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G_2–instantons over twisted connected sums

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We introduce a method to construct G_2–instantons over compact G_2–manifolds arising as the twisted connected sum of a matching pair of building blocks. Our construction is based on gluing G_2–instantons obtained from holomorphic vector bundles over the building blocks via the first author's work. We require natural compatibility and transversality conditions which can be interpreted in terms of certain Lagrangian subspaces of a moduli space of stable bundles on a $K3$ surface.

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1 Introduction

A G_2–manifold (Y, g) is a Riemannian 7–manifold whose holonomy group $\mathrm{Hol}(g)$ is contained in the exceptional Lie group G_2 or equivalently, a 7–manifold Y together with a torsion-free G_2–structure, that is, a nondegenerate 3–form ϕ satisfying a certain nonlinear partial differential equation; see eg Joyce [14, Part I]. An important method to produce examples of compact G_2–manifolds with $\mathrm{Hol}(g) = G_2$ is the *twisted connected sum construction*, suggested by Donaldson, pioneered by Kovalev [16] and later extended and improved by Kovalev and Lee [17] and Corti, Haskins, Nordström and Pacini [2]. Here is a brief summary of this construction: A *building block* consists of a smooth projective 3–fold Z and a smooth anticanonical $K3$ surface $\Sigma \subset Z$ with trivial normal bundle; see Definition 2.8. Given a choice of a hyperkähler structure $(\omega_I, \omega_J, \omega_K)$ on Σ such that $\omega_J + i\omega_K$ is of type $(2,0)$ and $[\omega_I]$ is the restriction of a Kähler class on Z, one can make $V := Z \setminus \Sigma$ into an asymptotically cylindrical (ACyl) Calabi–Yau 3–fold, that is, a noncompact Calabi–Yau 3–fold with a tubular end modeled on $\mathbb{R}_+ \times S^1 \times \Sigma$; see Haskins, Hein and Nordström [12]. Then $Y := S^1 \times V$ is an ACyl G_2–manifold with a tubular end modeled on $\mathbb{R}_+ \times T^2 \times \Sigma$.

Definition 1.1 Given a pair of building blocks $(Z_\pm, \Sigma_\pm)$, we have the following. A collection

$$\boldsymbol{m} = \{(\omega_{I,\pm}, \omega_{J,\pm}, \omega_{K,\pm}), \mathfrak{r}\}$$

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consisting of a choice of hyperkähler structures on $\Sigma_\pm$ such that $\omega_{J,\pm} + i\omega_{K,\pm}$ is of type $(2,0)$ and $[\omega_{I,\pm}]$ is the restriction of a Kähler class on $Z_\pm$ as well as a hyperkähler rotation $\mathfrak{r}\colon \Sigma_+ \to \Sigma_-$ is called *matching data* and $(Z_\pm, \Sigma_\pm)$ are said to *match* via $\boldsymbol{m}$. Here a *hyperkähler rotation* is a diffeomorphism $\mathfrak{r}\colon \Sigma_+ \to \Sigma_-$ such that

$$\mathfrak{r}^*\omega_{I,-} = \omega_{J,+}, \quad \mathfrak{r}^*\omega_{J,-} = \omega_{I,+} \quad \text{and} \quad \mathfrak{r}^*\omega_{K,-} = -\omega_{K,+}. \tag{1.2}$$

Given a matching pair of building blocks, one can glue $Y_\pm$ by interchanging the S^1–factors at infinity and identifying $\Sigma_\pm$ via $\mathfrak{r}$. This yields a simply connected compact 7–manifold Y together with a family of torsion-free G_2–structures $(\phi_T)_{T\geq T_0}$; see Kovalev [16, Section 4]. From the Riemannian viewpoint (Y, ϕ_T) contains a "long neck" modeled on $[-T,T]\times T^2\times\Sigma_+$; one can think of the twisted connected sum as reversing the degeneration of the family of G_2–manifolds that occurs as the neck becomes infinitely long.

If (Z,Σ) is a building block and $\mathcal{E}\to Z$ is a holomorphic vector bundle such that $\mathcal{E}|_\Sigma$ is stable, then $\mathcal{E}|_\Sigma$ carries a unique ASD instanton compatible with the holomorphic structure; see Donaldson [5]. The first author showed that in this situation $\mathcal{E}|_V$ can be given a Hermitian–Yang–Mills (HYM) connection asymptotic to the ASD instanton on $\mathcal{E}|_\Sigma$ [10]. The pullback of an HYM connection over V to $S^1\times V$ is a G_2*–instanton*, ie a connection A on a G–bundle over a G_2–manifold such that $F_A\wedge\psi = 0$ with $\psi := *\phi$. It was pointed out by Simon Donaldson and Richard Thomas in their seminal article on gauge theory in higher dimensions [9] that, formally, G_2–instantons are rather similar to flat connections over 3–manifolds. In particular, they are critical points of a Chern–Simons-type functional and there is hope that counting them could lead to an enumerative invariant for G_2–manifolds not unlike the Casson invariant for 3–manifolds; see Donaldson and Segal [8, Section 6] and the second author [22, Chapter 6]. The main result of this article is the following theorem, which gives conditions for a pair of such G_2–instantons over $Y_\pm = S^1\times V_\pm$ to be glued to give a G_2–instanton over (Y,ϕ_T).

Theorem 1.3 *Let $(Z_\pm,\Sigma_\pm)$ be a pair of building blocks that match via $\boldsymbol{m}$. Denote by Y the compact 7–manifold and by $(\phi_T)_{T\geq T_0}$ the family of torsion-free G_2–structures obtained from the twisted connected sum construction. Let $\mathcal{E}_\pm\to Z_\pm$ be a pair of holomorphic vector bundles such that the following hold:*

- *$\mathcal{E}_\pm|_{\Sigma_\pm}$ is stable. Denote the corresponding ASD instanton by $A_{\infty,\pm}$.*
- *There is a bundle isomorphism $\bar{\mathfrak{r}}\colon \mathcal{E}_+|_{\Sigma_+}\to\mathcal{E}_-|_{\Sigma_-}$ covering the hyperkähler rotation $\mathfrak{r}$ such that $\bar{\mathfrak{r}}^*A_{\infty,-} = A_{\infty,+}$.*

- *There are no infinitesimal deformations of* $\mathcal{E}_\pm$ *fixing the restriction to* $\Sigma_\pm$*:*

$$H^1(Z_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm)(-\Sigma_\pm)) = 0. \tag{1.4}$$

- *Denote by* $\mathrm{res}_\pm\colon H^1(Z_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm)) \to H^1(\Sigma_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm|_{\Sigma_\pm}))$ *the restriction map and by* $\lambda_\pm\colon H^1(Z_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm)) \to H^1_{A_{\infty,\pm}}$ *the composition of* $\mathrm{res}_\pm$ *with the isomorphism from Remark 1.6. The images of* λ_+ *and* $\overline{\mathfrak{r}}^* \circ \lambda_-$ *intersect trivially in* $H^1_{A_{\infty,+}}$*:*

$$\mathrm{im}(\lambda_+) \cap \mathrm{im}(\overline{\mathfrak{r}}^* \circ \lambda_-) = \{0\}. \tag{1.5}$$

Then there exists a nontrivial $\mathbb{P}\mathrm{U}(n)$*–bundle* E *over* Y*, a constant* $T_1 \geq T_0$ *and for each* $T \geq T_1$ *an irreducible and unobstructed*[1] G_2*–instanton* A_T *on* E *over* (Y, ϕ_T)*.*

Remark 1.6 If A is an ASD instanton on a $\mathbb{P}\mathrm{U}(n)$–bundle E over a Kähler surface Σ corresponding to a holomorphic vector bundle $\mathcal{E}$, then

$$H^1_A := \ker(\mathrm{d}_A^* \oplus \mathrm{d}_A^+\colon \Omega^1(\Sigma, \mathfrak{g}_E) \to (\Omega^0 \oplus \Omega^+)(\Sigma, \mathfrak{g}_E)) \cong H^1(\Sigma, \mathcal{E}nd_0(\mathcal{E}));$$

see Donaldson and Kronheimer [7, Section 6.4]. Here $\mathfrak{g}_E$ denotes the adjoint bundle associated with E.

Remark 1.7 If

$$H^1(\Sigma_+, \mathcal{E}nd_0(\mathcal{E}_+|_{\Sigma_+})) = \{0\}, \tag{1.8}$$

then (1.5) is vacuous. If, moreover, the topological bundles underlying $\mathcal{E}_\pm$ are isomorphic, then the existence of $\overline{\mathfrak{r}}$ is guaranteed by a theorem of Mukai; see Huybrechts and Lehn [13, Theorem 6.1.6].

