

The chromatic splitting conjecture at $n = p = 2$

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We show that the strongest form of Hopkins' chromatic splitting conjecture, as stated by Hovey, cannot hold at chromatic level $n = 2$ at the prime $p = 2$. More precisely, for $V(0)$, the mod 2 Moore spectrum, we prove that $\pi_k L_1 L_{K(2)} V(0)$ is not zero when k is congruent to -3 modulo 8. We explain how this contradicts the decomposition of $L_1 L_{K(2)} S$ predicted by the chromatic splitting conjecture.

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1 Introduction

Fix a prime p . Let S be the p -local sphere spectrum, and $L_n S$ be the Bousfield localization of S at the Johnson–Wilson spectrum $E(n)$. Let $K(n)$ be Morava K -theory. There is a homotopy pullback square called the chromatic fracture square:

$$\begin{array}{ccc} L_n S & \longrightarrow & L_{K(n)} S \\ \downarrow & & \downarrow \\ L_{n-1} S & \xrightarrow{\iota} & L_{n-1} L_{K(n)} S \end{array}$$

Let F_n be the fiber of the map $L_n S \rightarrow L_{K(n)} S$. Note that F_n is weakly equivalent to the fiber of ι . It was shown by Hovey [12, Lemma 4.1] that F_n is weakly equivalent to the function spectrum $F(L_{n-1} S, L_n S)$. Hopkins' chromatic splitting conjecture, as stated by Hovey [12, Conjecture 4.2], stipulates that ι is the inclusion of a wedge summand, so that

$$(1-1) \quad L_{n-1} L_{K(n)} S \simeq L_{n-1} S \vee \Sigma F_n.$$

We will call this the *weak* form of the chromatic splitting conjecture. However, [12, Conjecture 4.2] also gives an explicit decomposition of ΣF_n as a wedge of suspensions of spectra of the form $L_i S_p$ for $0 \leq i < n$. We will call this the *strong* form of the chromatic splitting conjecture.

The conjectured decomposition comes from the connection between the $K(n)$ -local category and the cohomology of a certain group called the Morava stabilizer group \mathbb{G}_n .

Let \mathbb{S}_n be the group of automorphisms of the formal group law of $K(n)$ over \mathbb{F}_{p^n} . Then \mathbb{G}_n is the extension of \mathbb{S}_n by the Galois group $\text{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$. Let \mathbb{W} be the Witt vectors on \mathbb{F}_{p^n} . There is a spectral sequence

(1-2)
$$H^s(\mathbb{G}_n, (E_n)_t) \implies \pi_{t-s} L_{K(n)} S.$$

Note that \mathbb{W} sits naturally in $(E_n)_0 \cong \mathbb{W} \llbracket u_1, \dots, u_{n-1} \rrbracket$. The inclusion induces a map

(1-3)
$$H^*(\mathbb{G}_n, \mathbb{W}) \rightarrow H^*(\mathbb{G}_n, (E_n)_0).$$

Morava proves in [16, Remark 2.2.5], using the work of Lazard, that

$$H^*(\mathbb{G}_n, \mathbb{W}) \otimes \mathbb{Q}_p \cong E(e_1, \dots, e_n)$$

for classes e_i of degree $2i - 1$. Therefore, $H^*(\mathbb{G}_n, \mathbb{W})$ contains an exterior algebra $E(x_1, \dots, x_n)$ for appropriate integral multiples x_i of the generators e_i . The chromatic splitting conjecture stipulates that, for some choice of x_1, \dots, x_n , the exterior algebra $E(x_1, \dots, x_n)$ injects into $H^*(\mathbb{G}_n, (E_n)_0)$ under the map (1-3), and that the nonzero products $x_{i_1} \cdots x_{i_j}$ survive in (1-2) to nontrivial elements in $\pi_{-2(\sum i_k)+j} L_{K(n)} S$. Further, it states that there is a factorization

$$\begin{array}{ccc} S_p^{-2(\sum i_k)+j} & \longrightarrow & L_{n-\max(i_k)} S_p^{-2(\sum i_k)+j} \\ \downarrow & & \downarrow \\ L_{K(n)} S & \longrightarrow & \Sigma F_n \end{array}$$

where S_p^m is the p -completion of S^m , and that these maps decompose ΣF_n as

(1-4)
$$\Sigma F_n \simeq \bigvee_{\substack{1 \leq j \leq n \\ 1 \leq i_1 < \dots < i_j \leq n}} L_{n-\max(i_k)} S_p^{-2(\sum i_k)+j}.$$

The chromatic splitting conjecture has been shown for $n \leq 2$ and for all primes p , except in the case $n = p = 2$. For $n = 1$, it follows immediately from a computation of $\pi_* L_1 S_p$; see Ravenel [19, Theorems 8.10 and 8.15]. At $n = 2$ and $p \geq 5$, it is due to Hopkins, and follows from Shimomura and Yabe’s computations [23]. The proof can be found in Behrens’ account of their work [4, Remark 7.8]. At $n = 2$ and $p = 3$, the conjecture was proved recently by Goerss, Henn and Mahowald [9].

In this paper, we show that the chromatic splitting conjecture as stated above cannot hold for $n = p = 2$. More precisely, we show that [12, Conjecture 4.2(iv)] fails in this case. At $n = 2$, (1-1) and (1-4) imply that

(1-5)
$$L_1 L_{K(2)} S \simeq L_1 S_p \vee L_1 S_p^{-1} \vee L_0 S_p^{-3} \vee L_0 S_p^{-4}.$$

We show that the right-hand side of (1-5) has too few homotopy groups for the equivalence to hold. However, our results do not contradict the possibility that ι is the inclusion of a wedge summand. Giving an alternative description for the fiber in this case is work in progress.

That our methods could disprove (1-5) was first suggested to the author by Paul Goerss. He and Mark Mahowald had been studying the computations of Shimomura and Wang [22] and Shimomura [21] and noticed that these suggest that the right-hand side of (1-5) is too small.

Statement of the results Let $V(0)$ be the cofiber of multiplication by p on S . Note that for any p -local spectrum X , there is a cofiber sequence

$$X \xrightarrow{p} X \rightarrow X \wedge V(0).$$

Since Bousfield localization of spectra preserves exact triangles, it follows that

$$L_E V(0) \simeq L_E S \wedge V(0)$$

for any spectrum E . This has the following consequence.

