Equivariant covers for hyperbolic groups

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We prove an equivariant version of the fact that word-hyperbolic groups have finite asymptotic dimension. This is important in connection with our forthcoming proof of the Farrell–Jones conjecture for $K_*(RG)$ for every word-hyperbolic group G and every coefficient ring R.

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1 Introduction

The asymptotic dimension of a metric space X was introduced by Gromov in [11, p 29]. It can be defined as the smallest number N such that for every $\alpha > 0$ there exists an open cover $\mathcal{U}$ of X with the following properties:

- $\dim \mathcal{U} \leq N$;
- The Lebesgue number of $\mathcal{U}$ is at least α, ie, for every $x \in X$ there is $U \in \mathcal{U}$ such that $x^\alpha \subseteq U$, where x^α is the open ball of radius α around x;
- The members of $\mathcal{U}$ have uniformly bounded diameters.

Recall that a cover $\mathcal{U}$ is of dimension $\leq N$ if every $x \in X$ is contained in no more then $N+1$ members of $\mathcal{U}$. The asymptotic dimension of a finitely generated group is its asymptotic dimension as a metric space with respect to any word metric. An important result of Yu [19] asserts that the Novikov conjecture holds for groups of finite asymptotic dimension. This can be viewed as an injectivity result for the assembly map in L–theory (after inverting 2). Further injectivity results for assembly maps for groups with finite asymptotic dimension can be found in Bartels [1], Carlsson and Goldfarb [6] and Bartels and Rosenthal [4]. On the other hand no surjectivity statement of assembly maps is known for all groups of finite asymptotic dimension and this is very much related to the absence of any equivariance condition for the cover $\mathcal{U}$ as above.

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Definition 1.1 Let G be a group and Z be a G–space. Let $\mathcal{F}$ be a collection of subgroups of G. An open cover $\mathcal{U}$ of Z is called an *$\mathcal{F}$–cover* if the following two conditions are satisfied.

(i) For $g \in G$ and $U \in \mathcal{U}$ we have either $g(U) = U$ or $g(U) \cap U = \varnothing$;

(ii) For $g \in G$ and $U \in \mathcal{U}$ we have $g(U) \in \mathcal{U}$;

(iii) For $U \in \mathcal{U}$ the subgroup $G_U := \{g \in G \mid g(U) = U\}$ is a member of $\mathcal{F}$.

Let G be a word-hyperbolic group. Fix a set of generators S. Let d_G be the word metric on G with respect to S. Let X be a hyperbolic complex with an isometric G–action in the sense of Mineyev [14]; see Section 6.1. Let ∂X be the Gromov boundary of X. (This boundary can be described as a quotient of the set of geodesic rays in X, where two such rays are identified if they are asymptotic [5, III.H.3].) Let $\bar{X} := X \cup \partial X$ be the compactification of X [5, III.H.3]). Let $\mathcal{VC}\mathrm{yc}$ denote the collection of virtually cyclic subgroups of G, that is of subgroups that have a cyclic subgroup of finite index. The following is our main result and should be thought of as an equivariant version of the (much easier) fact that hyperbolic groups have finite asymptotic dimension [11, p 31; 17].

Theorem 1.2 *Let G be word-hyperbolic and let X be a hyperbolic complex. Suppose that there is a simplicial proper cocompact G–action on X. Equip $G \times \bar{X}$ with the diagonal G–action. Then there exists a natural number $N = N(G, \bar{X})$ depending only on G and $\bar{X}$ such that the following holds: For every $\alpha > 0$ there exists an open $\mathcal{VC}\mathrm{yc}$–cover $\mathcal{U}$ of $G \times \bar{X}$ satisfying*

(i) $\dim(\mathcal{U}) \leq N$;

(ii) *For $g_0 \in G$ and $c \in \bar{X}$ there exists $U \in \mathcal{U}$ such that $g_0^\alpha \times \{c\} \subseteq U$, where g_0^α is the open ball with center g_0 and radius α with respect to the word metric d_G;*

(iii) *$G\backslash\mathcal{U}$ is finite.*

This result plays an important role in our proof of the Farrell–Jones conjecture for $K_*(RG)$ for every word-hyperbolic group G and every coefficient ring R [3].

The conclusion of Theorem 1.2 is formally similar to the definition of finite asymptotic dimension discussed above. The price we have to pay for the equivariance of the cover $\mathcal{U}$ is the space $\bar{X}$. For the application it will be very important that $\bar{X}$ is compact. (If we replace $\bar{X}$ by a finite dimensional G–CW–complex all whose isotropy groups lie in $\mathcal{VC}\mathrm{yc}$, then the conclusion follows easily from the fact that G has finite asymptotic dimension.) The members of $\mathcal{U}$ are only large in the G–coordinate; in the $\bar{X}$–coordinate they may be very small. Similar covers have been used in a slightly different situation

where $\bar{X}$ is replaced by a probability space with a measure preserving action of G; compare Gromov [12, p 300] and Sauer [18]. It would be interesting to know if there is a version of Theorem 1.2 in this situation.

It seems reasonable to hope that the class of groups G for which there is a compact G–space $\bar{X}$ such that the conclusion of Theorem 1.2 holds is bigger than the class of hyperbolic groups.

The proof of Theorem 1.2 is quite involved and uses a generalization of techniques used and developed by Farrell–Jones in [7]. Firstly, we study flows on metric spaces and prove the existence of long and thin covers; see Theorem 1.4. This generalizes the long and thin cells from [7, Proposition 7.2]. Secondly, we use a variant $FS(X)$ of Mineyev's half open symmetric join $*\bar{X}$ [14]. This space is a substitute for the sphere bundle of a negatively curved manifold and is equipped with a flow ϕ_τ (corresponding to the geodesic flow on the sphere bundle). In Theorem 1.5 we improve upon Mineyev's flow estimate [14, Theorem 57 on page 468]. The required cover is then produced by pulling back a long and thin cover of $FS(X)$ by the composition of the flow ϕ_τ for large τ with an embedding $G\times\bar{X}\to FS(X)$. A more detailed discussion follows in Sections 1.1 and 1.2.

1.1 Long thin covers

The existence of long thin covers will be proven in the following situation.

Convention 1.3 Let

- G be a discrete group;
- X be a metrizable topological space with a proper cocompact G–action on X;
- $\Phi\colon X\times\mathbb{R}\to X$ be a flow.

Assume that the following conditions are satisfied:

- Φ is G–equivariant;
- The number of closed orbits, which are not stationary and whose period is $\leq C$, of the flow induced on $G\backslash X$ is finite for every $C>0$;
- $X-X^{\mathbb{R}}$ is locally connected (notation explained below);
- If we put

$$k_G := \sup\{|H| \mid H\subseteq G \text{ subgroup with finite order } |H|\},$$
$$d_X := \dim(X-X^{\mathbb{R}}),$$

then $k_G<\infty$ and $d_X<\infty$.