Since $H^2(Z_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm)) \cong H^1(Z_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm)(-\Sigma_\pm))$ vanish by (1.4), there is a short exact sequence

$$0 \to H^1(Z_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm)) \xrightarrow{\mathrm{res}_\pm} H^1(\Sigma_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm|_{\Sigma_\pm})) \to H^2(Z_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm)(-\Sigma_\pm)) \to 0.$$

This sequence is self-dual under Serre duality. Tyurin [20, page 176ff] pointed out that this implies that

$$\mathrm{im}\,\lambda_\pm \subset H^1_{A_{\infty,\pm}}$$

is a complex Lagrangian subspace with respect to the complex symplectic structure induced by $\Omega_\pm := \omega_{J,\pm} + i\omega_{K,\pm}$ or equivalently, Mukai's complex symplectic structure

[1] See Definition 3.12.

on $H^1(Z_\pm, \mathcal{E}nd_0(\mathcal{E}_\pm))$. Under the assumptions of Theorem 1.3 the moduli space $\mathcal{M}(\Sigma_\pm)$ of holomorphic vector bundles over $\Sigma_\pm$ is smooth near $[\mathcal{E}_\pm|_{\Sigma_\pm}]$ and so are the moduli spaces $\mathcal{M}(Z_\pm)$ of holomorphic vector bundles over $Z_\pm$ near $[\mathcal{E}_\pm]$. Locally, $\mathcal{M}(Z_\pm)$ embeds as a complex Lagrangian submanifold into $\mathcal{M}(\Sigma_\pm)$. Since $\mathfrak{r}^*\omega_{K,-} = -\omega_{K,+}$, both $\mathcal{M}(Z_+)$ and $\mathcal{M}(Z_-)$ can be viewed as Lagrangian submanifolds of $\mathcal{M}(\Sigma_+)$ with respect to the symplectic form induced by $\omega_{K,+}$. Equation (1.5) asks for these Lagrangian submanifolds to intersect transversely at the point $[\mathcal{E}_+|_{\Sigma_+}]$. If one thinks of G_2–manifolds arising via the twisted connected sum construction as analogues of 3–manifolds with a fixed Heegaard splitting, then this is much like the geometric picture behind Atiyah–Floer conjecture in dimension three; see Atiyah [1].

Remark 1.9 The hypothesis (1.5) appears natural in view of the above discussion. Assuming (1.8) instead would slightly simplify the proof; see Remark 3.38. However, it would also substantially restrict the applicability of Theorem 1.3 and, hence, the chance of finding new examples of G_2–instantons because (1.8) is a very strong assumption.

Remark 1.10 There are as of yet no examples of new G_2–instantons constructed using Theorem 1.3. We plan to address this issue in future work.

Outline We recall the salient features of the twisted connected sum construction in Section 2. The expert reader may wish to skim through it to familiarize with our notation. The objective of Section 3 is to prove Theorem 3.24, which describes hypotheses under which a pair of G_2–instantons over a matching pair of ACyl G_2–manifolds can be glued. Finally, in Section 4 we explain how these hypotheses can be verified for G_2–instantons obtained via the first author's construction. Theorem 1.3 is then proved by combining Theorems 3.24 and 4.2 with Proposition 4.3.

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2 The twisted connected sum construction

In this section we review the twisted connected sum construction using the language introduced by Corti, Haskins, Nordström and Pacini [2].

2.1 Gluing ACyl G_2–manifolds

We begin with gluing matching pairs of ACyl G_2–manifolds.

Definition 2.1 Let (Z, ω, Ω) be a compact Calabi–Yau 3–fold. Here ω denotes the Kähler form and Ω denotes the holomorphic volume form. A G_2–manifold (Y, ϕ) is called *asymptotically cylindrical (ACyl) with asymptotic cross section* (Z, ω, Ω) if there exist a constant $\delta < 0$, a compact subset $K \subset Y$, a diffeomorphism $\pi\colon Y \setminus K \to \mathbb{R}_+ \times Z$ and a 2–form ρ on $\mathbb{R}_+ \times Z$ such that

$$\pi_*\phi = \mathrm{d}t \wedge \omega + \operatorname{Re}\Omega + \mathrm{d}\rho \quad \text{and} \quad \nabla^k \rho = O(e^{\delta t})$$

for all $k \in \mathbb{N}_0$. Here t denotes the coordinate on $\mathbb{R}_+$.

Remark 2.2 Unfortunately, Z is the customary notation both for building blocks and asymptotic cross sections of ACyl G_2–manifolds. To avoid confusion we point out that, unlike asymptotic cross sections, building blocks always come in pair with a divisor, eg (Z, Σ).

Definition 2.3 A pair of ACyl G_2–manifolds $(Y_\pm, \phi_\pm)$ with asymptotic cross sections $(Z_\pm, \omega_\pm, \Omega_\pm)$ is said to *match* if there exists a diffeomorphism $f\colon Z_+ \to Z_-$ such that

$$f^*\omega_- = -\omega_+ \quad \text{and} \quad f^* \operatorname{Re}\Omega_- = \operatorname{Re}\Omega_+.$$

Let $(Y_\pm, \phi_\pm)$ be a matching pair of ACyl G_2–manifolds. Fix $T \geq 1$ and define $F\colon [T, T+1] \times Z_+ \to [T, T+1] \times Z_-$ by

$$F(t, z) := (2T + 1 - t, f(z)).$$

Denote by Y_T the compact 7–manifold obtained by gluing together

$$Y_{T,\pm} := K_\pm \cup \pi_\pm^{-1}((0, T+1] \times Z_\pm)$$

via F. Fix a nondecreasing smooth function $\chi\colon \mathbb{R} \to [0, 1]$ with $\chi(t) = 0$ for $t \leq 0$ and $\chi(t) = 1$ for $t \geq 1$. Define a 3–form $\tilde{\phi}_T$ on Y_T by

$$\tilde{\phi}_T := \phi_\pm - \mathrm{d}[\pi_\pm^*(\chi(t - T + 1)\rho_\pm)]$$

on $Y_{T,\pm}$. If $T \gg 1$, then $\tilde{\phi}_T$ defines a closed G_2–structure on Y_T. Clearly, all the Y_T for different values of T are diffeomorphic; hence, we often drop the T from the notation. The G_2–structure $\tilde{\phi}_T$ is not torsion free yet, but can be made so by a small perturbation:

Theorem 2.4 (Kovalev [16, Theorem 5.34]) *In the above situation there exists a constant $T_0 \geq 1$ and for each $T \geq T_0$ there exists a 2–form η_T on Y_T such that $\phi_T := \tilde{\phi}_T + \mathrm{d}\eta_T$ defines a torsion-free G_2–structure; moreover, for some $\delta < 0$*

$$\|\mathrm{d}\eta_T\|_{C^{0,\alpha}} = O(e^{\delta T}). \tag{2.5}$$

2.2 ACyl Calabi–Yau 3–folds from building blocks

The twisted connected sum is based on gluing ACyl G_2–manifolds arising as the product of ACyl Calabi–Yau 3–folds with S^1.

Definition 2.6 Let $(\Sigma, \omega_I, \omega_J, \omega_K)$ be a hyperkähler surface. A Calabi–Yau 3–fold (V, ω, Ω) is called *asymptotically cylindrical (ACyl) with asymptotic cross section* $(\Sigma, \omega_I, \omega_J, \omega_K)$ if there exist a constant $\delta < 0$, a compact subset $K \subset V$, a diffeomorphism $\pi\colon V \setminus K \to \mathbb{R}_+ \times S^1 \times \Sigma$, a 1–form ρ and a 2–form σ on $\mathbb{R}_+ \times S^1 \times \Sigma$ such that

$$\begin{aligned}
\pi_*\omega &= \mathrm{d}t \wedge \mathrm{d}\alpha + \omega_I + \mathrm{d}\rho, \\
\pi_*\Omega &= (\mathrm{d}\alpha - i\,\mathrm{d}t) \wedge (\omega_J + i\omega_K) + \mathrm{d}\sigma \quad \text{and} \\
\nabla^k \rho &= O(e^{\delta t}) \quad \text{as well as} \quad \nabla^k \sigma = O(e^{\delta t})
\end{aligned}$$

for all $k \in \mathbb{N}_0$. Here t and α denote the respective coordinates on $\mathbb{R}_+$ and S^1.

