ divides $\boldsymbol{m}$. In particular for a product of n monomials $\boldsymbol{m} = \boldsymbol{m}_1 \cdots \boldsymbol{m}_n$ such that $\boldsymbol{m}_i \in J_G$ for all $i = 1, \dots, n$ and $\deg_{\boldsymbol{m}} x_j \leq n-1$ for all $j = 1, \dots, d$, we may assume that each $\boldsymbol{m}_i$ is square-free.

2.5. Lemma. *Let G be a graph with vertex set V_G of cardinality bigger than* 4. *Let J_G be the cover ideal of G in the polynomial ring $\mathsf{k}[V_G]$. Assume that there are $x_1, x_2, x_3, x_4 \in V_G$ such that $N(x_2) = \{x_1, x_3\}$ and $N(x_3) = \{x_2, x_4\}$. Assume further that, for a given positive integer n, $\mathfrak{m}_G \in \operatorname{Ass}(R/J_G^n)$ with witness $\boldsymbol{w}$. If* $\deg_{\boldsymbol{w}} x_1 = n-1$, *then* $\deg_{\boldsymbol{w}} x_2 + \deg_{\boldsymbol{w}} x_3 = n$.

Proof. Since $\boldsymbol{w}$ is a witness for the ideal J_G^n, we know that $\deg_{\boldsymbol{w}} x_2 + \deg_{\boldsymbol{w}} x_3 \geq n$ by the adjacency assumption and Corollary 2.2.

Since $\boldsymbol{w}$ is a witness for $\mathfrak{m}_G$, we have $x_2\boldsymbol{w} = \boldsymbol{m}_1 \cdots \boldsymbol{m}_n$, where $\boldsymbol{m}_1, \dots, \boldsymbol{m}_n \in J_G$. By Lemma 2.1, $\deg_{\boldsymbol{w}} x_i \leq n-1$, so we may assume that the monomial $\boldsymbol{m}_j$ is square-free for all $j = 1, \dots, n$; see 2.4.

Suppose for contradiction that $\deg_{\boldsymbol{w}} x_2 + \deg_{\boldsymbol{w}} x_3 \geq n+1$, which implies that $\deg_{x_2\boldsymbol{w}} x_2 + \deg_{x_2\boldsymbol{w}} x_3 \geq n+2$.

By Corollary 2.2, both x_2, and x_3 are in $\operatorname{supp} \boldsymbol{w}$. This implies that x_3^2 divides $x_2\boldsymbol{w}$, as $\deg_{x_2\boldsymbol{w}} x_2 \leq n$, and therefore there exist two integers i_1 and i_2 such that x_2 and x_3 belong to $\operatorname{supp} \boldsymbol{m}_{i_1}$ and $\operatorname{supp} \boldsymbol{m}_{i_2}$. If also x_1 belongs to $\operatorname{supp} \boldsymbol{m}_{i_j}$ for some $j = 1, 2$, then $\boldsymbol{m}_{i_j}/x_2 \in J_G$, since x_1x_3 divides $\boldsymbol{m}_{i_j}/x_2$. Thus, in this case,

$$\boldsymbol{w} = \frac{x_2\boldsymbol{m}}{x_2} = \boldsymbol{m}_1 \cdots \frac{\boldsymbol{m}_{i_j}}{x_2} \cdots \boldsymbol{m}_n \in J_G^n,$$

a contradiction, since $\boldsymbol{w}$ is a witness. Thus we may assume that x_1 does not divide $\boldsymbol{m}_{i_1}$ and $\boldsymbol{m}_{i_2}$, which implies that $\deg_{\boldsymbol{w}} x_1 < n-1$, contradicting the hypothesis. □

The careful analysis of the degrees of the witnesses allows us to draw useful conclusions about when $\mathfrak{m}_G$ is an associated prime after contracting a vertex.

2.6. Lemma. *Let G be a graph with vertex set V_G. Let J_G be the cover ideal of G in the polynomial ring $R = \mathsf{k}[V_G]$. Assume $x_1, y_1, y_2, x_2 \in V_G$ such that $N(y_1) = \{x_1, x_2\}$ and $N(y_2) = \{y_1, x_2\}$. Assume that $\mathfrak{m}_G \in \operatorname{Ass}(R/J_G^n)$ for some integer n and that there exists a witness $\boldsymbol{w}$ such that $\deg_{\boldsymbol{w}} x_1 = n-1$. Obtain G' by contracting y_1 and y_2. Then $\mathfrak{m}_{G'}$ belongs to $\operatorname{Ass}(\mathsf{k}[V_{G'}]/J_{G'}^n)$.*

Proof. Set $a_1 = \deg_{\boldsymbol{w}} y_1$ and let $a_2 = \deg_{\boldsymbol{w}} y_2$. We prove that the monomial $\boldsymbol{w}' = \boldsymbol{w}/(y_1^{a_1} y_2^{a_2})$ is a witness for the ideal $\mathfrak{m}_{G'}$, and thus $\mathfrak{m}_{G'}$ is an element of $\operatorname{Ass}(R/J_{G'}^k)$.