Recall that a G–action is *proper* if for every $x \in X$ there exists an open neighborhood U such that the set $\{g \in G \mid U \cap gU \neq \varnothing\}$ is finite. Recall that X is *locally connected* if for each $x \in X$ and each open neighborhood U of x we can find a connected open neighborhood U' of x with $U' \subseteq U$. A G–space X is called *cocompact* if $G\backslash X$ is compact. The *dimension of a collection of subsets* $\{U_i \mid i \in I\}$ is $\leq d$, if every point is contained in at most $d+1$ members of the U_i. The *covering dimension of a space* X is $\leq d$ if every open covering has an open refinement whose dimension is less or equal to d. One may replace the covering dimension d_X of $X - X^{\mathbb{R}}$ appearing above by the supremum of the covering dimensions of compact subsets of $X - X^{\mathbb{R}}$. Recall that an *equivariant flow* $\Phi\colon \mathbb{R} \times X \to X$ is a continuous $\mathbb{R}$–action, such that $\Phi_\tau(gx) = g\Phi_\tau(x)$ holds for all $g \in G$, $\tau \in \mathbb{R}$ and $x \in X$. We denote by $X^{\mathbb{R}}$ the $\mathbb{R}$–fixed point set, ie, the set of points $x \in X$ for which $\Phi_\tau(x) = x$ for all $\tau \in \mathbb{R}$. The period of a closed orbit of Φ which is not stationary is the smallest number $\tau > 0$ such that $\Phi_\tau(x) = x$ holds for all x in this orbit.

The following is our main result in this situation.

Theorem 1.4 *There exists a natural number N depending only on k_G, d_X and the action of G on an arbitrary small neighborhood of $X^{\mathbb{R}}$ such that for every $\alpha > 0$ there is an $\mathcal{VC}yc$–cover $\mathcal{U}$ of X with the following two properties:*

(i) $\dim \mathcal{U} \leq N$;

(ii) *For every $x \in X$ there exists $U \in \mathcal{U}$ such that*

$$\Phi_{[-\alpha,\alpha]}(x) := \{\Phi_\tau(x) \mid \tau \in [-\alpha,\alpha]\} \subseteq U;$$

(iii) *$G\backslash\mathcal{U}$ is finite.*

The main difference between Theorem 1.4 and [7, Proposition 7.2] is that we deal with metric spaces rather than manifolds. This requires a different type of general position argument (compare Section 3) and forces us to work with open covers rather than cell structures. While cell structures of a manifold are automatically finite dimensional, in our situation more care is needed to establish the bound on the dimension of $\mathcal{U}$ and our bound is much larger then the dimension of the metric space X. Finally, we deal with an honest proper action and do not require a torsion free subgroup of finite index, as is used in [8].

The proof of Theorem 1.4 will be given in Section 5 and depends on Sections 2, 3 and 4.

1.2 The flow space

Let G be a hyperbolic group. Fix a set of generators S. Let d_G be the word metric on G with respect to S. Let X be a hyperbolic complex and $\bar{X} = X \cup \partial X$ be its compactification as before. Assume that G acts isometrically on X. In Section 6 we introduce the metric space $(FS(X), d_{FS})$. This space is equipped with an isometric G–action and a G–equivariant flow ϕ_τ.

Our main flow estimate is the following.

Theorem 1.5 *There exists a continuous G–equivariant (with respect to the diagonal G–action on the source) map $j\colon G\times\bar{X} \to FS(X)$ such that for every $\alpha > 0$ there exists a number $\beta = \beta(\alpha)$ such that the following holds:*

If $g, h \in G$ with $d_G(g,h) \le \alpha$ and $c \in \bar{X}$ then there is $\tau_0 \in \mathbb{R}$ with $|\tau_0| \le \beta$ such that for all $\tau \in \mathbb{R}$

$$d_{FS}(\phi_\tau j(g,c), \phi_{\tau+\tau_0} j(h,c)) \le f_\alpha(\tau).$$

Here $f_\alpha\colon \mathbb{R} \to [0,\infty)$ is a function that depends only on α and has the property that $\lim_{\tau\to\infty} f_\alpha(\tau) = 0$.

An important ingredient of the proof of this result is Theorem 7.1 which is an improvement of Mineyev's [14, Theorem 57 on page 468]. The main differences are that we consider points not necessary on the same horosphere, and that we consider the action of the flow ϕ_τ and not translation by length. In addition, Mineyev's estimate is in terms of a pseudo-metric, not in terms of the metric d_{FS}.

In order to apply Theorem 1.4 to $FS(X)$ we need further properties of the flow space and G.

Proposition 1.6

(i) *The order of finite subgroups in G is bounded.*

(ii) *$FS(X) - FS(X)^{\mathbb{R}}$ is locally connected and has finite covering dimension.*

(iii) *If the action of G on X is cocompact and proper, then action of G on $FS(X)$ is also cocompact and proper.*

(iv) *If the action of G on X is cocompact and proper, then the number of closed orbits, which are not stationary and whose period is $\le C$, of the flow induced on $G\backslash FS(X)$ is finite for every $C > 0$.*

The proof of Theorem 1.5 will be given in Section 8 and depends only on Sections 6 and 7. The proof of Proposition 1.6 will be given in Section 9 and depends only on Sections 2 and 6.

1.3 Construction of the cover

Using the results from Sections 1.1 and 1.2 we can now give the proof of Theorem 1.2. During this proof we will use the following notation: if A is a subset of a metric space Z and $\delta > 0$, then A^δ denotes the set of all points $z \in Z$ for which $d(z, A) < \delta$; compare Definition 3.1.