Proposition 1.1 *The strong form of the chromatic splitting conjecture at $n = 2$ implies that $L_1 L_{K(2)} V(0) \simeq L_1 V(0) \vee L_1 \Sigma^{-1} V(0)$.*

We now fix our attention to the case when $p = 2$. Since $L_0 V(0)$ is contractible, it follows from the chromatic fracture square that $L_1 V(0) \simeq L_{K(1)} V(0)$. Computing $\pi_* L_{K(1)} V(0)$ is a routine exercise using the spectral sequence

(1-6)
$$E_2^{s,t} = H^s(\mathbb{G}_1, (E_1)_* V(0)) \implies \pi_{t-s} L_{K(1)} V(0).$$

The E_∞ -term is given in Figure 1. At $p = 2$, we have that $V(0)$ is not a ring spectrum. This manifests itself by the fact that $\pi_* L_{K(1)} V(0)$ is not a ring. In fact,

$$\pi_* L_{K(1)} V(0) = (\mathbb{Z}_2[\eta, \beta^{\pm 1}, \zeta_1]/(2\eta, \eta^3, \zeta_1^2))\{e_0, v_1 e_0\}/(2e_0, 2v_1 e_0 - \eta^2 e_0),$$

where $\eta \in \pi_1$ is the Hopf map, $\beta \in \pi_8$ is the v_1 -self-map detected by v_1^4 , and $\zeta_1 \in \pi_{-1}$ is detected by a generator of $H^1(\mathbb{G}_1, \mathbb{Z}_2) \cong H^1(\mathbb{Z}_2^\times, \mathbb{Z}_2)$. The element $e_0 \in \pi_0$ represents the inclusion of the bottom cell $S^0 \hookrightarrow V(0)$, and $v_1 e_0 \in \pi_2$ is a lift of $\Sigma \eta$ to the top cell:

$$\begin{array}{ccccc} & & S^2 & & \\ & \swarrow v_1 e_0 & \downarrow \Sigma \eta & & \\ S^0 & \xrightarrow{e_0} & V(0) & \longrightarrow & S^1 \xrightarrow{2} S^1 \end{array}$$

The following result is a consequence of Proposition 1.1.

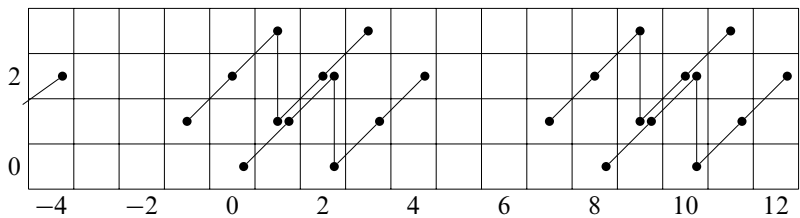


Figure 1: The E_∞ -term of (1-6) computing $\pi_* L_{K(1)} V(0)$. Vertical lines denote extensions by multiplication by 2, and lines of slope one denote multiplication by η .

Corollary 1.2 *The chromatic splitting conjecture implies that $\pi_k L_1 L_{K(2)} V(0)$ is zero when $k \equiv -3$ modulo 8.*

However, in this paper, we prove the following result.

Theorem 1.3 *There are nontrivial homotopy classes $\beta^t x$ in $\pi_{8t-3} L_1 L_{K(2)} V(0)$ and $\zeta_2 \beta^t x$ in $\pi_{8t-4} L_1 L_{K(2)} V(0)$.*

This has the following immediate consequence.

Theorem 1.4 *The homotopy group $\pi_k L_1 L_{K(2)} V(0)$ is nonzero when $k \equiv -3$ modulo 8. Therefore, the decomposition (1-5) of the chromatic splitting conjecture does not hold when $n = 2$ and $p = 2$.*

The broad strokes of the proof of Theorem 1.3 when $t = 0$ are as follows. Let $G_{24} \cong Q_8 \rtimes C_3$ be a representative of the unique conjugacy class of maximal finite subgroups of \mathbb{S}_2 . Let C_6 be a subgroup of G_{24} of order 6. Let \mathbb{S}_2^1 be the norm one subgroup so that $\mathbb{S}_2 \cong \mathbb{S}_2^1 \rtimes \mathbb{Z}_2$ (see Section 2). It follows from the duality resolution techniques of Goerss, Henn, Mahowald and Rezk and the work of Bobkova [6] that, for any X , there is a spectral sequence

$$E_1^{p,t} = \pi_t(\mathcal{E}_p \wedge X) \implies \pi_{t-p}(E_2^{h\mathbb{S}_2^1} \wedge X),$$

where \mathcal{E}_p are spectra such that $\mathcal{E}_0 \simeq E_2^{hG_{24}}$, $\mathcal{E}_p \simeq E_2^{hC_6}$ if $p = 1, 2$ and $(E_2)_* \mathcal{E}_3 \cong (E_2)_* E_2^{hG_{24}}$ as Morava modules. Localizing at $E(1)$, we obtain a spectral sequence

(1-7)
$$E_1^{p,t} = \pi_t L_1(\mathcal{E}_p \wedge X) \implies \pi_{t-p} L_1(E_2^{h\mathbb{S}_2^1} \wedge X).$$

We use this spectral sequence to show that $\pi_{-3} L_1(E_2^{h\mathbb{S}_2^1} \wedge V(0)) \cong \mathbb{F}_4$, in Lemma 4.1 and Proposition 4.2. After taking Galois invariants, we obtain a nonzero element x in $\pi_{-3} L_1(E_2^{h\mathbb{G}_2^1} \wedge V(0))$. In the cofiber sequence

$$L_1 L_{K(2)} V(0) \rightarrow L_1(E_2^{h\mathbb{G}_2^1} \wedge V(0)) \rightarrow L_1(E_2^{h\mathbb{G}_2^1} \wedge V(0)),$$

which is obtained from the cofiber sequence $L_{K(2)}S \rightarrow E_2^{h\mathbb{G}_2^1} \rightarrow E_2^{h\mathbb{G}_2^1}$ by smashing with $V(0)$ and localizing at $E(1)$; this class gives rise to nonzero elements $x \in \pi_{-3}L_1L_{K(2)}V(0)$ and $\zeta_2x \in \pi_{-4}L_1L_{K(2)}V(0)$.

Warning 1.5 We use the notation ζ_2 to denote the homotopy class defined by

$$\begin{array}{ccccccc} & & S^0 & & & & \\ & & \downarrow 1 & \searrow \zeta_2 & & & \\ L_{K(2)}S & \longrightarrow & E_2^{h\mathbb{G}_2^1} & \longrightarrow & E_2^{h\mathbb{G}_2^1} & \longrightarrow & \Sigma L_{K(2)}S \end{array}$$

Experts will notice that this clashes with Ravenel [17, Lemma 2.1], but this is the natural generalization of what is now commonly denoted by ζ_n at odd primes.

