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Slowly converging Yamabe flows

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We characterize the rate of convergence of a converging volume-normalized Yamabe flow in terms of Morse-theoretic properties of the limiting metric. If the limiting metric is an integrable critical point for the Yamabe functional (for example, this holds when the critical point is nondegenerate), then we show that the flow converges exponentially fast. In general, we make use of a suitable Łojasiewicz–Simon inequality to prove that the slowest the flow will converge is polynomially. When the limit metric satisfies an Adams–Simon-type condition we prove that there exist flows converging to it exactly at a polynomial rate. We conclude by constructing explicit examples of this phenomenon. These seem to be the first examples of a slowly converging solution to a geometric flow.

1 Introduction

Let M be an arbitrary smooth closed manifold of dimension $n \geq 3$ and set

$$N = \frac{2n}{n-2}.$$

In this article we study the quantitative rate of convergence of the volume-normalized Yamabe flow

$$\frac{\partial g}{\partial t} = -(R_g - r_g)g,$$

describing a one-parameter deformation of complete Riemannian metrics $g = g(t)$, $t \geq 0$, on M. Here R_g is the scalar curvature and r_g is its average (with respect to the Riemannian volume form associated to g). This is a flow on a volume-normalized conformal class on M. It arises as the gradient flow of the Einstein–Hilbert functional and thus is a fundamental tool in the study of scalar curvature deformations, mostly in connection with the celebrated Yamabe problem. Motivated by the well-known uniformization theorem, the problem asks whether for any given Riemannian manifold (M_0, g_0) one can find a positive function w such that the conformal metric $w^{N-2}g_0$

has constant scalar curvature. An affirmative answer to this question was obtained by the combined efforts of Yamabe [31], Trudinger [30], Aubin [2] and Schoen [24]; we refer the reader to the survey article by Lee and Parker [18].

In unpublished work, Hamilton introduced the Yamabe flow as a possible alternative method for solving the Yamabe problem and showed that the flow existed for all time $t \geq 0$. However, the problem of convergence turns out to be highly nontrivial. For conformally flat metrics with positive Ricci curvature, Chow showed that the flow converged as $t \to \infty$ to a metric of constant scalar curvature [14]. Ye removed the Ricci curvature condition [32] and, subsequently, Schwetlick and Struwe showed that the flow converged in dimensions $3 \leq n \leq 5$ under the assumption that the the Yamabe energy of the initial metric $g(0)$ was "not too large" [26]. The energy assumption was then removed by Brendle to establish unconditional convergence of the flow in dimensions $3 \leq n \leq 5$ in [7], and convergence in dimensions $n \geq 6$ under a technical hypothesis on the conformal class in [8]. We refer to Brendle [10] for a survey concerning these and related results.

Our work complements these contributions by showing that, depending on certain Morse-theoretic properties of the limit metric, the rate of convergence has either exponential or polynomial upper bounds, and in the latter case the polynomial rate of convergence cannot in general be improved; therefore, it gives an essentially complete description of the rate of convergence for this flow. Perhaps the most novel outcome of this work is that there exist slowly converging geometric flows.

1.1 Main results

We now list our main results (we will define the precise terminology below), starting with the following statement concerning general upper bounds on the rate of convergence of the Yamabe flow. Integrability is defined in Definition 8.

Theorem 1 *Assume $g(t)$ is a Yamabe flow that converges in $C^{2,\alpha}(M, g_\infty)$ to g_∞ as $t \to \infty$ for some $\alpha \in (0, 1)$. Then there is a $\delta > 0$ depending only on g_∞ so that:*

(i) *If g_∞ is an integrable critical point, then the convergence occurs at an exponential rate, that is*

$$\|g(t) - g_\infty\|_{C^{2,\alpha}(M,g_\infty)} \leq Ce^{-\delta t}$$

for some constant $C > 0$ depending on $g(0)$.

(ii) *In general, the rate of convergence cannot be worse than polynomial, that is*

$$\|g(t) - g_\infty\|_{C^{2,\alpha}(M,g_\infty)} \leq C(1 + t)^{-\delta}$$

for some constant $C > 0$ depending on $g(0)$.

The problem of determining of the rate of convergence of the flow was raised by Ye [32, page 36]. In general, the polynomial rate of convergence cannot be improved, as we discuss below.

Two previous results on this question are worth mentioning. First, Struwe's method of showing that the Yamabe flow on the 2–sphere (which agrees with the Ricci flow in this case) converges exponentially fast [29] can in fact be extended to prove that a Yamabe flow converging to the standard round metric on the sphere (in all dimensions) converges exponentially fast. (This also follows from the work of Brendle [9].) We remark that this is a special case of Theorem 1(i), as the round metric is integrable by Obata's theorem. Second, the convergence statement in case (ii) of Theorem 1 can in some sense be regarded as an implicit corollary of the arguments of [7]. (Because we are assuming the metric converges, there cannot be any bubbling phenomena; thus one may use the remaining arguments in [7], which may be verified to apply in any dimension, to conclude.) The proof we give for Theorem 1 is self-contained and applies, in a unified framework, to both settings. Moreover, the method we use directly applies to other gradient flows (eg the Calabi flow; we refer to [23, Section 4.3.4] for a discussion and references) although we do not go into the details of such applications in this article.

The integrability condition is a nearly sharp condition for exponential convergence, as is shown in Theorem 2. The Adams–Simon positivity condition (that we denote by AS_p for some p) is defined in Definition 10.

Theorem 2 *Assume that g_∞ is a nonintegrable critical point of the Yamabe energy with order of integrability $p \geq 3$. If g_∞ satisfies the Adams–Simon positivity condition AS_p, then there exists a metric $g(0)$ conformal to g_∞ such that the Yamabe flow $g(t)$*

starting from $g(0)$ exists for all time and converges in $C^\infty(M, g_\infty)$ to g_∞ as $t \to \infty$. The convergence occurs "slowly" in the sense that

$$C^{-1}(1+t)^{-\frac{1}{p-2}} \leq \|g(t) - g_\infty\|_{C^{2,\alpha}(M, g_\infty)} \leq C(1+t)^{-\frac{1}{p-2}}$$

for some constant $C > 0$.

The bulk of this article is devoted to the proof of Theorem 2. Our proof is based on an adaptation of the remarkable tools developed by Simon and Adams–Simon in the study of isolated singularities of minimal surfaces and harmonic maps to the parabolic setting and, more specifically, to the Yamabe flow. As stated, three conditions need to be checked for a critical point g_∞ to be a limit point of a slowly converging Yamabe flow. Integrability and degeneracy are defined in Definition 8 and Lemma 9. The degeneracy can be studied by looking at the spectrum of the Laplace operator Δ_{g_∞}. For a degenerate metric, determining integrability (or lack thereof) depends on understanding the set of constant scalar curvature metrics near g_∞. For instance, if g_∞ is isolated and degenerate, it must be nonintegrable. Lastly, the Adams–Simon condition AS_p (see Definition 10) concerns the first nontrivial term in the analytic expansion of the Lyapunov–Schmidt reduction F of the Yamabe functional at g_∞, whose order we denote by p.

We give two criteria, of different nature, to check the condition AS_p:

- The condition AS_3 is satisfied whenever

$$F_3(v) = -2(N-1)(N-2) R_{g_\infty} \int_M v^3 \, dV_{g_\infty}$$

  does not vanish identically on the nullspace Λ_0, which is the linear span of functions v such that $(n-1)\Delta_{g_\infty} v + R_{g_\infty} v = 0$. Of course, this can in principle be computed once Λ_0 is explicitly known.

- When $p > 3$, the condition AS_p holds if g_∞ is both degenerate and a strict local minimum of the Yamabe functional.

The relevance of the second criterion is related to the solution of the Yamabe problem: if $(M, [g])$ is not the round sphere with the associated conformal structure, then the Yamabe functional $\mathcal{Y}$ is coercive and thus has a global minimum $g_{\min}$. Hence, the existence of polynomially converging flows is guaranteed by Theorem 2 whenever $g_{\min}$ is isolated but degenerate (which is simply a condition on the spectrum of the Laplacian of $g_{\min}$).

We give examples of the applicability of these criteria in the following two propositions. Denote by $\mathbb{S}^k(R)$ the round sphere of radius R in $\mathbb{R}^{k+1}$.

Proposition 3 *Fix integers $n, m > 1$ and a closed m–dimensional Riemannian manifold (M^m, g_M) with constant scalar curvature $R_{g_M} \equiv 4(n+1)(m+n-1)$. We denote the complex projective space equipped with the Fubini–Study metric by $(\mathbb{P}^n, g_{FS})$, where the normalization of g_{FS} is fixed so that $\mathbb{S}^{2n+1}(1) \to (\mathbb{P}^n, g_{FS})$ is a Riemannian submersion. Then the product metric $(M^m \times \mathbb{P}^n, g_M \oplus g_{FS})$ is a degenerate critical point satisfying AS_3.*

We remark that any closed manifold (M, g_M) whose scalar curvature is a positive constant may be rescaled to satisfy the conditions of the previous proposition.

Proposition 4 *Let $n > 2$. The product metric on $\mathbb{S}^1(1/\sqrt{n-2}) \times \mathbb{S}^{n-1}(1)$ is a nonintegrable critical point satisfying AS_p for some $p \geq 4$.*

There are not many examples of degenerate critical points of geometric functionals where nonintegrability can be checked; see [1, Section 5]. In fact, it seems that our second example is the first of a critical point which satisfies AS_p for $p > 3$ (cf [1, Remark 1.19], where the authors explain a method for checking AS_3 that does not work for $p > 3$).

In conclusion, we may construct examples of slowly converging Yamabe flows in a range of conformal classes and in any dimension greater than 2. For example, Proposition 3 yields examples which are not conformally flat, while the metrics in Proposition 4 are locally conformally flat.

Corollary 5 *There exists a Yamabe flow in the conformal class of the metrics described in Propositions 3 and 4 that converges to the given metrics exactly at a polynomial rate, as in Theorem 2.*

This seems to be the first construction of a slowly converging flow in the setting of geometric flows of parabolic type. We expect that our methods can be adapted (possibly with the added difficulty of a large gauge group) to produce slowly converging flows for other (possibly degenerate) parabolic flows.

1.2 Outline of the proof of Theorem 2

The proof of Theorem 2 appears in Section 4. It is rather long and, at times, quite technical, so we take the opportunity here to outline its structure.

Ultimately, we would like to construct a solution $u(t)$ to a quasilinear parabolic equation that converges at a rate $O((1+t)^{-1/(p-2)})$ to the constant function 1. Intuitively, one expects the slow convergence to be "generated" infinitesimally by the nonintegrable

directions, namely from Λ_0. At the same time, in order to use a fixed-point argument to generate a polynomially converging flow it is most convenient to have a guess as to what the leading order behavior of the flow ought to be, and then show that the actual solution is a small perturbation, of order $o((1+t)^{-1/(p-2)})$, of this guess. Fortunately, the Adams–Simon condition precisely furnishes such an ansatz: the function $\varphi(t)$ of Lemma 15. The proof of Theorem 2 thus amounts to showing that we can find $u(t)$ solving the Yamabe flow with

$$(1) \qquad\qquad u(t) - \varphi(t) = o\big((1+t)^{-\frac{1}{p-2}}\big).$$

We now explain the different steps to derive this estimate.

Step 1 Firstly, we would like to understand separately the behavior of the flow in Λ_0 (the kernel of the linearized Yamabe operator) and $\Lambda_0^{\perp}$ (the orthogonal complement) directions. Thus, in Proposition 17 we prove that the Yamabe flow is equivalent to two flows: the *kernel-projected flow* that takes place on Λ_0 and the *kernel orthogonal-projected flow* that takes place on $\Lambda_0^{\perp}$. There are two key points about this result that make it useful. First, the kernel orthogonal-projected flow (see (12)) is a perturbation of a *linear* parabolic equation. In other words, (12) is, of course, nonlinear, but it can be considered as a linear equation since we prove an a priori estimate on the error term. Second, the kernel-projected flow (see (11)) is a perturbation of a system of ODEs. Again, we have a precise a priori estimate on the error term. That these estimates are sufficiently strong will play a crucial role in a contraction mapping argument discussed in Step 4 below.

The proof of Proposition 17 involves some rather tedious computations. First, in Lemma 16, we reduce the Yamabe flow to the situation of a gradient flow. Indeed, the Yamabe flow is certainly a gradient flow on the level of metrics but that is not quite the case on the level of conformal factors. After this preparatory step, we project the flow onto Λ_0 and its orthogonal complement, and try to reduce the resulting equations to (12) and (11). This involves multiple applications of Taylor's theorem, the Lyapunov–Schmidt reduction (Proposition 7) and the estimates on $D^3 \mathcal{Y}$ (Appendix A). It also uses the fact that the ansatz $\varphi(t)$ allows an important cancellation that considerably simplifies the kernel-projected piece, so that it can be reduced to a system of ODEs.

Step 2 We obtain a solution to the kernel-projected flow in a weighted Hölder norm in Lemma 18. This norm precisely captures a polynomial rate of decay of the solution.

Step 3 We obtain a solution to the kernel orthogonal-projected flow, again in a weighted norm, in Lemma 19. Here some care is needed since we must again work with parabolic Hölder norms.

Step 4 Finally, in Proposition 20, we set up a fixed-point argument in a Banach space that uses the two different weighted norms (on Λ_0 and $\Lambda_0^{\perp}$). Here one needs to be quite careful with the order of decay of the error terms collected in the previous three steps, in order to show that the map is a contraction. Once this is acheived, we have the existence of a Yamabe flow. Moreover, the flow satisfies the estimate (1) because we define the weighted norms so that any function which is a perturbation of $\varphi(t)$ in the given norm will fall off at a rate faster than $\varphi(t)$.

1.3 Structure of the article

Section 2 is devoted to fixing the notation and recalling some basic facts about the (volume-constrained) Yamabe functional, its analyticity and the Lyapunov–Schmidt reduction near a critical point. In Section 3, we use the Łojasiewicz–Simon inequality to prove Theorem 1. Then, in Section 4 we study polynomial convergence phenomena for nonintegrable critical points, and in Section 5 we prove Propositions 3 and 4. The computation of the third variation of the reduced Yamabe energy (namely of the formula for F_3 given above) is contained in Appendix A. Appendix B contains a proof of a technical lemma needed for the proof of Proposition 4.

2 Definitions and preliminaries

The *Yamabe functional* is defined by

$$\mathcal{Y}(g) := \operatorname{Vol}(M, g)^{-\frac{2}{N}} \int_M R_g \, dV_g,$$

where dV_g is the Riemannian volume form associated to g, R_g denotes the scalar curvature of g and $N = 2n/(n-2)$.

If $g = w^{N-2} g_\infty$ for some positive $w \in C^2(M)$ and smooth metric g_∞, then an alternative expression for the Yamabe functional (restricted to the conformal class of g) is

$$\mathcal{Y}(w) = \frac{\int_M \left((N+2) |\nabla_{g_\infty} w|_{g_\infty}^2 + R_{g_\infty} w^2 \right) dV_{g_\infty}}{\left(\int_M w^N \, dV_{g_\infty} \right)^{\frac{2}{N}}},$$

since $R_{w^{N-2} g_\infty} = w^{1-N} (R_{g_\infty} w - (N+2) \Delta_{g_\infty} w)$.

Consider the *unit volume conformal class* associated to the metric g_∞,

$$[g_\infty]_1 := \left\{ w^{N-2} g_\infty : w \in C^{2,\alpha}(M), w > 0, \int_M w^N \, dV_{g_\infty} = 1 \right\}.$$

In order to avoid ambiguities, we define the following notation: for $k \in \mathbb{N}$, we denote the kth differential of the Yamabe functional on $[g_\infty]_1$ at the point w in the directions $v_1, \ldots, v_k$ by

$$D^k \mathcal{Y}(w)[v_1, \ldots, v_k].$$

As we will see from (5), the functional $v \mapsto D^k \mathcal{Y}(w)[v_1, \ldots, v_{k-1}, v]$ is in the image of $L^2(M, g_\infty)$ under the natural embedding into $C^{2,\alpha}(M, g_\infty)'$. Therefore, we will also write

$$D^k \mathcal{Y}(w)[v_1, \ldots, v_{k-1}]$$

for this element of $L^2(M, g_\infty)$. When $k = 1$, we will drop the (second) brackets, and thus consider $D\mathcal{Y}(w) \in L^2(M, g_\infty)$.

We may write the differential of $\mathcal{Y}$ restricted to $[g_\infty]_1$ as

$$\tfrac{1}{2} D\mathcal{Y}(w)[v] = \int_M \left[-(N+2)\Delta_{g_\infty} w + R_{g_\infty} w - r_{w^{N-2} g_\infty} w^{N-1} \right] v \, dV_{g_\infty}$$

$$= \int_M \left(R_{w^{N-2} g_\infty} - r_{w^{N-2} g_\infty} \right) w^{N-1} v \, dV_{g_\infty},$$

for $v \in C^{2,\alpha}(M, g_\infty)$. Here

$$r_g = \mathrm{Vol}(M, g)^{-1} \int_M R_g \, dV_g.$$

Regarded as an element of $L^2(M, g_\infty)$, we have that

(2) $$\tfrac{1}{2} D\mathcal{Y}(w) = -(N+2)\Delta_\infty w + R_{g_\infty} w - r_{w^{N-2} g_\infty} w^{N-1}.$$

As above, we have associated the metric $w^{N-2} g_\infty$ to the function w. This is clearly a bijection, so we will continue to do so throughout. Thus, a unit volume metric g_∞ is a critical point for the Yamabe energy $\mathcal{Y}$ restricted to $[g_\infty]_1$ exactly when g_∞ has constant scalar curvature.