Proof Consider any $\alpha > 0$. Let $\beta = \beta(\alpha)$ be the number appearing in Theorem 1.5. It follows from Proposition 1.6 that Theorem 1.4 can be applied to $FS(X)$. Thus there is a number N (independent of α) such that there exists an $\mathcal{VC}$yc–cover $\mathcal{V}$ of $FS(X)$ of dimension no more than N with the following property: For every $\xi \in FS(X)$ there exists $V_\xi \in \mathcal{V}$ such that

$$\phi_{[-2\beta,2\beta]}(\xi) = \{\phi_\tau(\xi) \mid \tau \in [-2\beta, 2\beta]\} \subseteq V_\xi.$$

Since $\phi_{[-2\beta,2\beta]}(\xi)$ is compact, V_ξ is open and $\phi_{[-2\beta,2\beta]}(\xi) \subseteq V_\xi$, we can find $\delta_\xi > 0$ (depending on ξ and β, V_ξ) such that

$$\big(\phi_{[-2\beta,2\beta]}(\xi)\big)^{\delta_\xi} \subseteq V_\xi.$$

Because G acts by isometries, we can arrange that $\delta_\xi = \delta_{g\xi}$ holds for all $g \in G$. In particular we get $g \cdot (\phi_{[-2\beta,2\beta]}(\xi))^{\delta_\xi} = (\phi_{[-2\beta,2\beta]}(g\xi))^{\delta_{g\xi}}$. For $\xi \in FS(X)$ pick $\epsilon_\xi > 0$ such that

$$0 < e^\beta \epsilon_\xi < \delta_\xi/2.$$

Again we arrange that $\epsilon_{g\xi} = \epsilon_\xi$ holds for all $g \in G$. Obviously the collection

$$\Big\{\big(\phi_{[-\beta,\beta]}(\xi)\big)^{\epsilon_\xi} \mid \xi \in FS(X)\Big\}$$

is an open covering of $FS(X)$. Since G acts cocompactly, we can find finitely many points ξ_i for $i = 0, 1, 2, \ldots, I$ for some positive natural number I such that the G–cofinite collection

$$\Big\{\big(\phi_{[-\beta,\beta]}(g\xi_i)\big)^{\epsilon_{g\xi_i}} \mid g \in G, i \in \{0, 1, 2 \ldots, I\}\Big\}$$

is an open covering of $FS(X)$. Consider $\xi \in FS(X)$. Then we can find $i = i(\xi) \in \{0, 1, 2 \ldots, I\}$ and $g = g(\xi) \in G$ such that $\xi \in \big(\phi_{[-\beta,\beta]}(g\xi_i)\big)^{\epsilon_{g\xi_i}}$. In particular, there is $\tau \in [-\beta, \beta]$ such that $d_{FS}(\xi, \phi_\tau(g\xi_i)) < \epsilon_{g\xi_i}$. Let

$$\delta := \min\{\delta_{\xi_i}/2 \mid i = 0, 1, 2 \ldots, I\}.$$

Consider $\zeta \in \big(\phi_{[-\beta,\beta]}(\xi)\big)^\delta$. Choose $\sigma \in [-\beta, \beta]$ satisfying $d_{FS}(\zeta, \phi_\sigma(\xi)) < \delta$. In the following estimate we will use Lemma 7.2. (In this lemma the more careful notation

d_{FS,x_0} is used for d_{FS}.)

$$\begin{aligned} d_{FS}(\zeta, \phi_{\sigma+\tau}(g\xi_i)) &\leq d_{FS}(\zeta, \phi_\sigma(\xi)) + d_{FS}(\phi_\sigma(\xi), \phi_{\sigma+\tau}(g\xi_i)) \\ &< \delta + e^{|\sigma|} \cdot d_{FS}(\xi, \phi_\tau(g\xi_i)) \\ &< \delta + e^{\beta} \cdot \epsilon_{g\xi_i} \\ &< \delta_{g\xi_i} \end{aligned}$$

Since $\sigma + \tau \in [-2\beta, 2\beta]$, this implies

$$\big(\phi_{[-\beta,\beta]}(\xi)\big)^{\delta} \subseteq \big(\phi_{[-2\beta,2\beta]}(g\xi_i)\big)^{\delta_{g\xi_i}} \subseteq V_{g\xi_i}.$$

Thus we have found $\delta > 0$ such that for every $\xi \in FS(X)$ there exists $V_\xi \in \mathcal{V}$ such that

$$\big(\phi_{[-\beta,\beta]}(\xi)\big)^{\delta} \subseteq V_\xi. \tag{1.7}$$

We will construct the desired open covering $\mathcal{U}$ of $G \times \bar{X}$ by pulling back $\mathcal{V}$ with the composition

$$G \times \bar{X} \xrightarrow{j} FS(X) \xrightarrow{\phi_\tau} FS(X)$$

for an appropriate real number τ, where j is the map from Theorem 1.5. Obviously $\mathcal{U}$ has for every choice of τ all the desired properties except for the property that there exists $U_{(g_0,c)} \in \mathcal{U}$ such that $g_0^\alpha \times \{c\} \subseteq U_{(g_0,c)}$ for every $c \in \bar{X}$ and every $g_0 \in G$.

We conclude from Theorem 1.5 for $\tau \in \mathbb{R}$ and the function f_α appearing in Theorem 1.5

$$\phi_\tau \circ j(g,c) \in \big(\phi_{[-\beta,\beta]}(\phi_\tau \circ j(g_0,c))\big)^{f_\alpha(\tau)}$$

for all $c \in \bar{X}$ and all $g \in G$ with $d_G(g_0, g) < \alpha$. By Theorem 1.5 there is τ such that $f_\alpha(\tau) < \delta$. For such a choice of τ we conclude from (1.7) that

$$\phi_\tau \circ j(g,c) \in \big(\phi_{[-\beta,\beta]}(\phi_\tau \circ j(g_0,c))\big)^{\delta} \subset V_{\phi_\tau \circ j(g_0,c)}$$

for all $c \in \bar{X}$ and all $g \in G$ with $d_G(g, g_0) < \alpha$. This finishes the proof of Theorem 1.2. □

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2 Boxes

Convention 2.1 Throughout this section we consider

- a discrete group G;
- a metrizable topological space X;
- a proper cocompact G–action on X;
- a G–equivariant flow $\Phi\colon X \times \mathbb{R} \to X$ such that $X - X^{\mathbb{R}}$ is locally connected.

2.1 Basics about boxes

In this subsection we introduce and study the notion of a box.

Definition 2.2 Let B be a subset of a G–space. Define a subgroup of G by

$$G_B := \{g \in G \mid gB = B\},$$

where $gB := \{gb \mid b \in B\}$.

A subset B of a topological G–space is called an *$\mathcal{F}$–subset* for a collection $\mathcal{F}$ of subgroups of G, if G_B belongs to $\mathcal{F}$ and for all $g \in G$ we have the implication $gB \cap B \neq \varnothing \Rightarrow B = gB$.

Notice that $gB = B$ does *not* imply that $gb = b$ holds for all $b \in B$. We denote by $\mathcal{F}\mathrm{in}$ the collection of finite subgroups.