First, we show by contradiction that $\boldsymbol{w}' \notin J_{G'}^n$; toward this end, suppose that $\boldsymbol{w}' = \boldsymbol{m}_1 \cdots \boldsymbol{m}_n$ such that $\boldsymbol{m}_i \in J_{G'}$. For every $x \in V_{G'} \subset V_G$, we have $\deg_{\boldsymbol{w}'} x = \deg_{\boldsymbol{w}} x \leq n-1$, where the inequality is the content of Lemma 2.1. Therefore, by 2.4, we may assume that, for each $x \in V_{G'}$, x^2 does not divide $\boldsymbol{m}_j$ for $j = 1, \ldots, n$. For $1 \leq i \leq n$, define the monomial $\boldsymbol{n}_i$ as

$$\boldsymbol{n}_i = \begin{cases} \boldsymbol{m}_i & \text{if } x_1, x_2 \in \operatorname{supp} \boldsymbol{m}_i, \\ y_1 \boldsymbol{m}_i & \text{if } x_1 \notin \operatorname{supp} \boldsymbol{m}_i, \\ y_2 \boldsymbol{m}_i & \text{if } x_2 \notin \operatorname{supp} \boldsymbol{m}_i. \end{cases}$$

Since $\boldsymbol{m}_i \in J_{G'}$ and $\{x_1, x_2\}$ is an edge of the graph G', each $\boldsymbol{m}_i$ is divisible by at least one of x_1 or x_2, so that our construction of $\boldsymbol{n}_i$ is well-defined. Moreover, for the same reason, for each i such that $1 \leq i \leq n$, if $y_1 \in \operatorname{supp} \boldsymbol{n}_i$ or $y_2 \in \operatorname{supp} \boldsymbol{n}_i$ then $\boldsymbol{n}_i \in J_G$.

Denote by $\boldsymbol{w}''$ the product $\boldsymbol{n}_1 \cdots \boldsymbol{n}_n$ and set $b_i = \deg_{\boldsymbol{w}''} y_i$ for $i = 1, 2$. There are $n - b_1 - b_2$ monomials among the $\boldsymbol{n}_i$ such that $y_1, y_2 \notin \operatorname{supp} \boldsymbol{n}_i$ and therefore there are $n - b_1 - b_2$ monomials among the $\boldsymbol{n}_i$ such that $\operatorname{ver}(\boldsymbol{n}_i)$ are not covers of G as $\{y_1, y_2\}$ is an edge in G. We may assume, by renaming the $\boldsymbol{n}_i$, that

$$\begin{cases} \boldsymbol{n}_i \notin J_G, & i = 1, \ldots, n - b_1 - b_2, \\ \boldsymbol{n}_i \in J_G, & i = n - b_1 - b_2 + 1, \ldots, n. \end{cases}$$

Since $\deg_{\boldsymbol{m}_i} x \leq 1$ for every $x \in V_{G'}$, we have $\deg_{\boldsymbol{w}''} y_j = n - \deg_{\boldsymbol{w}''} x_j$ for $j = 1, 2$. In particular,

$$b_j = \deg_{\boldsymbol{w}''} y_j = n - \deg_{\boldsymbol{w}''} x_j = n - \deg_{\boldsymbol{w}} x_j \leq \deg_{\boldsymbol{w}} y_j = a_i,$$

where the inequality follows from Corollary 2.2, the fact that $\boldsymbol{w}$ is a witness for $\mathfrak{m}_G$ in $\operatorname{Ass}(R/J_G^n)$, and the assumption that $\{x_j, y_j\}$ is an edge of G for $j = 1, 2$. As $\deg_{\boldsymbol{w}''} x = \deg_{\boldsymbol{w}} x$ for all $x \in V_{G'}$, we know $\boldsymbol{w}''$ divides $\boldsymbol{w}$ and $\boldsymbol{w} = y_1^{a_1 - b_1} y_2^{a_2 - b_2} \boldsymbol{w}''$.