Organization of the paper In Section 2, we specialize to the case $n = 2$ and $p = 2$ and describe the duality resolution spectral sequence and its $E(1)$ –localization. In Section 3, we compute the E_1 –page of this spectral sequence for $V(0)$. In Section 4, we prove Theorem 1.3.

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2 The $E(1)$ –local duality resolution spectral sequence

We take the point of view that, at height 2, the Honda formal group law may be replaced by the formal group law of a supersingular elliptic curve. This was carefully explained in [3, Section 1]. (The reader who wants to ignore this subtlety may take $\mathbb{S}_\mathcal{C}$, $\mathbb{G}_\mathcal{C}$ and $E_\mathcal{C}$ to mean \mathbb{S}_2 , \mathbb{G}_2 and E_2 , respectively.)

Let $\mathbb{S}_\mathcal{C}$ be the group of automorphisms of the formal group law of the supersingular elliptic curve

$$\mathcal{C}: \quad y^2 + y = x^3$$

of height two over \mathbb{F}_4 ; see [3, Section 3] for the comparison. It admits an action of the Galois group $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Define

$$\mathbb{G}_C = \mathbb{S}_C \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2).$$

Let E_C be the spectrum which classifies the deformations of the formal group law of C over \mathbb{F}_4 as described, for example, in Rezk [20]. It can be chosen to be a complex oriented ring spectrum with

$$(E_C)_* = \mathbb{W}[[u_1]][u^{\pm 1}]$$

for $|u_1| = 0$, $|u| = -2$, whose formal group law is the formal group law of the curve

(2-1)

$$\mathcal{C}_U : \quad y^2 + 3u_1xy + (u_1^3 - 1)y = x^3.$$

It admits an action of \mathbb{G}_C , and for any finite spectrum X ,

$$L_{K(2)}X \simeq E_C^{h\mathbb{G}_C} \wedge X \simeq (E_C \wedge X)^{h\mathbb{G}_C};$$

see Behrens and Davis [5, page 5]. The group of automorphisms $\mathrm{Aut}(C)$ of C is of order 24 and injects into \mathbb{S}_C . We let G_{24} denote the image of $\mathrm{Aut}(C)$. We note that

$$G_{24} \cong Q_8 \rtimes C_3,$$

where Q_8 is a quaternion subgroup and C_3 a cyclic group of order 3. The group \mathbb{S}_C contains a central subgroup of order 2, which we denote by C_2 . We define

$$C_6 = C_2 \times C_3.$$

There is a surjective homomorphism $N: \mathbb{S}_C \rightarrow \mathbb{Z}_2^\times/(\pm 1) \cong \mathbb{Z}_2$, which we call the *norm*. It is constructed using the determinant of a representation $\rho: \mathbb{S}_C \rightarrow GL_2(\mathbb{W})$; see [3, Section 3]. Further, it can be extended to \mathbb{G}_C . We let \mathbb{S}_C^1 and \mathbb{G}_C^1 be the kernels of the norms, and note that the elements of finite order in \mathbb{S}_C and \mathbb{G}_C are contained in \mathbb{S}_C^1 and \mathbb{G}_C^1 respectively. Further,

(2-2)

$$\mathbb{S}_C \cong \mathbb{S}_C^1 \rtimes \mathbb{Z}_2 \quad \text{and} \quad \mathbb{G}_C \cong \mathbb{G}_C^1 \rtimes \mathbb{Z}_2.$$

The formal group law $F_{\mathcal{C}_U}$ of \mathcal{C}_U , is not 2–typical. Nonetheless, it is strictly isomorphic to a 2–typical formal group law classified by a map $BP_* \rightarrow (E_C)_*$. Further, $[2]_{F_{\mathcal{C}_U}}(x) \equiv u_1u^{-1}x^2 \text{ modulo } (2, x^4)$; see [3, Section 6.1] for details on $F_{\mathcal{C}_U}$. The strict isomorphism between $F_{\mathcal{C}_U}$ and its 2–typification preserves this identity. Hence, v_1 is mapped to u_1u^{-1} modulo (2). Since we are working primarily modulo (2), we abuse notation and let $v_1 = u_1u^{-1} \in (E_C)_2$.

We will need the following result, which can be found in Henn [11, Theorem 13] and is also discussed in greater detail in Bobkova [6]. We restate it here using our notation for convenience.

Theorem 2.1 (Goerss, Henn, Mahowald, Rezk and Bobkova) *There is a resolution of spectra in the $K(2)$ -local category given by*

$$\begin{array}{ccccccc} E_C^{hS^1_c} & \longrightarrow & E_C^{hG_{24}} & \longrightarrow & E_C^{hC_6} & \longrightarrow & E_C^{hC_6} \longrightarrow \mathcal{E}_3 \\ & & \parallel & & \parallel & & \parallel \\ & & \mathcal{E}_0 & \longrightarrow & \mathcal{E}_1 & \longrightarrow & \mathcal{E}_2 \longrightarrow \mathcal{E}_3 \end{array}$$

where $(E_C)_*\mathcal{E}_3 \cong (E_C)_*E_C^{hG_{24}}$ as Morava modules. Further, for any spectrum X , the resolution gives rise to a tower of fibrations spectral sequence

(2-3)
$$E_1^{p,t} = \pi_t(\mathcal{E}_p \wedge X) \xrightarrow{SS_1} \pi_{t-p}(E_C^{hS^1_c} \wedge X)$$

with differentials $d_r\colon E_r^{p,t} \rightarrow E_r^{p+r,t+r-1}$.

We call the resolution of Theorem 2.1 the *duality resolution*. Let π generate \mathbb{Z}_2 in the decompositions (2-2), and let $G'_{24} = \pi G_{24} \pi^{-1}$. Recall from [3] or [2] that there is also an *algebraic* duality resolution:

(2-4)
$$\begin{array}{ccccccccc} 0 \rightarrow \mathbb{Z}_2[\![S^1_c/G'_{24}]\!] & \rightarrow & \mathbb{Z}_2[\![S^1_c/C_6]\!] & \rightarrow & \mathbb{Z}_2[\![S^1_c/C_6]\!] & \rightarrow & \mathbb{Z}_2[\![S^1_c/G_{24}]\!] & \rightarrow & \mathbb{Z}_2 \rightarrow 0 \\ & & \parallel & & \parallel & & \parallel & & \parallel \\ & & \mathcal{C}_3 & \longrightarrow & \mathcal{C}_2 & \longrightarrow & \mathcal{C}_1 & \longrightarrow & \mathcal{C}_0 \end{array}$$