Given an ACyl Calabi–Yau 3–fold (V, ω, Ω), taking the product with S^1 with coordinate β, yields an ACyl G_2–manifold

$$(Y := S^1 \times V, \phi := \mathrm{d}\beta \wedge \omega + \mathrm{Re}\,\Omega)$$

with asymptotic cross section

$$(T^2 \times \Sigma, \mathrm{d}\alpha \wedge \mathrm{d}\beta + \omega_K, (\mathrm{d}\alpha - i\,\mathrm{d}\beta) \wedge (\omega_J + i\omega_I)).$$

Let $V_\pm$ be a pair of ACyl Calabi–Yau 3–folds with asymptotic cross section $\Sigma_\pm$ and suppose that $\mathfrak{r}\colon \Sigma_+ \to \Sigma_-$ is a hyperkähler rotation; see (1.2). Then $Y_\pm := V_\pm \times S^1$ match via the diffeomorphism $f\colon T^2 \times \Sigma_+ \to T^2 \times \Sigma_-$ defined by

$$f(\alpha, \beta, x) := (\beta, \alpha, \mathfrak{r}(x)).$$

Remark 2.7 If f did not interchange the S^1–factors, then Y would have infinite fundamental group and, hence, could not carry a metric with holonomy equal to G_2; see Joyce [15, Proposition 10.2.2].

ACyl Calabi–Yau 3–folds can be obtained from the following building blocks:

Definition 2.8 Corti, Haskins, Nordström and Pacini [3, Definition 5.1] A *building block* is a smooth projective 3–fold Z together with a projective morphism $f\colon Z \to \mathbb{P}^1$ such that the following hold:

- The anticanonical class $-K_Z \in H^2(Z)$ is primitive.
- $\Sigma := f^{-1}(\infty)$ is a smooth $K3$ surface and $\Sigma \sim -K_Z$.
- If N denotes the image of $H^2(Z)$ in $H^2(\Sigma)$, then the embedding $N \hookrightarrow H^2(\Sigma)$ is primitive.
- $H^3(Z)$ is torsion free.

Remark 2.9 The existence of the fibration $f\colon Z \to \mathbb{P}^1$ is equivalent to Σ having trivial normal bundle. This is crucial because it means that $Z \setminus \Sigma$ has a cylindrical end. The last two conditions in the definition of a building block are not essential; they have been made to facilitate the computation of certain topological invariants in [3].

In his original work Kovalev [16] used building blocks arising from Fano 3–folds by blowing up the base locus of a generic anticanonical pencil. This method was extended to the much larger class of semi-Fano 3–folds (a class of weak Fano 3–folds) by Corti, Haskins, Nordström and Pacini [2]. Kovalev and Lee [17] construct building blocks starting from $K3$ surfaces with nonsymplectic involutions, by taking the product with $\mathbb{P}^1$, dividing by $\mathbb{Z}_2$ and blowing up the resulting singularities.

Theorem 2.10 (Haskins, Hein and Nordström [12, Theorem D]) *Let (Z,Σ) be a building block and let $(\omega_I,\omega_J,\omega_K)$ be a hyperkähler structure on Σ such that $\omega_J + i\omega_K$ is of type $(2,0)$. If $[\omega_I] \in H^{1,1}(\Sigma)$ is the restriction of a Kähler class on Z, then there is an ACyl Calabi–Yau structure (ω,Ω) on $V := Z \setminus \Sigma$ with asymptotic cross section $(\Sigma,\omega_I,\omega_J,\omega_K)$.*

Remark 2.11 This result was first claimed by Kovalev in [16, Theorem 2.4]; see the discussion in [12, Section 4.1].

Combining the results of Kovalev and Haskins, Hein and Nordström, each matching pair of building blocks (see Definition 1.1) yields a one-parameter family of G_2–manifolds. This is called the *twisted connected sum construction*.

3 Gluing G_2–instantons over ACyl G_2–manifolds

In this section we discuss when a pair of G_2–instantons over a matching pair of ACyl G_2–manifolds $Y_\pm$ can be glued to give a G_2–instanton over (Y,ϕ_T).

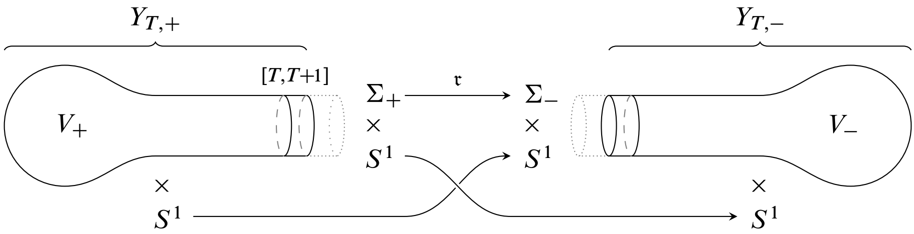

Figure 1: The twisted connected sum of a matching pair of building blocks

3.1 Linear analysis on ACyl manifolds

We recall some results about linear analysis on ACyl Riemannian manifolds. The references for the material in this subsection are Mazya and Plamenevskiĭ [19] and Lockhart and McOwen [18].

3.1.1 Translation-invariant operators on cylindrical manifolds Let $E \to X$ be a Riemannian vector bundle over a compact Riemannian manifold. By slight abuse of notation we also denote by E its pullback to $\mathbb{R} \times X$. Denote by t the coordinate function on $\mathbb{R}$. For $k \in \mathbb{N}_0$, $\alpha \in (0,1)$ and $\delta \in \mathbb{R}$ we define

$$\|\cdot\|_{C^{k,\alpha}_\delta} := \|e^{-\delta t}\cdot\|_{C^{k,\alpha}}$$

and denote by $C^{k,\alpha}_\delta(\mathbb{R}\times X, E)$ the closure of $C^\infty_0(\mathbb{R}\times X, E)$ with respect to this norm. We set $C^\infty_\delta := \bigcap_k C^{k,\alpha}_\delta$.

Let $D\colon C^\infty(X,E) \to C^\infty(X,E)$ be a linear self-adjoint elliptic operator of first order. The operator

$$L_\infty := \partial_t - D$$

extends to a bounded linear operator $L_{\infty,\delta}\colon C^{k+1,\alpha}_\delta(\mathbb{R}\times X, E) \to C^{k,\alpha}_\delta(\mathbb{R}\times X, E)$.

Theorem 3.1 (Mazja and Plamenevskiĭ [19, Theorem 5.1]) *The linear operator $L_{\infty,\delta}$ is invertible if and only if $\delta \notin \operatorname{spec}(D)$.*

Elements $a \in \ker L_\infty$ can be expanded as

$$a = \sum_{\delta\in\operatorname{spec} D} e^{\delta t} a_\delta, \tag{3.2}$$

where a_δ are δ–eigensections of D; see Donaldson [6, Section 3.1]. One consequence of this is the following result:

Proposition 3.3 *Denote by* λ_+ *and* λ_- *the first positive and negative eigenvalue of* D*, respectively. If* $a \in \ker L_\infty$ *and*

$$a = O(e^{\delta t}) \quad \text{as } t \to \infty$$

with $\delta < \lambda_+$*, then there exists* $a_0 \in \ker D$ *such that*

$$\nabla^k (a - a_0) = O(e^{\lambda_- t}) \quad \text{as } t \to \infty$$

for all $k \in \mathbb{N}_0$*. If* $a \in L^\infty(\mathbb{R} \times X, E)$*, then* $a = a_0$*.*

3.1.2 Asymptotically translation-invariant operators on ACyl manifolds Let M be a Riemannian manifold together with a compact set $K \subset M$ and a diffeomorphism $\pi\colon M \setminus K \to \mathbb{R}_+ \times X$ such that the pushforward of the metric on M is asymptotic to the metric on $\mathbb{R}_+ \times X$, this means here and in what follows that their difference and all of its derivatives are $O(e^{\delta t})$ as $t \to \infty$ with $\delta < 0$. Let F be a Riemannian vector bundle and let $\bar{\pi}\colon F|_{M \setminus K} \to E$ be a bundle isomorphism covering π such that the pushforward of the metric on F is asymptotic to the metric on E. Denote by $t\colon M \to [1, \infty)$ a smooth positive function which agrees with $t \circ \pi$ on $\pi^{-1}([1, \infty) \times X)$. We define

$$\| \cdot \|_{C^{k,\alpha}_\delta} := \| e^{-\delta t} \cdot \|_{C^{k,\alpha}}$$

and denote by $C^{k,\alpha}_\delta(M, F)$ the closure of $C^\infty_0(M, F)$ with respect to this norm.

Let $L\colon C^\infty_0(M, E) \to C^\infty_0(M, E)$ be an elliptic operator asymptotic to $L_\infty = \partial_t - D$, ie the coefficients of the pushforward of L to $\mathbb{R}_+ \times X$ are asymptotic to the coefficients of L_∞. The operator L extends to a bounded linear operator $L_\delta\colon C^{k+1,\alpha}_\delta(M, E) \to C^{k,\alpha}_\delta(M, E)$.

