Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial

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msp

Vol. 16 No. 1 2018

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vol. 16, no. 1, 2018
dx.doi.org/10.2140/iig.2018.16.159

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SIMPLE GROUPS OVER LOCAL FIELDS

by

J. Tits

This is a report on results obtained during the Institute, and which will be the subject of a forthcoming paper by F. Bruhat and J. Tits. Detailed proofs have not yet been written down, and several steps must still be checked, so that the theorems are presented here with a certain degree of incertainty. Discussions with other participants, in particular N. Iwahori, M. Kneser, T. Springer and R. Steinberg, have been of great help.

1. Notations. Terminology.

k is a field, K a Galois extension, Γ the Galois group, v a discrete valuation of K invariant under Γ, $\bar{k}$ and $\bar{K}$ the residue fields of k and K with respect to v, G a simply connected simple group defined over k and split over K, O the ring of integers in K, Δ the set of vertices of the "extended Dynkin diagram" of G (i.e. the set consisting of the simple roots and the opposite of the maximal root). An algebraic group and the group of its rational K-points are denoted by the same symbol. The field $\bar{k}$ is assumed to be perfect, although this hypothesis may be unnecessary.

Let there be given a "standard" (Chevalley) O-structure on G, let T be a maximal torus which splits over O, and let B be the inverse image under the reduction $\rho: G_0 \longrightarrow G_{\bar{K}}$ of a Borel subgroup of $G_{\bar{K}}$

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containing $\rho(T_0)$. We call Iwahori subgroups the conjugates of B in G and parahoric subgroups all proper subgroups containing an Iwahori subgroup. A parahoric subgroup P and a maximal torus T' are associated if there is an element $g \in G$ such that ${}^gB \subset P$ and ${}^gT = T'$; if $\bar{K}$ has "enough" elements, this is equivalent to saying that $P \cap T' = T'_0 = \{t \mid t \in T', \chi(t) \in O \text{ for all } \chi \in X^*(T')\}$.

The conjugacy classes of parahoric subgroups are in 1 - 1 correspondence with the proper subsets of Δ, the Iwahori subgroups corresponding to the empty set. Through its action on these conjugacy classes, Γ operates on Δ. Two parahoric subgroups, corresponding to two subsets Δ_1 and Δ_2 invariant by Γ will be called k-similar, if there is a bijection $\Delta_1 \longrightarrow \Delta_2$ compatible with the action of Γ, and which is an isomorphism of the Dynkin diagrams having the elements of Δ_1 and Δ_2 as vertices. Par abus de langage, a parahoric subgroup invariant under Γ will be called a k-parahoric subgroup.

2. Results.

Theorem 1.

a) To each maximal k-torus which splits over K is associated a k-parahoric subgroup.

b) All minimal k-parahoric subgroups are k-similar.

c) More precisely, if P_1 and P_2 are two minimal k-parahoric subgroups, there exists a bijection of the set of k-parahoric subgroups containing P_1 onto the set of k-parahoric subgroups containing P_2, which preserves inclusions and such that corresponding subgroups are k-similar. If Δ_1 is the subset of Δ associated with the conjugacy class of P_1, the number of maximal k-parahoric subgroups containing P_1 equals the number of orbits of Γ in $\Delta - \Delta_1$.

It is likely that in b), "k-similar" may be replaced by "conjugate over K".

Theorem 2. Suppose k complete, let L be the maximal unramified extension of k contained in K, assume that G is quasi-split over L and that K is the minimal extension of L over which G splits. Then:

a) G contains at least one k-parahoric subgroup. Every such subgroup is associated with a torus containing a maximal k-split torus.

b) Two minimal k-parahoric subgroups are conjugate by an element of G_k.

c) If G is anisotropic over k, there is a unique k-parahoric subgroup and it contains G_k. In particular, if Δ^o is the subset of Δ associated to the conjugacy class of this subgroup, $\Delta - \Delta^o$ consists of a single orbit of Γ.

Corollary 1. Under the assumptions of theorem 2, G possesses a maximal torus, defined over k, which splits over K.

Remarks. Keep the assumptions of theorem 2, let P be a minimal k-parahoric subgroup associated with a maximal k-torus containing the maximal k-split torus S, and let N = N(S) be the normalizer of S. It is to be expected that P_k, N_k form a BN-pair in G_k, whose Weyl group is an affine Weyl group, extension of the relative Weyl group. At the present stage, we seem to dispose of a method to prove this fact, but have

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not yet gone through particulars. In the non-complete case, the analogous result seems to be that, perhaps under suitable additional conditions (the conjugacy over k of the minimal k-parahoric subgroups?) the double cosets of P_k (P a minimal k-parahoric subgroup) in G_k have all the formal properties of the double cosets of the B of a BN-pair, but that in general, there exists no suitable N (this situation can be axiomatized and gives rise to a generalisation of BN-pairs).

