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Generalized quadrangles, Laguerre planes and shift planes of odd order

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We characterize the Miquelian Laguerre planes, and thus the classical orthogonal generalized quadrangles $Q(4, q)$, of odd order q by the existence of shift groups in affine derivations.

Introduction

A finite Laguerre plane $\mathcal{L} = (P, \mathcal{C}, \mathcal{G})$ of order n consists of a set P of $n(n+1)$ points, a set $\mathcal{C}$ of n^3 circles and a set $\mathcal{G}$ of $n+1$ generators, where both circles and generators are subsets of P, such that the following three axioms are satisfied:

(G) $\mathcal{G}$ partitions P and each generator contains n points.

(C) Each circle intersects each generator in precisely one point.

(J) Three points of which no two are on the same generator are joined by a unique circle.

Circles through x are called *touching in x* if they are equal or have no other point in common. The set of all circles through a given point x is denoted by $\mathcal{C}_x$. The *derived affine plane* $\mathbb{A}_x = (P \setminus [x], \mathcal{C}_x \cup \mathcal{G} \setminus \{[x]\})$ at a point $x \in P$ has the collection of all points not on the generator $[x]$ through x as its point set and, as lines, all circles passing through x (without the point x) and all generators apart from $[x]$. The axioms above easily yield that $\mathbb{A}_x$ is an affine plane. We refer to the generators as *vertical lines* in $\mathbb{A}_x$. Circles that touch each other in x give parallel lines in $\mathbb{A}_x$. A line W is introduced to obtain the projective completion $\mathbb{P}_x$ of $\mathbb{A}_x$; the common point of the verticals will be denoted by $v \in W$.

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The group $\mathrm{Aut}(\mathcal{L})$ of all automorphisms of a Laguerre plane $\mathcal{L}$ acts on the set $\mathcal{G}$ of generators. We call $\mathcal{L}$ an *elation Laguerre plane* if the kernel Δ of that action acts transitively on the set $\mathcal{C}$ of circles. It is known (see [Steinke 1991, 1.3]) that in every finite elation Laguerre plane the group Δ has a (unique) regular normal subgroup E called the *elation group*. For more details on elation Laguerre planes, we refer the reader to the introduction in [Steinke and Stroppel 2013].

In the present note, we only use a weaker transitivity assumption on Δ but combine this with additional assumptions. Our results can (and will) be applied to elation Laguerre planes with additional homogeneity assumptions, e.g., in [Steinke and Stroppel 2018] (see Theorem 2.3 below).

Finite Laguerre planes of *odd* order q are equivalent to antiregular generalized quadrangles of order q (i.e., with parameters (q, q)); see [Thas et al. 2006, Theorem 2.4.2]. Derivation at an antiregular point of a generalized quadrangle of odd order q produces a Laguerre plane of order q. Conversely, the Lie geometry of a Laguerre plane of odd order yields a generalized quadrangle with an antiregular point. Thus this generalized quadrangle is antiregular; see [Thas et al. 2006, Theorem 2.4.6]. However, this construction does not work when q is even.

On the other hand, a finite elation Laguerre plane of order q (regardless of whether q is even or odd) is equivalent to a generalized oval (or pseudo-oval) with $q+1$ points and thus to a translation generalized quadrangle of order q; see [Casse et al. 1985] or [Thas et al. 2006].

The elation group E is a $3m$-dimensional vector space over some field $\mathbb{F}$, and the stabilizer E_x of each point x is a $2m$-dimensional vector subspace of E. Under a duality the E_x yield a family of $q+1$ vector subspaces of dimension m in $\mathbb{F}^{3m}$. Changing to projective notation one sees that, geometrically, a finite elation Laguerre plane of order q is equivalent to a $(q+1)$-set of $(m-1)$-dimensional subspaces in the $(3m-1)$-dimensional projective space over $\mathbb{F}$; compare [Casse et al. 1985]. In [Thas et al. 2006] such a set is called a generalized oval. In fact, a generalized oval is just a 4-gonal family of type (q, q) in an abelian group; see [Thas et al. 2006, 3.2.2]. One obtains a translation generalized quadrangle of order q from a generalized oval, and on the other hand, every translation generalized quadrangle of order q arises from a generalized oval in this way; see [Thas et al. 2006, Theorem 3.5.1] or [Payne and Thas 2009, 8.7.1].

With the correspondence between Laguerre planes and certain generalized quadrangles as above, our results on Laguerre planes have corresponding formulations in generalized quadrangles, but we mainly use the language of Laguerre planes.

1. Translation planes

Theorem 1.1. *Let $\mathbb{P}$ be a finite projective plane of order n. Assume that a subgroup $D \leq \mathrm{Aut}(\mathbb{P})$ fixes each point on some line L. If n^2 divides the order of D then D*

contains a subgroup T of order n^2 consisting of elations with axis L. In particular, the plane $\mathbb{P}$ is a translation plane, and the order n is a prime power.