We now fix g_∞ to be a unit volume, constant scalar curvature metric. We denote by $\mathcal{CSC}_1$ the set of unit volume, constant scalar curvature metrics in $[g_\infty]_1$ and further define the *linearized Yamabe operator at g_∞*, $\mathcal{L}_\infty$, by means of the formula

$$-(N-2) \int_M w \mathcal{L}_\infty v \, dV_{g_\infty} := \tfrac{1}{2} D^2 \mathcal{Y}(g_\infty)[v, w]$$

$$= \tfrac{1}{2} \frac{d}{ds}\Big|_{s=0} D\mathcal{Y}\big((1+sv)^{N-2} g_\infty\big)[w]$$

for $v \in C^2(M)$. A computation shows that

$$\mathcal{L}_\infty v = (n-1)\Delta_{g_\infty} v + R_{g_\infty} v.$$

We define $\Lambda_0 := \ker \mathcal{L}_\infty \subset L^2(M, g_\infty)$.

Spectral theory shows that Λ_0 is finite-dimensional, since it is the eigenspace of the Laplacian for the eigenvalue $R_{g_\infty}/(n-1)$. We will write $\Lambda_0^\perp$ for the $L^2(M, g_\infty)$ orthogonal complement. It is crucial throughout this work that the Yamabe functional is an analytic map. Here we mean analytic in the sense of [33, Definition 8.8].

Lemma 6 *Fix a metric g_∞. The Yamabe functional is an analytic functional on $\{u \in C^{2,\alpha}(M, g_\infty) : u > 0\}$ in the sense that for each $w_0 \in C^{2,\alpha}(M, g_\infty)$ with $w_0 > 0$, there is an $\epsilon > 0$ and bounded multilinear operators*

$$\mathcal{Y}^{(k)} \colon C^{2,\alpha}(M, g_\infty)^{\times k} \to \mathbb{R} \quad \text{for each } k \geq 0$$

such that if $\|w - w_0\|_{C^{2,\alpha}} < \epsilon$, then $\sum_{k=0}^\infty \|\mathcal{Y}^{(k)}\| \cdot \|w - w_0\|_{C^{2,\alpha}}^k < \infty$ and

$$\mathcal{Y}(w) = \sum_{k=0}^\infty \mathcal{Y}^{(k)} \underbrace{(w - w_0, w - w_0, \ldots, w - w_0)}_{k \text{ times}} \quad \text{in } C^{2,\alpha}(M, g_\infty).$$

It is not hard to verify this, by simply expanding the denominator of $\mathcal{Y}$ in a power series around $(\int_M w_0^N \, dV_{g_\infty})^{-2/N}$ and noting that the numerator is already a bilinear function in w. Now, by a standard Lyapunov–Schmidt reduction (see [33, Theorem 4.H], [28, Section 3]), we may use the implicit function theorem to show the following.

Proposition 7 *There is $\epsilon > 0$ and an analytic map*

$$\Phi \colon \Lambda_0 \cap \{v : \|v\|_{L^2} < \epsilon\} \to C^{2,\alpha}(M, g_\infty) \cap \Lambda_0^\perp$$

such that $\Phi(0) = 0$, $D\Phi(0) = 0$,

$$(3) \qquad \sup_{\|v\|_{L^2} < \epsilon, \, \|w\|_{L^2} \leq 1} \left\| D\Phi(v)[w] \right\|_{L^2} < 1,$$

and such that defining $\Psi(v) = 1 + v + \Phi(v)$, then $\Psi(v) > 0$, $\mathrm{Vol}(M, \Psi(v)^{N-2} g_\infty) = 1$ and

$$\mathrm{proj}_{\Lambda_0^\perp}[D\mathcal{Y}(\Psi(v))] = \mathrm{proj}_{\Lambda_0^\perp}\left[(R_{\Psi(v)^{N-2}g_\infty} - r_{\Psi(v)^{N-2}g_\infty})\Psi(v)^{N-1}\right] = 0.$$

Furthermore,

$$\mathrm{proj}_{\Lambda_0}[D\mathcal{Y}(\Psi(v))] = \mathrm{proj}_{\Lambda_0}\left[(R_{\Psi(v)^{N-2}g_\infty} - r_{\Psi(v)^{N-2}g_\infty})\Psi(v)^{N-1}\right] = DF,$$

where $F \colon \Lambda_0 \cap \{v : \|v\|_{L^2} \leq \epsilon\} \to \mathbb{R}$ is defined by $F(v) = \mathcal{Y}(\Psi(v))$. Finally, the intersection of $\mathcal{CSC}_1$ with a small $C^{2,\alpha}(M, g_\infty)$ neighborhood of 1 coincides with

$$\mathcal{S}_0 := \{\Psi(v) : v \in \Lambda_0, \|v\|_{L^2} < \epsilon, DF(v) = 0\},$$

which is a real analytic subvariety (possibly singular) of the following $(\dim \Lambda_0)-$ dimensional real analytic submanifold of $C^{2,\alpha}(M, g_\infty)$:

$$\mathcal{S} := \{\Psi(v) : v \in \Lambda_0, \|v\|_{L^2} < \epsilon\}.$$

This follows in the usual way from the analytic implicit function theorem; see [33, Corollary 4.23]. We will refer to $\mathcal{S}$ as the *natural constraint* for the problem.

Definition 8 For $g_\infty \in \mathcal{CSC}_1$, we say that g_∞ is *integrable* if for all $v \in \Lambda_0$, there is a path $w(t) \in C^2((-\epsilon, \epsilon) \times M, g_\infty)$ such that $w(t)^{N-2} g_\infty \in \mathcal{CSC}_1$ and $w(0) = 1$, $w'(0) = v$. Equivalently, g_∞ is integrable if and only if $\mathcal{CSC}_1$ agrees with $\mathcal{S}$ in a small neighborhood of 1 in $C^{2,\alpha}(M, g_\infty)$.

For our purposes, the following equivalent characterization of integrability is crucial:

Lemma 9 [1, Lemma 1] *Integrability as defined above is equivalent to the functional F (as defined in Proposition 7, the Lyapunov–Schmidt reduction) being constant on a neighborhood of 0 inside Λ_0.*

We remark that if $\Lambda_0 = 0$, ie if $\mathcal{L}_\infty$ is injective, it is standard to call g_∞ a *nondegenerate* critical point; if this holds, g_∞ is automatically integrable in the above sense. On the other hand, if Λ_0 is nonempty, then we call g_∞ *degenerate*. We emphasize that there are many examples of degenerate metrics; see for example [5].

Now suppose that g_∞ is a nonintegrable critical point. Because $F(v)$ is analytic (it is the composition of two analytic functions), we may expand it in a power series

$$F(v) = F(0) + \sum_{j \geq p} F_j(v),$$

where F_j is a degree-j homogeneous polynomial on Λ_0 and p is chosen so that F_p is nonzero. We will call p the *order of integrability* of g_∞. We will also need a further hypothesis for nonintegrable critical points, introduced in [1].

Definition 10 We say that g_∞ satisfies the *Adams–Simon positivity condition*, AS_p for short (here p is the order of integrability of g_∞), if it is nonintegrable and $F_p|_{\mathbb{S}^k}$ attains a positive maximum for some $\hat{v} \in \mathbb{S}^k \subset \Lambda_0$. Recall that F_p is the lowest-degree nonconstant term in the power series expansion of $F(v)$ around 0 and $\mathbb{S}^k$ is the unit sphere[1] in Λ_0.

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[1]Here we are using the inner product induced on Λ_0 coming from the L^2 inner product on $T_1[g_\infty]_1$.

The Adams–Simon positivity condition is ultimately needed for the construction of the function φ in Lemma 15, which serves as an approximate solution to the Yamabe flow converging at a polynomial rate. It is an interesting question whether or not AS_p is a necessary condition for the existence of slowly converging examples in the elliptic and parabolic settings.

An important observation is that when the order of integrability p is odd, the Adams–Simon positivity condition is always satisfied. Moreover, the order of integrability (at a critical point of $\mathcal{Y}$) always satisfies $p \geq 3$ as we recall in Appendix A. Furthermore, we show there that

$$(4) \qquad F_3(v) = -2(N-1)(N-2)R_{g_\infty} \int_M v^3 \, dV_{g_\infty}.$$

3 The Łojasiewicz–Simon inequality and rate of convergence

One of the tools for controlling the rate of convergence of the Yamabe flow will be the Łojasiewicz–Simon inequality. This was first proven for a certain class of geometric functionals by Simon [27], who showed that the classical Łojasiewicz inequality for analytic functions in finite dimensions could be extended to a Banach space setting.

Definition 11 (Łojasiewicz–Simon inequality) Suppose that $\mathcal{B}$ is a Banach space and $U \subset \mathcal{B}$ is an open subset. Fix a functional $E \in C^2(U, \mathbb{R})$, and denote by $DE \in C^1(U, \mathcal{B}')$ its first derivative. (Here $\mathcal{B}'$ is the dual Banach space to $\mathcal{B}$.) We will additionally fix a Banach space $\mathcal{W}$ with a continuous embedding $\mathcal{W} \hookrightarrow \mathcal{B}'$. For $x_0 \in U$ a critical point of E, ie $DE(x_0) = 0$, we say that E satisfies the *Łojasiewicz–Simon inequality* with exponent $\theta \in (0, \frac{1}{2}]$ near x_0 if there exists a neighborhood $x_0 \in V \subset U$ as well as constants $C > 0$ such that

$$|E(x) - E(x_0)|^{1-\theta} \leq C \|DE(x)\|_{\mathcal{W}} \quad \text{for all } x \in V.$$

Notice that if $\mathcal{B} = \mathcal{W} = \mathbb{R}^n$, this reduces to the classical Łojasiewicz inequality [20]. The Łojasiewicz–Simon inequality has recently received much attention; we will apply the following general result to show that it holds in our setting:

Proposition 12 [12, Theorem 3.10] *Fix $\mathcal{B}$, $U \subset \mathcal{B}$, $E \in C^2(U, \mathbb{R})$, $\mathcal{W} \hookrightarrow \mathcal{B}'$ and $x_0 \in U$ with $DE(x_0) = 0$ as in the previous definition. We also define the second derivative $\mathcal{L} := D^2 E \in C(U, \mathcal{B}(\mathcal{B}, \mathcal{B}'))$, where $\mathcal{B}(\mathcal{B}, \mathcal{B}')$ is the space of continuous maps between the Banach spaces $\mathcal{B}$ and $\mathcal{B}'$. We will suppose that the following hypotheses are satisfied:*

(A) The kernel $\ker \mathcal{L}(x_0) \subset \mathcal{B}$ is complemented in $\mathcal{B}$, ie there exists a projection $P \in \mathcal{B}(\mathcal{B}, \mathcal{B})$ such that range $P = \ker \mathcal{L}(x_0)$. It follows from this that $\mathcal{B} = \ker \mathcal{L}(x_0) \oplus \ker P$ is a topological direct sum. Denote by $P' \in \mathcal{B}(\mathcal{B}', \mathcal{B}')$ the adjoint map.

(B1) The map $\mathcal{W} \hookrightarrow \mathcal{B}'$ is a continuous embedding.

(B2) The adjoint projection P' leaves $\mathcal{W}$ invariant.

(B3) The map DE is in $C^1(U, \mathcal{W})$.

(B4) We have range $\mathcal{L}(x_0) = \ker P' \cap \mathcal{W}$.

Under these hypotheses, we may find a neighborhood U_0 of 0 in $\ker \mathcal{L}(x_0)$ and a neighborhood U_1 of 0 in $\ker P$ as well as a function $H \in C^1(U_0, U_1)$ parametrizing the natural constraint, ie

$$\{x \in U_0 + U_1 : DE(x_0 + x) \in (\ker \mathcal{L}(x_0))'\} = \{x + H(x) : x \in U_0\}.$$

Recall that the natural constraint is then

$$S := \{x_0 + x + H(x) : x \in U_0\}.$$

Finally, suppose that the following holds:

(C) The function $E(x_0 + \cdot\,)$ satisfies the Łojasiewicz inequality on the natural constraint S with exponent $\theta \in (0, \frac{1}{2}]$. More precisely, we assume that

$$|E(x_0 + x + H(x)) - E(x_0)|^{1-\theta} \le C \|DE(x_0 + x + H(x))\|_{\mathcal{W}}$$

for all $x \in U_0$.

Then the functional E satisfies the Łojasiewicz–Simon inequality near x_0 with the same exponent $\theta \in (0, \frac{1}{2}]$.

Proposition 13 Suppose that g_∞ is a unit volume, constant scalar curvature metric. There are $\theta \in (0, \frac{1}{2}]$, $\epsilon > 0$ and $C > 0$ only depending on n and g_∞ such that for $u \in C^{2,\alpha}(M, g_\infty)$ with $\|u - 1\|_{C^{2,\alpha}(M, g_\infty)} < \epsilon$ and $\mathrm{Vol}(M, u^{N-2}g_\infty) = 1$, then

$$|r_{u^{N-2}g_\infty} - r_{g_\infty}|^{1-\theta} \le C \|D\mathcal{Y}(u^{N-2}g_\infty)\|_{L^2(M, g_\infty)}.$$

If g_∞ is an integrable critical point, then $\theta = \frac{1}{2}$. If g_∞ is nonintegrable, then $\theta = 1/p$, where p is the order of integrability of g_∞.

Proof To verify this, we will show that the hypotheses of Proposition 12 are satisfied for the Yamabe energy $\mathcal{Y}$. We work with the Banach spaces $\mathcal{B} := C^{2,\alpha}(M, g_\infty)$ and $\mathcal{W} := L^2(M, g_\infty)$, and fix U a small enough ball around 1 in $C^{2,\alpha}(M, g_\infty)$ so that Proposition 7 is applicable in U.

Hypothesis (A) is the statement that $\Lambda_0 = \ker \mathcal{L}_\infty$ is complemented in $C^{2,\alpha}(M, g_\infty)$, which is immediate by the following argument. One first checks that the L^2 projection map $\mathrm{proj}_{\Lambda_0}$ restricts to a continuous map from $C^{2,\alpha}(M, g_\infty)$ onto Λ_0 (since, of course, $C^{2,\alpha}(M, g_\infty) \hookrightarrow L^2(M, g_\infty)$ is a continuous embedding); from this, it follows (see [12, page 580]) that Λ_0' is complemented (by the map $\mathrm{proj}_{\Lambda_0}'$) in the dual space $C^{2,\alpha}(M, g_\infty)'$, and its complement $\Lambda_0'^{\perp}$ may be canonically identified with $(\Lambda_0^\perp)'$.

Hypothesis (B) is satisfied as follows: Consider the map

$$(5) \qquad \mathcal{W} := L^2(M, g_\infty) \hookrightarrow C^{2,\alpha}(M, g_\infty)', \quad f \mapsto \left(\varphi \mapsto \int_M f\varphi \, dV_{g_\infty} \right).$$

(B1) This map is continuous.

(B2) The map $\mathrm{proj}_{\Lambda_0}' \in \mathcal{B}(C^{2,\alpha}(M, g_\infty)')$ leaves $L^2(M, g_\infty)$ invariant; of course, here we are considering the composition

$$\mathrm{proj}_{\Lambda_0} \colon C^{2,\alpha}(M, g_\infty) \to \Lambda_0 \hookrightarrow C^{2,\alpha}(M, g_\infty).$$

(B3) That $D\mathcal{Y} \in C^1(U, L^2(M, g_\infty))$ follows from the explicit form of $D\mathcal{Y}$ given above.

(B4) Finally, we must verify that $\mathrm{range}\, \mathcal{L}_\infty = \Lambda_0'^{\perp} \cap L^2(M, g)$. The fact that $\mathrm{range}\, \mathcal{L}_\infty \subseteq \Lambda_0'^{\perp} \cap L^2(M, g)$ is obvious because $\mathcal{L}_\infty$ is formally self-adjoint on L^2. The other inclusion follows from the L^2 spectral decomposition of $\mathcal{L}_\infty$.

Thus to prove the Łojasiewicz–Simon inequality with exponent $\theta \in (0, \frac{1}{2}]$, it is enough to check (C), ie that the Yamabe energy restricted to the natural constraint satisfies the Łojasiewicz–Simon inequality with exponent $\theta \in (0, \frac{1}{2}]$. Recall that in Proposition 7 we have defined $F(v) = \mathcal{Y}(\Psi(v))$. In the integrable case, clearly $F(v) \equiv F(0)$, so F satisfies the Łojasiewicz–Simon inequality for $\theta = \frac{1}{2}$.

In general, F is an analytic function whose power series has its first nonzero term of degree p, by definition. Thus we may conclude (see [12, Proposition 2.3(b)]) that F satisfies the Łojasiewicz–Simon inequality with exponent $\theta = 1/p$.

The claim follows from this; we have replaced $\mathcal{Y}(u)$ with $r_{u^{N-2}g_\infty}$ in the left-hand side of the inequality by using the assumption that $u^{N-2}g_\infty$ has unit volume. $\hspace{2cm} \square$

Now we show how the Łojasiewicz–Simon inequality yields quantitative estimates on the rate of convergence of the Yamabe flow.