Now, let X be a finite spectrum. Resolving (2-4) into a double complex of projective S^1_c -modules and applying the functor $\mathrm{Hom}_{\mathbb{Z}_2[\![S^1_c]\!]}(-, (E_C)_t X)$ gives rise to a spectral sequence

(2-5)
$$E_1^{p,q,t} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![S^1_c]\!]}(\mathcal{C}_p, (E_C)_t X) \xrightarrow{SS_2} H^{p+q}(S^1_c, (E_C)_t X)$$

with differentials $d_r\colon E_r^{p,q,t} \rightarrow E_r^{p+q,r,t+r-1}$. Further, in each fixed degree p , there are spectral sequences

(2-6)
$$E_1^{p,q,t} = \mathrm{Ext}^q_{\mathbb{Z}_2[\![S^1_c]\!]}(\mathcal{C}_p, (E_C)_t X) \xrightarrow{SS_3} \pi_{t-q}(\mathcal{E}_p \wedge X)$$

with differentials $d_r\colon E_r^{p,q,t} \rightarrow E_r^{p+q,r,t+r-1}$. Finally, there is also a spectral sequence

(2-7)
$$E_2^{s,t} = H^s(S^1_c, (E_C)_t X) \xrightarrow{SS_4} \pi_{t-s}(E_C^{hS^1_c} \wedge X)$$

with differentials $d_r\colon E_r^{s,t} \rightarrow E_r^{s+r,t+r-1}$. Thus, for X finite, we obtain a diagram of spectral sequences:

(2-8)

$$\begin{array}{ccc} \mathrm{Ext}_{\mathbb{Z}_2[[\mathbb{S}_C^1]]}^q(\mathcal{C}_p, (E_C)_t X) & \xrightarrow{SS_2} & H^{p+q}(\mathbb{S}_C^1, (E_C)_t X) \\ \downarrow SS_3 & & \downarrow SS_4 \\ \pi_{t-q}(\mathcal{C}_p \wedge X) & \xrightarrow{SS_1} & \pi_{t-(p+q)}(E_C^{h\mathbb{S}_C^1} \wedge X) \end{array}$$

Remark 2.2 For elements of Adams–Novikov filtration $s = 0$ in $E_1^{p,t}(SS_1)$, the differentials d_1 are related to the d_1 –differentials in the algebraic duality resolution spectral sequence SS_2 in the following way. If X is finite, as in [10, Proposition 2.4 and (2.7)], for G a closed subgroup of \mathbb{G}_C , there are isomorphisms of Morava modules

(2-9) $(E_C)_t(E_C^{hG} \wedge X) \cong \mathrm{Hom}^c(\mathbb{G}_C/G, (E_C)_t X) \cong \mathrm{Hom}_{\mathbb{Z}_2}(\mathbb{Z}_2[[\mathbb{G}_C/G]], (E_C)_t X).$

Let

$$E_1(SS_1)^{p,t} \cong \pi_t(\mathcal{C}_p \wedge X) \xrightarrow{h} H^0(\mathbb{G}_C, (E_C)_t(\mathcal{C}_p \wedge X)) \cong E_1^{p,0,t}(SS_2)$$

be the edge homomorphism for the spectral sequence

$$H^s(\mathbb{G}_C, (E_C)_t(\mathcal{C}_p \wedge X)) \implies \pi_{t-s}(\mathcal{C}_p \wedge X).$$

The spectral sequence SS_1 is constructed so that the following diagram commutes:

$$\begin{array}{ccc} E_1^{p,t}(SS_1) & \xrightarrow{h} & E_1^{p,0,t}(SS_2) \\ d_1 \downarrow & & \downarrow d_1 \\ E_1^{p+1,t}(SS_1) & \xrightarrow{h} & E_1^{p+1,0,t}(SS_2) \end{array}$$

When both horizontal maps h are injective, one can deduce information in SS_1 from information in SS_2 .

For the statement of the next result, recall that for any closed subgroup F of \mathbb{G}_C and finite spectrum X , there is a spectral sequence

(2-10)

$$E_2^{s,t}(F, X) = H^s(F, (E_C)_t X) \implies \pi_{t-s}(E_C^{hF} \wedge X).$$

The author learned the proof of the following result from Paul Goerss.

Lemma 2.3 *Let S a closed subgroup of \mathbb{S}_C which is invariant under the action of $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$. Let $G \cong S \rtimes \mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ be the corresponding closed subgroup of \mathbb{G}_C . Then for any finite X and any $2 \leq r \leq \infty$,*

$$E_r^{s,t}(S, X) \cong \mathbb{W} \otimes_{\mathbb{Z}_2} E_r^{s,t}(G, X),$$

and the differentials of the spectral sequence $E_r^{s,t}(S, X)$ are \mathbb{W} –linear.

Proof The action of $\mathrm{Gal}(\mathbb{F}_4/\mathbb{F}_2)$ on $(E_C)_*X$ is semilinear over \mathbb{W} , so there is an isomorphism $E_2^{*,*}(S, X) \cong \mathbb{W} \otimes_{\mathbb{Z}_2} E_2^{*,*}(G, X)$. Now consider, $E_r^{s,t}(\mathbb{S}_C, S^0)$. We have $E_2^{0,0}(\mathbb{S}_C, S^0) \cong \mathbb{W}$ and the subring \mathbb{Z}_2 of \mathbb{W} consists of permanent cycles. The spectral sequence $E_r^{*,*}(\mathbb{S}_C, S^0)$ is multiplicative, so the differentials $d_r: E_r^{0,0} \rightarrow E_r^{r,r-1}$ are \mathbb{Z}_2 -derivations. Since \mathbb{W} is an étale extension of \mathbb{Z}_2 , for any r , the \mathbb{Z}_2 -derivations from \mathbb{W} to the \mathbb{W} -module $E_r^{r,r-1}$ are zero. Hence, $E_2^{0,0}(\mathbb{S}_C, S^0) \cong \mathbb{W}$ consists of permanent cycles and the differentials are \mathbb{W} -linear. Since the spectral sequence $E_r^{*,*}(S, X)$ is one of modules over $E_r^{*,*}(\mathbb{S}_C, S^0)$, the differentials of $E_r^{*,*}(S, X)$ are also \mathbb{W} -linear, and the result follows. \square

In what follows, we will use the following remark.

Remark 2.4 Let X be a finite spectrum and F be a closed subgroup of \mathbb{G}_C . As noted by Devinatz in the proof of [7, Lemma 3.5], it follows from the fact that E_C^{hF} is $(K_C)_*$ -local E_C -nilpotent, (see Devinatz and Hopkins [8, Proposition A.3]) that the descent spectral sequence (2-10) has a horizontal vanishing line.