Notice that for each $i = 1, \ldots, n - b_1 - b_2$, and for each $j = 1, 2$, the monomial $y_j \boldsymbol{n}_i$ is in J_G. Since $a_1 + a_2 = n$ by Lemma 2.5, $y_1^{a_1 - b_1} y_2^{a_2 - b_2} \boldsymbol{n}_1 \cdots \boldsymbol{n}_{n - b_1 - b_2} \in J_G^{n - b_1 - b_2}$, so that $\boldsymbol{w} = y_1^{a_1} y_2^{a_2} \boldsymbol{w}'' = y_1^{a_1 - b_1} y_2^{a_2 - b_2} \boldsymbol{n}_1 \cdots \boldsymbol{n}_k \in J_G^n$, a contradiction to our assumption about $\boldsymbol{w}$ being a witness. Thus, we conclude that $\boldsymbol{w}'$ could not have been in $J_{G'}^n$ to begin with, completing the first section of the proof.

Next, we show that for $x \in V_{G'}$, we have $x\boldsymbol{w}' \in J_{G'}^n$. But $x\boldsymbol{w} \in J_G^n$, and in particular $x\boldsymbol{w} = \boldsymbol{m}_1 \cdots \boldsymbol{m}_n$, where $\boldsymbol{m}_i \in J_G$ for $1 \le i \le n$. Since $a_1 + a_2 = n$ by Lemma 2.5, and since each $\boldsymbol{m}_i$ must be divisible by at least one of y_1 or y_2 (since $\{y_1, y_2\} \in E_G$), it must be the case that each $\boldsymbol{m}_i$ contains precisely one of y_1 or y_2. This implies that $y_1 \in \operatorname{supp} \boldsymbol{m}_i$ if and only if $x_2 \in \operatorname{supp} \boldsymbol{m}_i$, and $y_2 \in \operatorname{supp} \boldsymbol{m}_i$ if and only if $x_1 \in \operatorname{supp} \boldsymbol{m}_i$, since $\operatorname{ver}(\boldsymbol{m}_i)$ is a cover for G. Thus either x_1 or x_2 belong to $\operatorname{supp}(\boldsymbol{m}_i)$ for every $i = 1, \ldots, n$. For this reason the monomials defined as

$$\boldsymbol{m}_i' = \begin{cases} \boldsymbol{m}_i / y_1 & \text{if } y_1 \in \operatorname{supp} \boldsymbol{m}_i, \\ \boldsymbol{m}_i / y_2 & \text{if } y_2 \in \operatorname{supp} \boldsymbol{m}_i \end{cases}$$

have the property that $\operatorname{ver}(\boldsymbol{m}_i')$ is a cover for G' for all $i = 1, \ldots, n$. Therefore we have $x\boldsymbol{w}' = \boldsymbol{m}_1' \cdots \boldsymbol{m}_n' \in J_{G'}^n$, as desired. $\square$

The following lemma gives instances for which a variable appears with maximal degree in a witness.

2.7. Lemma. *Let G be a graph with vertex set V_G. Let J_G be the cover ideal for G in the polynomial ring $R = \mathsf{k}[V_G]$. Assume that there exists a positive integer n such that $\mathfrak{m}_G \in \operatorname{Ass}(R/J_G^n)$ with witness $\boldsymbol{w}$. Suppose G contains a proper induced subgraph K that is a complete graph in $n+1$ vertices with one edge $\{y_1, y_2\}$ removed. Then* $\deg_{\boldsymbol{w}}(y_1) = n - 1$.

Proof. Label the vertices in V_K as $y_1, y_2, \ldots, y_{n+1}$. By Lemma 2.1, we know $\deg_{\boldsymbol{w}}(y_1) \le n - 1$, so it remains to show that $\deg_{\boldsymbol{w}}(y_1) \ge n - 1$. Suppose for the sake of contradiction that $\deg_{\boldsymbol{w}}(y_1) < n - 1$, and let x be a vertex of G but not a vertex of the proper subgraph H. Since $x\boldsymbol{w} \in J_G^n$, we can write $x\boldsymbol{w} = \boldsymbol{m}_1 \cdots \boldsymbol{m}_n$, with $\boldsymbol{m}_i \in J_G$. This implies that for each $i = 1, \ldots, n$, $\operatorname{ver}(\boldsymbol{m}_i)$ is a cover of G and therefore a cover for K.