Proof of Theorem 1 We consider a Yamabe flow $g(t) = u(t)^{N-2} g_\infty$ which converges to g_∞ in $C^2(M, g_\infty)$ as $t \to \infty$. We may assume without loss of generality that g_∞ and thus $g(t)$ have unit volume. In Proposition 13 we have shown there is a Łojasiewicz–Simon inequality near g_∞ for some $\theta \in (0, \frac{1}{2}]$. We emphasize that we are regarding $D\mathcal{Y}(g(t))$ as an element of $L^2(M, g_\infty)$, ie $D\mathcal{Y}(g(t)) = 2(R_{g(t)} - r_{g(t)})u(t)^{N-1}$.

Choose t_0 so that for $t \geq t_0$, $\|u(t) - 1\|_{C^0(M, g_\infty)} \leq \frac{1}{2}$. We then have that

$$\frac{d}{dt}(r_{g(t)} - r_{g_\infty}) = -\frac{2}{N-2} \int_M (R_{g(t)} - r_{g(t)})^2 u(t)^N \, dV_{g_\infty}$$

$$\leq -c \int_M (R_{g(t)} - r_{g(t)})^2 u(t)^{2N-2} \, dV_{g_\infty}$$

$$= -c\|D\mathcal{Y}(g(t))\|^2_{L^2(M, g_\infty)} \leq -c(r_{g(t)} - r_{g_\infty})^{2-2\theta},$$

where $c > 0$ is a constant depending only on n and g_∞ (that may vary at each step). Let us first assume that the Łojasiewicz–Simon inequality is satisfied with $\theta = \frac{1}{2}$, ie that we are in the integrable case. The previous inequality yields $r_{g(t)} - r_{g_\infty} \leq Ce^{-2\delta t}$, for $\delta > 0$ depending only on n and g_∞ and $C > 0$ depending on $g(0)$ (chosen so that this actually holds for all $t \geq 0$). On the other hand, if the Łojasiewicz–Simon inequality holds with $\theta \in (0, \frac{1}{2})$ then the same argument shows that $r_{g(t)} - r_{g_\infty} \leq C(1+t)^{1/(2\theta-1)}$.

Recall that the evolution equation for the conformal factor $u = u(t)$ is given by

$$\frac{\partial u}{\partial t} = -\frac{u}{N-2}(R_{g(t)} - r_{g(t)}).$$

Thus, exploiting the fact that the flow converges in C^2, we may use the Łojasiewicz–Simon inequality to compute

$$\frac{d}{dt}(r_{g(t)} - r_{g_\infty})^\theta = \theta(r_{g(t)} - r_{g_\infty})^{\theta-1} \frac{d}{dt}(r_{g(t)} - r_{g_\infty})$$

$$\leq -c\theta(r_{g(t)} - r_{g_\infty})^{\theta-1} \|D\mathcal{Y}(g(t))\|^2_{L^2(M, g_\infty)}$$

$$\leq -c\theta\|D\mathcal{Y}(g(t))\|_{L^2(M, g_\infty)} \leq -c\theta \left\| \frac{\partial u}{\partial t} \right\|_{L^2(M, g_\infty)}.$$

Thus, if $\theta = \frac{1}{2}$ (recall $\lim_{t \to \infty} u(t) = 1$),

$$\|u(t) - 1\|_{L^2(M, g_\infty)} \leq \int_t^\infty \left\| \frac{\partial u}{\partial s} \right\|_{L^2(M, g_\infty)} ds$$

$$\leq -c \int_t^\infty \frac{d}{ds}[(r_{g(s)} - r_{g_\infty})^{\frac{1}{2}}] \, ds = c(r_{g(t)} - r_{g_\infty})^{\frac{1}{2}} \leq Ce^{-\delta t}.$$

If $\theta \in (0, \frac{1}{2})$, a similar computation yields $\|u(t) - 1\|_{L^2(M,g_\infty)} \le C(1+t)^{-\theta/(1-2\theta)}$.

To obtain C^2 estimates, we may interpolate between $L^2(M, g)$ and $W^{k,2}(M, g)$ for k large enough: interpolation [4, Theorem 6.4.5] and Sobolev embedding yield some constant $\eta \in (0, 1)$ such that

$$\|u(t) - 1\|_{C^{2,\alpha}(M,g_\infty)} \le \|u(t) - 1\|_{L^2(M,g_\infty)}^{\eta} \|u(t) - 1\|_{W^{k,2}(M,g_\infty)}^{1-\eta}.$$

Because $u(t)$ is converging to 1 in $C^{2,\alpha}$ (and thus in C^∞ by parabolic Schauder estimates and bootstrapping), the second term is uniformly bounded. Thus, exponential (polynomial) decay of the L^2 norm give exponential (polynomial) decay of the $C^{2,\alpha}$ norm as well. □

Remark 14 The assumption in Theorem 1 that $u(t)$ converges in $C^{2,\alpha}$ to the constant function 1 can be weakened to assuming merely that the Yamabe flow converges in $L^N(M, g_0)$. Indeed, it is possible to show that the latter already implies the flow has a smooth limit to which it converges in $C^{2,\alpha}$. The L^N convergence is equivalent to saying that the flow converges in the Ebin L^2 metric on the space of Riemannian metrics, restricted to the conformal class [15, Section 4].

4 Slowly converging Yamabe flows

In this section, we show that given a nonintegrable critical point g_∞ satisfying a particular hypothesis, there exists a Yamabe flow $g(t)$ such that $g(t)$ converges to g_∞ exactly at a polynomial rate. This shows that the conditions in Theorem 1 are nearly sharp. We will do so by adapting the arguments of Adams and Simon [1] to the parabolic setting. (In [1, Section 6], the authors remark that their results should extend to the parabolic setting, but this requires some serious work. Moreover, the Yamabe functional does not completely fit into their framework because of the volume normalization term.)

4.1 Projecting the Yamabe flow with estimates

The goal of this subsection is to obtain an equivalent formulation of the Yamabe flow in terms of *two* flows: one taking place on the finite-dimensional space Λ_0 (the "kernel-projected flow"), and the other on the infinite-dimensional complement $\Lambda_0^\perp$ (the "kernel orthogonal-projected flow"). The a priori estimates of Proposition 17 make this possible.

The next lemma will provide a function which approximately solves the Yamabe flow. The remaining parts of this section will be devoted to perturbing it to an exact solution

of the flow. Here and in the sequel we will always use $f'(t)$ to denote the time derivative of a function $f(t)$. The constant T should be thought of as a large, but fixed, parameter. Because none of the constants in the bounds that we will derive in Proposition 17 depend implicitly on T, we will be allowed to take T large in the final step of the proof of Theorem 2.

Lemma 15 *Assume that g_∞ satisfies AS_p as defined in Definition 10, ie $F_p|_{\mathbb{S}^{k-1}}$ achieves a positive maximum for some point $\hat{v}$ in the unit sphere $\mathbb{S}^{k-1} \subset \Lambda_0$. Then, for any fixed $T \geq 0$, the function*

$$(6) \qquad \varphi(t) := \varphi(t, T) = (T+t)^{-\frac{1}{p-2}} \left(\frac{2(N-2)}{p(p-2)F_p(\hat{v})} \right)^{\frac{1}{p-2}} \hat{v}$$

solves $2(N-2)\varphi' + DF_p(\varphi) = 0$.

Proof Assume that $F_p|_{\mathbb{S}^{k-1}}$ achieves a positive maximum at $\hat{v}$. Then, for any $\lambda \in \mathbb{R}$,

$$(7) \qquad DF_p(\lambda \hat{v}) = p|\lambda|^{p-1} F_p(\hat{v})\hat{v}.$$

The reason for this is that F_p is p–homogeneous, so it is some function on $\mathbb{S}^{k-1}$ times r^p. The $\mathbb{S}^{k-1}$ part has zero gradient at $\hat{v}$ by assumption, so the gradient must be radial. The exact form follows from differentiating the r^p part along with scaling. Thus

$$DF_p(\varphi) = p(T+t)^{-1-\frac{1}{p-2}} \left(\frac{2(N-2)}{p(p-2)F_p(\hat{v})} \right)^{1+\frac{1}{p-2}} F_p(\hat{v})\hat{v} = \frac{2(N-2)}{p-2}(T+t)^{-1}\varphi(t).$$

Since $\varphi'(t) = -(p-2)^{-1}(T+t)^{-1}\varphi(t)$, the claim follows. $\qquad \square$

In the next result and subsequently in this section, we will always denote by $\|f(t)\|_{C^{k,\alpha}}$ the parabolic $C^{k,\alpha}$ norm on $(t, t+1) \times M$. More precisely, for $\alpha \in (0,1)$, we define the seminorm

$$|f(t)|_{C^{0,\alpha}} = \sup_{\substack{(s_i, x_i) \in (t, t+1) \times M \\ (s_1, x_1) \neq (s_2, t_2)}} \frac{|f(s_1, x_1) - f(s_2, x_2)|}{\left(d_{g_\infty}(x_1, x_2)^2 + |t_1 - t_2| \right)^{\frac{\alpha}{2}}}$$

and for $k \geq 0$ and $\alpha \in (0, 1)$, we define the norm

$$(8) \quad \|f(t)\|_{C^{k,\alpha}} = \sum_{|\beta| + 2j \leq k} \sup_{(t, t+1) \times M} |D_x^\beta D_t^j f| + \sum_{|\beta| + 2j = k} |D_x^\beta D_t^j f|_{C^{0,\alpha}},$$

where the norm and derivatives in the sum are taken with respect to g_∞. When we mean an alternative norm, we will *always* indicate the domain.

The reason that we have chosen these norms is that they will be needed to close the fixed-point iteration argument in Proposition 20; showing that a certain map is a contraction map will use parabolic Schauder estimates (shown in Lemma 19), which require the chosen norms. Hence, we will use these norms throughout this section.

First, we prove a preliminary lemma, which allows us to estimate the difference between $D\mathcal{Y}(u)$ and the term $u^{2-N} D\mathcal{Y}(u)$ that appears on the right-hand side of the evolution equation (13) for the conformal factor under the Yamabe flow. This will allow us to reduce this evolution equation to a gradient flow. Intuitively, this is clear since u is approximately 1, but the point is to show that the difference $(u^{2-N} - 1)D\mathcal{Y}(u)$ is sufficiently small with respect to certain weighted norms that will be used in the proof of the contraction mapping argument in Section 4.4.

Lemma 16 *There exists $T_0 > 0$, $\epsilon_0 > 0$ and $c > 0$, all depending on g_∞ and $\hat{v}$, such that the following holds: Fix $T > T_0$. Then, for $\varphi(t)$ as in Lemma 15 and $w \in C^{2,\alpha}(M \times [0, \infty))$, defining $u := \Psi(\varphi + w^\top) + w^\perp$, the term*

$$E_0^\top(w) := \mathrm{proj}_{\Lambda_0}\big(D\mathcal{Y}(u)u^{2-N} - D\mathcal{Y}(u)\big)$$

satisfies

$$\|E_0^\top(w)\|_{C^{0,\alpha}} \le c\big((T+t)^{-1-\frac{1}{p-2}} + \|w^\top\|_{C^{0,\alpha}}^{p-1} + \|w^\perp\|_{C^{2,\alpha}}\big)\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big),$$

$$\begin{aligned}
\|E_0^\top(w_1) - E_0^\top(w_2)\|_{C^{0,\alpha}} \\
\le c\big((T+t)^{-1-\frac{1}{p-2}} + \|w_1^\top\|_{C^{0,\alpha}}^{p-1} + \|w_2^\top\|_{C^{0,\alpha}}^{p-1} + \|w_1^\perp\|_{C^{2,\alpha}} + \|w_2^\perp\|_{C^{2,\alpha}}\big) \\
\times \|w_1 - w_2\|_{C^{2,\alpha}} \\
+ c\big((T+t)^{-\frac{1}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\big(\|w_1^\top\|_{C^{0,\alpha}}^{p-2} + \|w_2^\top\|_{C^{0,\alpha}}^{p-2}\big) \\
\times \|w_1^\top - w_2^\top\|_{C^{0,\alpha}} \\
+ c\big((T+t)^{-\frac{1}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\|w_1^\perp - w_2^\perp\|_{C^{2,\alpha}}.
\end{aligned}$$

Identical estimates hold for $E_0^\perp(w) := \mathrm{proj}_{\Lambda_0^\perp}(D\mathcal{Y}(u)u^{2-N} - D\mathcal{Y}(u))$. Here, we are using the parabolic Hölder norms on $(t, t+1) \times M$ as defined above; the bounds hold for each fixed $t \ge 0$, with the constants independent of T and t.

Proof First,

$$u^{2-N} = 1 + \int_0^1 \frac{d}{ds}\Big[\big(1 + s\underbrace{(\varphi + w^\top + \Phi(\varphi + w^\top) + w^\perp)}_{:=\phi}\big)^{2-N}\Big]\,ds$$

$$= 1 + (2-N)\int_0^1 (1 + s\phi)^{1-N}\phi\,ds.$$

So, letting $E_0(w) := D\mathcal{Y}(u)u^{2-N} - D\mathcal{Y}(u)$, we have that

$$(9) \quad \|E_0(w)\|_{C^{0,\alpha}}$$
$$\leq c\|D\mathcal{Y}(u)\|_{C^{0,\alpha}}\big(\|\varphi\|_{C^{0,\alpha}} + \|w^\top\|_{C^{0,\alpha}} + \|\Phi(\varphi + w^\top)\|_{C^{0,\alpha}} + \|w^\perp\|_{C^{0,\alpha}}\big)$$
$$\leq c\|D\mathcal{Y}(u)\|_{C^{0,\alpha}}\big((T+t)^{-\frac{1}{p-2}} + \|w^\top\|_{C^{0,\alpha}} + \|w^\perp\|_{C^{0,\alpha}}\big).$$

We have used the fact that $\Phi(0) = 0$ and $\Phi \colon \Lambda_0 \to C^{2,\alpha}(M, g_\infty)$ is a differentiable map. Taylor's theorem shows that for $\psi_{s,r} := 1 + r(\varphi + w^\top + \Phi(\varphi + w^\top) + sw^\perp)$,

$$D\mathcal{Y}(u) = D\mathcal{Y}(\Psi(\varphi + w^\top)) + \int_0^1 D^2\mathcal{Y}(\psi_{s,1})[w^\perp]\,ds$$
$$= D\mathcal{Y}(\Psi(\varphi + w^\top)) - 2(N-2)\mathcal{L}_\infty w^\perp$$
$$+ \int_0^1 \int_0^s D^3\mathcal{Y}(\psi_{s,\tilde{s}})[w^\perp, \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\perp]\,d\tilde{s}\,ds$$
$$= \mathrm{proj}_{\Lambda_0} D\mathcal{Y}(\Psi(\varphi + w^\top)) - 2(N-2)\mathcal{L}_\infty w^\perp$$
$$+ \int_0^1 \int_0^s D^3\mathcal{Y}(\psi_{s,\tilde{s}})[w^\perp, \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\perp]\,d\tilde{s}\,ds$$
$$= DF(\varphi + w^\top) - 2(N-2)\mathcal{L}_\infty w^\perp$$
$$+ \int_0^1 \int_0^s D^3\mathcal{Y}(\psi_{s,\tilde{s}})[w^\perp, \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\perp]\,d\tilde{s}\,ds.$$

In the last line, we used the bound (27) on $D^3\mathcal{Y}$ discussed in Appendix A. Now, observe that $DF(0) = D^2F(0) = \cdots = D^{p-1}F(0) = 0$, by definition of p, the order of integrability. Therefore, Taylor's theorem shows that

$$\|DF(\varphi + w^\top)\|_{C^{0,\alpha}} \leq c\|\varphi + w^\perp\|_{C^{0,\alpha}}^{p-1} \leq c\big((T+t)^{-1-\frac{1}{p-2}} + \|w^\top\|_{C^{0,\alpha}}^{p-1}\big).$$

Bounding the other two terms in the above expression for $D\mathcal{Y}(u)$ (using again the bound (27) on $D^3\mathcal{Y}$), we have that

$$(10) \quad \|D\mathcal{Y}(u)\|_{C^{0,\alpha}} \leq c\big((T+t)^{-1-\frac{1}{p-2}} + \|w^\top\|_{C^{0,\alpha}}^{p-1} + \|w^\perp\|_{C^{2,\alpha}}\big).$$

We define

$$E_0^\top(w) := \mathrm{proj}_{\Lambda_0} E_0(w), \quad E_0^\perp(w) := \mathrm{proj}_{\Lambda_0^\perp} E_0(w).$$

The asserted bounds for $E_0^\top(w)$ follow from the bound (9) on $E_0(w)$, the estimate (10) and the continuity of the map $\mathrm{proj}_{\Lambda_0} \colon C^{0,\alpha}(M, g_\infty) \to \Lambda_0$,

$$\|\mathrm{proj}_{\Lambda_0} f\|_{C^{0,\alpha}(M,g_\infty)} \leq c\|\mathrm{proj}_{\Lambda_0} f\|_{L^2(M,g_\infty)}$$
$$\leq c\|f\|_{L^2(M,g_\infty)} \leq c\|f\|_{C^{0,\alpha}(M,g_\infty)},$$

where the first inequality follows because of the finite dimensionality of Λ_0. Note that this is a spatial bound, so it does not include the t–Hölder norm, but the desired space-time norm bound follows easily from it: if f is time-dependent,

$$
\begin{aligned}
\|(\mathrm{proj}_{\Lambda_0} f)(t_1) - (\mathrm{proj}_{\Lambda_0} f)(t_2)\|_{C^{0,\alpha}(M,g_\infty)} &= \|\mathrm{proj}_{\Lambda_0}(f(t_1) - f(t_2))\|_{C^{0,\alpha}(M,g_\infty)} \\
&\leq c\|f(t_1) - f(t_2)\|_{C^{0,\alpha}(M,g_\infty)}.
\end{aligned}
$$

Dividing by $|t_1 - t_2|^{\alpha/2}$ and taking the supremum over all such $t_1, t_2 \in (t, t+1)$, the asserted bound follows. The bound for $E_0^\top(w_1) - E_0^\top(w_2)$ follows similarly. This, combined with the bound (9) on $E_0(w)$ and the estimate (10), implies by the triangle inequality that identical estimates hold also for $E_0^\perp(w)$. $\qquad\square$

The next result reduces the Yamabe flow to two flows, one on Λ_0 and the other on $\Lambda_0^\perp$.