Now, recall that the telescope conjecture holds at height $n = 1$. This was proved at odd primes by Miller [15] and at $p = 2$ by Mahowald [14]. In particular, we have the following result.

Theorem 2.5 (Mahowald and Miller) *Let Y admit a v_1 -self-map $v_1^k: \Sigma^{2k}Y \rightarrow Y$. Then*

$$L_1Y \simeq L_{K(1)}Y \simeq v_1^{-1}Y,$$

where

$$v_1^{-1}Y := \mathrm{colim}(\cdots \xrightarrow{v_1^k} \Sigma^{2k}Y \xrightarrow{v_1^k} Y \xrightarrow{v_1^k} \cdots).$$

Proposition 2.6 *For any finite type-1 spectrum X , with self map $v_1^k: \Sigma^{2k}X \rightarrow X$, there is a diagram of strongly convergent spectral sequences:*

$$\begin{array}{ccc} v_1^{-1} \mathrm{Ext}_{\mathbb{Z}_2[[\mathbb{S}_C^1]]}^q(\mathcal{E}_p, (E_C)_tX) & \xrightarrow{L_1SS_2} & v_1^{-1}H^{p+q}(\mathbb{S}_C^1, (E_C)_tX) \\ \downarrow L_1SS_3 & & \downarrow L_1SS_4 \\ \pi_{t-q}L_1(\mathcal{E}_p \wedge X) & \xrightarrow{L_1SS_1} & \pi_{t-(p+q)}L_1(E_C^{h\mathbb{S}_C^1} \wedge X) \end{array}$$

Proof The spectral sequence L_1SS_2 is obtained from SS_2 by inverting the element $v_1^k \in (E_C)_{2k}X$, and L_1SS_1 is obtained by the applying L_1 to the tower of fibrations which gives rise to SS_1 . The spectral sequences L_1SS_3 and L_1SS_4 are obtained by inverting the algebraic element v_1^k in the spectral sequences SS_3 or SS_4 , and using the fact that

$$v_1^{-1}\pi_*(\mathcal{E}_p \wedge X) \cong \pi_*L_1(\mathcal{E}_p \wedge X).$$

With regards to the strong convergence of the four spectral sequences, note that localization with respect to v_1 is exact. Therefore, the localized spectral sequences will converge strongly if they have horizontal vanishing lines at the E_∞ -term. The spectral sequences SS_1 and SS_2 have a vanishing line at $p = 4$ for all $r \geq 1$. As noted in [Remark 2.4](#), the descent spectral sequences SS_3 and SS_4 have horizontal vanishing lines. Therefore, the spectral sequences $L_1 SS_i$ exist and converge. \square

Remark 2.7 As in [Remark 2.2](#), the differentials d_1 in $L_1 SS_1$ and $L_1 SS_2$ commute with the edge homomorphisms

$$E_1(L_1 SS_1)^{p,t} \cong \pi_t L_1(\mathcal{E}_p \wedge X) \xrightarrow{h} v_1^{-1} H^0(\mathbb{G}_C, (E_C)_t(\mathcal{E}_p \wedge X)) \cong E_1^{p,0,t}(L_1 SS_2).$$

Remark 2.8 For X as in [Proposition 2.6](#), the element $v_1^{2k} \in (E_C)_{2k} X$ can be chosen to be Galois invariant. Therefore, the results of [Lemma 2.3](#) also hold for the localized spectral sequences. That is, let

$$v_1^{-1} E_2^{s,t}(F, X) \cong v_1^{-1} H^s(F, (E_C)_t X) \implies \pi_{t-s} L_1(E_C^{hF} \wedge X).$$

Then for S and G as in [Lemma 2.3](#), we have

$$v_1^{-1} E_r^{s,t}(S, X) \cong \mathbb{W} \otimes_{\mathbb{Z}_2} v_1^{-1} E_r^{s,t}(G, X)$$

for $2 \leq r \leq \infty$, and the differentials are \mathbb{W} -linear.

3 The homotopy of $L_1(E_C^{hG^{24}} \wedge V(0))$ and $L_1(E_C^{hC^6} \wedge V(0))$

The spectrum $V(0)$ has a self map

$$\beta \colon \Sigma^8 V(0) \xrightarrow{v_1^4} V(0),$$

and in this section, we give the E_1 -term for

$$E_1^{p,q}(L_1 SS_1) = \pi_q L_1(\mathcal{E}_p \wedge V(0)) \xrightarrow{L_1 SS_1} \pi_{q-p} L_1(E_C^{hS^1} \wedge V(0)).$$

In order to do so, we must compute $\pi_* L_1(E_C^{hG^{24}} \wedge V(0))$ and $\pi_* L_1(E_C^{hC^6} \wedge V(0))$. We do this using the descent spectral sequences

$$v_1^{-1} H^s(G, (E_C)_t V(0)) \implies \pi_{t-s} L_1(E_C^{hG} \wedge V(0)).$$

Notation 3.1 We use the following conventions. First,

$$v_1 = u_1 u^{-1}, \quad v_2 = u^{-3} \quad \text{and} \quad j_0 = u_1^3.$$

The element Δ is the discriminant of \mathcal{C}_U , and hence is given by

$$\Delta = 27v_2(v_1^3 - v_2)^3 \equiv v_2(v_1^3 + v_2)^3 \pmod{2},$$

and

$$c_4 = 9v_1^4 + 72v_1v_2 \equiv v_1^4 \pmod{2}.$$

The j -invariant is

$$j = c_4^3 \Delta^{-1} \equiv v_1^{12} \Delta^{-1} \pmod{2}.$$

These identities can be computed using Silverman [24, Section III.1]; see also [3, Section 4.2]. We abuse notation and let

$$\eta = \delta(v_1),$$

where δ is the Bockstein associated to

$$0 \rightarrow (E_C)_*/2 \xrightarrow{2} (E_C)_*/4 \rightarrow (E_C)_*/2 \rightarrow 0.$$

This is justified by the fact that $\delta(v_1)$ detects the homotopy class η (see [3, Section 4.1]).