Definition 2.3 A *box* B is a subset $B \subseteq X$ with the following properties:

(i) B is a compact $\mathcal{F}\mathrm{in}$–subset;

(ii) There exists a real number $l = l_B > 0$, called the *length* of the box B, with the property that for every $x \in B$ there exists real numbers $a_-(x) \leq 0 \leq a_+(x)$ and $\epsilon(x) > 0$ satisfying

$$\begin{aligned} l &= a_+(x) - a_-(x); \\ \Phi_\tau(x) &\in B \ \text{ for } \tau \in [a_-(x), a_+(x)]; \\ \Phi_\tau(x) &\notin B \ \text{ for } \tau \in (a_-(x) - \epsilon(x), a_-(x)) \cup (a_+(x), a_+(x) + \epsilon(x)). \end{aligned}$$

Definition 2.4 Let $B \subseteq X$ be a box. Then the following data are associated to it:

- The *length* $l_B > 0$;
- Let $G_B \subseteq G$ be the finite subgroup $\{g \in G \mid gB = B\}$;

- We denote by B° the (topological) interior and by ∂B the (topological) boundary of $B \subseteq X$;
- Let $S_B \subseteq B$ be the set of points $\{x \in B \mid a_-(x) + a_+(x) = 0\}$. We call S_B the *central slice* of B;
- Let $\partial_\pm B$ be the set of points $\{x \in B \mid a_\pm(x) = 0\} = \{\phi_{a_\pm(x)}(x) \mid x \in S_B\}$. We call $\partial_- B$ the *bottom* and $\partial_+ B$ the *top* of B. Define the *open bottom* and *open top* $\partial_\pm B^\circ := \{\phi_{a_\pm(x)}(x) \mid x \in S_B \cap B^\circ\}$;
- Let $\pi_B\colon B \to S_B$ be the retraction onto the central slice which sends x to $\Phi_{(a_+(x)+a_-(x))/2}(x)$.

Remark 2.5 A box does not intersect $X^{\mathbb{R}}$ but may intersect a closed orbit. A box does never contain a closed orbit. It may happen that a nonclosed orbit meets the central slice infinitely many times, but whenever it meets the central slice it has to leave the box before it comes back to the central slice. We do not require that the central slice is connected. We have for $x \in B$, $\tau \in [a_-(x), a_+(x)]$ that $\Phi_\tau(x) \in B$ and

$$a_-(\Phi_\tau(x)) = a_-(x) - \tau;$$
$$a_+(\Phi_\tau(x)) = a_+(x) - \tau.$$

Lemma 2.6 *Let $B \subseteq X$ be a box of length $l = l_B$. Then the following holds:*

(i) *We get for $g \in G_B$ and $x \in X$*

$$a_-(gx) = a_-(x);$$
$$a_+(gx) = a_+(x).$$

The bottom $\partial_- B$, the open bottom $\partial_- B^\circ$, the top $\partial_+ B$, the open top $\partial_+ D^\circ$, the central slice S_B and the interior B° are $\mathcal{F}in$–subsets of G and satisfy (unless they are empty)

$$G_B = G_{B^\circ} = G_{\partial_- B} = G_{\partial_- B^\circ} = G_{\partial_+ B} = G_{\partial_+ B^\circ} = G_{S_B};$$

(ii) *The maps*

$$a_\pm\colon B \to \mathbb{R}, \quad x \mapsto a_\pm(x)$$

are continuous;

(iii) *The maps*

$$\mu\colon S_B \times [-l/2, l/2] \xrightarrow{\cong} B, \quad (x, \tau) \mapsto \Phi_\tau(x)$$

and $\mu^{-1}\colon B \xrightarrow{\cong} S_B \times [-l/2, l/2], \quad x \mapsto \left(\Phi_{\frac{a_-(x)+a_+(x)}{2}}(x), l/2 - a_+(x)\right)$

are to one another inverse G_B–homeomorphisms, where $G_B = G_{S_B}$ acts on $S_B \times [-l/2, l/2]$ by $g \cdot (s,t) = (gs,t)$.
We have

$$\begin{aligned} B^\circ &= \mu((S_B \cap B^\circ) \times (-l/2, l/2)); \\ \partial B &= \mu((S_B \cap \partial B) \times [-l/2, l/2] \cup (S_B \times \{-l/2, l/2\})); \\ \partial_\pm B &= \mu(S_B \times \{\pm l/2\}); \\ \partial_\pm B^\circ &= \mu((S_B \cap B^\circ) \times \{\pm l/2\}); \end{aligned}$$

(iv) *The space $S_B \cap B^\circ$ is locally connected;*

(v) *There exists $\epsilon_B > 0$ depending only on B such that the numbers $\epsilon(x)$ appearing in Definition 2.3 can be chosen so that $\epsilon(x) \geq \epsilon_B$ holds for all $x \in B$.*

Proof (i) For $x \in B$ and $g \in G_B$ we have $\Phi_\tau(x) \in B \Leftrightarrow g \cdot \Phi_\tau(x) = \Phi_\tau(gx) \in B$. This implies $a_\pm(gx) = a_\pm(x)$ for $x \in B$ and $g \in G_B$. We conclude from the definition of the bottom $\partial_- B$, the open bottom $\partial_- B^\circ$, the top $\partial_+ B$, the open top $\partial_+ B^\circ$, the central slice S_B and the interior B° that these sets are G_B invariant and contained in the $\mathcal{F}$in–subset B. Hence they are themselves $\mathcal{F}$in–subsets of X and satisfy $G_B = G_{B^\circ} = G_{\partial_- B} = G_{\partial_+ B} = G_{S_B}$ if nonempty.

(ii) Consider $x \in B$ and $\epsilon > 0$ with $\epsilon < \epsilon(x)$, where $\epsilon(x)$ is the number appearing in Definition 2.3. The points $\Phi_{a_\pm(x) \pm \epsilon}(x)$ lie outside B. Since B is compact and X is a Hausdorff space, we can find an open neighborhood $V_\pm$ of $\Phi_{a_\pm(x)\pm\epsilon}(x)$ such that $V_\pm$ does not meet B. Put $U_\pm = (\Phi_{a_\pm(x)\pm\epsilon})^{-1}(V_\pm)$. Then $x \in U_\pm$ and $\Phi_{a_\pm(x)\pm\epsilon}(u)$ does not lie in B for $u \in U_\pm$. This implies $a_+(u) < a_+(x) + \epsilon$ for $u \in U_+ \cap B$ and $a_-(x) - \epsilon < a_-(u)$ for $u \in U_- \cap B$. Put $U = U_- \cap U_+ \cap B$. Then $U \subseteq B$ is an open neighborhood of x in B such that $a_-(x) - \epsilon < a_-(u)$ and $a_+(u) < a_+(x) + \epsilon$ holds for $u \in U$. Since $a_+(u) - a_-(u) = l$ for all $u \in U$, we conclude $a_\pm(u) \in (a_\pm(x) - \epsilon, a_\pm(x) + \epsilon)$ for all $u \in U$. Hence $a_\pm$ is continuous.

(iii) The maps μ and μ^{-1} are continuous since Φ and by assertion (ii) the maps a_+ and a_- are continuous. One easily checks that they are inverse to one another.