Proposition 3.4 [12, Proposition 2.4] *If* $\delta \notin \operatorname{spec}(D)$*, then* L_δ *is Fredholm.*

Elements in the kernel of L still have an asymptotic expansion analogous to (3.2). We need the following result which extracts the constant term of this expansion.

Proposition 3.5 *There is a constant* $\delta_0 > 0$ *such that, for all* $\delta \in [0, \delta_0]$, $\ker L_\delta = \ker L_0$ *and there is a linear map* $\iota\colon \ker L_0 \to \ker D$ *such that*

$$\nabla^k (\bar{\pi}_* a - \iota(a)) = O(e^{-\delta_0 t}) \quad \text{as } t \to \infty$$

for all $k \in \mathbb{N}_0$*; in particular,*

$$\ker \iota = \ker L_{-\delta_0}.$$

Proof Let $\lambda_\pm$ be the first positive/negative eigenvalue of D. Then pick $0 < \delta_0 < \min(\lambda_+, -\lambda_-)$ such that the decay conditions made above hold with $-2\delta_0$ instead of δ. Given $a \in \ker L_{\delta_0}$, set $\tilde{a} := \chi(t)\bar{\pi}_* a_\pm$ with χ as in Section 2.1. Then $L_\infty \tilde{a} \in C^\infty_{-\delta_0}$. By Theorem 3.1 there exists a unique $b \in C^\infty_{-\delta_0}$ such that $L_\infty(\tilde{a} - b) = 0$. By Proposition 3.3 $(\tilde{a} - b)_0 \in \ker D$ and $\tilde{a} - b - (\tilde{a} - b)_0 = O(e^{\lambda_- t})$ as t tends to infinity. From this it follows that $a \in \ker L_0$; hence, the first part of the proposition. With $\iota(a) := (\tilde{a} - b)_0$ the second part also follows. □

3.2 Hermitian–Yang–Mills connections over Calabi–Yau 3–folds

Suppose (Z, ω, Ω) is a Calabi–Yau 3–fold and $(Y := \mathbb{R} \times Z, \phi := \mathrm{d}t \wedge \omega + \operatorname{Re}\Omega)$ is the corresponding cylindrical G_2–manifold. In this section we relate translation-invariant G_2–instantons over Y with Hermitian–Yang–Mills connections over Z. Let G denote a compact semisimple Lie group.

Definition 3.6 Let (Z, ω) be a Kähler manifold and let E be a G–bundle over Z. A connection A on E is a *Hermitian–Yang–Mills (HYM) connection* if

$$F_A^{0,2} = 0 \quad \text{and} \quad \Lambda F_A = 0. \tag{3.7}$$

Here Λ is the dual of the Lefschetz operator $L := \omega \wedge \cdot$.

Remark 3.8 We are mostly interested in the special case of $\mathrm{U}(n)$–bundles; however, for $G = \mathrm{U}(n)$, (3.7) is too restrictive as it forces $c_1(E) = 0$. There are two customary ways to circumnavigate this issue. One is to change (3.7) and instead of the second part require that ΛF_A be equal to a constant in $\mathfrak{u}(1)$, the center of $\mathfrak{u}(n)$, which is determined by the degree of $\det E$; the other one is to work with the induced $\mathbb{P}\mathrm{U}(n)$–bundle. These viewpoints are essentially equivalent and we adopt the latter.

Remark 3.9 By the first part of (3.7) an HYM connection induces a holomorphic structure on E. If Z is compact, then there is a one-to-one correspondence between gauge equivalence classes of HYM connections on E and isomorphism classes of polystable holomorphic $G^{\mathbb{C}}$–bundles $\mathcal{E}$ whose underlying topological bundle is E; see Donaldson [5] and Uhlenbeck and Yau [21].

On a Calabi–Yau 3–fold (3.7) is equivalent to

$$F_A \wedge \operatorname{Im}\Omega = 0 \quad \text{and} \quad F_A \wedge \omega \wedge \omega = 0;$$

hence, using $\psi = *\phi = *(\mathrm{d}t \wedge \omega + \operatorname{Re}\Omega) = \frac{1}{2}\omega \wedge \omega - \mathrm{d}t \wedge \operatorname{Im}\Omega$ one easily derives:

Proposition 3.10 [10, Proposition 8] *Denote by $\pi_Z \colon Y \to Z$ the canonical projection. A is an HYM connection if and only if $\pi_Z^* A$ is a G_2–instanton.*

In general, if A is a G_2–instanton on a G–bundle E over a G_2–manifold (Y,ϕ), then the moduli space $\mathcal{M}$ of G_2–instantons near $[A]$, ie the space of gauge equivalence classes of G_2–instantons near $[A]$, is the space of small solutions $(\xi,a)\in(\Omega^0\oplus\Omega^1)(Y,\mathfrak{g}_E)$ of the system of equations

$$\mathrm{d}_A^*a=0 \quad\text{and}\quad \mathrm{d}_{A+a}\xi+*(F_{A+a}\wedge\psi)=0$$

modulo the action of $\Gamma_A\subset\mathcal{G}$, the stabilizer of A in the gauge group of E, assuming Y is compact or appropriate assumptions are made regarding the growth of ξ and a. The linearization $L_A\colon(\Omega^0\oplus\Omega^1)(Y,\mathfrak{g}_E)\to(\Omega^0\oplus\Omega^1)(Y,\mathfrak{g}_E)$ of this equation is

$$L_A:=\begin{pmatrix} & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi\wedge\mathrm{d}_A)\end{pmatrix}. \tag{3.11}$$

It controls the infinitesimal deformation theory of A.

Definition 3.12 A is called *irreducible and unobstructed* if L_A is surjective.

If A is irreducible and unobstructed, then $\mathcal{M}$ is smooth at $[A]$. If Y is compact, then L_A has index zero; hence, is surjective if and only if it is invertible; therefore, irreducible and unobstructed G_2–instantons form isolated points in $\mathcal{M}$. If Y is noncompact, the precise meaning of $\mathcal{M}$ and L_A depends on the growth assumptions made on ξ and a, and $\mathcal{M}$ may very well be positive-dimensional.

Proposition 3.13 *If A is an HYM connection on a bundle E over a G_2–manifold $Y:=\mathbb{R}\times Z$ as in Proposition 3.10, then the operator $L_{\pi_Z^*A}$ defined in (3.11) can be written as*

$$L_{\pi_Z^*A}=\tilde{I}\partial_t+D_A,$$

where

$$\tilde{I}:=\begin{pmatrix} & -1 & \\ 1 & & \\ & & I\end{pmatrix}$$

and $D_A\colon(\Omega^0\oplus\Omega^0\oplus\Omega^1)(Z,\mathfrak{g}_E)\to(\Omega^0\oplus\Omega^0\oplus\Omega^1)(Z,\mathfrak{g}_E)$ is defined by

$$D_A:=\begin{pmatrix} & & \mathrm{d}_A^* \\ & & \Lambda\mathrm{d}_A \\ \mathrm{d}_A & -I\mathrm{d}_A & -*(\operatorname{Im}\Omega\wedge\mathrm{d}_A)\end{pmatrix}. \tag{3.14}$$

(Note that $TY=\underline{\mathbb{R}}\oplus\pi_Z^*TZ$.)

Proof Plugging $\psi = \frac{1}{2}\omega\wedge\omega - \mathrm{d}t\wedge \operatorname{Im}\Omega$ into the definition of $L_{\pi_Z^* A}$ and using the fact that the complex structure acts via

$$I = \tfrac{1}{2} * (\omega\wedge\omega\wedge\cdot) \tag{3.15}$$

on $\Omega^1(Z,\mathfrak{g}_E)$ the assertion follows by a direct computation. □

Definition 3.16 Let A be an HYM connection on a G–bundle E over a Kähler manifold (Z,ω). Set

$$\mathcal{H}_A^i := \ker\big(\bar\partial_A \oplus \bar\partial_A^*\colon \Omega^{0,i}(Z,\mathfrak{g}_E^{\mathbb{C}}) \to (\Omega^{0,i+1}\oplus\Omega^{0,i-1})(Z,\mathfrak{g}_E^{\mathbb{C}})\big).$$

We call $\mathcal{H}_A^0$ the space of *infinitesimal automorphisms* of A and $\mathcal{H}_A^1$ the space of *infinitesimal deformations* of A.

Remark 3.17 If Z is compact and A is a connection on a $\mathbb{P}\mathrm{U}(n)$–bundle E corresponding to a holomorphic vector bundle $\mathcal{E}$, then $\mathcal{H}_A^i \cong H^i(Z,\mathcal{E}nd_0(\mathcal{E}))$.