The v_1 -torsion-free elements of $H^*(G_{24}, (E_C)_*V(0))$ generate a submodule isomorphic to

$$\mathbb{F}_4\llbracket j \rrbracket[v_1, \eta, \Delta^{\pm 1}, k]/(\eta^4 - v_1^4 k, j\Delta - v_1^{12})$$

for elements of degrees (s, t) , where s is the cohomological grading, t is the internal grading, and

$$|v_1| = (0, 2), \quad |\eta| = (1, 2), \quad |\Delta| = (0, 24), \quad |k| = (4, 0), \quad |j| = (0, 0);$$

see Section 4.2 or the appendix of [3]. On the other hand, $H^*(C_6, (E_C)_*V(0))$ is v_1 -torsion-free and is isomorphic to

$$\mathbb{F}_4\llbracket j_0 \rrbracket[v_1, \eta, v_2^{\pm 1}, h]/(\eta - v_1 h, j_0 v_2 - v_1^3),$$

where $|v_2| = (0, 6)$, $|h| = (1, 0)$ and $|j_0| = (0, 0)$; see Section 4.2 of [3].

The next proposition is an immediate consequence of these results. In its statement, we let $\mathbb{F}_4((x))$ denote the Laurent series on x .

Proposition 3.2 *There are isomorphisms*

$$v_1^{-1} H^*(G_{24}, (E_C)_*V(0)) \cong \mathbb{F}_4((j))[v_1^{\pm 1}, \eta]$$

and

$$v_1^{-1} H^*(C_6; (E_C)_*V(0)) \cong \mathbb{F}_4((j_0))[v_1^{\pm 1}, \eta].$$

The degrees (s, t) are given by $|v_1| = (0, 2)$, $|\eta| = (1, 2)$, $|j| = (0, 0)$ and $|j_0| = (0, 0)$. The restriction associated to the inclusion of C_6 in G_{24} maps j to $j_0^4(1 + j_0)^{-3}$.

Proof This follows from [3, Section 4.2] after inverting v_1 . □

To compute the differentials, we will use the following observation.

Remark 3.3 There is a class α_3 in $\mathrm{Ext}_{BP_*BP}^{1,6}(BP_*, BP_*)$ (see Ravenel [18, page 430]) such that $d_3(\alpha_3) = \eta^4$. Further, α_3 reduces to ηv_1^2 in $\mathrm{Ext}_{BP_*BP}^{1,6}(BP_*, BP_*V(0))$, so $\eta d_3(v_1^2) = \eta^4$.

In general, for a 2-local BP -algebra spectrum E , the E -Adams spectral sequence for any spectrum X is a module over $\mathrm{Ext}_{BP_*BP}(BP_*, BP_*)$. There is a universal d_3 -differential $d_3(\alpha_3 z) = \eta^4 z + \alpha_3 d_3(z)$. Further, if 2 annihilates $E_*(X)$, this reduces to $d_3(\eta v_1^2 z) = \eta^4 z + \eta v_1^2 d_3(z)$. If there is no η -torsion on the E_3 -term as in our examples below, this gives a universal differential $d_3(v_1^2 z) = \eta^3 z + v_1^2 d_3(z)$.

Lemma 3.4 *Let G be a closed subgroup of \mathbb{G}_C . Let X be a $K(2)$ -local spectrum such that $(E_C)_*X \cong (E_C)_*E_C^{hG}$. Then the $K(2)$ -local, E_C -Adams spectral sequence computing π_*X has E_2 -term isomorphic to $H^*(G, (E_C)_*)$.*

Proof We first prove that the E_2 -term is isomorphic to $H^*(\mathbb{G}_C, (E_C)_*X)$. This can be deduced directly from Barthel and Heard [1, Theorem 4.3]. Nonetheless, we sketch the proof here. The assumption on $(E_C)_*X$ implies that it is profree as an $(E_C)_*$ -module. An inductive argument using [13, Proposition 8.4] and [10, Proposition 2.4] shows that

$$\pi_*L_{K(2)}(E_C^{\wedge k} \wedge X) \cong \mathrm{Hom}^c(\mathbb{G}_C^{k-1}, (E_C)_*X),$$

which allows us to identify the E_2 -term as $H^*(\mathbb{G}_C, (E_C)_*X)$. Now, using the fact that $(E_C)_*X \cong (E_C)_*E_C^{hG}$ as Morava modules, (2-9) and Shapiro’s lemma imply that $H^*(\mathbb{G}_C, (E_C)_*X) \cong H^*(G, (E_C)_*)$. □

Lemma 3.5 *Let X be a $K(2)$ -local spectrum such that $(E_C)_*X \cong (E_C)_*E_C^{hG_{24}}$ as Morava modules. Then the $K(2)$ -local, E_C -Adams spectral sequence computing $\pi_*(X \wedge V(0))$ has E_2 -term isomorphic to $H^*(G_{24}, (E_C)_*V(0))$. Further, in this spectral sequence, the elements Δ^k and $v_1 \Delta^k$ are d_3 -cycles for all k .*

Proof The identification of the E_2 -term follows from Lemma 3.4 and the five lemma. There are no d_2 -differentials, so all elements survive to the E_3 -term. Let $\epsilon = 0, 1$. It follows from [2, Theorem 4.2.2], that $d_3(v_\epsilon \Delta^k) = v_1^{10+\epsilon} \eta^3 p(j) \Delta^{k-1}$ for $p(j) \in \mathbb{F}_4\llbracket j \rrbracket$. Suppose that $p(j)$ is not zero. Then $p(j) = j^r p_0(j)$ for $r \geq 0$ and $p_0(j) \in \mathbb{F}_4\llbracket j \rrbracket$ such that $p_0(j) \equiv \ell$ modulo (j) for some $\ell \in \mathbb{F}_4^\times$. Using the fact that the differentials

are η - and v_1^4 -linear (since $X \wedge V(0)$ has a v_1^4 -self map), [Remark 3.3](#) and the identity $j = v_1^{12} \Delta^{-1}$, we have

$$\begin{aligned} 0 &= d_3(v_1^{10+\epsilon} \eta^3 p(j) \Delta^{k-1}) \\ &= v_1^{12r+8} \eta^3 d_3(v_1^{2+\epsilon} p_0(j) \Delta^{k-r-1}) \\ &= v_1^{12r+8+\epsilon} \eta^6 p_0(j) \Delta^{k-r-1} + v_1^{12r+10} \eta^3 d_3(v_1^\epsilon p_0(j) \Delta^{k-r-1}). \end{aligned}$$

Again, by [\[2, Theorem 4.2.2\]](#), $H^3(G_{24}, (E_C)_t V(0))$ is $\mathbb{F}_4[v_1, \eta]$ -torsion-free in degrees $t \equiv 6 + 2\epsilon$ modulo (24), so we can conclude that

$$\eta^3 p_0(j) \Delta^{k-r-1} = v_1^{2-\epsilon} d_3(v_1^\epsilon p_0(j) \Delta^{k-r-1}).$$