Since the flow is compatible with the G–action and $G_B = G_{S_B}$, the map μ is $G_{S_B} = G_B$–equivariant.

Next we prove

$$\mu\left(\left(S_B \cap B^\circ\right) \times (-l/2, l/2)\right) \subseteq B^\circ; \tag{2.7}$$

$$\mu((S_B \cap \partial B) \times [-l/2, l/2] \cup (S_B \times \{-l/2, l/2\})) \subseteq \partial B. \tag{2.8}$$

Consider $(x,\tau) \in (S_B \cap B^\circ) \times (-l/2, l/2)$. Since a_- and a_+ are continuous by assertion (ii) and $a_-(x) = -l/2$ and $a_+(x) = l/2$, we can find an open neighborhood

$U \subseteq B^\circ$ of x such that $\tau \in (a_-(u), a_+(u))$ holds for all $u \in U$. Hence $\Phi_\tau(U)$ is contained in B. Since $\Phi_\tau(U)$ is an open subset of X, we have $\Phi_\tau(U) \subseteq B^\circ$. Since $\mu(x,\tau) = \Phi_\tau(x)$ lies in $\Phi_\tau(U)$, the inclusion (2.7) is proven.

Consider $x \in S_B$. Let $U \subseteq X$ be an open neighborhood of $\Phi_{l/2}(x)$. Since $\mathbb{R} \to X$, $\tau \mapsto \Phi_\tau(x)$ is a continuous map, there is an ϵ with $0 < \epsilon < l/2$ such that $\Phi_\tau(x) \in U$ holds for $\tau \in (l/2-\epsilon, l/2+\epsilon)$. Since $\{\Phi_\tau(x) \mid \tau \in (l/2-\epsilon, l/2)\}$ is contained in B and $\{\Phi_\tau(x) \mid \tau \in (l/2, l/2+\epsilon)\}$ is contained in $X - B$, the open neighborhood U of $\Phi_{l/2}(x)$ intersects both B and $X - B$. This shows $\Phi_{l/2}(x) = \mu(x, l/2) \in \partial B$. Analogously one proves $\Phi_{-l/2}(x) = \mu(x, -l/2) \in \partial B$.

Consider $x \in S_B \cap \partial B$ and $\tau \in (-l/2, l/2)$. We want to show $\mu(x,\tau) \in \partial B$. Suppose the converse. Since $\mu(x,\tau) = \Phi_\tau(x)$ belongs to B, there must be an open neighborhood U of $\Phi_\tau(x)$ such that $U \subseteq B$. Since the functions a_- and a_+ are continuous by assertion (ii) and $a_-(x) = -l/2 < -\tau < a_+(x) = l/2$, we can arrange by making U smaller that $-\tau \in (a_-(u), a_+(u))$ holds for all $u \in U$. Hence $\Phi_{-\tau}(U)$ is an open subset of X which is contained in B and contains x. This contradicts $x \in \partial B$. This finishes the proof of (2.8). Now assertion (iii) follows from (2.7) and (2.8).

(iv) Since B° is an open subset of the locally connected space $X - X^{\mathbb{R}}$, it is itself locally connected. Because of assertion (iii) the space $S_B \cap B^\circ \times (-l/2, l/2)$ is locally connected. Since the projection $S_B \cap B^\circ \times (-l/2, l/2) \to S_B \cap B^\circ$ is an open continuous map and the image of a connected set under a continuous maps is again connected, $S_B \cap B^\circ$ is locally connected.

(v) Suppose that such ϵ_B does not exists. Then we can find a sequence $(x_n)_{n\geq 0}$ of elements in B and a sequence $(\tau_n)_{n\geq 0} > 0$ of positive real numbers with $\lim_{n\to\infty} \tau_n = 0$ such that one of the following holds for $n \geq 0$:

(a) $\Phi_{a_-(x_n)-\tau_n}(x_n) \in B$ and $\Phi_{a_-(x_n)-\tau}(x_n) \notin B$ for $\tau \in (0, \tau_n)$

(b) $\Phi_{a_+(x_n)+\tau_n}(x_n) \in B$ and $\Phi_{a_+(x_n)+\tau}(x_n) \notin B$ for $\tau \in (0, \tau_n)$

By passing to a subsequence we can arrange that x_n converges to some point $x \in B$ and (a) holds for all $n \geq 0$ or (b) holds for all $n \geq 0$. We only treat the case (a), where $\Phi_{a_-(x_n)-\tau_n}(x_n) \in B$ and $\Phi_{a_-(x_n)-\tau}(x_n) \notin B$ for $\tau \in (0, \tau_n)$ holds for all $n \geq 0$, the proof in the other case (b) is analogous. Put $y_n = \Phi_{a_-(x_n)-\tau_n}(x_n)$. Then $y_n \in \partial_+ B$ for all $n \geq 0$ since $y_n \in B$ and $\Phi_\tau(y_n) = \Phi_{a_-(x_n)-\tau_n+\tau}(x_n) \notin B$ holds for $\tau \in (0, \tau_n)$. We conclude $\lim_{n\to\infty} a_-(x_n) = a_-(x)$ from assertion (ii). Hence $\lim_{n\to\infty} y_n = \Phi_{a_-(x)}(x)$. Since $y_n \in \partial_+ B$ for $n \geq 0$, we have $\lim_{n\to\infty} y_n \in \partial_+ B$. This contradicts $\Phi_{a_-(x)}(x) \in \partial_- B$ since $l_B > 0$. □

We mention that in general S_B itself is not locally connected.

Remark 2.9 It is a little bit surprising that the function $a_\pm$ is continuous as stated in Lemma 2.6 (ii) since there seem to be no link between different flow lines entering the box. The point here is that we require the box to be compact. If G is trivial and we consider the flow $\phi_\tau(x,y) = (x+\tau, y)$ on $\mathbb{R}^2$, the subset of $\mathbb{R}^2$ given by

$$B := \{(x,y) \mid x, y \in [-1,1], y \neq 0\} \cup \{(x,0) \mid x \in [0,2]\}$$

satisfies all the requirements of a box of length 2 except for compactness and the functions $a_\pm$ are not continuous at $(0,0)$.

Definition 2.10 Consider a box B of length l_B. Let $V \subseteq S_B$ be a nonempty closed $\mathcal{F}$in–subset of the G_{S_B}–space S_B and a, b real numbers with $-l_B/2 \leq a < b \leq l_B/2$. Define a new box of length $b-a$ by

$$B(V;a,b) := \Phi_{[a,b]}(V) := \{\Phi_\tau(v) \mid v \in V, \tau \in [a,b]\}.$$

If $a = -v/2$ and $b = v/2$ for some $v \in [0,w]$ we abbreviate

$$B(V;v) := B(V;-v/2, v/2).$$

If $a = -l_B/2$ and $b = l_B/2$, we abbreviate

$$B(V) := B(V;-l_B/2, l_B/2),$$

and call $B(V)$ the *restriction* of B to V.