Proposition 3.18 *If (Z,ω,Ω) is a compact Calabi–Yau 3–fold and A is an HYM connection on a G–bundle $E\to Z$, then*

$$\ker D_A \cong \mathcal{H}_A^0 \oplus \mathcal{H}_A^1$$

with D_A as in (3.14).

Proof If $s\in\mathcal{H}_A^0$ and $\alpha\in\mathcal{H}_A^1$, then $D_A(\operatorname{Re} s,\operatorname{Im} s,\alpha+\bar\alpha)=0$. Conversely, if $(\xi,\eta,a)\in\ker D_A$, then applying d_A^* (resp. $\mathrm{d}_A^*\circ I$) to

$$\mathrm{d}_A\xi - I\,\mathrm{d}_A\eta - *(\operatorname{Im}\Omega\wedge \mathrm{d}_A a) = 0,$$

using (3.15), taking the L^2 inner product with ξ (resp. η) and integrating by parts yields $\mathrm{d}_A\xi=0$ (resp. $\mathrm{d}_A\eta=0$). Thus $\xi+i\eta\in\mathcal{H}_A^0$ and

$$\mathrm{d}_A^* a = 0,\quad \Lambda \mathrm{d}_A a = 0 \quad\text{and}\quad \operatorname{Im}\Omega\wedge\mathrm{d}_A a = 0,$$

which implies $\alpha := a^{0,1}\in\mathcal{H}_A^1$ because $\mathrm{d}_A^* = \partial_A^* + \bar\partial_A^*$ and $\Lambda\mathrm{d}_A = -i\,\partial_A^* + i\,\bar\partial_A^*$. □

3.3 G_2–instantons over ACyl G_2–manifolds

Definition 3.19 Let (Y,ϕ) be an ACyl G_2–manifold with asymptotic cross section (Z,ω,Ω). Let A_∞ be an HYM connection on a G–bundle $E_\infty\to Z$. A G_2–instanton A on a G–bundle $E\to Y$ is called *asymptotic to* A_∞ if there exist a constant

$\delta < 0$ and a bundle isomorphism $\overline{\pi}\colon E|_{Y\setminus K} \to E_\infty$ covering $\pi\colon Y \setminus K \to \mathbb{R}_+ \times Z$ such that

$$\nabla^k(\overline{\pi}_* A - A_\infty) = O(e^{\delta t}) \tag{3.20}$$

for all $k \in \mathbb{N}_0$. Here by a slight abuse of notation we also denote by E_∞ and A_∞ their respective pullbacks to $\mathbb{R}_+ \times Z$.

Definition 3.21 Let (Y, ϕ) be an ACyl G_2–manifold and let A be a G_2–instanton on a G–bundle over (Y, ϕ) asymptotic to A_∞. For $\delta \in \mathbb{R}$ we set

$$\mathcal{T}_{A,\delta} := \ker L_{A,\delta} = \bigl\{\underline{a} \in \ker L_A \bigm| \nabla^k \overline{\pi}_* \underline{a} = O(e^{\delta t}) \text{ for all } k \in \mathbb{N}_0\bigr\},$$

where $\underline{a} = (\xi, a) \in (\Omega^0 \oplus \Omega^1)(Y, \mathfrak{g}_E)$. Set $\mathcal{T}_A := \mathcal{T}_{A,0}$.

Proposition 3.22 *Let (Y, ϕ) be an ACyl G_2–manifold and let A be a G_2–instanton asymptotic to A_∞. Then there is a constant $\delta_0 > 0$ such that for all $\delta \in [0, \delta_0]$, $\mathcal{T}_{A,\delta} = \mathcal{T}_A$ and there is a linear map $\iota\colon \mathcal{T}_A \to \mathcal{H}^0_{A_\infty} \oplus \mathcal{H}^1_{A_\infty}$ such that*

$$\nabla^k(\overline{\pi}_* \underline{a} - \iota(\underline{a})) = O(e^{-\delta_0 t})$$

for all $k \in \mathbb{N}_0$; in particular,

$$\ker \iota = \mathcal{T}_{A,-\delta_0}.$$

Proof By Proposition 3.13, L_A is asymptotic to $\tilde{I}(\partial_t - \tilde{I} D_A)$. Since $\tilde{I} D_A$ is self-adjoint and $\ker \tilde{I} D_A = \ker D_A$, we can apply Proposition 3.5 to obtain a linear map $\iota\colon \mathcal{T}_A \to \ker D_{A_\infty}$ and use the isomorphism $\ker D_{A_\infty} \cong \mathcal{H}^0_{A_\infty} \oplus \mathcal{H}^1_{A_\infty}$ from Proposition 3.18. □

Proposition 3.23 *Let (Y, ϕ) be an ACyl G_2–manifold and let A be a G_2–instanton asymptotic to A_∞. Then*

$$\dim \operatorname{im} \iota = \tfrac{1}{2} \dim(\mathcal{H}^0_{A_\infty} \oplus \mathcal{H}^1_{A_\infty})$$

and, if $\mathcal{H}^0_{A_\infty} = 0$, then $\operatorname{im} \iota \subset \mathcal{H}^1_{A_\infty}$ is Lagrangian with respect to the symplectic structure on $\mathcal{H}^1_{A_\infty}$ induced by ω.

Proof By Lockhart and McOwen [18, Theorem 7.4] for $0 < \delta \ll 1$

$$\dim \operatorname{im} \iota = \operatorname{index} L_{A,\delta} = \tfrac{1}{2} \dim \ker D_{A_\infty}.$$

Suppose $\mathcal{H}^0_{A_\infty} = 0$. If $(\xi, a) \in \mathcal{T}_A$, then $\mathrm{d}_A^* \mathrm{d}_A \xi = 0$ and, by Proposition 3.22, ξ decays exponentially. Integration by parts shows that $\mathrm{d}_A \xi = 0$; hence $\xi = 0$. Therefore $\mathcal{T}_A \subset \Omega^1(Y, \mathfrak{g}_E)$.

We show that $\operatorname{im}\iota$ is isotropic: for $a, b \in \mathcal{T}_A$,

$$\frac{1}{2}\int_Z \langle \iota(a) \wedge \iota(b)\rangle \wedge \omega \wedge \omega = \int_Y \mathrm{d}(\langle a \wedge b\rangle \wedge \psi) = 0$$

because $\mathrm{d}_A a \wedge \psi = \mathrm{d}_A b \wedge \psi = 0$. □

3.4 Gluing G_2–instantons over ACyl G_2–manifolds

In the situation of Proposition 3.23, if $\ker\iota = 0$ and $\mathcal{H}^0_{A_\infty} = 0$, then one can show that the moduli space $\mathcal{M}(Y)$ of G_2–instantons near $[A]$ which are asymptotic to some HYM connection is smooth. Although the moduli space $\mathcal{M}(Z)$ of HYM connections near $[A_\infty]$ is not necessarily smooth, formally, it still makes sense to talk about its symplectic structure and view $\mathcal{M}(Y)$ as a Lagrangian submanifold. The following theorem shows, in particular, that transverse intersections of a pair of such Lagrangians give rise to G_2–instantons.

Theorem 3.24 *Let $(Y_\pm, \phi_\pm)$ be a pair of ACyl G_2–manifolds which match via $f\colon Z_+ \to Z_-$. Denote by $(Y_T, \phi_T)_{T \geq T_0}$ the resulting family of compact G_2–manifolds arising from the construction in Section 2.1. Let $A_\pm$ be a pair of G_2–instantons on $E_\pm$ over $(Y_\pm, \phi_\pm)$ asymptotic to $A_{\infty,\pm}$. Suppose that the following hold:*

- *There is a bundle isomorphism $\bar{f}\colon E_{\infty,+} \to E_{\infty,-}$ covering f such that $\bar{f}^* A_{\infty,-} = A_{\infty,+}$.*
- *The maps $\iota_\pm\colon \mathcal{T}_{A_\pm} \to \ker D_{A_{\infty,\pm}}$ constructed in Proposition 3.22 are injective and their images intersect trivially:*

$$\operatorname{im}(\iota_+) \cap \operatorname{im}(\bar{f}^* \circ \iota_-) = \{0\} \subset \mathcal{H}^0_{A_{\infty,+}} \oplus \mathcal{H}^1_{A_{\infty,+}}. \tag{3.25}$$

Then there exists $T_1 \geq T_0$ and for each $T \geq T_1$ there exists an irreducible and unobstructed G_2–instanton A_T on a G–bundle E_T over (Y_T, ϕ_T).