Proposition 17 *There exists $T_0 > 0$, $\epsilon_0 > 0$ and $c > 0$, all depending on g_∞ and $\hat{v}$, such that the following holds: Fix $T > T_0$. Then, for $\varphi(t)$ as in Lemma 15 and $w \in C^{2,\alpha}(M \times [0,\infty))$, there are functions $E^\top(w)$ and $E^\perp(w)$ such that $u := \Psi(\varphi + w^\top) + w^\perp$ is a solution to the Yamabe flow if and only if*

$$
(11) \qquad\qquad 2(N-2)(w^\top)' + D^2 F_p(\varphi)w^\top = E^\top(w),
$$

$$
(12) \qquad\qquad (w^\perp)' - \mathcal{L}_\infty w^\perp = E^\perp(w).
$$

Here, as long as $\|w\|_{C^{2,\alpha}} \leq \epsilon_0$, the error terms $E^\top$ and $E^\perp$ satisfy

$$
\begin{aligned}
\|E^\top&(w)\|_{C^{0,\alpha}} \\
&\leq c\big((T+t)^{-1-\frac{1}{p-2}} + \|w^\top\|_{C^{0,\alpha}}^{p-1} + \|w^\perp\|_{C^{2,\alpha}}\big)\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big) \\
&\quad + c(T+t)^{-\frac{p}{p-2}} + c(T+t)^{-\frac{p-1}{p-2}}\|w^\top\|_{C^{0,\alpha}} + c(T+t)^{-\frac{p-3}{p-2}}\|w^\top\|_{C^{0,\alpha}}^2 \\
&\quad + c\|w^\top\|_{C^{0,\alpha}}^{p-1} + c\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big)\|w^\perp\|_{C^{2,\alpha}},
\end{aligned}
$$

$$
\begin{aligned}
\|E^\top&(w_1) - E^\top(w_2)\|_{C^{0,\alpha}} \\
&\leq c\big((T+t)^{-1-\frac{1}{p-2}} + \|w_1^\top\|_{C^{0,\alpha}}^{p-1} + \|w_2^\top\|_{C^{0,\alpha}}^{p-1} + \|w_1^\perp\|_{C^{2,\alpha}} + \|w_2^\perp\|_{C^{2,\alpha}}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \|w_1 - w_2\|_{C^{2,\alpha}} \\
&\quad + c\big((T+t)^{-\frac{1}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\big(\|w_1^\top\|_{C^{0,\alpha}}^{p-2} + \|w_2^\top\|_{C^{0,\alpha}}^{p-2}\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times \|w_1^\top - w_2^\top\|_{C^{0,\alpha}} \\
&\quad + c\big((T+t)^{-\frac{1}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\|w_1^\perp - w_2^\perp\|_{C^{2,\alpha}}
\end{aligned}
$$

$$+ c\big((T+t)^{-\frac{p-3}{p-2}}(\|w_1^\top\|_{C^{0,\alpha}} + \|w_2^\top\|_{C^{0,\alpha}}) + \|w_1^\top\|_{C^{0,\alpha}}^{p-2} + \|w_2^\top\|_{C^{0,\alpha}}^{p-2}\big)$$
$$\times \|w_1^\top - w_2^\top\|_{C^{0,\alpha}}$$
$$+ c(T+t)^{-\frac{p-1}{p-2}}\|w_1^\top - w_2^\top\|_{C^{0,\alpha}},$$

$$\|E^\perp(w)\|_{C^{0,\alpha}}$$
$$\leq c\big((T+t)^{-1-\frac{1}{p-2}} + \|w^\top\|_{C^{0,\alpha}}^{p-1} + \|w^\perp\|_{C^{2,\alpha}}\big)\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big)$$
$$+ c\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big)\|w^\perp\|_{C^{2,\alpha}}$$
$$+ c\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big)\big((T+t)^{-1-\frac{1}{p-2}} + \|w'\|_{C^{0,\alpha}}\big),$$

$$\|E^\perp(w_1) - E^\perp(w_2)\|_{C^{0,\alpha}}$$
$$\leq c\big((T+t)^{-1-\frac{1}{p-2}} + \|w_1^\top\|_{C^{0,\alpha}}^{p-1} + \|w_2^\top\|_{C^{0,\alpha}}^{p-1} + \|w_1^\perp\|_{C^{2,\alpha}} + \|w_2^\perp\|_{C^{2,\alpha}}\big)$$
$$\times \|w_1 - w_2\|_{C^{2,\alpha}}$$
$$+ c\big((T+t)^{-\frac{1}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\big(\|w_1^\top\|_{C^{0,\alpha}}^{p-2} + \|w_2^\top\|_{C^{0,\alpha}}^{p-2}\big)$$
$$\times \|w_1^\top - w_2^\top\|_{C^{0,\alpha}}$$
$$+ c\big((T+t)^{-\frac{1}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\|w_1^\perp - w_2^\perp\|_{C^{2,\alpha}}$$
$$+ c\big((T+t)^{-\frac{1}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\|w_1' - w_2'\|_{C^{0,\alpha}}$$
$$+ c\big((T+t)^{-1-\frac{1}{p-2}} + \|w_1'\|_{C^{0,\alpha}} + \|w_2'\|_{C^{0,\alpha}}\big)\|w_1 - w_2\|_{C^{2,\alpha}}.$$

Here we are using the parabolic Hölder norms on $(t, t+1) \times M$ as defined above; the bounds hold for each fixed $t \geq 0$, with the constants independent of T and t.

Proof Recall that u is a solution to the Yamabe flow if and only if

$$(13) \quad (N-2)\frac{\partial u}{\partial t} = -\tfrac{1}{2}D\mathcal{Y}(u)u^{2-N} = (N+2)u^{2-N}\Delta_\infty u - R_\infty u^{3-N} + r_u N^{-2} g_\infty u,$$

where, as always in this article, $\mathcal{Y}$ is defined on the unit volume conformal class and $D(\cdot)$ is the corresponding constrained differential. We now project the Yamabe flow equation onto Λ_0 and $\Lambda_0^\perp$, so u solves the Yamabe flow if and only if the following two equations are satisfied:

$$2(N-2)(\varphi + w^\top)'$$
$$= -\operatorname{proj}_{\Lambda_0}\big[D\mathcal{Y}\big(1 + \varphi + w^\top + \Phi(\varphi + w^\top) + w^\perp\big)u(t)^{2-N}\big] + E_0^\top(w),$$
$$2(N-2)\big(\Phi(\varphi + w^\top) + w^\perp\big)'$$
$$= -\operatorname{proj}_{\Lambda_0^\perp}\big[D\mathcal{Y}\big(1 + \varphi + w^\top + \Phi(\varphi + w^\top) + w^\perp\big)u(t)^{2-N}\big] + E_0^\perp(w).$$

Here we emphasize that (as in the previous section) we are considering $D\mathcal{Y}(w)$ as a *function* on M, via the $L^2(M, g_\infty)$ pairing. In other words, we are using (2). Now we claim that we may use Taylor's theorem to show that

$$\operatorname{proj}_{\Lambda_0} D\mathcal{Y}\big(1 + \varphi + w^\top + \Phi(\varphi + w^\top) + w^\perp\big)$$
$$= \operatorname{proj}_{\Lambda_0} D\mathcal{Y}\big(1 + \varphi + w^\top + \Phi(\varphi + w^\top)\big) + E_1^\top(w),$$

with the bounds

$$\|E_1^\top(w)\|_{C^{0,\alpha}} \le c\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big)\|w^\perp\|_{C^{2,\alpha}},$$
$$\|E_1^\top(w_1) - E_1^\top(w_2)\|_{C^{0,\alpha}} \le c\big(\|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\|w_1^\perp - w_2^\perp\|_{C^{2,\alpha}}.$$

These follow from the integral form of the remainder in Taylor's theorem. Defining $\psi_{s,r} := 1 + r(\varphi + w^\top + \Phi(\varphi + w^\top) + sw^\perp)$, we have

$$E_1^\top(w) = \int_0^1 \frac{d}{ds} \operatorname{proj}_{\Lambda_0} D\mathcal{Y}\big(1 + \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\perp\big)\, ds$$

$$= \int_0^1 \operatorname{proj}_{\Lambda_0} \frac{d}{ds} D\mathcal{Y}(\psi_{s,1})\, ds$$

$$= \int_0^1 \operatorname{proj}_{\Lambda_0} D^2\mathcal{Y}(\psi_{s,1}^{N-2} g_\infty)[w^\perp]\, ds$$

$$= \int_0^1 \operatorname{proj}_{\Lambda_0} D^2\mathcal{Y}(g_\infty)[w^\perp]\, ds$$
$$\qquad + \int_0^1 \int_0^s \operatorname{proj}_{\Lambda_0} \frac{d}{d\tilde{s}} D^2\mathcal{Y}(\psi_{\tilde{s}}^{N-2} g_\infty)[w^\perp]\, d\tilde{s}\, ds$$

$$= -2(N-2) \int_0^1 \operatorname{proj}_{\Lambda_0} \mathcal{L}_\infty w^\perp\, ds$$
$$\qquad + \int_0^1 \int_0^s \operatorname{proj}_{\Lambda_0} \frac{d}{d\tilde{s}} D^2\mathcal{Y}(\psi_{s,\tilde{s}}^{N-2} g_\infty)[w^\perp]\, d\tilde{s}\, ds$$

$$= \int_0^1 \int_0^s \operatorname{proj}_{\Lambda_0} \frac{d}{d\tilde{s}} D^2\mathcal{Y}(\psi_{s,\tilde{s}}^{N-2} g_\infty)[w^\perp]\, d\tilde{s}\, ds$$

$$= \int_0^1 \int_0^s \operatorname{proj}_{\Lambda_0} D^3\mathcal{Y}(\psi_{s,\tilde{s}}^{N-2} g_\infty)[w^\perp, \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\perp]\, d\tilde{s}\, ds.$$

The $C^{0,\alpha}$ norm of $D^3\mathcal{Y}(\psi_{s,\tilde{s}}^{N-2} g_\infty)[w^\perp, \varphi + w^\top + \Phi(\varphi + w^\top) + sw^\perp]$ is uniformly bounded by

$$c\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big)\|w^\perp\|_{C^{2,\alpha}}$$

(as long as we choose $T \geq T_0$ large enough and $\|w\|_{C^{2,\epsilon}} \leq \epsilon_0$ small enough to ensure that $\psi_{s,\tilde{s}}$ is sufficiently close to 1 in $C^{2,\alpha}$). This is discussed at the end of Appendix A, along with the corresponding bound for $E_1^\top(w_1) - E_1^\top(w_2)$.

Recall that $F(v) := \mathcal{Y}(\Psi(v))$ and that

$$\mathrm{proj}_{\Lambda_0} D\mathcal{Y}\big(1 + \varphi + w^\top + \Phi(\varphi + w^\top)\big) = DF(\varphi + w^\top),$$

using the Lyapunov–Schmidt reduction (Proposition 7). Furthermore, by analyticity (Lemma 6 and Proposition 7) DF has a convergent power series representation around 0 with lowest-order term of order $p - 1$. Thus, as long as $\varphi + w^\top$ is small enough, we may write

$$DF(\varphi + w^\top) = DF(\varphi) + D^2 F(\varphi)(w^\top) + E_2^\top(w^\top),$$

where

$$\|E_2^\top(w^\top)\|_{C^{0,\alpha}} \leq c\big((T+t)^{-\frac{p-3}{p-2}} + \|w^\top\|_{C^{0,\alpha}}^{p-3}\big)\|w^\top\|_{C^{0,\alpha}}^2,$$

$$\|E_2^\top(w_1^\top) - E_2^\top(w_2^\top)\|_{C^{0,\alpha}}$$
$$\leq c\big((T+t)^{-\frac{p-3}{p-2}}(\|w_1^\top\|_{C^{0,\alpha}} + \|w_2^\top\|_{C^{0,\alpha}}) + \|w_1^\top\|_{C^{0,\alpha}}^{p-2} + \|w_2^\top\|_{C^{0,\alpha}}^{p-2}\big)$$
$$\times \|w_1^\top - w_2^\top\|_{C^{0,\alpha}}.$$

We prove this in the case that Λ_0 has dimension equal to one, namely for $k = 1$; the higher-dimensional case follows from a similar argument using multi-index notation. We have that

$$DF(z) = \sum_{j=p-1}^{\infty} c_j z^j.$$

Thus

$$\|E_2^\top(w^\top)\|_{C^{0,\alpha}} = \left\| \sum_{j=p-1}^{\infty} c_j \big[(\varphi + w^\top)^j - \varphi^j - j\varphi^{j-1} w^\top\big] \right\|_{C^{0,\alpha}}$$

$$= \left\| \sum_{j=p-1}^{\infty} \sum_{l=2}^{j} c_j \binom{j}{l} \varphi^{j-l} (w^\top)^l \right\|_{C^{0,\alpha}}$$

$$\leq \sum_{j=p-1}^{\infty} \sum_{l=2}^{j} |c_j| \binom{j}{l} \|\varphi\|_{C^{0,\alpha}}^{j-l} \|w^\top\|_{C^{0,\alpha}}^l$$

$$\leq \|w^\top\|_{C^{0,\alpha}}^2 \sum_{j=p-1}^{\infty} \sum_{l=2}^{j} |c_j| \binom{j}{l} \big(\|\varphi\|_{C^{0,\alpha}}^{j-2} + \|w^\top\|_{C^{0,\alpha}}^{j-2}\big)$$

$$\leq \|w^\top\|_{C^{0,\alpha}}^2 \sum_{j=p-1}^\infty |c_j| 2^j \left(\|\varphi\|_{C^{0,\alpha}}^{j-2} + \|w^\top\|_{C^{0,\alpha}}^{j-2} \right)$$

$$\leq 2^{p-1} \|\varphi\|_{C^{0,\alpha}}^{p-3} \|w^\top\|_{C^{0,\alpha}}^{p-1} \sum_{j=p-1}^\infty |c_j| 2^{j+1-p} \|\varphi\|_{C^{0,\alpha}}^{j+1-p}$$

$$+ 2^{p-1} \|w^\top\|_{C^{0,\alpha}}^{p-1} \sum_{j=p-1}^\infty |c_j| 2^{j+1-p} \|w^\top\|_{C^{0,\alpha}}^{j+1-p}.$$

Because $D^{p-1}F(z)$ has an absolutely convergent power series for every z sufficiently close to 0, choosing ϵ_0 small enough, T_0 large enough, and using Lemma 15, we may guarantee that both sums are bounded above by 1. The asserted bound on $E_2^\top(w^\top)$ follows. A similar argument yields the other bound.

These arguments show that the Λ_0–component of the Yamabe flow may be written as

$$2(N-2)(\varphi' + (w^\top)') = -DF(\varphi) - D^2 F(\varphi)(w^\top) + E_1^\top(w) - E_2^\top(w^\top).$$

Now, expanding F in a power series, $F = F(0) + \sum_{j=p}^\infty F_j$, we may write the above expression as

$$2(N-2)(\varphi' + (w^\top)') = -DF_p(\varphi) - D^2 F_p(\varphi)(w^\top) + \underbrace{E_1^\top(w) - E_2^\top(w^\top) + E_3^\top(w)}_{:=E^\top(w)},$$

where

$$E_3^\top(w) = \sum_{j\geq p+1} (DF_j(\varphi) + D^2 F_j(\varphi)w^\top).$$

By analyticity, this converges in eg $C^{0,\alpha}$ for $\|\varphi\|_{C^{2,\alpha}}$ and $\|w\|_{C^{2,\alpha}}$ small enough. Because each term in the sum is a homogeneous polynomial, we get the following error bound by using the formula for φ:

$$\|E_3^\top(w)\|_{C^{0,\alpha}} \leq c \left((T+t)^{-\frac{p}{p-2}} + (T+t)^{-\frac{p-1}{p-2}} \|w^\top\|_{C^{0,\alpha}} \right),$$

$$\|E_3^\top(w_1) - E_3^\top(w_2)\|_{C^{0,\alpha}} \leq c(T+t)^{-\frac{p-1}{p-2}} \|w_1^\top - w_2^\top\|_{C^{0,\alpha}}.$$

Therefore, by Lemma 15, $w^\top$ needs to satisfy the equation

$$2(N-2)(w^\top)' + D^2 F_p(\varphi)w^\top = E^\top(w),$$

where $E^\top(w)$ satisfies the asserted bounds.