Proof. For each nontrivial element $\delta \in D$ there is a (unique) center c_δ, i.e., a point c_δ such that δ fixes each line through c_δ ([Baer 1946], see [Hughes and Piper 1973, Theorem 4.9]). The elations in D are just those in the set

$$T := \{\mathrm{id}\} \cup \{\tau \in D \setminus \{\mathrm{id}\} \mid c_\tau \in L\};$$

that set forms a normal subgroup of D (see [Hughes and Piper 1973, Theorem 4.13]).

For any point x outside L, the stabilizer D_x consists of id and elements with center x. The order of any element of D_x divides $n-1$. So the order of D_x and the number n^2 of points outside L are coprime, and D acts transitively on the set A of points outside L. For each $\delta \in D \setminus T$ we have $c_\delta \notin L$, and $\delta \in D_{c_\delta}$ yields that the order of δ divides $n-1$, and is coprime to n^2.

Let $\mathcal{B}$ denote the set of T-orbits in A. Then D acts on $\mathcal{B}$, and so does D/T because $T \trianglelefteq D$ acts trivially on $\mathcal{B}$. Transitivity of D on A implies that D/T is transitive on $\mathcal{B}$. Now $|\mathcal{B}| = n^2/|T|$ divides $|D/T|$. The latter order is coprime to n^2 because each member (distinct from T) of the quotient has a representative of order coprime to n^2. So $|\mathcal{B}| = 1$, and transitivity of T is proved. □

Theorem 1.2. *Let $\mathcal{L}$ be a Laguerre plane of finite order n with kernel Δ. If ∞ is a point such that n^2 divides the order of the stabilizer Δ_∞ then the derived projective plane $\mathbb{P}_\infty$ is a dual translation plane, and the order n is a prime power.*

Proof. The group D induced by Δ_∞ on the dual $\mathbb{P}$ of $\mathbb{P}_\infty$ satisfies the assumptions of Theorem 1.1. □

Theorem 1.3. *Let $\mathcal{L}$ be a Laguerre plane of finite order n, and assume that there is a point ∞ such that n^2 divides the order of the stabilizer Δ_∞. If there exist a circle $K \in \mathcal{C}_\infty$ and a subgroup $H \le \mathrm{Aut}(\mathcal{L})_\infty$ such that H fixes each circle touching K in ∞ and H acts transitively on $K \setminus \{\infty\}$, then $\mathbb{P}_\infty$ has Lenz type V (at least), and is coordinatized by a semifield.*

Proof. From Theorem 1.2 we know that $\mathbb{P}_\infty$ is a dual translation plane. The translation axis in the dual of $\mathbb{P}_\infty$ is the common point v for the generators in the projective closure of $\mathbb{A}_\infty$. The elations of $\mathbb{P}_\infty$ with center v and axis W form a group of order n; we denote that group by V and note that V is a group of translations of $\mathbb{A}_\infty$.

Our assumptions on H secure that H induces a group of translations of $\mathbb{A}_\infty$; the common center is the point at infinity for the "horizontal line" $K \setminus \{\infty\}$. We obtain a transitive group HV of translations on $\mathbb{A}_\infty$. So $\mathbb{P}_\infty$ is also a translation plane, and has Lenz type V at least. □

2. Shift groups

Recall that a shift group on a projective plane is a group of automorphisms fixing an incident point-line pair (x, Y) and acting regularly both on the set of points outside Y and on the set of lines not through x.

Theorem 2.1. *Let $\mathcal{L}$ be a finite Laguerre plane of odd order, and assume that there exists a point u and a subgroup $S \leq \mathrm{Aut}(\mathcal{L})_u$ such that S induces a transitive group of translations on the affine plane $\mathbb{A}_u$.*

(1) *If $s \in [u] \smallsetminus \{u\}$ is fixed by S then S induces a shift group on $\mathbb{P}_s$.*

(2) *If S fixes a point t of $\mathcal{L}$ and induces a transitive group of translations on $\mathbb{A}_t$ then $t = u$.*

Proof. Let n denote the order of $\mathcal{L}$. Assume that $s \in [u] \smallsetminus \{u\}$ is fixed by S. Then S induces a group of automorphisms of $\mathbb{P}_s$; we have to exhibit an incident point-line pair (x, Y) such that S acts regularly both on the set of points outside Y and on the set of lines not through x.

It is obvious that S acts regularly on the set of affine points in $\mathbb{P}_s$ because that set coincides with the set of points of $\mathbb{A}_u$. We let the line W at infinity play the role of Y. Also, the set of vertical lines (induced by generators) is invariant under S, so we let their point at infinity play the role of x (so $x \in W$ is the point v at infinity of vertical lines).

It remains to show that S acts regularly on the set of nonvertical lines of $\mathbb{A}_s$; these lines are induced by the circles through s. Assume that $\tau \in S$ fixes a circle C through s. Our assumption that n be odd implies that the translation of $\mathbb{A}_u$ induced by τ does not have any orbit of length 2, and we obtain that τ is trivial if there is a set of one or two points outside $[u]$ invariant under τ.