Proof The proof proceeds in three steps. We first produce an approximate G_2–instanton $\tilde{A}_T$ by an explicit cut-and-paste procedure. This reduces the problem to solving the nonlinear partial differential equation

$$\mathrm{d}^*_{\tilde{A}_t} a = 0 \quad \text{and} \quad \mathrm{d}_{\tilde{A}_T + a}\xi + *_T(F_{\tilde{A}_T + a} \wedge \psi_T) = 0 \tag{3.26}$$

for $a \in \Omega^1(Y_T, \mathfrak{g}_{E_T})$ and $\xi \in \Omega^0(Y_T, \mathfrak{g}_{E_T})$, where $\psi_T := *\phi_T$. Under the hypotheses of Theorem 3.24 we will show that we can solve the linearization of (3.26) in a uniform fashion. The existence of a solution of (3.26) then follows from a simple application of Banach's fixed-point theorem.

Step 1 There exists a $\delta < 0$ and for each $T \geq T_0$ there exists a connection $\tilde{A}_T$ on a G–bundle E_T over Y_T such that

$$\|F_{\tilde{A}_T} \wedge \psi_T\|_{C^{0,\alpha}} = O(e^{\delta T}). \tag{3.27}$$

The bundle E_T is constructed by gluing $E_\pm|_{Y_{T,\pm}}$ via $\bar{f}$ and the connection $\tilde{A}_T$ is defined by

$$\tilde{A}_T := A_\pm - \bar{\pi}_\pm^*[\chi(t - T + 1)a_\pm]$$

over $Y_{T,\pm}$ with

$$a_\pm := \bar{\pi}_{\pm,*}A_\pm - A_{\infty,\pm},$$

$\bar{\pi}_\pm$ is as in Definition 3.19 and χ is as in Section 2.1. Then (3.27) is a straightforward consequence of (2.5) and (3.20).

Step 2 Define a linear operator $L_T\colon C^{1,\alpha} \to C^{0,\alpha}$ by (3.11) with $A = \tilde{A}_T$ and $\phi = \phi_T$. Then there exist constants $\tilde{T}_1, c > 0$ such that for all $T \geq \tilde{T}_1$ the operator L_T is invertible and

$$\|L_T^{-1}\underline{a}\|_{C^{1,\alpha}} \leq ce^{(|\delta|/4)T}\|\underline{a}\|_{C^{0,\alpha}}. \tag{3.28}$$

Step 2.1 There exists a constant $c > 0$ such that for all $T \geq T_0$

$$\|\underline{a}\|_{C^{1,\alpha}} \leq c(\|L_T\underline{a}\|_{C^{0,\alpha}} + \|\underline{a}\|_{L^\infty}). \tag{3.29}$$

This is an immediate consequence of standard interior Schauder estimates because of (2.5) and (3.20).

Step 2.2 There exist constants $\tilde{T}_1 \geq T_0$ and $c > 0$ such that for $T \in [\tilde{T}_1, \infty)$

$$\|\underline{a}\|_{L^\infty} \leq ce^{(|\delta|/4)T}\|L_T\underline{a}\|_{C^{0,\alpha}}. \tag{3.30}$$

Suppose not; then there exist a sequence (T_i) tending to infinity and a sequence $(\underline{a}_i)$ such that

$$\|\underline{a}_i\|_{L^\infty} = 1 \quad \text{and} \quad \lim_{i\to\infty} e^{(|\delta|/4)T_i}\|L_{T_i}\underline{a}_i\|_{C^{0,\alpha}} = 0. \tag{3.31}$$

Then by (3.29),

$$\|\underline{a}_i\|_{C^{1,\alpha}} \leq 2c. \tag{3.32}$$

Hence, by Arzelà–Ascoli we can assume (by passing to a subsequence) that the sequence $\underline{a}_i|_{Y_{T_i,\pm}}$ converges in $C^{1,\alpha/2}_{\mathrm{loc}}$ to some section $\underline{a}_{\infty,\pm}$ of $(\Lambda^0 \oplus \Lambda^1) \otimes \mathfrak{g}_{E_\pm}$ over $Y_\pm$, which is bounded and satisfies

$$L_{A_\pm} \underline{a}_{\infty,\pm} = 0$$

because of (2.5) and (3.20). Using standard elliptic estimates, $\underline{a}_{\infty,\pm} \in \mathcal{T}_{A_\pm}$.

Proposition 3.33 *In the above situation,*

$$\lim_{i\to\infty} \|(\underline{a}_i|_{Y_{T_i,\pm}}) - (\underline{a}_{\infty,\pm}|_{Y_{T_i,\pm}})\|_{L^\infty(Y_{T_i,\pm})} = 0.$$

The proof of this proposition will be given at the end of this section. Accepting it as a fact for now, it follows immediately that

$$\iota_+(\underline{a}_{\infty,+}) = \bar{f}^* \circ \iota_-(\underline{a}_{\infty,-})$$

because $Y_{T_i,+} \cap Y_{T_i,-} = [T_i, T_i+1] \times Z_+$. Now, by (3.25) we must have $\iota_\pm(\underline{a}_{\infty,\pm}) = 0$; hence, $\underline{a}_{\infty,\pm} = 0$, since $\iota_\pm$ are injective.

However, by (3.31) there exist $x_i \in Y_{T_i}$ such that $|\underline{a}_{T_i}|(x_i) = 1$. By passing to a further subsequence and possibly changing the roles of $+$ and $-$ we can assume that each $x_i \in Y_{T_i,+}$; hence, by Proposition 3.33, $\underline{a}_{\infty,+} \neq 0$, contradicting what was derived above. This proves (3.30).

Step 2.3 We complete the proof of Step 2.

Combining (3.29) and (3.30) yields

$$\|\underline{a}\|_{C^{1,\alpha}} \le c e^{(|\delta|/4)T} \|L_T \underline{a}\|_{C^{0,\alpha}}.$$

Therefore, L_T is injective; hence, also surjective since L_T is formally self adjoint.

Step 3 There exists a constant $T_1 \ge \tilde{T}_1$ and for each $T \ge T_1$ a smooth solution $\underline{a} = \underline{a}_T$ of (3.26) such that $\lim_{T\to\infty} \|\underline{a}_T\|_{C^{1,\alpha}} = 0$.

We can write (3.26) as

$$L_T \underline{a} + Q_T(\underline{a}) + \varepsilon_T = 0 \tag{3.34}$$

where $Q_T(\underline{a}) := \frac{1}{2} *_T([a \wedge a] \wedge \psi_T) + [a, \xi]$ and $\varepsilon_T := *_T(F_{\tilde{A}_T} \wedge \psi_T)$. We make the ansatz $\underline{a} = L_T^{-1} \underline{b}$. Then (3.34) becomes

$$\underline{b} + \tilde{Q}_T(\underline{b}) + \varepsilon_T = 0 \tag{3.35}$$

with $\tilde{Q}_T = Q_T \circ L_T^{-1}$. By (3.28),

$$\|\tilde{Q}_T(\underline{b}_1) - \tilde{Q}_T(\underline{b}_2)\|_{C^{0,\alpha}} \le ce^{(|\delta|/2)T}(\|\underline{b}_1\|_{C^{0,\alpha}} + \|\underline{b}_2\|_{C^{0,\alpha}})\|\underline{b}_1 - \underline{b}_2\|_{C^{0,\alpha}}$$

for some constant $c > 0$ independent of $T \ge \tilde{T}_1$. By Step 1, $\|\varepsilon_T\|_{C^{0,\alpha}} = O(e^{\delta T})$. Now, Lemma 3.36 yields the desired solution of (3.35) and thus of (3.26) provided $T \ge T_1$ for a suitably large $T_1 \ge \tilde{T}_1$. By elliptic regularity $\underline{a}$ is smooth. □

Lemma 3.36 (Donaldson and Kronheimer [7, Lemma 7.2.23]) *Let X be a Banach space and let $T\colon X \to X$ be a smooth map with $T(0) = 0$. Suppose there is a constant $c > 0$ such that*

$$\|Tx - Ty\| \le c(\|x\| + \|y\|)\|x - y\|.$$

If $y \in X$ satisfies $\|y\| \le \frac{1}{10c}$, then there exists a unique $x \in X$ with $\|x\| \le \frac{1}{5c}$ solving

$$x + Tx = y.$$

Moreover, this $x \in X$ satisfies $\|x\| \le 2\|y\|$.

To complete the proof of Theorem 3.24 it now remains to prove Proposition 3.33 for which we require the following result.