Since $\deg_{\boldsymbol{w}}(y_1) < n - 1$, suppose without loss of generality that $y_1 \nmid \boldsymbol{m}_{n-1}$ and $y_1 \nmid \boldsymbol{m}_n$. Then $y_j \in \operatorname{supp}(\boldsymbol{m}_i)$ for $3 \le j \le n+1$ and $i = n-1, n$ since $\{y_1, y_j\}$ is an edge of H and therefore G. In particular $y_3 \cdots y_{n+1} \mid \boldsymbol{m}_{n-1}$ and $y_3 \cdots y_{n+1} \mid \boldsymbol{m}_n$. Again by Lemma 2.1, we know that $\deg_{\boldsymbol{w}}(y_j) \le n - 1$, so y_j can divide at most $n - 3$ of the monomials $\boldsymbol{m}_1, \ldots, \boldsymbol{m}_{n-2}$ for $3 \le j \le n+1$. Thus,

$$\sum_{j=3}^{n+1} \sum_{i=1}^{n-2} \deg_{\boldsymbol{m}_i}(y_j) \le \sum_{j=3}^{n+1} (n-3) = n^2 - 4n + 3.$$

On the other hand, each $\boldsymbol{m}_i$ must cover H and so contains at least all but one of $y_3, \ldots, y_{n+1}$, whence

$$\sum_{i=1}^{n-2} \sum_{j=3}^{n+1} \deg_{\boldsymbol{m}_i}(y_j) \ge \sum_{i=1}^{n-2} (n-2) = n^2 - 4n + 4,$$

which is obviously a contradiction. Thus we conclude that $\deg_{\boldsymbol{w}}(y_1) = n-1$, as desired. □

In the rest of the paper, given a finite set S, we denote by $|S|$ its cardinality.

2.8. Lemma. *Let G be an h-wheel with rim R^G and center C^G. Let k be its radial number and $\ell_1, \ldots, \ell_k$ its radial lengths. If W is a vertex cover for G that contains all the vertices in C^G, then*

$$|W| \geq \tfrac{1}{2}(|G| - h + 1) + h.$$

If W is a vertex cover for G missing one vertex from C^G, then

$$|W| \geq k + h - 1 + \left\lfloor \tfrac{1}{2}(\ell_1 - 1) \right\rfloor + \cdots + \left\lfloor \tfrac{1}{2}(\ell_k - 1) \right\rfloor.$$

Moreover,

$$k + h - 1 + \left\lfloor \tfrac{1}{2}(\ell_1 - 1) \right\rfloor + \cdots + \left\lfloor \tfrac{1}{2}(\ell_k - 1) \right\rfloor \geq \tfrac{1}{2}(|G| - h + 1) + h.$$

Proof. Assume that W contains C^G. The vertex set $W \cap R^G$ has to be a vertex cover for R^G. Since R^G is an odd hole, the cardinality of $W \cap R^G$ has to be at least

$$\tfrac{1}{2}(|R^G| + 1) = \tfrac{1}{2}(|G| - h + 1).$$

Therefore the cardinality of W is at least

$$\tfrac{1}{2}(|G| - h + 1) + h.$$

Assume now that W does not contain all the center vertices. If G were a 1-wheel, we know from [Kesting et al. 2011, Lemma 2.1] that the cover not containing the center would have cardinality of at least

$$k + \left\lfloor \tfrac{1}{2}(\ell_1 - 1) \right\rfloor + \cdots + \left\lfloor \tfrac{1}{2}(\ell_k - 1) \right\rfloor,$$

which is also the number of vertices that W needs to have to cover the subgraph induced by the 1-wheel with the center not in W. The cover W needs to contain further the other $h-1$ centers, so that the following inequality holds:

$$|W| \geq k + h - 1 + \left\lfloor \tfrac{1}{2}(\ell_1 - 1) \right\rfloor + \cdots + \left\lfloor \tfrac{1}{2}(\ell_k - 1) \right\rfloor.$$