We have to show that $B(V;a,b)$ is again a box. Since V is a closed subset of the compact set S_A, it is compact. Hence $V \times [a,b]$ is compact. We conclude that $B(V,a,b)$ as the image of a compact set under the continuous map $\Phi\colon X \times \mathbb{R} \to X$ is compact. From Lemma 2.6 we get $G_B = G_{S_B}$ and the G_B–equivariant homeomorphism

$$\mu_B\colon S_B \times [-l_B/2, l_B/2] \xrightarrow{\cong} B, \quad (x,\tau) \mapsto \Phi_\tau(x).$$

The subset V of the G_B–space S_B is a $\mathcal{F}$in–subset. Hence $B(V;a,b) = \mu_B(V \times [a,b])$ is a $\mathcal{F}$in–subset of the G_B–space B. Since B is a $\mathcal{F}$in–subset of the G–space X, $B(V;a,b)$ is a $\mathcal{F}$in–subset of the G–space X. Consider $x \in B(V;a,b)$. We can write it as $x = \Phi_\tau(v)$ for $v \in S_B$ and $\tau \in [a,b]$. Put $a_-(x) = a - \tau$ and $a_+(x) = b - \tau$. Let $\epsilon(x)$ for $x \in B$ be the number appearing in the Definition 2.3 of a box for B. Now one easily checks that the collections $a_\pm(x)$ and $\epsilon(x)$ have the desired properties of

Definition 2.3 for $B(V;a,b)$. This shows that $B(V;a,b)$ is a box. We have

$$\begin{aligned}\partial_- B(V;a,b) &= \Phi_a(V);\\ \partial_+ B(V;a,b) &= \Phi_b(V);\\ \partial_- B(V;a,b)^\circ &= \Phi_a(V\cap B^\circ);\\ \partial_+ B(V;a,b) &= \Phi_b(V\cap B^\circ);\\ S_{B(V;a,b)} &= \Phi_{\frac{a+b}{2}}(V).\end{aligned}$$

In particular $B(V;v)$ is a box of length v with central slice V and $B(V)$ is box of length l_B with central slice V.

2.2 Constructing boxes

Lemma 2.11 *For every $x\in X-X^{\mathbb{R}}$ there exists a nonequivariant box whose interior contains x.*

The following proof is a variation of an argument used in [16, Theorem 1.2.7].

Proof of Lemma 2.11 Because the G–action on the metrizable space X is proper and cocompact X is locally compact; compare [13, Theorem 1.38 on page 27]. Let $\alpha>0$ with $\Phi_\alpha(x)\neq x$. Let $W_\pm$ be a closed neighborhood of $\Phi_{\pm\alpha}(x)$ that does not contain x. By continuity there exist compact neighborhoods $U'\subset U$ of x and $\epsilon>0$ such that $\Phi_{[\pm\alpha-\epsilon,\pm\alpha+\epsilon]}U\subset W_\pm$ and $\Phi_{[-\epsilon,\epsilon]}U'\subset U$ and U is disjoint from $W_-\cup W_+$. Let $f\colon X\to[0,\infty)$ be a continuous function with $f(y)=1$ for $y\in U$ and $f(y)=0$ for $y\in W_-\cup W_+$. Define $\psi\colon U\to\mathbb{R}$ by

$$\psi(y)=\ln\left(\int_{-\alpha}^{\alpha} f(\Phi_\tau(y))e^{-\tau}d\tau\right).$$

(The logarithm makes sense because the integrant is nonnegative, continuous, and positive for $\tau=0$.) Let $y\in U'$. If $s\in[\pm\alpha-\epsilon,\pm\alpha+\epsilon]$ then $\Phi_s(y)\in W_\pm$ and therefore $f(\Phi_s(y))=0$. Using this we compute for $\delta\in[-\epsilon,\epsilon]$

$$\begin{aligned}\psi(\Phi_\delta(y)) &= \ln\left(\int_{-\alpha}^{\alpha} f(\Phi_{\tau+\delta}(y))e^{-\tau}d\tau\right)\\ &= \ln\left(\int_{-\alpha+\delta}^{\alpha+\delta} f(\Phi_\tau(y))e^{-\tau+\delta}d\tau\right)\\ &= \ln\left(e^{\delta}\int_{-\alpha+\delta}^{\alpha+\delta} f(\Phi_\tau(y))e^{-\tau}d\tau\right)\end{aligned}$$

$$= \delta + \ln\left(\int_{-\alpha}^{\alpha} f(\Phi_\tau(y))e^{-\tau}d\tau\right)$$
$$= \delta + \psi(y).$$

Define $S := U' \cap \psi^{-1}(\psi(x))$. Then $B := \{\Phi_\tau(s) \mid s \in S, \tau \in [-\epsilon/2, \epsilon/2]\}$ is a box whose interior contains x. □

Definition 2.12 Let C be a box of length l_C. Let

$$\mu_C\colon S_C \times [-l_C/2, l_C/2] \to C, \quad (x, \tau) \mapsto \Phi_\tau(x)$$

be the homeomorphism appearing in Lemma 2.6 (iii).

Consider a subset $S \subseteq C$. It is called *transversal to the flow with respect to* C if $\mu_C^{-1}(S) \cap \{x\} \times [-l_C/2, l_C/2]$ consists of at most one point for every $x \in S_C$.

Lemma 2.13 *Let C be box of length l_C. Let B be a box with $B \subseteq C$. Then we can find for every $x \in S_B$ a closed neighborhood $U \subseteq S_B$ of x satisfying*

(i) *U is a G_x–invariant $\mathcal{F}$in–subset of the G_B–space S_B;*

(ii) *U is transversal to the flow with respect to C.*

Proof Let $\tau_{S_B}\colon S_B \to [-l_C/2, l_C/2]$ be the continuous function given by the restriction to S_B of the composite of the projection $S_C \times [-l_C/2, l_C/2] \to [-l_C/2, l_C/2]$ and μ_C^{-1}. Let $U_1 \subseteq S_B$ be a closed neighborhood of $x \in S_B$ such that $|\tau_{S_B}(u) - \tau_{S_B}(x)| < l_B/2$ holds for $u \in U_1$. Choose a closed neighborhood $U_2 \subseteq S_B$ of x such that $gU_2 \cap U_2 \neq \varnothing \Rightarrow g \in G_x$ holds for $g \in G_B$. Put $U = \bigcap_{g \in G_x} g(U_1 \cap U_2)$. This is a closed neighborhood of x in S_B which is G_x–invariant, a $\mathcal{F}$in–subset of the G_B–space S_B and satisfies $|\tau_{S_B}(u) - \tau_{S_B}(x)| \leq l_B/2$ for $u \in U$.