Now we turn to the $\Lambda_0^\perp$ portion of the Yamabe flow. First, recall that by Proposition 7,

$$\text{proj}_{\Lambda_0^\perp} D\mathcal{Y}(\Psi(\varphi + w^\top)) = 0.$$

Combined with the fact that $D\text{proj}_{\Lambda_0^\perp} D\mathcal{Y}$ at 1 equals $-2(N-2)\text{proj}_{\Lambda_0^\perp} \mathcal{L}_\infty$ (this follows because D and $\text{proj}_{\Lambda_0^\perp}$ commute), we thus may use Taylor's theorem to write (using the fact that $\mathcal{L}_\infty$ is linear)

$$\text{proj}_{\Lambda_0^\perp} D\mathcal{Y}\big(\Psi(\varphi + w^\top) + w^\perp\big) = -2(N-2)\mathcal{L}_\infty w^\perp - E_1^\perp(w),$$

where

$$\|E_1^\perp(w)\|_{C^{0,\alpha}} \le c\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big)\|w^\perp\|_{C^{2,\alpha}},$$

$$\|E_1^\perp(w_1) - E_1^\perp(w_2)\|_{C^{0,\alpha}} \le c\big(\|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\|w_1^\perp - w_2^\perp\|_{C^{2,\alpha}}$$
$$+ c\big(\|w_1^\perp\|_{C^{2,\alpha}} + \|w_2^\perp\|_{C^{2,\alpha}}\big)\|w_1 - w_2\|_{C^{2,\alpha}}.$$

To check this, we write

$$\text{proj}_{\Lambda_0^\perp} D\mathcal{Y}\big(\Psi(\varphi + w^\top) + w^\perp\big)$$

$$= \text{proj}_{\Lambda_0^\perp} D\mathcal{Y}\big(\Psi(\varphi + w^\top)\big) + \int_0^1 \frac{d}{ds} \text{proj}_{\Lambda_0^\perp} D\mathcal{Y}\big(\Psi(\varphi + w^\top) + sw^\perp\big)\,ds$$

$$= \int_0^1 \text{proj}_{\Lambda_0^\perp} D^2\mathcal{Y}\big(\Psi(\varphi + w^\top) + sw^\perp\big)[w^\perp]\,ds$$

$$= -2(N-2)\mathcal{L}_\infty w^\perp$$
$$+ \int_0^1 \big[\text{proj}_{\Lambda_0^\perp} D^2\mathcal{Y}\big(\Psi(\varphi + w^\top) + sw^\perp\big)[w^\perp] - \text{proj}_{\Lambda_0^\perp} D^2\mathcal{Y}(1)[w^\perp]\big]\,ds.$$

Given this, we may control the asserted $C^{0,\alpha}$ bounds by the $C^{2,\alpha}$ norm of φ and w, by an argument similar to that used for $E_1^\top$. (The derivative of Ψ is uniformly bounded near 0 by Proposition 7.)

We also consider $\Phi(\varphi + w^\top)' := E_2^\perp(w)$ as an error term, as it satisfies

$$\|E_2^\perp(w)\|_{C^{0,\alpha}} \le c\big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big)\big((T+t)^{-1-\frac{1}{p-2}} + \|w'\|_{C^{0,\alpha}}\big),$$

$$\|E_2^\perp(w_1) - E_2^\perp(w_2)\|_{C^{0,\alpha}}$$

$$\le c\big((T+t)^{-\frac{1}{p-2}} + \|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)\|w_1' - w_2'\|_{C^{0,\alpha}}$$

$$+ c\big((T+t)^{-1-\frac{1}{p-2}} + \|w_1'\|_{C^{0,\alpha}} + \|w_2'\|_{C^{0,\alpha}}\big)\|w_1 - w_2\|_{C^{2,\alpha}}.$$

Here we have used (3) and have controlled $\|w\|_{L^2}$ by the $C^{2,\alpha}$ norm. Thus the kernel-orthogonal component of the Yamabe flow is

$$(w^\perp)' = \mathcal{L}_\infty w^\perp + E^\perp(w),$$

where $E^\perp(w)$ satisfies the asserted bounds. Combining the Λ_0 equation with the $\Lambda_0^\perp$ equation finishes the proof. $\square$

4.2 Solving the kernel-projected flow with polynomial decay estimates

In this subsection we solve the kernel-projected flow (11). First, from the definition of φ in (6) and the fact that $D^2 F_p$ is homogeneous of degree $p - 2$,

$$D^2 F_p(\varphi) = (T + t)^{-1} \left(\frac{2(N-2)}{p(p-2) F_p(\hat{v})} \right) D^2 F_p(\hat{v}).$$

Diagonalize the Hessian term, and denote by $\mu_1, \ldots, \mu_k$ the eigenvalues of

$$\frac{2(N-2)}{p(p-2) F_p(\hat{v})} D^2 F_p(\hat{v}).$$

Let e_i be an orthonormal basis in which this Hessian is diagonalized. Thus, the kernel-projected flow is equivalent to the following system of ODEs for[2] $v_i := w^\top \cdot e_i$.

(14) $$(N - 2) v_i' + \frac{\mu_i}{T + t} v_i = E_i^\top := E^\top \cdot e_i, \quad i = 1, \ldots, k.$$

Fix for the rest of this subsection a number γ fulfilling $\gamma \notin \left\{ \frac{\mu_1}{2(N-2)}, \ldots, \frac{\mu_k}{2(N-2)} \right\}$. Define the weighted norms

$$\|u\|_{C_\gamma^{0,\alpha}} := \sup_{t>0} \left[(T + t)^\gamma \|u(t)\|_{C^{0,\alpha}} \right] \quad \text{and} \quad \|u\|_{C_{1,\gamma}^{0,\alpha}} := \|u\|_{C_\gamma^{0,\alpha}} + \|u'\|_{C_{1+\gamma}^{0,\alpha}}.$$

We recall that these Hölder norms are space-time norms on the interval $(t, t + 1) \times M$, as defined in (8).

Given γ as above, let $\Pi_0 = \Pi_0(\gamma)$ denote the linear subspace of Λ_0 generated by the eigenvectors of $2(N - 2)/(p(p - 2) F_p(\hat{v})) D^2 F_p(\hat{v})$ whose eigenvalue, say μ, satisfies $\mu > 2(N-2)\gamma$. Moreover, let $\text{proj}_{\Pi_0} : \Lambda_0 \to \Pi_0$ be the corresponding linear projector.

The next lemma concerns the system (14).

Lemma 18 *For any $T > 0$ such that $\|E^\top\|_{C_{1+\gamma}^{0,\alpha}} < \infty$, there is a unique u with $u(t) \in \Lambda_0$, $t \in [0, \infty)$, satisfying $\|u\|_{C_\gamma^0} < \infty$, $\text{proj}_{\Pi_0}(u(0)) = 0$, and such that $v_i := u \cdot e_i$ solves the system (14). Furthermore, we have the bound*

$$\|u\|_{C_{1,\gamma}^{0,\alpha}} \leq C \|E^\top\|_{C_{1+\gamma}^{0,\alpha}}.$$

Proof Letting

$$w_j := (T + t)^{\frac{\mu_j}{2(N-2)}} v_j,$$

---

[2]Using, as above, the natural L^2 inner product on Λ_0 regarded as a subset of $T_1[g_\infty]_1$.

the system (14) is equivalent to

$$w_j' = \frac{1}{2(N-2)}(T+t)^{\frac{\mu_j}{2(N-2)}} E_j^\top, \quad j = 1, \dots, k.$$

Suppose that j is such that $\gamma > \mu_j/(2(N-2))$. Then, we claim that we may solve the jth ODE as

$$w_j(t) = \alpha_j - (2(N-2))^{-1} \int_t^\infty (T+\tau)^{\frac{\mu_j}{2(N-2)}} E_j^\top(\tau) \, d\tau,$$

which would give

$$u_j(t) = (T+t)^{-\frac{\mu_j}{2(N-2)}} \alpha_j - (2(N-2))^{-1}(T+t)^{-\frac{\mu_j}{2(N-2)}} \int_t^\infty (T+\tau)^{\frac{\mu_j}{2(N-2)}} E_j^\top(\tau) \, d\tau.$$

This amounts to checking that the integral converges under our assumptions on $E^\top$:

$$\left| (T+t)^{-\frac{\mu_j}{2(N-2)}} \int_t^\infty (T+\tau)^{\frac{\mu_j}{2(N-2)}} E_i^\top(\tau) \, d\tau \right|$$

$$\leq (T+t)^{-\frac{\mu_j}{2(N-2)}} \| E_j^\top \|_{C_{1+\gamma}^0} \int_t^\infty (T+\tau)^{\frac{\mu_i}{2(N-2)}-\gamma-1} \, d\tau$$

$$= \left(\gamma - \frac{\mu_j}{2(N-2)} \right)^{-1} (T+t)^{-\frac{\mu_j}{2(N-2)}} \| E_i^\top \|_{C_{1+\gamma}^0} (T+t)^{\frac{\mu_j}{2(N-2)}-\gamma}$$

$$= C_i(T+t)^{-\gamma} \| E_j^\top \|_{C_{1+\gamma}^0}.$$

The previous estimate also shows that, since by assumption $\gamma > \mu_j/(2(N-2))$, to have $\|u\|_{C_\gamma^{0,\alpha}} < \infty$, it must hold that $\alpha_j = 0$.

On the other hand, if $\gamma < \mu_j/(2(N-2))$, we may solve the ODE as

$$w_j(t) = \alpha_j + (2(N-2))^{-1} \int_0^t (T+\tau)^{\frac{\mu_j}{2(N-2)}} E_j^\top(\tau) \, d\tau.$$

By requiring $\text{proj}_{\Pi_0} u(0) = 0$, we see that $\alpha_j = 0$. As a result, the bounds for $\|u_j\|_{C_\gamma^0}$ follow from a similar calculation as before.

Combining these two cases proves existence, uniqueness and the $\|u\|_{C_\gamma^0}$ bound. It thus remains to prove the inequality $\|u\|_{C_{1,\gamma}^{0,\alpha}} \leq C \|E^\top\|_{C_{1+\gamma}^{0,\alpha}}$.

By finite dimensionality, the (spatial) $C^{0,\alpha}(M)$ Hölder norms of each basis element in Λ_0 are uniformly bounded. Thus, it remains to show that the desired inequality holds for the Hölder norm in the time direction, along with the same thing for $u'(t)$.

(The general space-time norm will then be bounded by the triangle inequality.) Suppose that j is such that $\gamma > \mu_j/(2(N-2))$. Then, we have seen above that

$$u_j(t) = -(2(N-2))^{-1}(T+t)^{-\frac{\mu_j}{2(N-2)}} \int_t^\infty (T+\tau)^{\frac{\mu_j}{2(N-2)}} E_j^\top(\tau)\, d\tau.$$

Notice that

$$u_j'(t) = \mu_j(T+t)^{-\frac{\mu_j}{2(N-2)}-1} \int_t^\infty (T+\tau)^{\frac{\mu_j}{2(N-2)}} E_j^\top(\tau)\, d\tau - (2(N-2))^{-1} E_j^\top(t).$$

Thus

$$
\begin{aligned}
\|u_j'\|_{C^{0,\alpha}} &\le C \left\| (T+t)^{-\frac{\mu_j}{2(N-2)}-1} \int_t^\infty (T+\tau)^{\frac{\mu_j}{2(N-2)}} E_j^\top(\tau)\, d\tau \right\|_{C^1} + C\|E_j^\top(t)\|_{C^{0,\alpha}} \\
&\le C \left\| (T+t)^{-\frac{\mu_j}{2(N-2)}-2} \int_t^\infty (T+\tau)^{\frac{\mu_j}{2(N-2)}} E_j^\top(\tau)\, d\tau \right\|_{C^0} + C\|E_j^\top(t)\|_{C^{0,\alpha}} \\
&\le C(T+t)^{-1-\gamma} \|E_j^\top\|_{C^{0,\alpha}_{1+\gamma}}.
\end{aligned}
$$

We may use the C^0 bound on u_j' to obtain a Hölder estimate for u_j. From this, the claimed inequality follows. □

4.3 Solving the kernel-orthogonal projected flow

Define the weighted norms

$$\|u\|_{L^2_q} = \sup_{t\in[0,\infty)} \left[(T+t)^q \|u(t)\|_{L^2(M)} \right],$$

where the L^2 norm is the spatial norm of $u(t)$ on M, taken with respect to g_∞, and

$$\|u\|_{C^{2,\alpha}_q} = \sup_{t\ge 0} \left[(T+t)^q \|u(t)\|_{C^{2,\alpha}} \right],$$

where, as usual, the Hölder norms are the space-time norms defined in (8). Also, let

$$
\begin{aligned}
\Lambda_\uparrow &:= \mathrm{span}\{\varphi \in C^\infty(M) : \mathcal{L}_\infty \varphi + \delta\varphi = 0, \delta < 0\}, \\
\Lambda_\downarrow &:= \overline{\mathrm{span}\{\varphi \in C^\infty(M) : \mathcal{L}_\infty \varphi + \delta\varphi = 0, \delta > 0\}}^{L^2}.
\end{aligned}
$$

From the spectral theory of the Laplacian, $L^2(M, g_\infty) = \Lambda_\uparrow \oplus \Lambda_0 \oplus \Lambda_\downarrow$ and $\Lambda_\uparrow$ and Λ_0 are finite-dimensional. Write the nonnegative integers as an ordered union $\mathbb{N} = K_\uparrow \cup K_0 \cup K_\downarrow$, where the ordering of the indices comes from an ordering of the eigenfunctions of the Laplacian Δ_{g_∞} and the partitioning of $\mathbb{N}$ corresponds to which of $\Lambda_\downarrow, \Lambda_0$ or $\Lambda_\uparrow$ the kth eigenfunction of Δ_{g_∞} lies in.

Lemma 19 For any $T > 0$ and $q < \infty$ such that $\|E^\perp\|_{L_q^2} < \infty$, there is a unique $u(t)$ with $u(t) \in \Lambda_0^\perp$, $t \in [0, \infty)$, satisfying $\|u\|_{L_q^2} < \infty$, $\mathrm{proj}_{\Lambda_\downarrow}(u(0)) = 0$ and

$$(15) \qquad\qquad u' = \mathcal{L}_\infty u + E^\perp.$$

Furthermore, $\|u\|_{L_q^2} \le C\|E^\perp\|_{L_q^2}$ and $\|u\|_{C_q^{2,\alpha}} \le C\|E^\perp\|_{C_q^{0,\alpha}}$.

Proof Recall that

$$(16) \qquad\qquad (N-2)\mathcal{L}_\infty = (N+2)\Delta_\infty + (N-2)R_\infty.$$

Let φ_i be an eigenfunction (with eigenvalue δ_i) of $\frac{1}{2}\mathcal{L}_\infty$ which is orthogonal to the kernel Λ_0. The flow equation (15) reduces to the system

$$(17) \qquad\qquad u_i' + \delta_i u_i = E_i^\perp,$$

where $E_i^\perp = \langle E^\perp, \varphi_i \rangle$ and $u_i = \langle u, \varphi_i \rangle$. This is equivalent to

$$(18) \qquad\qquad (e^{\delta_i t} u_i)' = e^{\delta_i t} E_i^\perp.$$

Thus, we may represent the components of the solution as

$$u_i^\perp(t) = \begin{cases} \beta_i e^{-\delta_i t} + e^{-\delta_i t} \int_0^t e^{\delta_i \tau} E_i^\perp(\tau)\, d\tau & \text{for } i \in K_\downarrow, \\[2mm] \beta_i e^{-\delta_i t} - e^{-\delta_i t} \int_t^\infty e^{\delta_i \tau} E_i^\perp(\tau)\, d\tau & \text{for } i \in K_\uparrow. \end{cases}$$

In particular, we have that

$$u(t) = \sum_{j \in K_\downarrow} \left(\beta_j e^{-\delta_j t} + e^{-\delta_j t} \int_0^t e^{\delta_j \tau} E_j^\perp(\tau)\, d\tau \right) \varphi_j$$
$$+ \sum_{j \in K_\uparrow} \left(\beta_j e^{-\delta_j t} - e^{-\delta_j t} \int_t^\infty e^{\delta_j \tau} E_j^\perp(\tau)\, d\tau \right) \varphi_j.$$

This sum is in an L^2 sense (but then elliptic regularity guarantees that the sum converges uniformly on compact time intervals). We note that for $i \in K_\uparrow$, if $\|u\|_{L_q^2} < \infty$ then necessarily $\beta_i = 0$. Furthermore, by requiring that $\mathrm{proj}_{\Lambda_\downarrow} u(0) = 0$, we also have $\beta_i = 0$ for $i \in K_\downarrow$.