We now need to show that this value is greater than $\frac{1}{2}(|G| - h + 1) + h$. Denote by C a subgraph of G isomorphic to a 1-wheel. We know that

$$k + \left\lfloor \tfrac{1}{2}(\ell_1 - 1) \right\rfloor + \cdots + \left\lfloor \tfrac{1}{2}(\ell_k - 1) \right\rfloor \geq \tfrac{1}{2}|C| + 1,$$

as shown in [Kesting et al. 2011]. This implies

$$k + \left\lfloor \tfrac{1}{2}(\ell_1 - 1) \right\rfloor + \cdots + \left\lfloor \tfrac{1}{2}(\ell_k - 1) \right\rfloor \geq \tfrac{1}{2}(|G| - h + 1) + 1,$$

as $|G|-h+1$ is the cardinality of a subgraph of G isomorphic to a 1-wheel. It follows that

$$k+h-1+\left\lfloor \tfrac{1}{2}(\ell_1-1)\right\rfloor+\cdots+\left\lfloor \tfrac{1}{2}(\ell_k-1)\right\rfloor \geq \tfrac{1}{2}(|G|-h+1)+h. \qquad \square$$

3. Main theorems

We first prove that if G is an h-wheel then $\mathfrak{m}_G$ appears as an associated prime of low powers of the cover ideal.

3.1. Theorem. *Let G be an h-wheel, and let J_G be the cover ideal of G in the ring $R=\mathsf{k}[V_G]$. Then $\mathfrak{m}_G \notin \operatorname{Ass}(R/J_G^n)$ if $n \leq h+1$.*

Proof. Let $y_1, \ldots, y_h$ label the vertices in C^G, let $x_1, x_2, \ldots, x_k$ label the radial vertices, and let ℓ_i be the radial lengths for $i=1,\ldots,k$. Denote by x_{ij}, for $j=1,\ldots,\ell_i-1$, the vertices between x_i and x_{i+1} if $i<k$ and the vertices between x_k and x_1 if $i=k$.

Because the centers and one radial vertex form a complete graph in $h+1$ vertices, Lemma 2.3 implies that $G \notin \operatorname{Ass}(R/J^n)$ for every integer n such that $n \leq h$.

We next show that $G \notin \operatorname{Ass}(R/J_G^{h+1})$, and to do so we consider two cases.

<u>Case 1</u>: Assume that there are two radial vertices, say x_t and x_{t+1}, such that $\{x_t, x_{t+1}\}$ is an edge. In this case we can conclude that $G \notin \operatorname{Ass}(R/J^{h+1})$ by a direct application of Lemma 2.3 since x_t, x_{t+1}, and the centers of the h-wheel G form a complete $(h+2)$-graph.

<u>Case 2</u>: Assume that G is an h-wheel with no two radial vertices adjacent. We know by the definition of an h-wheel that there exist an x_t and an x_{t+1} such that the path from x_t to x_{t+1} is odd. By relabeling the vertices of G we may assume that $t=1$. Suppose for a contradiction that there exists a witness $\boldsymbol{w}$ for the maximal ideal $\mathfrak{m}_G$ to be in $\operatorname{Ass}(R/J^{h+1})$. Using Lemma 2.7 with K being the induced subgraph by C^G, and the vertices x_1, x_2, we can conclude that the $\deg_{\boldsymbol{w}} x_1 = h$. Thus from Lemma 2.5, we have that $\deg_w x_{11} + \deg_w x_{12} = h+1$. Further, by an application of Lemma 2.6, we can contract x_{11} and x_{12} to form a new graph G' such that $\mathfrak{m}_{G'} \in \operatorname{Ass}(\mathsf{k}[V_{G'}]/J_{G'}^{h+1})$. Because the path from x_1 to x_2 along the subgraph induced by R^G is odd, we can perform this operation until x_1 is adjacent to x_2 and conclude the proof by an application of Case 1. $\square$

3.2. Theorem. *Let G be an h-wheel and let J_G be the cover ideal of G in the ring $R=\mathsf{k}[V_G]$. Then $\mathfrak{m}_G \in \operatorname{Ass}(R/J_G^{h+2})$.*

Proof. Label with $y_1, \ldots, y_h$ the vertices in C^G, and with $x_1, \ldots, x_k$ the radial vertices, where k is the radial number. Let ℓ_i denote the radial lengths for $i=1,\ldots,k$. Label by x_{ij}, for $j=1,\ldots,\ell_i-1$, the vertices between x_i and x_{i+1} if

$i < k$ and the vertices between x_k and x_1 if $i = k$. The subgraph R^G is an odd cycle. We set d to be the size of R^G. Notice that $\ell_1 + \cdots + \ell_k = d$.

We prove that $\mathfrak{m}_G$ is in $\operatorname{Ass}(R/J_G^{h+2})$ by providing a witness. Let $\boldsymbol{w}$ be the monomial

$$\boldsymbol{w} = \Bigl(\prod_{i=1,\dots,h} y_i^{h+1}\Bigr)\Bigl(\prod_{i=1,\dots,k} x_i^{h+1}\Bigr)\Bigl(\prod_{\substack{i=1,\dots,k\\ j=1,\dots,\ell_i-1}} x_{ij}^{a}\Bigr),$$

where $a = 1$ if j is odd, and $a = h+1$ if j is even.