Since $\epsilon = 0$ or 1 , the right-hand side is divisible by v_1 , while the left-hand side is not, a contradiction. Therefore, we must have $p(j) = 0$. \square

In the next two propositions, we let

$$R(-) = \mathbb{W}((-))[\beta^{\pm 1}, \eta]/(2\eta, \eta^3).$$

Proposition 3.6 *Let X be as in [Lemma 3.5](#). The $E(1)$ -localization of the $K(2)$ -local, E_C -Adams spectral sequence*

$$E_2^{s,t} = v_1^{-1} H^s(\mathbb{G}_C, (E_C)_t(X \wedge V(0))) \implies \pi_{t-s} L_1(X \wedge V(0))$$

satisfies

$$E_\infty^{s,t} \cong R(j)\{x, v_1 x\}/(2 \cdot x, 2v_1 x)$$

for x in $(0, 0)$ and $v_1 x \in (0, 2)$. Further, $\pi_{8t} L_1(X \wedge V(0)) \cong \mathbb{F}_4((j))\{\beta^t\}$ and the edge homomorphisms

$$h: \pi_{8t} L_1(X \wedge V(0)) \rightarrow v_1^{-1} H^0(G_{24}, (E_C)_{8t} V(0))$$

are isomorphisms.

Proof By [Lemma 3.5](#) and naturality, $E_2^{s,t}$ is isomorphic to $v_1^{-1} H^s(G_{24}, (E_C)_t V(0))$ and $j^k = v_1^{12k} \Delta^{-k}$ and $v_1 j^k$ are d_3 -cycles. By [Remark 3.3](#), there are differentials $d_3(v_1^2 j^k) = \eta^3 j^k$ and $d_3(v_1^3 j^k) = v_1 \eta^3 j^k$. This, together with the fact that the differentials are v_1^4 -linear, determines all d_3 -differentials. The E_4 -term has a horizontal vanishing line at $s = 3$. Therefore, there cannot be any higher differentials. Letting x be the element detected by $1 \in H^0(G_{24}, (E_C)_0 V(0))$, $v_1 x$ the element detected by $v_1 \in H^0(G_{24}, (E_C)_2 V(0))$ and β^t the element detected by v_1^{4t} , we obtain the desired description of the E_∞ -term. For degree reasons, $\pi_{8t} L_1(X \wedge V(0)) \cong \mathbb{F}_4((j))\{\beta^t\}$. That the edge homomorphisms are isomorphisms in degrees $8t$ follows since $v_1^{-1} H^0(G_{24}, (E_C)_{8t} V(0)) \cong \mathbb{F}_4((j))\{v_1^{4t}\}$ and $h(j^k \beta^t) = j^k v_1^{4t}$. \square

Remark 3.7 When $X = V(0)$, the class x can be described as the composite $S^0 \rightarrow L_1 E_C^h G_{24} \xrightarrow{1 \wedge e_0} L_1(E_C^h G_{24} \wedge V(0))$, where the first map is the unit and e_0 is the

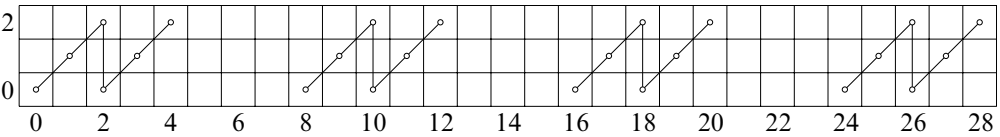


Figure 2: This picture is both an illustration of the homotopy groups $\pi_*L_1(E_C^{hG_{24}} \wedge V(0))$ and of the homotopy groups $\pi_*L_1(E_C^{hC_6} \wedge V(0))$. For the former, a \circ denotes a copy of $\mathbb{F}_4((j))$, and for the latter, it denotes a copy of $\mathbb{F}_4((j_0))$.

inclusion of the bottom cell. In $\pi_*V(0)_{(2)}$, there is a relation $2v_1e_0 = \eta^2e_0$ for v_1e_0 detected by $v_1 \in BP_2V(0)$ in the Adams–Novikov spectral sequence. This then implies that $2v_1x = \eta^2x$ in $\pi_*L_1(E_C^{hG_{24}} \wedge V(0))$, so

$$\pi_*L_1(E_C^{hG_{24}} \wedge V(0)) \cong R(j)\{x, v_1x\}/(2 \cdot x, 2v_1x - \eta^2x).$$

With some work, one can show that the relation $2v_1x = \eta^2x$ holds for arbitrary X satisfying the condition of Lemma 3.5. However, this fact is not needed here.

Proposition 3.8 *There is an isomorphism*

$$\pi_*L_1(E_C^{hC_6} \wedge V(0)) \cong R(j_0)\{y, v_1y\}/(2 \cdot y, 2v_1y - \eta^2y)$$

for y in $(0, 0)$ and $v_1y \in (0, 2)$; see Figure 2. Hence, $\pi_*L_1(E_2^{hC_6} \wedge V(0))$ is 8–periodic with periodicity generator β . Further, the edge homomorphisms

$$h: \pi_{8t}L_1(E_C^{hC_6} \wedge V(0)) \rightarrow v_1^{-1}H^0(C_6, (E_C)_{8t}V(0))$$

are isomorphisms.

Proof We prove that j_0^k is a d_3 –cycle for all integers k . Then an argument similar to that of Proposition 3.6 finishes the computation of the E_∞ –term, where we let y be the element detected by $1 \in H^0(C_6, (E_C)_0V(0))$ and v_1y be the element detected by $v_1y \in H^0(C_6, (E_C)_2V(0))$. The extension is obtained as in Remark 3.7.