We can write the following integral bounds for u:

$$
\left\| \sum_{j \in K_\downarrow} u_j(t)\varphi_j \right\|_{L^2}^2 \leq \sum_{j \in K_\downarrow} \left(\int_0^t e^{\delta_j(\tau-t)} E_j^\perp(\tau)\, d\tau \right)^2
$$

$$
\leq \sum_{j \in K_\downarrow} \left(\int_0^t e^{\delta_{\min}(\tau-t)} E_j^\perp(\tau)\, d\tau \right)^2
$$

$$
\leq \left\| \int_0^t e^{\delta_{\min}(\tau-t)} E^\perp(\tau)\, d\tau \right\|_{L^2}^2 ,
$$

where $\delta_{\min} = \min_{j \in K_\downarrow} \delta_j$ and in the last step we made use of the Parseval identity. Taking square roots gives

$$
\left\| \sum_{j \in K_\downarrow} u_j(t)\varphi_j \right\|_{L^2} \leq \left\| \int_0^t e^{\delta_{\min}(\tau-t)} E^\perp(\tau)\, d\tau \right\|_{L^2} \leq \int_0^t e^{\delta_{\min}(\tau-t)} \| E^\perp \|_{L^2}\, d\tau
$$

hence we can finally make use of our decay assumption on $E^\perp$ to get

$$
\left\| \sum_{j \in K_\downarrow} u_j(t)\varphi_j \right\|_{L^2} \leq \| E^\perp \|_q \int_0^t e^{\delta_{\min}(\tau-t)} (T+\tau)^{-q}\, d\tau.
$$

We bound the integral as

$$
\int_0^t e^{\delta_{\min}(\tau-t)} (T+\tau)^{-q}\, d\tau
$$

$$
= \int_0^{t/2} e^{\delta_{\min}(\tau-t)} (T+\tau)^{-q}\, d\tau + \int_{t/2}^t e^{\delta_{\min}(\tau-t)} (T+\tau)^{-q}\, d\tau
$$

$$
\leq T^{-q} \int_0^{t/2} e^{\delta_{\min}(\tau-t)}\, d\tau + (T+t/2)^{-q} \int_{t/2}^t e^{\delta_{\min}(\tau-t)}\, d\tau
$$

$$
\leq \delta_{\min}^{-1} T^{-q} (e^{-\delta_{\min}t/2} - e^{-\delta_{\min}t}) + \delta_{\min}^{-1}(T+t/2)^{-q}(1 - e^{-\delta_{\min}t/2}).
$$

From this we see that

$$
\left\| \sum_{j \in K_\downarrow} u_j(t)\varphi_j \right\|_{L^2} \leq C \| E^\perp \|_{L_q^2} (T+t)^{-q}.
$$

A similar argument holds for the $K_\uparrow$ terms. From this, the asserted bounds for $\|u\|_{L_q^2}$ follow readily.

The rest of the proof is devoted to showing that the $C_q^{2,\alpha}$ bounds follow from the L_q^2 bounds. By interior parabolic Schauder estimates [19, Theorem 4.9], we have that

for $t \geq 1$,

$$\|u(t)\|_{C^{2,\alpha}} \leq C\Big(\sup_{s \in (t-1,t+1) \times M} |u(s,x)| + \|E^\perp\|_{C^{0,\alpha}((t-1,t+1) \times M)} \Big).$$

We emphasize that the $C^{2,\alpha}$ norm on the left-hand side is the space-time norm on $(t, t+1) \times M$, as defined in (8).

We claim that for $\epsilon > 0$, there exists $c(\epsilon) > 0$ such that for any function $\varphi \in C^{0,\alpha}(M)$,

$$\sup_{x \in M} |\varphi(x)| \leq c(\epsilon)\|\varphi\|_{L^2(M)} + \epsilon\|\varphi\|_{C^{0,\alpha}(M)}.$$

This follows immediately from an argument by contradiction in conjunction with the Arzelà–Ascoli theorem. Using this in the Schauder estimate (bounding the supremum of the spatial $C^{0,\alpha}(M)$ norm over $t \in (t-1, t+1)$ by the space-time Hölder norm on $(t-1, t+1) \times M$) we get

$$\|u(t)\|_{C^{2,\alpha}} \leq C\Big(\sup_{s \in (t-1,t+1)} \|u(s,x)\|_{L^2(M)} + \|E^\perp\|_{C^{0,\alpha}((t-1,t+1) \times M)} \Big)$$
$$+ C\epsilon \|u(t)\|_{C^{0,\alpha}((t-1,t+1) \times M)}.$$

Multiplying by $(T+t)^q$ and taking the supremum over $t \geq 1$ yields

$$\sup_{t \geq 1} \big[(T+t)^q \|u(t)\|_{C^{2,\alpha}} \big]$$
$$\leq C\Big(\sup_{t \geq 0} \big[(T+t)^q \|u(s,x)\|_{L^2(M)} \big] + \sup_{t \geq 0} \big[(T+t)^q \|E^\perp\|_{C^{0,\alpha}((t,t+1) \times M)} \big] \Big)$$
$$+ C\epsilon \sup_{t \geq 0} \big[(T+t)^q \|u(t)\|_{C^{0,\alpha}((t-1,t+1) \times M)} \big]$$
$$= C(\|u\|_{L_q^2} + \|E^\perp\|_{C_q^{0,\alpha}}) + C\epsilon \|u\|_{C_q^{0,\alpha}}$$
$$\leq C(\|E^\perp\|_{L_q^2} + \|E^\perp\|_{C_q^{0,\alpha}}) + C\epsilon \|u\|_{C_q^{0,\alpha}}$$
$$\leq C\|E^\perp\|_{C_q^{0,\alpha}} + C\epsilon \|u\|_{C_q^{0,\alpha}}.$$

To finish the proof, it remains to extend the supremum up to $t = 0$, because then we may absorb the second term back into the left-hand side of the inequality by choosing ϵ sufficiently small. This may be achieved via global Schauder estimates [19, Theorem 4.28]:

$$\|u(t)\|_{C^{2,\alpha}((0,1) \times M)} \leq C\Big(\sup_{s \in (0,1)} \|u(s,x)\|_{L^2(M)} + \epsilon\|u\|_{C^{0,\alpha}((0,1) \times M)}$$
$$+ \|E^\perp\|_{C^{0,\alpha}((0,2) \times M)} + \|u(0)\|_{C^{2,\alpha}(M)} \Big)$$

Note that

$$u(0) = -\sum_{j \in K_\uparrow} \left(\int_0^\infty e^{\delta_j \tau} E_j^\perp(\tau)\, d\tau \right) \varphi_j.$$

The space $\Lambda_\uparrow$ is finite-dimensional, so there must be a uniform constant $C > 0$ such that $\|\varphi_j\|_{C^{2,\alpha}(M)} \leq C\|\varphi_j\|_{L^2(M)}$ for all $j \in K_\uparrow$. Using this we have that

$$\|u(0)\|_{C^{2,\alpha}(M)}^2 \leq C \sum_{j \in K_\uparrow} \left(\int_0^\infty e^{\delta_j \tau} E_j^\perp(\tau)\, d\tau \right)^2 \|\varphi_j\|_{C^{2,\alpha}(M)}^2$$

$$\leq C \sum_{j \in K_\uparrow} \left(\int_0^\infty e^{\delta_j \tau} E_j^\perp(\tau)\, d\tau \right)^2 \|\varphi_j\|_{L^2(M)}^2 = C \|u(0)\|_{L^2(M)}^2.$$

Using the L_q^2 bound obtained above, we may extend the supremum to $t \geq 0$ and get the desired Hölder bounds (absorbing, as commented before, the $C^{0,\alpha}$ norms of u into the left-hand side, by choosing ϵ small). $\qquad\square$

4.4 Construction of a slowly converging flow

To proceed further, we define the norm

$$\|f\|_\gamma^* := \|\mathrm{proj}_{\Lambda_0} f\|_{C_{1,\gamma}^{0,\alpha}} + \|\mathrm{proj}_{\Lambda_0^\perp} f\|_{C_{1+\gamma}^{2,\alpha}}.$$

Recall that

$$\|u\|_{C_{1,\gamma}^{0,\alpha}} = \sup_{t \geq 0} \left[(T+t)^\gamma \|u(t)\|_{C^{0,\alpha}} \right] + \sup_{t \geq 0} \left[(T+t)^{1+\gamma} \|u'(t)\|_{C^{0,\alpha}} \right],$$

$$\|u\|_{C_{1+\gamma}^{2,\alpha}} = \sup_{t \geq 0} \left[(T+t)^{1+\gamma} \|u(t)\|_{C^{2,\alpha}} \right],$$

where the Hölder norms are the space-time Hölder norms defined in (8). For γ to be specified below, we define X to be the Banach space of functions f with $\|f\|_\gamma^* < \infty$.

Proposition 20 *Assume that g_∞ satisfies AS_p. We may thus fix a point where $F_p|_{\mathbb{S}^{k-1}}$ achieves a positive maximum and denote it by $\hat{v}$. Define*

$$\varphi(t) = (T+t)^{-\frac{1}{p-2}} \left(\frac{2(N-2)}{p(p-2)F_p(\hat{v})} \right)^{\frac{1}{p-2}} \hat{v},$$

as in Lemma 15. Then, there exists $C > 0$, $T > 0$, $(p-2)^{-1} < \gamma < 2(p-2)^{-1}$ and $u(t) \in C^\infty(M \times (0,\infty))$ such that $u(t) > 0$ for all $t > 0$, $g(t) := u(t)^{N-2} g_\infty$ is a solution to the Yamabe flow and

$$\left\| w^\top(t) + \Phi(\varphi(t) + w^\top(t)) + w^\perp(t) \right\|_\gamma^* = \|u(t) - \varphi(t) - 1\|_\gamma^* \leq C.$$

Proof We fix $1/(p-2) < \gamma < 2/(p-2)$ so that $\gamma \notin \left\{ \dfrac{\mu_1}{2(N-2)}, \ldots, \dfrac{\mu_k}{2(N-2)} \right\}$. By Proposition 17, it is enough to solve

$$2(N-2)(w^\top)' + D^2 F_p(\varphi)w^\top = E^\top(w), \quad (w^\perp)' - \mathcal{L}_\infty w^\perp = E^\perp(w),$$

for $w(t)$ with $\|w\|_\gamma^* < C$. To do so, we will use the contraction mapping method. We define a map

$$S \colon \{w \in X : \|w\|_\gamma^* \le 1\} \to X = \{w : \|w\|_\gamma^* < \infty\},$$

by defining $u := \operatorname{proj}_{\Lambda_0} S(w)$ to be the solution of

$$2(N-2)u' + D^2 F_p(\varphi)u = E^\top(w)$$

and $v := \operatorname{proj}_{\Lambda_0^\perp} S(w)$ to be the solution of

$$v' - \mathcal{L}_\infty v = E^\perp(w).$$

Thus, we have defined the map $S(w)$ by its orthogonal projections onto Λ_0 and $\Lambda_0^\perp$. These solutions exist, in the right function spaces, by combining the bounds for the error terms in Proposition 17 with Lemmas 18 and 19. Furthermore, we have the explicit bound

$$\|\operatorname{proj}_{\Lambda_0} S(w)\|_{C_{1,\gamma}^{0,\alpha}} \le c \|E^\top(w)\|_{C_{1+\gamma}^{0,\alpha}}$$

$$\le c \sup_{t \ge 0} (T+t)^{1+\gamma} \big((T+t)^{-1 - \frac{1}{p-2}} + \|w^\top\|_{C^{0,\alpha}}^{p-1} + \|w^\perp\|_{C^{2,\alpha}} \big)$$
$$\times \big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}} \big)$$
$$+ c \sup_{t \ge 0} \big((T+t)^{\gamma - \frac{2}{p-2}} + (T+t)^{\gamma - \frac{1}{p-2}} \|w^\top\|_{C^{2,\alpha}} \big)$$
$$+ c \sup_{t \ge 0} \big((T+t)^{\gamma + \frac{1}{p-2}} \|w^\top\|_{C^{2,\alpha}}^2 + (T+t)^{1+\gamma} \|w^\top\|_{C^{2,\alpha}}^{p-1} \big)$$
$$+ c \sup_{t \ge 0} (T+t)^{1+\gamma} \|w^\perp\|_{C^{2,\alpha}}^2$$
$$\le c \big(T^{\gamma - \frac{2}{p-2}} + (T^{-\frac{1}{p-2}} + T^{(p-2)(\frac{1}{p-2} - \gamma)}) \|w\|_\gamma^* \big).$$

Here, we have absorbed powers of $(T+t)$ into the various w norms and bounded the result by $\|w\|_\gamma^*$. Note that the $w^\perp$ terms in $\|w\|_\gamma^*$ are multiplied by $(T+t)^{1+\gamma}$, but the $w^\top$ term in $\|w\|_\gamma^*$ is only multiplied by $(T+t)^\gamma$, so we cannot absorb as high of a power of $(T+t)$ into it. (Fortunately, the $w^\top$ terms are all raised to a large power, or already multiplied by an appropriately decaying power of $(T+t)$, as is

easily checked.) In the last step, we have used the bound

$$\|w^\top\|_{C^{2,\alpha}((t,t+1)\times M)} \le c\big(\|w^\top\|_{C^{0,\alpha}((t,t+1)\times M)} + \|(w^\top)'\|_{C^{0,\alpha}((t,t+1)\times M)}\big).$$

This is a consequence of the fact that Λ_0 is finite-dimensional (so any two norms on it are uniformly equivalent)[3] and that the parabolic $C^{2,\alpha}$ Hölder norms only contain at most one time derivative (which does not come paired with any spatial derivatives). Similarly,

$$\begin{aligned}
\|\mathrm{proj}_{\Lambda_0^\perp} S(w)\|_{C^{2,\alpha}_{1+\gamma}} &\le \|E^\perp(w)\|_{C^{0,\alpha}_{1+\gamma}} \\
&\le c \sup_{t\ge 0}\Big[(T+t)^{1+\gamma}\big((T+t)^{-1-\frac{1}{p-2}} + \|w^\top\|_{C^{0,\alpha}}^{p-1} + \|w^\perp\|_{C^{2,\alpha}}\big) \\
&\qquad\qquad\qquad\qquad \times \big((T+t)^{-\frac{1}{p-2}} + \|w\|_{C^{2,\alpha}}\big)\Big] \\
&\quad + c\sup_{t\ge 0}\Big[\big((T+t)^{1+\gamma-\frac{1}{p-2}} + (T+t)^{1+\gamma}\|w\|_{C^{2,\alpha}}\big)\|w^\perp\|_{C^{2,\alpha}}\Big] \\
&\quad + c\sup_{t\ge 0}\Big[\big((T+t)^{1+\gamma-\frac{1}{p-2}} + (T+t)^{1+\gamma}\|w\|_{C^{2,\alpha}}\big) \\
&\qquad\qquad\qquad\qquad \times \big((T+t)^{-1-\frac{1}{p-2}} + \|w'\|_{C^{0,\alpha}}\big)\Big] \\
&\le c\big(T^{\gamma-\frac{2}{p-2}} + (T^{-\frac{1}{p-2}} + T^{(p-2)(\frac{1}{p-2}-\gamma)})\|w\|_\gamma^*\big).
\end{aligned}$$

Thus, because $\gamma \in \big(1/(p-2), 2/(p-2)\big)$, by choosing T large enough we can ensure that S maps $\{w : \|w\|_\gamma^* \le 1\} \subset X$ into itself. Finally, we check that we can guarantee that S is a contraction mapping by taking T even larger if necessary. The following inequalities are proven by the same argument we have just used:

$$\|\mathrm{proj}_{\Lambda_0} S(w_1) - \mathrm{proj}_{\Lambda_0} S(w_2)\|_{C^0_{1,\gamma}} \le c(T^{-\frac{1}{p-2}} + T^{(p-2)(\frac{1}{p-2}-\gamma)})\|w_1 - w_2\|_\gamma^*,$$

$$\|\mathrm{proj}_{\Lambda_0^\perp} S(w_1) - \mathrm{proj}_{\Lambda_0^\perp} S(w_2)\|_{C^{2,\alpha}_{1+\gamma}} \le c(T^{-\frac{1}{p-2}} + T^{(p-2)(\frac{1}{p-2}-\gamma)})\|w_1 - w_2\|_\gamma^*.$$

Thus, by enlarging T if necessary, we have that S is a contraction map. This finishes the proof. $\qquad\square$

We now show how the previous proposition yields solutions converging at exactly a polynomial rate.

Proof of Theorem 2 From Proposition 17, we have constructed $\varphi(t)$ and $u(t)$ so that

$$\varphi(t) = (T+t)^{-\frac{1}{p-2}}\Big(\frac{2(N-2)}{p(p-2)F_p(\hat{v})}\Big)^{\frac{1}{p-2}}\hat{v},$$

---

[3]We note that the *space-time* $C^{k,\alpha}$ norms on Λ_0 are *not* all uniformly equivalent because time-dependence of the functions turns the space into an infinite-dimensional vector space. In our inequality, we used that *spatial* $C^{k,\alpha}$ norms of any element in Λ_0 are all equivalent to any other spatial $C^{k',\alpha'}$ norm.

$u(t)^{N-2} g_\infty$ is a solution to the Yamabe flow, and

$$u(t) = 1 + \varphi(t) + \widetilde{w}(t) := 1 + \varphi(t) + w^\top(t) + \Phi(\varphi(t) + w^\top(t)) + w^\perp(t),$$

where $\widetilde{w}(t)$ satisfies (in particular) $\|\widetilde{w}\|_{C^0} \leq C(1+t)^{-\gamma}$ for some $C > 0$ and all $t \geq 0$. We have arranged that $\gamma > 1/(p-2)$, which implies that $\varphi(t)$ is decaying slower than $\widetilde{w}(t)$. Thus

$$\|u(t) - 1\|_{C^0} \geq C(1+t)^{-\frac{1}{p-2}}$$

as $t \to \infty$. From this, the assertion follows.   □

5 Examples satisfying AS_p

In this section we provide explicit examples of metrics which satisfy AS_p for both $p = 3$ and $p \geq 4$. This allows us, via Theorem 2, to conclude the existence of slowly converging Yamabe flows.

5.1 Examples which satisfy AS_3

In this subsection we prove Proposition 3. Suppose that we are given integers $n, m > 1$ and a closed m–dimensional Riemannian manifold (M^m, g_M) with constant scalar curvature $R_{g_M} \equiv 4(n+1)(m+n-1)$. We denote the complex projective space equipped with the Fubini–Study metric by $(\mathbb{P}^n, g_{FS})$. (We normalize the Fubini–Study metric so the map $\mathbb{S}^{2n+1} \to \mathbb{P}^n$ from the unit sphere is a submersion.) We will show that the product metric $(M^m \times \mathbb{P}^n, g_M \oplus g_{FS})$ is a degenerate critical point satisfying AS_3. Recall that this implies that the metric is nonintegrable by Lemma 9.