To show that $\boldsymbol{w}$ is the desired monomial, we first prove that

$$\operatorname{tot\,deg}(\boldsymbol{w}) = hk + h(h+1) + n + h\bigl(\lfloor\tfrac{1}{2}(l_1-1)\rfloor + \cdots + \lfloor\tfrac{1}{2}(l_k-1)\rfloor\bigr).$$

In computing the $\deg(\boldsymbol{w})$, the contribution from the variables y_m and x_i, for $m = 1, \dots, h$ and $i = 1, \dots, k$, is given by $(h+1)h + (h+1)k$. For $i = 1, \dots, k-1$, between x_i and x_{i+1}, there are $\ell_i - 1$ vertices, and there are $\ell_k - 1$ vertices between x_k and x_1. Given an integer s, there are $\lfloor\frac{1}{2}s\rfloor$ even integers and $\lceil\frac{1}{2}s\rceil$ odd integers between 1 and s. Therefore, in computing $\operatorname{tot\,deg}(\boldsymbol{w})$, the contributions from the variables x_{ij} are given by

$$(h+1)\bigl(\lfloor\tfrac{1}{2}(l_1-1)\rfloor + \cdots + \lfloor\tfrac{1}{2}(l_k-1)\rfloor\bigr) + \lceil\tfrac{1}{2}(l_1-1)\rceil + \cdots + \lceil\tfrac{1}{2}(l_k-1)\rceil.$$

The total degree of the monomial $\boldsymbol{w}$ is therefore equal to

$$\begin{aligned}
\operatorname{tot\,deg}(\boldsymbol{w}) &= (h+1)k+(h+1)h+\sum_{i=1}^{k}\lceil\tfrac{1}{2}(\ell_i-1)\rceil+(h+1)\sum_{i=1}^{k}\lfloor\tfrac{1}{2}(\ell_i-1)\rfloor\\
&= (h+1)h+(h+1)k+h\sum_{i=1}^{k}\lfloor\tfrac{1}{2}(\ell_i-1)\rfloor+\sum_{i=1}^{k}\bigl(\lfloor\tfrac{1}{2}(\ell_i-1)\rfloor+\lceil\tfrac{1}{2}(\ell_i-1)\rceil\bigr)\\
&= hk+h(h+1)+k+\sum_{i=1}^{k}(\ell_i-1)+h\sum_{i=1}^{k}\lfloor\tfrac{1}{2}(\ell_i-1)\rfloor\\
&= hk+h(h+1)+\sum_{i=1}^{k}\ell_i+h\sum_{i=1}^{k}\lfloor\tfrac{1}{2}(\ell_i-1)\rfloor\\
&= hk+h(h+1)+d+h\sum_{i=1}^{k}\lfloor\tfrac{1}{2}(\ell_i-1)\rfloor.
\end{aligned}$$

To prove that $\boldsymbol{w}$ does not belong to J_G^{h+2}, we first show that

$$\operatorname{tot\,deg}(\boldsymbol{w}) < 2\bigl(\tfrac{1}{2}(|G|-h+1)+h\bigr)+h\Bigl(k+h-1+\sum_{i=1}^{k}\lfloor\tfrac{1}{2}(\ell_i-1)\rfloor\Bigr). \quad (3.2.1)$$

Supposing this inequality is not satisfied, we have

$$2\big(\tfrac{1}{2}(|G|-h+1)+h\big)+hk+h^2-h+h\sum_{i=1}^{k}\big\lfloor\tfrac{1}{2}(\ell_i-1)\big\rfloor \\ \leq hk+h^2+h+d+h\sum_{i=1}^{k}\big\lfloor\tfrac{1}{2}(\ell_i-1)\big\rfloor,$$

which implies

$$h+d \geq 2\big(\tfrac{1}{2}(|G|-h+1)+h\big)-h,$$

or $h+d \geq |G|+1$. But $|G|=|C^G|+h=d+h$. Thus

$$d+h \geq d+h+1,$$

which is impossible. Thus the inequality holds.