Note that no vertical line distinct from $[u]$ is fixed by τ when τ is not the identity. As τ induces a translation on $\mathbb{A}_u$, there exists $D \in \mathcal{C}_u$ such that τ fixes each circle touching D in u (these circles induce the parallels to the line induced by D on $\mathbb{A}_u$). Pick a point $z \in C \smallsetminus \{s\}$, and let D' be the circle through z touching D in u. Then τ leaves the intersection $D' \cap C$ invariant. This is a set with one or two elements, and we find that τ is trivial. So the orbit of C under S has length $|S| = n^2$, and fills all of $\mathcal{C}_s$. Thus S acts regularly on the set of nonvertical lines of $\mathbb{A}_s$, as required.

Now assume that S fixes t and induces a transitive group of translations on $\mathbb{A}_t$. Then $t \in [u]$ because S acts regularly on the set of points outside $[u]$. For any circle $C \in \mathcal{C}_t$, we pick two points $a, b \in C \smallsetminus \{t\}$. Then there exists $\tau \in S$ such that $\tau(a) = b$. As τ is a translation both of $\mathbb{A}_u$ and of $\mathbb{A}_t$, the orbit of a under $\langle\tau\rangle$ is contained both in the line C of $\mathbb{A}_t$ and in some line B of $\mathbb{A}_u$, that is, in some circle B through u. Since n is odd, that orbit has at least three points, and $B = C$. This yields $t = u$, as claimed. □

Theorem 2.2. *Assume that $\mathcal{L}$ is a finite Laguerre plane of odd order n, and let ∞ be a point. Let U denote the set of all points $u \in [\infty] \setminus \{\infty\}$ such that there exists a subgroup $S_u \leq \mathrm{Aut}(\mathcal{L})$ of order n^2 fixing both ∞ and u and acting as a group of translations on $\mathbb{A}_u$. Then the following hold*:

(1) *There are at least $|U|$ many different shift groups on $\mathbb{P}_\infty$.*

(2) *If $|U| > 1$ then $\mathbb{A}_\infty$ is a translation plane.*

(3) *If $\mathbb{A}_\infty$ is a translation plane and U is not empty then $\mathbb{P}_\infty$ has Lenz type V at least and can be coordinatized by a commutative semifield, and the middle nucleus of such a coordinatizing semifield has order at least $|U| + 1$.*

(4) *If $|U| > \sqrt{n}$ then $\mathbb{P}_\infty$ is Desarguesian.*

Proof. Using Theorem 2.1 we see for any $u \in U$ that S_u is a shift group on $\mathbb{P}_\infty$, and different points $t, u \in U$ yield different groups S_t and S_u. This gives the first assertion. All these shift groups have the same fixed flag in $\mathbb{P}_\infty$.

If a finite projective plane admits more than one shift group, it is a translation plane; see [Knarr and Stroppel 2009, 10.2]. If a translation plane admits at least one shift group then it can be coordinatized by a commutative semifield ([Knarr and Stroppel 2009, 9.12], [Spille and Pieper-Seier 1998]) and the different shift groups with the same fixed flag are parameterized by the nonzero elements of the middle nucleus of such a semifield; see [Knarr and Stroppel 2009, 9.4].

The additive group of the coordinatizing semifield forms a vector space over the middle nucleus (see [Hughes and Piper 1973, p. 170]). If the middle nucleus has more than $\sqrt{n}$ elements then that vector space has dimension 1, and the middle nucleus coincides with the semifield. This means that the semifield is a field, and the plane is Desarguesian. □

Theorem 2.2 is used in [Steinke and Stroppel 2018] to prove the following:

Theorem 2.3. *Let $\mathcal{L}$ be a finite elation Laguerre plane of odd order. If there exists a point ∞ such that $\mathrm{Aut}(\mathcal{L})_\infty$ acts two-transitively on $\mathcal{G} \setminus \{[\infty]\}$ then the affine plane $\mathbb{A}_\infty$ is Desarguesian, and $\mathcal{L}$ is Miquelian.* □

Remark 2.4. If $\mathbb{P}$ is a projective plane of even order then a shift group on $\mathbb{P}$ will never be elementary abelian; see [Knarr and Stroppel 2009, 1.5, 5.8]. Thus a shift group on such a plane will not act as a transitive group of translations on any other affine plane (of the same order).

With the correspondence between Laguerre planes and certain generalized quadrangles as mentioned in the introduction, Theorem 2.3 yields the following. Here we use the standard notation of $x^\perp$ for all points collinear to x in a generalized quadrangle $\mathcal{Q}$ and $\pi(x, y)$ for the affine plane obtained at an antiregular point x; see [Thas et al. 2006, Theorem 2.4.1] for a definition).

Corollary 2.5. *Let $\mathcal{Q}$ be a finite translation generalized quadrangle of odd order q with an antiregular base point x. If there exists a point y collinear to x such that the stabilizer $\mathrm{Aut}(\mathcal{Q})_{x,y}$ acts two-transitively on $x^\perp \setminus \{x, y\}^{\perp\perp}$, then the affine plane $\pi(x, y)$ is Desarguesian, and $\mathcal{Q}$ is the classical orthogonal generalized quadrangle $Q(4, q)$.*

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