It remains to show that U is transversal to the flow with respect to C. Suppose the converse. So we can find $u_0, u_1 \in U$, $v \in S_C$ and τ_0, τ_1 such that $u_0 = \Phi_{\tau_0}(v)$, $u_1 = \Phi_{\tau_1}(v)$ and $\tau_0 \neq \tau_1$. Note that $\tau_0 = \tau_{S_B}(u_0)$ and $\tau_1 = \tau_{S_B}(u_1)$. Since B is a box of length l_B, we can find for $i = 0, 1$ real numbers $\epsilon_i > 0$ such that $\Phi_{[-l_B/2, l_B/2]}(u_i) \subseteq B$ and $\Phi_\tau(u_i) \notin B$ for $\tau \in (-l_B/2 - \epsilon_i, -l_B/2) \cup (l_B/2, l_B/2 + \epsilon_i)$. This implies $|\tau_1 - \tau_0| > l_B$. But by the definition of U, $|\tau_1 - \tau_0| = |\tau_{S_B}(u_1) - \tau_{S_B}(u_0)| \leq |\tau_{S_B}(u_1) - \tau_{S_B}(x)| + |\tau_{S_B}(x) - \tau_{S_B}(u_0)| \leq l_B$. This is the required contradiction. □

Next we show for $x \in X$ that the existence of nonequivariant box containing x in its interior already implies the existence of an equivariant box containing x in its interior. The basic idea of proof is an averaging process in the time direction of the flow applied to the central slice of a nonequivariant box.

Lemma 2.14 *Suppose for the point $x \in X$ that there is a nonequivariant box whose interior contains x.*

Then there exists a box B in the sense of Definition 2.3 satisfying

(i) $G_B = G_{S_B} = G_x$;

(ii) $x \in S_B \cap B^\circ$;

(iii) *S_B is connected.*

Proof Let C be a nonequivariant box, ie a box in the sense of Definition 2.3 in the case, where the group G is trivial, such that $x \in C^\circ$. Let $l = l_C$ be the length of C. Since the G–action on X is proper by assumption and X is metrizable, we can find a closed neighborhood U of x such that U is a $\mathcal{F}$in–subset of X with $G_U = G_x$. We can assume without loss of generality that $x \in S_C \cap C^\circ$ and $C \subseteq U$ holds and for every $\tau \in [-l, -l/2] \cup [l/2, l]$ and $s \in S_C$ we have $\Phi_\tau(s) \notin C^\circ$, otherwise replace C by an appropriate restriction. Let S_C be the central slice of C. Let

$$\mu\colon S_C \times [-l/2, l/2] \to C, \quad (s, \tau) \mapsto \Phi_\tau(s)$$

be the homeomorphism of Lemma 2.6 (iii). Since S_C is compact, $C^\circ \subseteq X$ is open, G_x is finite and X is metrizable, we can find a compact neighborhood $S_0 \subseteq S_C \cap C^\circ$ of $x \in S_C \cap C^\circ$ such that $gS_0 \subseteq C^\circ$ holds for all $g \in G_x$. Define

$$S_1 = \bigcap_{g \in G_x} S_0 \cap \pi_C(\mu^{-1}(gS_0)),$$

where $\pi_C\colon S_C \times [-l/2, l/2] \to S_C$ is the projection. Then $S_1 \subseteq S_0$ is a compact neighborhood of x in S_0. By construction there exists for every $g \in G_x$ and $s \in S_1$ precisely one element $\tau_g(s) \in (-l/2, l/2)$ such that $\Phi_{\tau_g(s)}(s) \in gS_0$. The function $\tau_g(s)$ is continuous in s and has image in $(-l/2 + \delta, l/2 - \delta)$ for some δ with $0 < \delta < l/2$, since it is the restriction to S_1 of the continuous function with a compact source

$$\tau_g\colon S_0 \to (-l/2, l/2), \quad s \mapsto -\pi_C \circ \mu^{-1}(g^{-1}s).$$

Define the continuous function

$$\tau\colon S_1 \to (-l/2, l/2), \quad s \mapsto \frac{1}{|G_x|} \cdot \sum_{g \in G_x} \tau_g(s).$$

Put $$S_2 = \{\Phi_{\tau(s)}(s) \mid s \in S_1\}.$$

Next we show that $S_2 \subseteq C$ is G_x–invariant. Consider $g' \in G_x$ and $u \in S_2$. Write $u = \Phi_{\tau(s)}(s)$ for appropriate $s \in S_1$. Let $s' \in S_0$ be the element uniquely determined by $\Phi_{\tau_{(g')^{-1}}(s)}(s) = (g')^{-1}s'$. Then we get for $g \in G_x$

$$\Phi_{\tau_g(s)-\tau_{(g')^{-1}}(s)}(s') = \Phi_{\tau_g(s)} \circ \Phi_{-\tau_{(g')^{-1}}(s)}(s') = \Phi_{\tau_g(s)}(g's) = g'\Phi_{\tau_g(s)}(s) \in g'gS_0.$$

Since $g'gS_0 \subseteq C^\circ$ and $\tau_g(s)$ and $\tau_{(g')^{-1}}(s)$ belong to $(-l/2, l/2)$ and hence $\tau_g(s) - \tau_{(g')^{-1}}(s) \in (-l, l)$ we conclude $\tau_g(s) - \tau_{(g')^{-1}}(s) \in (-l/2, l/2)$. Hence $s' \in S_1$ and $\tau_{g'g}(s') = \tau_g(s) - \tau_{(g')^{-1}}(s)$. Since this implies $\tau(s') = \tau(s) - \tau_{(g')^{-1}}(s)$, we conclude

$$\begin{aligned} g' \cdot u &= g' \cdot \Phi_{\tau(s)}(s) = \Phi_{\tau(s)}(g's) = \Phi_{\tau(s')+\tau_{(g')^{-1}}(s)}(g's) \\ &= \Phi_{\tau(s')}\left(g' \cdot \Phi_{\tau_{(g')^{-1}}(s)}(s)\right) = \Phi_{\tau(s')}\left(g' \cdot (g')^{-1}s'\right) = \Phi_{\tau(s')}s' \in S_2. \end{aligned}$$

Since $S_1 \subseteq S_C \cap C^\circ$ is compact, S_2 is a compact G_x–invariant subset of C° with $x \in S_2$. Let S_3 be the component of S_2 which contains x. Then S_3 is a connected closed subset of S_2. Since $gS_3 \cap S_3$ contains x for $g \in G_x$, the subset S_3 is G_x–invariant. Thus $S_3 \subseteq C^\circ$ is a compact connected G_x–invariant subspace containing x.