Write $g := g_M \oplus g_{FS}$. Because $R_{g_{FS}} = 4n(n+1)$ [22, page 86], the scalar curvature of g satisfies $R_g = 4(n+1)(m+n-1) + 4n(n+1) = 4(n+1)(m+2n-1)$. The dimension of $M^m \times \mathbb{P}^n$ is $m + 2n$, so Λ_0 consists of eigenfunctions of Δ_g with eigenvalue $R_g/(m+2n-1) = 4(n+1)$. Because $\lambda_1(g_{FS}) = 4(n+1)$ [3, Proposition C.III.1], we see that $(M^m \times \mathbb{P}^n, g)$ is degenerate; for any first eigenfunction v on $\mathbb{P}^n$, the function $1 \otimes v$ on $M^m \times \mathbb{P}^n$ will be an eigenfunction of Δ_g with eigenvalue $4(n+1)$.

The eigenfunctions of $\Delta_{g_{FS}}$ may be explicitly constructed by considering polynomials on $\mathbb{C}^n$ which are homogeneous of degree k in both z and $\bar{z}$ and which are harmonic. These polynomials restrict to $\mathbb{S}^{2n+1}$ and are invariant under the natural $\mathbb{S}^1$ action, so they descend to the quotient. This is described in detail in [3, Proposition C.III.1]. By a recent observation of Kröncke [17, page 25], the harmonic polynomial $h(z, \bar{z}) := z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_3 + z_3\bar{z}_2 + z_3\bar{z}_1 + z_1\bar{z}_3$, defined on $\mathbb{C}^{n+1}$ for $n \geq 2$, descends to a first eigenfunction v of $\Delta_{g_{FS}}$ for which $\int_{\mathbb{P}^n} v^3 \, dV_{g_{FS}} \neq 0$. The function $1 \otimes v$ is an

eigenfunction of Δ_g with eigenvalue $4(n+1)$, so it is an element of Λ_0. Moreover, by Fubini's theorem, $\int_{M^m \times \mathbb{P}^n} (1 \otimes v)^3 \, dV_g = \text{Vol}(M^m, g_M) \int_{\mathbb{P}^n} v^3 \, dV_{g_{FS}} \neq 0$. Thus we see that $(M^m \times \mathbb{P}^n, g)$ is degenerate and by (4), the function F_3 is not everywhere zero on Λ_0. This shows that $(M^m \times \mathbb{P}^n, g)$ satisfies AS_3, as claimed.

5.2 Examples satisfying AS_p for $p \geq 4$

This subsection is devoted to the detailed study of the Yamabe problem on $\mathbb{S}^1(R) \times \mathbb{S}^{n-1}$. Our goal is to obtain examples in any dimension $n \geq 3$ of a nonintegrable critical point of $\mathcal{Y}$ which satisfies the condition AS_p for some $p \geq 4$, as defined in Definition 10. The study involves properties of a certain *period function* $\tau(\alpha)$ defined below. Here we start by giving an overview of Schoen's discussion [25], supplying detailed proofs. The main new observation is that these facts imply the existence of a constant scalar curvature metric satisfying the assumptions of Theorem 2. We observe that the same ODE which we analyze has been considered from a different perspective in [21], where the authors analyze moduli spaces of singular Yamabe metrics.

5.2.1 An ODE parametrizing all solutions of the Yamabe problem We consider a one-parameter family of conformal classes $[g_T]$ on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ represented by the natural product metric $\mathbb{S}^1(T/2\pi) \times \mathbb{S}^{n-1}(1)$. (Here $\mathbb{S}^k(r)$ is the k–sphere of radius r in $\mathbb{R}^{k+1}$.) We will write t for the coordinate on $\mathbb{S}^1(T/2\pi)$.

Proposition 21 [25] *Let $u_0 = ((n-2)/n)^{\frac{1}{4}(n-2)} = (2/N)^{1/(N-2)}$. Then there exists a map $\tau \colon (u_0, 1) \to \mathbb{R}_{>0}$ which parametrizes solutions to the Yamabe problem on $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ in the following sense: For a given $T > 0$, up to scaling the conformal factor, the complete list of constant scalar curvature metrics in $[g_T]$ is (i) the product metric and (ii) a metric of the form $u(t)^{N-2} g_T$, where $u(t)$ solves the ODE*

$$4u'' - (n-2)^2 u + n(n-2) u^{\frac{n+2}{n-2}} = 0,$$

with initial conditions $(u(t_0), u'(t_0)) = (\alpha, 0)$ for some $t_0 \in \mathbb{S}^1(T/2\pi)$. Here $\alpha \in (u_0, 1)$ is any solution of $\tau(\alpha) = T/k$, with k an arbitrary positive integer.

Proof We will follow Schoen and look for solutions to the Yamabe problem with constant scalar curvature $n(n-1)$ (equal to that of the unit sphere), and in doing so we drop the volume constraint. A crucial observation is that by a result of Caffarelli, Gidas and Spruck (following the classical work of Gidas, Ni and Nirenberg), a constant scalar curvature metric in $[g_T]$ must have conformal class only depending on the $\mathbb{S}^1$ variable t; see [11; 16]. This reduces the problem to studying an ODE rather than a PDE.

It will be convenient to lift the analysis to the universal cover $\mathbb{R} \times \mathbb{S}^{n-1}$ and use $(t, \xi) \in \mathbb{R} \times \mathbb{R}^n$ with $|\xi| = 1$ as coordinates. In particular, we will forget about g_T for now and consider instead the metric $g = dt^2 + g_{\mathbb{S}^{n-1}(1)}$ on $\mathbb{R} \times \mathbb{S}^{n-1}$. Then a solution to the Yamabe problem in $[g_T]$ will correspond to a function $u(t)$ on $\mathbb{R} \times \mathbb{S}^{n-1}$ (depending only on the first factor) with period T in t and for which $u(t)^{4/(n-2)} g$ has constant scalar curvature $n(n-1)$.

Now, $u(t)^{4/(n-2)} g$ having constant scalar curvature $n(n-1)$ is equivalent to the ODE

$$(19) \qquad 4u'' - (n-2)^2 u + n(n-2) u^{\frac{n+2}{n-2}} = 0$$

as $R_g = R_{g_{\mathbb{S}^{n-1}(1)}} = (n-1)(n-2)$ and

$$n(n-1) = R_{u^{N-2}g} = -(N+2) u^{-\frac{n+2}{n-2}} \left(u'' - \frac{1}{N+2} R_g u \right).$$

There is an obvious solution to (19) given by the constant $u(t) \equiv u_0 = ((n-2)/n)^{\frac{1}{4}(n-2)}$. This simply corresponds to the rescaling of g_T so that it has scalar curvature $n(n-1)$, as desired.

There is a second explicit solution to (19) obtained by considering $\mathbb{R} \times \mathbb{S}^{n-1}$ as the coordinate patch of $\mathbb{S}^n$ given by $\mathbb{S}^n - \{N, S\}$, the sphere minus two antipodal points. The restriction of the standard metric on $\mathbb{S}^n$ to $\mathbb{S}^n - \{N, S\}$, which has scalar curvature $n(n-1)$, then produces a solution to (19) as long as we can check that this metric is conformally related to g. To see this, notice that the map $\Psi \colon (\mathbb{R} \times \mathbb{S}^{n-1}, g) \to (\mathbb{R}^n - \{0\}, g_{\text{Euc}}), (t, \xi) \mapsto e^t \xi$ is conformal, because

$$\Psi^* g_{\text{Euc}} = \Psi^* (dr^2 + r^2 g_{\mathbb{S}^{n-1}}) = e^{2t} dt^2 + e^{2t} g_{\mathbb{S}^{n-1}} = e^{2t} g,$$

where g_{Euc} denotes the Euclidean metric on $\mathbb{R}^n \setminus \{0\}$. On the other hand, by stereographic projection the spherical coordinate patch on $\mathbb{R}^n \setminus \{0\}$ has the metric

$$g_{\mathbb{S}^n} = \frac{4 g_{\text{Euc}}}{(1+r^2)^2}, \quad \text{where } r = |x| \text{ with } x \in \mathbb{R}^n \setminus \{0\}.$$

Thus $\Psi^*(g_{\mathbb{S}^n}) = (4e^{2t}/(1 + e^{2t})^2) g = (\cosh t)^{-2} g$. Therefore we have another solution to (19) given by $u_1(t) = (\cosh t)^{-(n-2)/2}$. Of course, the metric $u_1(t)^{4/(n-2)} g$ does not descend to the quotient $\mathbb{S}^1(T/2\pi) \times \mathbb{S}^{n-1}$ (and it is not even a complete metric on $\mathbb{R} \times \mathbb{S}^{n-1}$) but it will prove useful in the sequel.

By setting $v = du/dt$, (19) can be converted to a first-order system

$$(20) \qquad \frac{d}{dt}(u, v) = X(u, v),$$

where the vector field X on the uv–plane is defined by

$$X(u, v) = \left(v, \frac{(n-2)^2 u - n(n-2) u^{\frac{n+2}{n-2}}}{4} \right).$$

We note that the second component of X is negative when $u > u_0$ and positive when $u < u_0$. From the above analysis, we know that

$$(21) \qquad (u(t), v(t)) = (u_1(t), u_1'(t)) = \left((\cosh t)^{-\frac{n-2}{2}}, \left(\frac{1}{4} - \frac{n}{8} \right) \frac{\sinh t}{(\cosh t)^{n/2}} \right)$$

is a solution to (20). Letting $t = 0$, we see that $(1, 0)$ is on this integral curve. Additionally, letting $t \to \pm\infty$ (note here that $n \geq 3$), we see that the curve tends to $(0, 0)$. Thus, the orbit associated to u_1 along with $(0, 0)$ encloses a region Ω with compact closure $\overline{\Omega}$, and such Ω is invariant under the flow since its boundary is a homoclinic cycle (ie a trajectory that limits to the same critical point at $t = \pm\infty$).

Claim 22 *Any periodic solution with $u > 0$ for all time must lie inside Ω.*

Proof By the previous comments, it suffices to consider a trajectory $\gamma(t) = (u(t), v(t))$ in $\mathbb{R}^2 \setminus \Omega$. Observe that

$$(22) \qquad\qquad 4v'(t) = n(n-2)u\left(u_0^{N-2} - u^{N-2}\right).$$

Thus, whenever $u(t) > u_0$ we have $v'(t) < 0$. We divide the proof into two cases.

Case 1 $u(0) > u_0$. In this case, since $\gamma(t)$ is defined globally for $t \in \mathbb{R}$ and is periodic, we claim that there exists O_0 such that $u(O_0) = u_0$ and $v'(t) < 0$ for $t \in (0, O_0)$. Indeed, if it were not the case, monotonicity of the second component of γ should immediately imply that the corresponding trajectory of $\gamma(\cdot)$ is unbounded, contradicting the periodicity assumption.

As a result, clearly $v(O_0) < 0$, hence the system implies $u'(O_0) < 0$ and thus it follows that $v(t) > v(O_0)$ for $t > O_0$. Since γ is global, it must remain in the right half-space (as $u > 0$) and by monotonicity of the second component it follows that γ must approach $(0, 0)$ as $t \nearrow \infty$. But then it is not periodic. (Note that in fact, by time-reversal symmetry, γ should be a homoclinic cycle.)

Case 2 $u(0) < u_0$. We can assume $v(0) > 0$, since otherwise the last part of the proof of Case 1 applies. Thus $v(0) > 0$, and therefore $u'(0) > 0$ and $v'(0) > 0$. It follows once again that γ crosses the vertical line $u = u_0$, and then we are in Case 1. $\qquad\square$

Claim 23 *Suppose that $\gamma_\alpha(t) = (u(t), v(t))$ solves (20) and that $\gamma_\alpha(0) = (\alpha, 0) \in \Omega$. Then either $\gamma_\alpha \equiv (u_0, 0)$ or γ_α is a smooth periodic orbit contained in $\Omega - \partial\Omega$. Furthermore, if $u_0 \leq \alpha_1 < \alpha_2$, then γ_{α_1} is enclosed by γ_{α_2}.*

Proof For the first statement, let us start by observing that (by compactness of $\overline{\Omega}$) any such solution $\gamma_\alpha(t)$ must be defined for all times and $\gamma_\alpha(t) = \gamma_\alpha(-t)$ for all $t \in \mathbb{R}$ by means of a standard ODE uniqueness argument (as $X(u, v) = X(u, -v)$). To proceed further, let us recall that the flow (20) is generated by the Hamiltonian (see [21, (2.3)])

$$(23) \qquad H(u, v) = 2v^2 + \frac{(n-2)^2(u^N - u^2)}{2}.$$

The corresponding conservation law (together with the fact that $H(u, 0) = 0$ implies $u \in \{0, 1\}$) rules out the existence of solutions γ_α such that $\lim_{t \to +\infty} \gamma_\alpha(t) = (0, 0)$ (and thus $\lim_{t \to -\infty} \gamma_\alpha(t) = (0, 0)$ as well) or $\lim_{t \to +\infty} \gamma_\alpha(t) = (1, 0)$ (and thus $\lim_{t \to -\infty} \gamma_\alpha(t) = (1, 0)$) whenever $\alpha \in (0, 1)$. Then the first claim follows from the fact that for $\alpha \in (0, 1)$, γ_α must intersect the u–axis (exactly) twice and away from the zeros of the vector field X. Then, by the time-reversal symmetry, we conclude that γ_α must be periodic, hence also smooth. These arguments show in particular that $\mathrm{Im}\,\gamma_\alpha$ for such α is diffeomorphic to $\mathbb{S}^1$. Uniqueness of solutions to ODEs then implies the last claim. $\square$

Finally, this allows us to conclude the general classification of constant scalar curvature metrics in $[g_T]$. Any constant curvature metric must depend only on the t variable and thus lift to a solution $u(t)$ to the ODE (19) on $\mathbb{R}$. The solution must be periodic of period T/k for some positive integer k, as the metric descends to $\mathbb{S}^1(T/2\pi) \times \mathbb{S}^{n-1}$. Thus, by the above claim, there exists $\alpha \in (u_0, 1)$ such that $u(t)$ solves the ODE with initial conditions $(\alpha, 0)$ (after possibly shifting $u(t)$ by a rotation of $\mathbb{S}^1$). Of course, if $k > 1$, then what we mean here is that the conformal factor on $\mathbb{S}^1(T/2\pi) \times \mathbb{S}^{n-1}$ is $u(t)$ concatenated k times. By definition $\tau(\alpha) = T/k$. This completes the proof of Proposition 21. $\square$

5.2.2 The period function

Lemma 24 [25] *The period function $\tau(\alpha)$ is continuous on the interval $(u_0, 1)$. Furthermore, it satisfies:*

(i) $\lim_{\alpha \nearrow 1} \tau(\alpha) = +\infty.$

(ii) $\lim_{\alpha \searrow u_0} \tau(\alpha) = (n-2)^{-1/2} 2\pi := T_0.$

Proof (i) Suppose that there is a sequence $\alpha_k \nearrow 1$ such that $\tau(\alpha_k) \leq C$ for some constant C. By possibly extracting a subsequence, we may assume $\lim_{k\to\infty} \tau(\alpha_k) = T_\infty < \infty$. Now consider the points $\gamma_{\alpha_k}(\frac{1}{2}T_\infty)$. By making use of the equation $H(\alpha_k, 0) = H(u_{\alpha_k}(\frac{1}{2}\tau(\alpha_k)), 0)$, we now claim that $u_{\alpha_k}(\frac{1}{2}\tau(\alpha_k)) \to 0$ as $k \to \infty$. Indeed, $u_{\alpha_k}(\frac{1}{2}\tau(\alpha_k)) \in [0, u_0]$, so we may assume that it converges to some value u_∞ by further extracting a subsequence. Thus, taking the limit of $H(u_{\alpha_k}(\frac{1}{2}\tau(\alpha_k)), 0) = H(\alpha_k, 0)$ as $k \to \infty$, we get that

$$(24) \qquad u_\infty^2(u_\infty^{N-2} - 1) = 0.$$

However, because $u_\infty \leq u_0$, the second factor must be negative, so $u_\infty = 0$. Thus, we see that $\gamma_{\alpha_k}(\frac{1}{2}T_\infty)$ must converge to $(0, 0)$. On the other hand, by continuous dependence of solutions to ODEs on their initial data, $\gamma_{\alpha_k}(\frac{1}{2}T_\infty)$ must converge to $\gamma_1(\frac{1}{2}T_\infty)$, which cannot be $(0, 0)$. This is a contradiction.

(ii) We will show this by proving that as $\alpha \searrow u_0$, if we rescale the solutions, then they converge to a solution of the linearized ODE around $(u_0, 0)$. We shift u_0 to the origin and blow up by defining $(\widetilde{u}, \widetilde{v}) = ((u - u_0)/(\alpha - u_0), v/(\alpha - u_0))$. Thus the ODE becomes

$$(25) \qquad \frac{d}{dt}(\widetilde{u}, \widetilde{v}) = \widetilde{X}_\alpha := \left(\widetilde{v}, \frac{n(n-2)}{4} \frac{(\alpha - u_0)\widetilde{u} + u_0}{\alpha - u_0}\left(u_0^{N-2} - ((\alpha - u_0)\widetilde{u} + u_0)^{N-2}\right)\right).$$

Notice that under the rescaling, the trajectory $\widetilde{\gamma}_\alpha$ encircles the origin and contains the point $(1, 0)$. Moreover, as $\alpha \searrow u_0$ the vector field $\widetilde{X}_\alpha$ converges to

$$(26) \qquad \widetilde{X}_{u_0} = \left(\widetilde{v}, -\frac{n(n-2)(N-2)}{4}u_0^{N-2}\widetilde{u}\right) = (\widetilde{v}, (2-n)\widetilde{u}).$$

Thus, the solution to the linearized equation is

$$\widetilde{\gamma}_{u_0} := \left(\cos((n-2)^{1/2}t), -(n-2)^{1/2}\sin((n-2)^{1/2}t)\right),$$

which is periodic with period given by $T_0 := (n-2)^{-1/2}2\pi$.