Now we show that this inequality implies $\boldsymbol{w} \notin J_G^{h+2}$. Assume otherwise. Then we can write $\boldsymbol{w}=\boldsymbol{h}\boldsymbol{m}_1\cdots\boldsymbol{m}_{h+2}$ such that for each $i=1,\dots,h+2$ not only the monomial $\boldsymbol{m}_i \in J_G$ but also $\operatorname{ver}(\boldsymbol{m}_i)$ is a minimal cover for G. The total degree of each $\boldsymbol{m}_i$ is equal to $|\operatorname{ver}(\boldsymbol{m}_i)|$. Therefore, by Lemma 2.8, we have

$$\operatorname{tot\,deg}(\boldsymbol{m}_i) \geq \tfrac{1}{2}(|C|-h+1)+h$$

if $\operatorname{ver}(\boldsymbol{m}_i)$ is a cover containing the vertices of C^G, or

$$\operatorname{tot\,deg}(\boldsymbol{m}_i) \geq k+h-1+\big\lfloor\tfrac{1}{2}(\ell_1-1)\big\rfloor+\cdots+\big\lfloor\tfrac{1}{2}(\ell_k-1)\big\rfloor$$

if $\operatorname{ver}(\boldsymbol{m}_i)$ is a cover that does not contain all vertices of C^G.

Notice that $\sum_{i=1}^{h}\deg_{\boldsymbol{w}} y_i = h(h+1)$. If $\operatorname{ver}(\boldsymbol{m}_i)$ is a cover that contains all the vertices of C^G for each $i=1,\dots,h-2$ then $\sum_{i=1}^{h}\deg_{\boldsymbol{w}} y_i \geq h(h+2)$, which is a contradiction. In particular, there are least h monomials among the monomials $\boldsymbol{m}_i$ that correspond to covers not containing all vertices in C^G. An application of Lemma 2.8, yields the inequality

$$\begin{aligned}\operatorname{tot\,deg}(\boldsymbol{w}) &= \operatorname{tot\,deg}(\boldsymbol{h})+\operatorname{tot\,deg}(\boldsymbol{m}_1)+\cdots+\operatorname{tot\,deg}(\boldsymbol{m}_{h+2})\\ &\geq 2\big(\tfrac{1}{2}(|C|-h+1)+h\big)+h\big(k+h-1+\big\lfloor\tfrac{1}{2}(l_1-1)\big\rfloor+\cdots+\big\lfloor\tfrac{1}{2}(l_k-1)\big\rfloor\big).\end{aligned}$$

This contradicts inequality (3.2.1) and shows that $\boldsymbol{w} \notin J_G^{h+2}$.

To finish the proof, we need to show that for every vertex $x \in V_G$ the monomial $x\boldsymbol{w}$ is in J_G^{h+2}.

For every $i=1,\dots,h$, let C_i be the induced subgraph isomorphic to the 1-wheel with center in y_i. In [Kesting et al. 2011, Theorem 2.2], the authors prove that

$$\boldsymbol{w}_i = y_i^2 \prod_{i=1,\dots,k} x_i^2 \prod_{j\,\text{odd}} x_{ij} \prod_{j\,\text{even}} x_{ij}^2 \tag{3.2.2}$$

is a witness for $\mathfrak{m}_{C_i} \in \operatorname{Ass}(\Bbbk[V_{C_i}]/J_{C_i}^3)$. Pick a vertex $x \in V_G$. Without loss of generality we may assume that $x \in V_{C_1}$. Then $x\boldsymbol{w}_1 \in J_{C_1}^3$, so $y_2^3 \cdots y_h^3 x\boldsymbol{w}_1 \in J_G^3$.

Define $\boldsymbol{m} = \prod_{i=1,\dots,k} x_i^2 \prod_{j \text{ odd}} x_{ij} \prod_{j \text{ even}} x_{ij}^2$ and notice that

$$\boldsymbol{w} = \frac{y_1^{h-1} y_2^{h+1} \cdots y_h^{h+1} \boldsymbol{w}_1 \cdot \boldsymbol{m}^{h-1}}{\prod_{i,j} x_i^{h-1} x_{ij}^{h-1}}.$$

Define

$$\boldsymbol{m}_i = \frac{y_1 \cdots y_{i-1} y_{i+1} \cdots y_h \cdot \boldsymbol{m}}{\prod_{i,j} x_i x_{ij}}$$

for each $i = 2, \dots, h$. It is easy to see that $\operatorname{ver}(\boldsymbol{m}_i)$ is a cover for G for every $i = 2, \dots, h$. The following equality shows that $x\boldsymbol{w} \in J_G^{h+2}$:

$$x\boldsymbol{w} = (y_2^3 \cdots y_h^3 x \boldsymbol{w}_1)\boldsymbol{m}_2 \cdots \boldsymbol{m}_h. \qquad \square$$

Finally we prove that if G is an h-wheel then $\mathfrak{m}_G$ is an associated prime in high powers of the cover ideal.