Proposition 3.37 *In the situation of Theorem 3.24, there is a $\gamma_0 > 0$ such that for each $\gamma \in (0, \gamma_0)$ the linear operator $L_{A_\pm}\colon C^{1,\alpha}_\gamma \to C^{0,\alpha}_\gamma$ has a bounded right inverse.*

Proof By Proposition 3.4, $L_{A_\pm}\colon C^{1,\alpha}_\gamma \to C^{0,\alpha}_\gamma$ is Fredholm whenever $\gamma > 0$ is sufficiently small. The cokernel of $L_{A_\pm}$ can be identified to be $\mathcal{T}_{A_\pm,-\gamma}$, which is trivial by hypothesis. □

Proof of Proposition 3.33 We restrict to the $+$ case; the $-$ case is identical. It follows from the construction of $\underline{a}_{\infty,+}$ that for each fixed compact subset $K \subset Y_+$

$$\lim_{i\to\infty} \|(\underline{a}_i|_K) - (\underline{a}_{\infty,+}|_K)\|_{L^\infty(K)} = 0.$$

To strengthen this to an estimate on all of $Y_{T_i,+}$ the factor $e^{\frac{|\delta|}{4}T}$ in (3.31) will be important, even though it is clearly not optimal.

With χ as in Section 2.1 define a cut-off function $\chi_T\colon Y_+ \to [0,1]$ by $\chi_T(x) := 1 - \chi(t_+(x) - \frac{3}{2}T)$. For each sufficiently small $\gamma > 0$ we have

$$\|L_{A_+}(\chi_{T_i}\underline{a}_i)\|_{C^{0,\alpha}_\gamma(Y_+)} = O(e^{-(3/2)\gamma T_i})$$

using the estimates (2.5), (3.20), (3.31) and (3.32). Using Proposition 3.37 we construct $\underline{b}_i \in C^{1,\alpha}_\gamma$ such that $\underline{a}^i_{\infty,+} := \chi_{T_i}\underline{a}_i + \underline{b}_i \in \mathcal{T}_{A_+,\gamma}$ and $\|\underline{b}_i\|_{C^{1,\alpha}_{0,\gamma}} = O(e^{-(3/2)\gamma T_i})$. Hence,

$$\|(\underline{a}_i|_{Y_{T_i,+}}) - (\underline{a}^i_{\infty,+}|_{Y_{T_i,+}})\|_{L^\infty(Y_{T_i,+})} = O(e^{-(1/2)\gamma T_i}).$$

Moreover, $\lim_{i\to\infty}\|(\underline{a}^i_{\infty,+}|_K) - (\underline{a}_{\infty,+}|_K)\|_{L^\infty(K)} = 0$ and since both $\|\cdot\|_{L^\infty(K)}$ and $\|\cdot\|_{L^\infty(Y_+)}$ are norms on the finite-dimensional vector space $\mathcal{T}_{A_+,\gamma} = \mathcal{T}_{A_+}$ it also follows that

$$\lim_{i\to\infty}\|\underline{a}^i_{\infty,+} - \underline{a}_{\infty,+}\|_{L^\infty(Y_+)} = 0.$$

Therefore,

$$\lim_{i\to\infty}\|(\underline{a}_i|_{Y_{T_i,+}}) - (\underline{a}_{\infty,+}|_{Y_{T_i,+}})\|_{L^\infty(Y_{T_i,+})} = 0. \qquad \square$$

Remark 3.38 The proof of Theorem 3.24 slightly simplifies assuming $\mathcal{H}^0_{A_{\infty,+}} \oplus \mathcal{H}^1_{A_{\infty,+}} = \{0\}$ instead of (3.25): We can directly conclude that $\iota_\pm(\underline{a}_{\infty,\pm}) = 0$ and, hence, $\underline{a}_{\infty,\pm} = 0$; thus making Proposition 3.33 unnecessary. In particular, (3.30) holds without the additional factor of $e^{(|\delta|/4)T}$.

4 From holomorphic vector bundles over building blocks to G_2–instantons over ACyl G_2–manifolds

We now discuss how to deduce Theorem 1.3 from Theorem 3.24.

Definition 4.1 Let (V,ω,Ω) be an ACyl Calabi–Yau 3–fold with asymptotic cross section $(\Sigma,\omega_I,\omega_J,\omega_K)$. Let A_∞ be an ASD instanton on a G–bundle E_∞ over Σ. An HYM connection A on a G–bundle E over V is called *asymptotic to* A_∞ if there exist a constant $\delta < 0$ and a bundle isomorphism $\bar{\pi}\colon E|_{V\setminus K} \to E_\infty$ covering $\pi\colon V\setminus K \to \mathbb{R}_+ \times S^1 \times \Sigma$ such that

$$\nabla^k(\bar{\pi}_* A - A_\infty) = O(e^{\delta t})$$

for all $k \in \mathbb{N}_0$. Here by a slight abuse of notation we also denote by E_∞ and A_∞ their respective pullbacks to $\mathbb{R}_+ \times S^1 \times \Sigma$.

The following theorem can be used to produce examples of HYM connections A on $\mathbb{P}\mathrm{U}(n)$–bundles over ACyl Calabi–Yau 3–folds asymptotic to ASD instantons A_∞; hence, by taking the product with S^1, examples of G_2–instantons $\pi_V^* A$ asymptotic to $\pi_\Sigma^* A_\infty$ over the ACyl G_2–manifold $S^1 \times V$. Here $\pi_V\colon S^1\times V \to V$ and $\pi_\Sigma\colon T^2 \times \Sigma \to \Sigma$ denote the canonical projections.

Theorem 4.2 (Sá Earp [10, Theorem 59]) *Let Z and Σ be as in Theorem 2.10 and let $(V := Z \setminus \Sigma, \omega, \Omega)$ be the resulting ACyl Calabi–Yau 3–fold. Let $\mathcal{E}$ be a holomorphic vector bundle over Z and let A_∞ be an ASD instanton on $\mathcal{E}|_\Sigma$ compatible with the holomorphic structure. Then there exists an HYM connection A on $\mathcal{E}|_V$ which is compatible with the holomorphic structure and asymptotic to A_∞.*

By slight abuse of notation we also denote by A_∞ the ASD instanton on the $\mathbb{P}\mathrm{U}(n)$–bundle associated with $\mathcal{E}|_\Sigma$ and by A the HYM connection on the $\mathbb{P}\mathrm{U}(n)$–bundle associated with $\mathcal{E}|_V$. Theorems 3.24 and 4.2 together with the following result immediately imply Theorem 1.3.

Proposition 4.3 *In the situation of Theorem 4.2, suppose $H^0(\Sigma, \mathcal{E}nd_0(\mathcal{E}|_\Sigma)) = 0$. Then*

$$\mathcal{H}^1_{\pi_\Sigma^* A_\infty} = H^1_{A_\infty}, \tag{4.4}$$

see Definition 3.16 and Remark 1.6, and for some small $\delta > 0$ there exist injective linear maps

$$\kappa_-\colon \mathcal{T}_{\pi_V^* A, -\delta} \to H^1(Z, \mathcal{E}nd_0(\mathcal{E})(-\Sigma)),$$
$$\kappa\colon \mathcal{T}_{\pi_V^* A} \to H^1(Z, \mathcal{E}nd_0(\mathcal{E}))$$

such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{T}_{\pi_V^* A,-\delta} \arrow[r] \arrow[d, "\kappa_-"] & \mathcal{T}_{\pi_V^* A} \arrow[r, "\iota"] \arrow[d, "\kappa"] & \mathcal{H}^1_{\pi_\Sigma^* A_\infty} \arrow[d, "\cong"] \\ H^1(Z, \mathcal{E}nd_0(\mathcal{E})(-\Sigma)) \arrow[r] & H^1(Z, \mathcal{E}nd_0(\mathcal{E})) \arrow[r] & H^1(\Sigma, \mathcal{E}nd_0(\mathcal{E}|_\Sigma)) \end{tikzcd} \tag{4.5}$$

Equation (4.4) is a direct consequence of $\mathcal{H}^0_{A_\infty} = 0$. The proof of the remaining assertions requires some preparation.

4.1 Comparing infinitesimal deformations of $\pi_V^* A$ and A

Proposition 4.6 *If A is an HYM connection asymptotic to A_∞, then there exists a $\delta_0 > 0$ such that for all $\delta \le \delta_0$*

$$\mathcal{T}_{\pi_V^* A, \delta} = \left\{ \underline{a} \in \ker D_A \;\middle|\; \nabla^k \bar{\pi}_* \underline{a} = O(e^{\delta t}) \text{ for all } k \in \mathbb{N}_0 \right\}, \tag{4.7}$$

with D_A as in (3.14).