The spectral sequence $H^*(C_6, (E_C)_*) \Rightarrow \pi_*E_C^{hC_6}$ is multiplicative; hence, in this spectral sequence, all elements of the form a^2 are d_3 –cycles. Note that j_0 lifts to an invariant in $H^0(C_6, (E_C)_0)$. This implies that $d_3(j_0^{2^r}) = 0$ and $d_3(j_0^{2^r+1}) = j_0^{2^r}d_3(j_0)$. Hence, it suffices to prove that j_0 is a d_3 –cycle. The restriction induced by the inclusion of C_6 in G_{24} , maps j to $j_0^4(1 + j_0)^{-3}$. By naturality, the element $d_3(j_0^4(1 + j_0)^{-3}) = 0$. However,

$$d_3(j_0^4(1 + j_0)^{-3}) = j_0^4(1 + j_0)^{-4}d_3(1 + j_0) = j_0^4(1 + j_0)^{-4}d_3(j_0),$$

which implies that $d_3(j_0) = 0$. □

4 Some elements in $\pi_* L_1 L_{K(2)} V(0)$

We now turn to examining the spectral sequence

$$E_1^{p,q}(L_1 SS_1) = \pi_q L_1(\mathcal{E}_p \wedge V(0)) \xrightarrow{L_1 SS_1} \pi_{q-p} L_1(E_C^{h\mathbb{S}_C^1} \wedge V(0)).$$

The idea is to use knowledge of the differentials in the spectral sequence

$$E_1^{p,q,t}(L_1 SS_2) = v_1^{-1} \operatorname{Ext}_{\mathbb{Z}_2[[\mathbb{S}_C^1]]}^q(\mathcal{E}_p, (E_C)_t V(0)) \xrightarrow{L_1 SS_2} v_1^{-1} H^{p+q}(\mathbb{S}_C^1, (E_C)_t V(0))$$

to deduce information about the differentials of $L_1 SS_1$.

Lemma 4.1 *In the spectral sequence $L_1 SS_1$, we have $E_2^{3,8t} \cong \mathbb{F}_4\{\beta^t\}$.*

Proof From [Section 3](#), we have that

$$E_1^{p,8t} \cong \begin{cases} \mathbb{F}_4((j))\{\beta^t\}, & p = 0, 3, \\ \mathbb{F}_4((j_0))\{\beta^t\}, & p = 1, 2. \end{cases}$$

From [Remark 2.7](#) and the fact that the edge homomorphisms are isomorphisms in these degrees, we obtain a commutative diagram

$$\begin{array}{ccccccc} E_1^{0,8t}(L_1 SS_1) & \xrightarrow{d_1} & E_1^{1,8t}(L_1 SS_1) & \xrightarrow{d_1} & E_1^{2,8t}(L_1 SS_1) & \xrightarrow{d_1} & E_1^{3,8t}(L_1 SS_1) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\ E_1^{0,0,8t}(L_1 SS_2) & \xrightarrow{d_1} & E_1^{1,0,8t}(L_1 SS_2) & \xrightarrow{d_1} & E_1^{2,0,8t}(L_1 SS_2) & \xrightarrow{d_1} & E_1^{3,0,8t}(L_1 SS_2) \end{array}$$

where β^{4t} maps to v_1^{4t} . Theorem 1.2.1 and Corollary 1.2.3 of [\[3\]](#) give a computation of the spectral sequence $L_1 SS_2$. In particular, it follows immediately from these results that

$$E_2^{3,0,8t}(L_1 SS_2) \cong \mathbb{F}_4((j))\{v_1^{4t}\}/(j) \cong \mathbb{F}_4\{v_1^{4t}\}.$$

The claim follows. □

Proposition 4.2 *If $k \equiv -3$ modulo 8, then $\pi_k L_1(E_C^{h\mathbb{S}_C^1} \wedge V(0)) \cong \mathbb{F}_4$.*

Proof We use the spectral sequence $E_r^{p,q} = E_r^{p,q}(L_1 SS_1)$. From [Proposition 3.6](#) applied to $X = \mathcal{E}_0$ and $X = \mathcal{E}_3$, and from [Proposition 3.8](#), it follows that for $r = 1, 2$ or 3 and for any p ,

$$E_1^{p,8t-r} = \pi_{8t-r} L_1(\mathcal{E}_p \wedge V(0)) = 0.$$

By [Lemma 4.1](#), $E_2^{3,8t} \cong \mathbb{F}_4\{\beta^{8t}\}$, which proves the claim. □

Proposition 4.3 *If $k \equiv -3$ modulo 8, then $\pi_k L_1(E_C^{h\mathbb{G}_C^1} \wedge V(0)) \cong \mathbb{F}_2$.*

Proof It follows from [Remark 2.8](#) that

$$v_1^{-1}E_{\infty}^{*,*}(\mathbb{S}_C^1, V(0)) \cong \mathbb{W} \otimes_{\mathbb{Z}_2} v_1^{-1}E_{\infty}^{*,*}(\mathbb{G}_C^1, V(0)).$$

Since $\pi_k L_1(E_C^{h\mathbb{S}_C^1} \wedge V(0)) \cong \mathbb{F}_4$, there is a unique $s_0 \geq 0$ such that $E_{\infty}^{s_0, k+s_0}(\mathbb{S}_C^1, V(0))$ is nonzero, and $E_{\infty}^{s_0, k+s_0}(\mathbb{S}_C^1, V(0)) \cong \mathbb{F}_4$. Therefore, $E_{\infty}^{s, k+s}(\mathbb{G}_C^1, V(0)) = 0$ if $s \neq s_0$ and $E_{\infty}^{s_0, k+s_0}(\mathbb{G}_C^1, V(0)) \cong \mathbb{F}_2$. \square

Definition 4.4 Define the class $x \in \pi_{-3} L_1(E_C^{h\mathbb{G}_C^1} \wedge V(0))$ to be the nonzero element.

Recall that

$$\mathbb{G}_C \cong \mathbb{G}_C^1 \rtimes \mathbb{Z}_2.$$

Let π be a topological generator of the subgroup \mathbb{Z}_2 in \mathbb{G}_C . There is a cofiber sequence

(4-1)
$$L_{K(2)}S \rightarrow E_C^{h\mathbb{G}_C^1} \xrightarrow{\pi-1} E_C^{h\mathbb{G}_C^1}.$$

We can now prove our main result.

Proof of Theorem 1.3 Since $L_{K(2)}S \wedge V(0) \simeq L_{K(2)}V(0)$ and localization preserves exact triangles, the fiber sequence (4-1) gives rise to a fiber sequence

(4-2)
$$L_1 L_{K(2)}V(0) \rightarrow L_1(E_C^{h\mathbb{G}_C^1} \wedge V(0)) \xrightarrow{\pi-1} L_1(E_C^{h\mathbb{G}_C^1} \wedge V(0)).$$

Since π acts by automorphisms and the only automorphism of \mathbb{F}_2 is the identity, the map $\pi - 1$ acts trivially on $\pi_{8t-3} L_1(E_C^{h\mathbb{G}_C^1} \wedge V(0))$. Therefore, in the long exact sequence on homotopy groups, the class $\beta^t x$ is in the kernel of $\pi - 1$, and the image of $\beta^t x$ under the map $L_1(E_C^{h\mathbb{G}_C^1} \wedge V(0)) \rightarrow \Sigma L_1 L_{K(2)}V(0)$ is nonzero. We denote it by $\zeta_2 \beta^t x$. \square

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