3.3. Theorem. *Let G be an h-wheel and let J_G be the cover ideal of G in the ring $R = \Bbbk[V_G]$. Then $\mathfrak{m}_G \in \operatorname{Ass}(R/J_G^n)$ for all $n \geq h+2$.*

Proof. Fix an integer $n \geq h+2$ and let t satisfy $n = h+2+t$. Let S be the cover of G that has all the vertices in C^G and every other vertex in R^G. In particular $|S| = h + \frac{1}{2}(|R^G|+1)$.

Consider the monomial $\tilde{\boldsymbol{w}} = (\boldsymbol{m})^t \boldsymbol{w}$, where $\boldsymbol{w}$ is the witness constructed in the proof of Theorem 3.2 and $\boldsymbol{m}$ is the squarefree monomial such that $\operatorname{ver}(\boldsymbol{m}) = S$. In particular, $\operatorname{tot\,deg} \boldsymbol{m} = h + \frac{1}{2}(|R^G|+1) = h + \frac{1}{2}(|G|-h+1)$. Using the inequality (3.2.1) we obtain

$$\begin{aligned}
&\operatorname{tot\,deg}(\tilde{\boldsymbol{w}}) \\
&\quad < t\big(\tfrac{1}{2}(|G|-h+1)+h\big) + 2\big(\tfrac{1}{2}(|G|-h+1)+h\big) + h\left(k+h-1+\sum_{i=1}^{k} \big\lfloor \tfrac{1}{2}(\ell_1-1)\big\rfloor\right) \\
&\quad = (n-h)\big(\tfrac{1}{2}(|G|-h+1)+h\big) + h\left(k+h-1+\sum_{i=1}^{k} \big\lfloor \tfrac{1}{2}(\ell_1-1)\big\rfloor\right).
\end{aligned}$$

We claim that $\tilde{\boldsymbol{w}}$ is a witness for $\mathfrak{m}_G \in \operatorname{Ass}(\Bbbk[V_G]/(J_G^n))$. If, toward contradiction, $\tilde{\boldsymbol{w}} \in J_G^n$, then we can write $\tilde{\boldsymbol{w}} = \boldsymbol{h}\boldsymbol{m}_1 \cdots \boldsymbol{m}_n$ such that, for each $i = 1, \dots, n$, not only the monomial $\boldsymbol{m}_i \in J_G$ but also $\operatorname{ver}(\boldsymbol{m}_i)$ is a minimal cover for G. As $\sum_{i=1}^{h} \deg_{\tilde{\boldsymbol{w}}} y_i = th + h(h+1) = (n-1)h$, there are at least h covers among $\operatorname{ver}(\boldsymbol{m}_i)$ that do not contain all of C^G. This implies

$$\begin{aligned}
\operatorname{tot\,deg}(\tilde{\boldsymbol{w}}) &= \operatorname{tot\,deg}(\boldsymbol{h}) + \operatorname{tot\,deg}(\boldsymbol{m}_1) + \cdots + \operatorname{tot\,deg}(\boldsymbol{m}_n) \\
&\geq (n-h)\big(\tfrac{1}{2}(|G|-h+1)+h\big) + h\left(k+h-1+\sum_{i=1}^{k} \big\lfloor \tfrac{1}{2}(\ell_1-1)\big\rfloor\right),
\end{aligned}$$

contradicting the inequality above. To finish, let $x \in V_G$. Then $x\tilde{\boldsymbol{w}} = (\boldsymbol{m})^t x\boldsymbol{w} \in J_G^{t+h+2}$, since $x\boldsymbol{w} \in J_G^{h+2}$, as we showed in the proof of Theorem 3.2, and $\boldsymbol{m} \in J_G$ by assumption. □

We conclude the paper with the following:

3.4. Corollary. *For every integer d there exists an ideal $I_d \subset \mathsf{k}[x_1, \dots, x_d]$ such that* $\operatorname{astab}(I_d) = d - 3$.

Proof. Consider the h-wheel with $h = d - 5$ such that the graph induced on R^G is a 5-cycle. Theorems 3.2 and 3.3 show that $\operatorname{astab}(I_d) = d - 5 + 2 = d - 3$. □

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