Now, we claim first that the $\tau(\alpha)$ are bounded as $\alpha \to u_0$, say $\tau(\alpha) \leq \frac{5}{2}T_0$. Suppose not, so there are $k \to \infty$ and $\alpha_k \searrow u_0$ such that $\tau(\alpha_k) > \frac{5}{2}T_0$. Now, on the one hand, by continuous dependence on initial data and due to the explicit formula of $\widetilde{\gamma}_{u_0}$, we have that for any fixed $t \in (\frac{1}{2}T_0, T_0)$ and for k large enough, the trajectory $\widetilde{\gamma}_{\alpha_k}(t)$ has $\widetilde{v}_{\alpha_k}(t) \geq \epsilon > 0$. On the other hand (by Claim 23), since we have assumed that $T_0 < \frac{2}{5}\tau(\alpha_k) < \frac{1}{2}\tau(\alpha_k)$, $\widetilde{\gamma}_{\alpha_k}(t)$ must always have $\widetilde{v}_{\alpha_k}(t) < 0$, because $\frac{1}{2}\tau(\alpha_k)$ is the first (positive) time when $\widetilde{\gamma}_{\alpha_k}$ crosses the $\widetilde{u}$–axis. This is a contradiction.

That being said, because $\tau(\alpha)$ is bounded for α close to u_0, for any $\alpha_k \searrow \alpha$ we may assume that $\tau(\alpha_k) \to \overline{T}$ for some $\overline{T}$. By continuous dependence of ODEs on their parameters, thus $\lim_{k\to\infty} \widetilde{\gamma}_{\alpha_k}(\frac{1}{2}\tau(\alpha_k)) = \widetilde{\gamma}_{u_0}(\frac{1}{2}\overline{T})$. Because all $\widetilde{\gamma}_{\alpha_k}(\frac{1}{2}\tau(\alpha_k))$ have

$\tilde{u} \le 0$ and lie on the $\tilde{v} = 0$ axis, we see that $\lim_{k \to \infty} \tilde{\gamma}_{\alpha_k}(\frac{1}{2}\tau(\alpha_k)) = (-1, 0)$ and, at the same time, necessarily $\overline{T} = (2q+1)T_0$ for some integer $q \in \mathbb{N}$. But if $q > 0$ then we would have $\overline{T} \ge 3T_0$, contradicting our previous argument which showed that $\overline{T} \le \frac{5}{2}T_0$ instead. Hence $q = 0$, so $\overline{T} = T_0$, and this completes the proof of (ii).

Continuity of $\tau(\alpha)$ follows by a similar argument as the one used in (i). □

5.2.3 Checking AS_p for $p \ge 4$ Proposition 4 follows from the next result.

Proposition 25 *The product metric g_∞ on $\mathbb{S}^1(T_0/2\pi) \times \mathbb{S}^{n-1}(1)$ is a degenerate critical point of the Yamabe functional. When $n > 2$ it is nonintegrable, a global minimum of the Yamabe energy, and satisfies AS_p for some even $p \ge 4$.*

Proof We start by proving degeneracy. Note that $R_{g_\infty} = R_{g_{\mathbb{S}^{n-1}(1)}} = (n-2)(n-1)$ so it suffices to show that $\lambda_1(g_\infty) = n-2$. The eigenvalues of g_∞ are the sums of those of each of its factors. Therefore

$$\lambda_1(g_\infty) = \min\{\lambda_1(\mathbb{S}^{n-1}(1)), \lambda_1(\mathbb{S}^1(T_0/2\pi))\} = \min\{n-1, (T_0/2\pi)^{-2}\} = n-2.$$

Monotonicity of the period function follows from the general result [6, Lemma 1.2] or from [13]. We review the proof in our special setting in Appendix B, as it seems not to be well-known to experts. Nonintegrability is now immediate since $\varphi_1(t) := \sin(\sqrt{n-2}t)$ is an eigenfunction of $\mathcal{L}_\infty$, while Proposition 21 and the fact that $\tau(\alpha)$ is strictly increasing imply that g_∞ is the only critical point of the Yamabe energy: because $\tau(\alpha)$ is strictly increasing and $\lim_{\alpha \searrow u_0} \tau(\alpha) = T_0$, there cannot be $\alpha \in (u_0, 1)$ and integers $k \ge 1$ such that $\tau(\alpha) = T_0/k$. Thus, Λ_0 is one-dimensional, but 1 is the *unique* critical point of $\mathcal{Y}$ in $[g_\infty]_1$, so g_∞ must be nonintegrable. Notice that because 1 is the unique critical point of $\mathcal{Y}$ in $[g_\infty]_1$, the solution of the Yamabe problem guarantees that it is the global minimum of the Yamabe energy on $[g_\infty]_1$.

Now, because g is a nonintegrable critical point, the function $F(v)$ defined on Λ_0 in Proposition 7 is necessarily nonconstant. Furthermore, because g is a unique global minimum for the Yamabe problem in its conformal class, we see that $\mathcal{Y}(1) < \mathcal{Y}(w)$ for any $w^{N-2}g_\infty \in [g_\infty]_1$ with $w \not\equiv 1$. In particular, this yields that if $v \ne 0$ then necessarily $F(0) < F(v)$. Thus, denoting by p the order of integrability of g, it is clear that F_p must be everywhere nonnegative (if it were not, we could take v small enough so that Taylor's theorem would imply that $F(v) < F(0)$, contradicting the previous argument). From this, it is clear that $p \ge 3$ and in fact has to be even. (We remark that one can directly check $p \ne 3$ because $D^3 F(0)[\varphi_1, \varphi_1, \varphi_1] = 0$ by using the explicit form of $F_3(v)$ given in (4).) □

Appendix A: Computing F_3

In this appendix we compute the term F_3 (see Proposition 7 and the subsequent discussion) for a metric g_∞ with constant scalar curvature. We believe this computation is of certain interest because, as the reader may check from what follows, the higher-order polynomials F_p for $p \geq 4$ cannot be determined explicitly, since this would require stronger information on the reduction map Φ (or, equivalently, on Ψ) at the linearization point than we actually have according to Proposition 7.

We denote by $\langle \cdot, \cdot \rangle$ the $L^2(M, g_\infty)$ pairing and refer the reader to Section 2 for the notation concerning differentials and gradients. First, we will show that $F_1(v) = F_2(v) = 0$. To check this, notice that $DF(w)[v] = D\mathcal{Y}(\Psi(w))\big[D\Psi(w)[v]\big]$. Thus, $DF(0) = 0$, since $D\mathcal{Y}(1) = 0$ as 1 is a critical point of the Yamabe functional (by assumption, $g_\infty \in \mathcal{CSC}_1$) and of course $\Psi(0) = 1$. Therefore, $F_1 = 0$. Similarly,

$$D^2 F(w)[v, u] = D^2 \mathcal{Y}(\Psi(w))\big[D\Psi(w)[u], D\Psi(w)[v]\big] + \langle D\mathcal{Y}(\Psi(w)), D^2\Psi(w)[v, u]\rangle.$$

When setting $w = 0$, we have $\Psi(0) = 1$, $D\Psi(0) = \mathrm{Id}$ and

$$\begin{aligned} D^2 F(0)[v, u] &= D^2 \mathcal{Y}(1)[u, v] + \langle D\mathcal{Y}(1), D^2\Psi(0)[v, u]\rangle \\ &= -2(N-2)\langle \mathcal{L}_\infty u, v\rangle + \langle D\mathcal{Y}(1), D^2\Psi(0)[v, u]\rangle. \end{aligned}$$

As before, the second term vanishes. The first term vanishes because v is in the kernel of the linearization of $\mathcal{L}_\infty$, by assumption.

As observed in [1, Remark 1.19], one may explicitly compute F_3 without explicit knowledge of Ψ (and this is what typically makes AS_3 simpler to check than AS_p with $p > 3$ in the examples). We will use this observation and check that to compute $D^3 F(0)$, one may in fact compute $D^3 \widetilde{F}(0)$, where $\widetilde{F} \colon \Lambda_0 \to \mathbb{R}$ is defined by $\widetilde{F}(v) = \mathcal{Y}(1 + v)$. We first compute $D^3 F$:

$$\begin{aligned} D^3 F(w)[v, u, z] = \; & D^3 \mathcal{Y}(\Psi(w))\big[D\Psi(w)[v], D\Psi(w)[u], D\Psi(w)[z]\big] \\ & + D^2 \mathcal{Y}(\Psi(w))\big[D^2\Psi(w)[u, z], D\Psi(w)[v]\big] \\ & + D^2 \mathcal{Y}(\Psi(w))\big[D\Psi(w)[u], D^2\Psi(w)[v, z]\big] \\ & + D^2 \mathcal{Y}(\Psi(w))\big[D\Psi(w)[z], D^2\Psi(w)[v, u]\big] \\ & + \langle D\mathcal{Y}(\Psi(w)), D^3\Psi(w)[v, u, z]\rangle \end{aligned}$$

Setting $w = 0$ and using similar considerations as before (in particular noting that $D^2\mathcal{Y}(1)[\cdot]$ is self-adjoint), we obtain $D^3 F(0)[v, u, z] = D^3 \mathcal{Y}(1)[v, u, z]$. Performing the same computation for $D^3 \widetilde{F}(0)$ yields the same result. Next, we compute $D^3 \widetilde{F}(0)$. Recall from Section 2 that the differential of the Yamabe energy is given

by $\frac{1}{2}D\mathcal{Y}(w)[v] = \int_M [-(N+2)\Delta_{g_\infty}w + R_{g_\infty}w - r_{w^{N-2}g_\infty}w^{N-1}]v\,dV_{g_\infty}$. The first two terms are linear in w, so when computing the third derivative of $\mathcal{Y}$ at 1, they will vanish. Let us then concentrate on the third term. Because $r_{w^{N-2}g_\infty} = \mathcal{Y}(w)$, we have already shown that the first and second directional derivatives of this expression in directions in Λ_0 vanish at $w = 1$. Hence we see that

$$D^3\mathcal{Y}(1)[u, z, v] = -2(N-1)(N-2)r_{g_\infty}\int_M uzv\,dV_{g_\infty}$$

for $u, z, v \in \Lambda_0$, proving (4).

In this final paragraph (contrary to the rest of this section) we will use the space-time $C^{k,\alpha}$ norms on an interval $(t, t+1) \times M$, as in Section 4. One can observe that, by repeating the argument used above for w such that $\|w-1\|_{C^{2,\alpha}} < 1$, it is clear that the $C^{0,\alpha}$ norm of $D^3\mathcal{Y}(w)[v, u]$, regarded (via the $L^2(M, g_\infty)$ pairing) as a function on M, can be bounded by a uniform constant times the $C^{2,\alpha}$ norm of u times that of v. More precisely,

$$(27) \qquad\qquad \|D^3\mathcal{Y}(w)[u, v]\|_{C^{0,\alpha}} \le C\|u\|_{C^{2,\alpha}}\|v\|_{C^{2,\alpha}}$$

for some uniform $C > 0$. Furthermore, for w_1, w_2 such that $\|w_i - 1\|_{C^{2,\alpha}} < 1$ ($i = 1, 2$), we have

$$\|D^3\mathcal{Y}(w_1)[v, v] - D^3\mathcal{Y}(w_2)[u, u]\|_{C^{0,\alpha}} \le C\big(\|w_1\|_{C^{2,\alpha}} + \|w_2\|_{C^{2,\alpha}}\big)$$
$$\times \big(\|u\|_{C^{2,\alpha}} + \|v\|_{C^{2,\alpha}}\big)\|u - v\|_{C^{2,\alpha}}$$

for some uniform $C > 0$. These facts are used in the proofs of Lemma 16 and Proposition 17.

Appendix B: Monotonicity of the period function

Here we review the proof of [6, Lemma 1.2] in our special setting. Recalling (23), define the "potential energy" $U(u) = \frac{1}{2}H - v^2 = (2/(N-2))^2(u^N - u^2)$. Its absolute minimum in the range $(0, 1)$ is attained at $u = u_0$. Denote by $\lambda_\beta(t) = (u(t), v(t))$ the solution of (20) with $\lambda_\beta(0) = (u_0, \beta) \in \Omega$, where $\beta \in [0, \sqrt{-U(u_0)})$ and $\lambda_0 = (u_0, 0)$. This solution intersects the u–axis at exactly two points that we denote by $(z_-(\beta), 0)$ and $(z_+(\beta), 0)$ with $z_-(\beta) < z_+(\beta)$.

Since $v = \dfrac{du}{dt}$ and $dt = \dfrac{du}{v} = \dfrac{du}{\sqrt{H/2 - U}}$, the half-period of $\lambda_\beta(t)$ is given by

$$\tfrac{1}{2}\tau(\beta) = \int_{u_0}^{z_+(\beta)} \frac{du}{\sqrt{H(\beta)/2 - U(u)}} - \int_{u_0}^{z_-(\beta)} \frac{du}{\sqrt{H(\beta)/2 - U(u)}},$$

where $H(\beta) = 2\beta^2 + 2U(u_0)$. Notice that $U(z_\pm(\beta)) = \frac{1}{2}H(\beta) = \beta^2 + U(u_0)$, whence differentiation in β gives $(\partial U/\partial u)(z_\pm(\beta))z'_\pm(\beta) = 2\beta$. Thus, by setting $a = \frac{1}{\beta}\sqrt{U(u) - U(u_0)}$, we get $\frac{1}{2}\tau(\beta) = \int_0^1 (z_+ - z_-)'(\beta t)(dt/\sqrt{1 - t^2})$. The advantage of this formula is its simple dependence on β: it suffices to show now that $z_+ - z_-$ is convex in $\beta \in (0, \sqrt{-U(u_0)})$, where we note that $-U(u_0) = (N-2)^{-1}(2/N)^{N/(N-2)}$. Geometrically, this means that the "width" of the domains enclosed by the image of λ_β is convex as a function of their "height" 2β. Differentiation in β yields

$$\frac{1}{2}z''_\pm(\beta) = \frac{U'^2 - 2U''(U - U(u_0))}{U'^3}(z_\pm(\beta)).$$

L'Hôpital's rule applied twice, using that $\lim_{\beta \to 0} z_\pm(\beta) = u_0$ and $U'(u_0) = 0$, immediately gives $\lim_{\beta \to 0} \frac{1}{2}z''_\pm(\beta) = -U'''(u_0)/(3U''^2(u_0)) =: A < 0$. The convexity claim follows if $\frac{1}{2}z''_+(\beta) \geq A \geq \frac{1}{2}z''_-(\beta)$, for $\beta \in (0, \sqrt{-U(u_0)})$. Since the sign of $U'(z_\pm)$ (which is the sign of the denominator of $z''_\pm$) is $\pm$, both inequalities follow if $F(u) := U'^2 - 2U''(U - U(u_0)) - AU'^3 \geq 0$ on $u \in (0, 1)$. Now, we have that $U''(u) = (2/(N-2))^2(N(N-1)u^{N-2} - 2)$ is negative on

$$(0, u_*) := \left(0, \left(\frac{2}{N(N-1)}\right)^{\frac{1}{N-2}}\right) \subset (0, 1),$$

so, as $U''' > 0$, $F' = -2U'''(U - U(u_0)) - 3AU'^2U'' \leq 0$, ie F does not increase in that range. Therefore, it suffices to show that $F \geq 0$ on $(u_*, 1)$. In that regime (where $U'' > 0$), consider the function $H := F/U''$, and compute

$$H' = \frac{U'^2U'''}{U''^2} \times \left[A\left(U' - \frac{3U''^2}{U'''}\right) - 1\right].$$

Denote the expression inside the brackets by K and note that the sign of H equals the sign of K. Now

$$K' = A\left(U'' - \frac{6U''U'''^2 - 3U''''U''^2}{U'''^2}\right) \quad \text{or} \quad K' = \frac{AU''}{U'''^2}(-5U'''^2 + 3U''''U''),$$

whose sign is opposite to the signs of the expressions in the parantheses, that we denote by L. But $L = -\frac{9}{2}(U'')^{8/3}((U'')^{-2/3})''$, and $(U'')^{-2/3}$ is seen to be convex on $(u_*, 1)$; thus $K' \geq 0$ there (as $U'' > 0$ there). Now, K vanishes at u_0, so $K \geq 0$ and $H' \geq 0$ on $(u_0, 1)$. But $F(u_0) = H(u_0) = 0$, so $H \geq 0$ and $F \geq 0$ on $(u_0, 1)$. Further, K must be negative on (u_*, u_0), as $K' \geq 0$ on $(u_*, 1)$ while $K(u_0) = 0$. Thus $H' \leq 0$ on (u_*, u_0), so H is nonincreasing there; but $H(u_0) = 0$, so we must have $H \geq 0$ also on (u_*, u_0). In conclusion, $F \geq 0$ on $(0, 1)$, as desired.

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