Proof We can write $L_A = \tilde{I}\partial_\beta + D_A$ where β denotes the coordinate on S^1. For $\delta \le 0$, (4.7) follows by an application of [23, Lemma A.1] by the second author. The right-hand side is contained in the left-hand side of (4.7) which, by Proposition 3.22, is independent of $\delta \in [0, \delta_0]$. $\square$

Proposition 4.8 *In the situation of Proposition 4.3, there exists a constant $\delta_0 > 0$ such that, for all $\delta \le \delta_0$, $\mathcal{H}^0_{A,\delta} = 0$ and*

$$\mathcal{T}_{\pi_V^* A,\delta} \cong \mathcal{H}^1_{A,\delta},$$

where

$$\mathcal{H}^i_{A,\delta} := \{\alpha \in \mathcal{H}^i_A \mid \nabla^k \overline{\pi}_* \alpha = O(e^{\delta t}) \text{ for all } k \in \mathbb{N}_0\}.$$

Proof If $\delta \le \delta_0$ (cf. Proposition 3.22) and $(\xi, \eta, a) \in \mathcal{T}_{A,\delta}$, then $\iota(\xi, \eta, a) \in \{0\} \oplus \mathcal{H}^1_{A_\infty}$. Hence ξ and η decay exponentially and one use can Proposition 4.6 and argue as in the proof of Proposition 3.18; it also follows that $\mathcal{H}^0_{A,\delta} = 0$. $\square$

4.2 Acyclic resolutions via forms of exponential growth/decay

In view of the above what is missing to prove Proposition 4.3 is a way to relate $\mathcal{H}^1_{A,\delta}$ with the cohomology of (twists of) $\mathcal{E}nd_0(\mathcal{E})$. This is what the following result provides.

Proposition 4.9 *Let (Z, Σ) be a building block and let $V := Z \setminus \Sigma$ be the ACyl Calabi–Yau 3–fold constructed via Theorem 2.10. Suppose that $\mathcal{E}$ is a holomorphic vector bundle over Z and suppose that A is an HYM connection on $\mathcal{E}$ compatible with the holomorphic structure and asymptotic to an ASD instanton on $\mathcal{E}|_\Sigma$.*

For $\delta \in \mathbb{R}$ define a complex of sheaves $(\mathcal{A}^\bullet_\delta, \overline{\partial})$ on Z by

$$\mathcal{A}^i_\delta(U) := \{\alpha \in \Omega^{0,i}(V \cap U, \mathcal{E}) \mid \nabla^k \overline{\pi}_* \alpha = O(e^{\delta t}) \text{ for all } k \in \mathbb{N}_0\}. \tag{4.10}$$

If $\delta \in \mathbb{R} \setminus \mathbb{Z}$, then the complex of sheaves $(\mathcal{A}^\bullet_\delta, \overline{\partial})$ is an acyclic resolution of $\mathcal{E}(\lfloor \delta \rfloor \Sigma)$. In particular, setting $\kappa^i_\delta(\alpha) := [\alpha]$ one obtains maps

$$\kappa^i_\delta \colon \mathcal{H}^i_{A,\delta} \to H^i(\Gamma(\mathcal{A}^\bullet_\delta), \overline{\partial}) \cong H^i(Z, \mathcal{E}(\lfloor \delta \rfloor \Sigma)).$$

Remark 4.11 In Proposition 4.9, $\lfloor \delta \rfloor$ denotes the largest integer not greater than δ; in particular, $\lfloor \delta \rfloor \Sigma$ is a divisor on Z.

Remark 4.12 We state Proposition 4.9 in dimension three; however, it works *mutatis mutandis* in all dimensions.

Proof of Proposition 4.9 The proof consists of three steps.

Step 1 The sheaves $\mathcal{A}_\delta^\bullet$ are C^∞–modules, hence, acyclic; see Demailly [4, Chapter IV, Corollary 4.19].

Step 2 We have $\mathcal{E}(\lfloor\delta\rfloor\Sigma) = \ker\big(\bar\partial\colon \mathcal{A}_\delta^0 \to \mathcal{A}_\delta^1\big)$.

Let $x \in Z$ and let $U \subset Z$ denote a small open neighborhood of x. An element $s \in \ker(\bar\partial\colon \Gamma(U,\mathcal{A}_\delta^0) \to \Gamma(U,\mathcal{A}_\delta^1))$ corresponds to a holomorphic section of $\mathcal{E}|_{V\cap U}$ such that $|z|^{-\delta}s$ stays bounded. Here z is a holomorphic function on U vanishing to first order along $\Sigma\cap U$, whose existence follows from Definition 2.8. Then $z^{-\lfloor\delta\rfloor}s$ is weakly holomorphic in U. By elliptic regularity $z^{-\lfloor\delta\rfloor}s$ extends across $U\cap\Sigma$ and thus s defines an element of $\Gamma(U,\mathcal{E}(\lfloor\delta\rfloor\Sigma))$. Conversely, it is clear that $\Gamma(U,\mathcal{E}(\lfloor\delta\rfloor\Sigma)) \subset \ker\big(\bar\partial\colon \Gamma\big(U,\mathcal{A}_\delta^0\big) \to \Gamma\big(U,\mathcal{A}_\delta^1\big)\big)$.

Step 3 The complex of sheaves $\big(\mathcal{A}_\delta^\bullet,\bar\partial\big)$ is exact.

Away from Σ the exactness follows from the usual $\bar\partial$–Poincaré Lemma. If $x\in\Sigma$, then since Z is fibred over $\mathbb{P}^1$, by Definition 2.8, there exist a small open neighborhood U of x in Z, a polydisc $D\subset\Sigma$ centered at x and a biholomorphic map $\pi\colon V\cap U \to \mathbb{R}_+\times S^1\times D$ such that the pushforward of the Kähler metric on $V\cap U$ via π is asymptotic to the metric induced by that on D. The necessary version of the $\bar\partial$–Poincaré lemma can now be proved along the lines of Griffiths and Harris [11, page 25] provided the linear operator

$$\bar\partial\colon C_\delta^\infty\Omega^0(\mathbb{R}\times S^1) \to C_\delta^\infty\Omega^{0,1}(\mathbb{R}\times S^1)$$

is invertible. This, however, is a simple consequence of Theorem 3.1 since $\bar\partial = \partial_t + i\,\partial_\alpha$ and the spectrum of $i\,\partial_\alpha$ on $S^1=\mathbb{R}/\mathbb{Z}$ is $\mathbb{Z}$. □

4.3 Proof of Proposition 4.3

In view of Proposition 4.8 we only need to establish (4.5) with $\mathcal{H}^1_{A,\delta}$ instead of $\mathcal{T}_{\pi_V^*A,\delta}$. By Proposition 4.9 applied to $\mathcal{E}nd_0(\mathcal{E})$, we have linear maps

$$\kappa_\delta^1\colon \mathcal{H}^1_{A,\delta} \to H^1(Z,\mathcal{E}nd_0(\mathcal{E})(\lfloor\delta\rfloor\Sigma)) \quad \text{for } \delta\in\mathbb{R}\setminus\mathbb{Z};$$

hence, linear maps

$$\kappa_-\colon \mathcal{H}^1_{A,-\delta} \to H^1(Z,\mathcal{E}nd_0(\mathcal{E})(-\Sigma)),$$
$$\kappa\colon \mathcal{H}^1_{A} = \mathcal{H}^1_{A,\delta} \to H^1(Z,\mathcal{E}nd_0(\mathcal{E}))$$

for some small $\delta > 0$ making the following diagram commute:

$$\begin{tikzcd}
\mathcal{H}^1_{A,-\delta} \arrow[r] \arrow[d, "\kappa_-"] & \mathcal{H}^1_A \arrow[r, "\iota"] \arrow[d, "\kappa"] & H^1_{A_\infty} \arrow[d, "\cong"] \\
H^1(Z, \mathcal{E}nd_0(\mathcal{E})(-\Sigma)) \arrow[r] & H^1(Z, \mathcal{E}nd_0(\mathcal{E})) \arrow[r] & H^1(\Sigma, \mathcal{E}nd_0(\mathcal{E}|_\Sigma))
\end{tikzcd}$$

The map κ_- is injective, because if $\kappa_- a = 0$, then $a = \bar\partial s$ for some $s \in \Gamma\big(Z, \mathcal{A}^0_{-\delta}\big)$ and thus

$$\int_V \|a\|^2 = \int_V \langle a, \bar\partial s\rangle = \int_V \langle \bar\partial^* a, s\rangle = 0.$$

Since $H^0(\Sigma, \mathcal{E}nd_0(\mathcal{E}|_\Sigma)) = 0$, the first map on the bottom is injective and because the rows are exact a simple diagram chase proves shows that κ is injective. □

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