Theorem 3. *Assume that* G *is quasi-split over an unramified extension* L *of* k *contained in* K, *and that it possesses a maximal* K-*split torus defined over* k. *Then, if the cohomological dimension of* $\bar{\bar{k}}$ *is* ≤ 1, *the minimal* k-*parahoric subgroups are Iwahori subgroups.*

Corollary 2. *If* k *is complete and dim* $\bar{k} \leq 1$, *every anisotropic absolutely simple group* G *is of type* A_n.

Proof. By theorem 2 c) and theorem 3, Γ must be transitive on the extended Dynkin diagram, and this can happen only for A_n.

Corollary 3. *If* k *is complete and dim* $\bar{k} \leq 1$, $H^1(k, G) = 0$.

This generalizes a theorem of M. Kneser. A simple proof of the fact that the existence of a k-Iwahori subgroup implies the nullity of H^1 has been indicated by Springer. There is an alternative proof, using corollary 1 and the classification.

3. Methods.

3.1. Geometric representation. Let T be a maximal K-split torus. There exists a canonical representation of the parahoric subgroups associated to T, by the simplexes (of all dimensions ≥ 0) of a cellular decomposition of an affine space A over $\mathbb{Q}$, which is a principal homogeneous space over $V = X^*(T) \otimes \mathbb{Q}$. If T is defined over k, Γ operates on A and leaves the cellular decomposition invariant. Since only a finite quotient of Γ operates on A, Γ has a fix point, which belong to a simplex representing a k-parahoric subgroup. This proves theorem 1 a).

If there exist two distinct k-parahoric subgroups associated with the given k-torus T, Γ has two fix points in A, therefore a line of fix points, and it has also a line of fix points in V from which follows that T cannot be anisotropic. This is the main argument in the proof of theorem 2c).

3.2. <u>Reduction.</u> Let P be a parahoric subgroup and Δ_1 be the subset of Δ which represents the conjugacy class of P. There exists an O-structure in G for which P is the group of integral points. After reduction, this gives a group $\bar{P}$ defined over $\bar{K}$, which is in general not semi-simple. The Dynkin diagram of the semi-simple quotient $\bar{\bar{P}}$ of P is the diagram whose set of vertices is Δ_1. The homomorphism ρ: $P \to \bar{\bar{P}}_{\bar{K}}$ can be defined "intrinsically", without reference to the O-structure mentioned above; it establishes a 1-1 correspondence between the parahoric subgroups contained in P and the parabolic subgroups of $\bar{\bar{P}}_{\bar{K}}$. One of the main tools is the Galois descent on the reduction ρ, when P is a k-parahoric subgroup. Roughly, theorem 3 is proved as follows: assume P is a k-parahoric subgroup; since $\bar{k}$ is of dimension ≤ 1, $\bar{\bar{P}}$ is quasi-split over $\bar{k}$, that is, it contains a $\bar{k}$-Borel subgroup whose inverse image in P is then a k-Iwahori subgroup of G. One main difficulty to be overcome is that, when K is ramified over k, some elements of Γ operate on $\bar{\bar{P}}$ as rational automorphisms.

3.3. Projective limit. An essential use is made of the fact that, in the complete case, parahoric subgroups can be viewed as projective limits of algebraic groups defined over $\bar{k}$. This is used for instance to show by a limiting process that k-parahoric subgroups are associated to k-maximal tori.

4. Dictionary. One notices a formal analogy between the usual theory of parabolic subgroups and the theory of parahoric subgroup, expressed by the following dictionary:

Parabolic	Parahoric
Borel	Iwahori
Connected	Simply connected
Dynkin diagram	Extended Dynkin diagram
Weyl group	Affine Weyl group
Finite fields	p-adic fields
Inseparability	Ramification
etc.	etc.

Innovations in Incidence Geometry

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Innovations in Incidence Geometry: Algebraic, Topological and Combinatorial (ISSN 2640-7345 electronic, 2640-7337 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840 is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

IIG peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Innovation in Incidence Geometry

Vol. 16 No. 1 2018

Complement to the Collected Works of Jacques Tits

edited by Bernhard Mühlherr and Hendrik Van Maldeghem

This volume contains 31 writings of Jacques Tits that were not included in his four-volume *Œuvres – Collected Works*, published by the European Mathematical Society in 2013 in the series Heritage of European Mathematics.