We can find δ with $0 < \delta < l/2$ such that $B \subseteq C^\circ$ holds for $B := \Phi_{[-\delta/2,\delta/2]}(S_3)$.

Next we show that B is a box of length δ. Since S_3 is G_x–invariant and the flow Φ commutes with the G–action the subset $B \subseteq X$ is G_x–invariant. Recall that $B \subseteq C$ holds and C is a $\mathcal{F}$in–subset of X with $G_C = G_x$. Hence B is a compact $\mathcal{F}$in–subset of X. Consider $y \in B$. There is precisely one element $s \in S_3$ and $\tau \in [-\delta/2, \delta/2]$ satisfying $y = \Phi_\tau(s)$ since S_C and hence S_3 is transversal to the flow with respect to C. Put $a_-(y) = -\delta/2 - \tau$, $a_+(y) = \delta/2 - \tau$, $\epsilon_-(y) = \epsilon_+(y) = l/2$. Then

$$\begin{aligned} \delta &= a_+(y) - a_-(y); \\ \Phi_\tau(y) &\in B \quad \text{for } \tau \in [a_-(y), a_+(y)]; \\ \Phi_\tau(y) &\notin B \quad \text{for } \tau \in (a_-(y) - \epsilon(y), a_-(y)) \cup (a_+(y), a_+(y) + \epsilon(y)). \end{aligned}$$

Hence B is a box with connected central slice $S_B = S_3$. We have $x \in S_3$. The projection $\pi_C\colon C \to S_C$ induces a homeomorphism $S_2 \to S_1$ and maps the component S_3 of S_2 to a component S_1' of S_1. Since S_1 is an open neighborhood of x in the space $S_C \cap C^\circ$ which is locally connected by Lemma 2.6 (iv), the component S_1' is a neighborhood of x in the space $S_C \cap C^\circ$. Since τ is continuous we conclude from Lemma 2.6 (iii) that x lies in the interior of B. □

Definition 2.15 For $x \in X$ define its *G–period*

$$\mathrm{per}_\Phi^G(x) = \inf\{\tau \mid \tau > 0,\ \exists\, g \in G \text{ with } \Phi_\tau(x) = gx\} \quad \in [0, \infty],$$

where the infimum over the empty set is defined to be ∞. If $L \subseteq X$ is an orbit of the flow Φ, define its *G–period* by

$$\mathrm{per}_\Phi^G(L) := \mathrm{per}_\Phi^G(x)$$

for any choice of $x \in X$ with $L = \Phi_{\mathbb{R}}(x)$.

For $r \geq 0$ put

$$X_{>r} := \{x \in X \mid \mathrm{per}_\Phi^G(x) > r\};$$
$$X_{\leq r} := \{x \in X \mid \mathrm{per}_\Phi^G(x) \leq r\}.$$

Consider $x \in X$. Then the G–period $\mathrm{per}_\Phi^G(x)$ vanishes if and only if $x \in X^{\mathbb{R}}$. We have $\mathrm{per}_\Phi^G(x) = \infty$ if and only if the orbit through x is not periodic and $g\Phi_{\mathbb{R}}(x) \cap \Phi_{\mathbb{R}}(x) = \varnothing$ holds for all $g \neq 1$, or, equivalently, the orbit through Gx in the quotient space $G\backslash X$ with respect to the induced flow is not periodic. If $0 < \mathrm{per}_\Phi^G(x) < \infty$, then the properness of the G–action implies the existence of $g \in G$ such that $\Phi_{\mathrm{per}_\Phi^G(x)}(x) = gx$ and $\mathrm{per}_\Phi^G(x)$ is the period of the periodic orbit through Gx in the quotient space $G\backslash X$ with respect to the induced flow.

Next we show for a point x that we can find an equivariant box around a given compact part of the flow line, where the compact part is as long as the G–orbit length allows. The idea of proof is to take an equivariant box which contains x in its interior, making its central slice very small by restriction and then prolonging the box along the flow line though x.

Lemma 2.16 *Suppose for the point $x \in X$ that there is a nonequivariant box whose interior contains x. Consider a real number l with $0 < l < \mathrm{per}_\Phi^G(x)$.*

Then we can find a box C which satisfies

- $l_C = l$;
- $G_C = G_x$;
- $x \in S_C \cap C^\circ$;
- *S_C is connected.*

Proof From Lemma 2.14 we conclude the existence of a box B in the sense of Definition 2.3 which satisfies $G_B = G_x$, S_B is connected and $x \in S_B \cap B^\circ$. Let l_B be the length of B. From Lemma 2.6 (v) we obtain a number $\epsilon_B > 0$ such that for every $y \in S_B$ and $\tau \in (-l_B/2 - \epsilon_B, -l_B/2) \cup (l_B/2, l_B/2 + \epsilon_B)$ the element $\Phi_\tau(y)$ does not belong to B. We can arrange by restricting B and diminishing ϵ_B that $l_B < l$ and $l + \epsilon_B < \mathrm{per}_\Phi^G(x)$ holds.

Next we show

$$\Phi_{[-l/2,l/2]}(x) \cap g\Phi_{[-l/2,l/2]}(x) \neq \varnothing \Rightarrow g \in G_x.$$

Namely, consider $y \in \Phi_{[-l/2,l/2]}(x) \cap g\Phi_{[-l/2,l/2]}(x)$. Then $y = \Phi_\tau(x) = g\Phi_\sigma(x)$ for appropriate $\tau, \sigma \in [-l/2, l/2]$. This implies $\Phi_{\tau-\sigma}(x) = gx$ and $|\tau - \sigma| \leq l < \mathrm{per}^G_\Phi(x)$. We conclude $\tau - \sigma = 0$ and hence $g \in G_x$.

Since the G–action on X is proper, we can find a closed neighborhood $V^1_x \subseteq X$ of x such that $\Phi_{[-l/2,l/2]}(V^1_x) \cap g\Phi_{[-l/2,l/2]}(V^1_x) \neq \varnothing \Rightarrow g \in G_x$ holds. From $l + \epsilon_B < \mathrm{per}^G_\Phi(x)$ we conclude that

$$\Phi_{[-l/2+l_B,l/2]}(x) \cap \Phi_{[-l/2-\epsilon_B,-l/2]}(x) = \varnothing;$$
$$\Phi_{[-l/2,l/2-l_B]}(x) \cap \Phi_{[l/2,l/2+\epsilon_B]}(x) = \varnothing.$$

Since Φ is continuous and $[-l/2 + l_B, l/2]$, $[-l/2 - \epsilon_B, -l/2]$, $[-l/2, l/2 - l_B]$ and $[l/2, l/2 + \epsilon_B]$ are compact, we can find a closed neighborhood $V^2_x \subseteq X$ of x such that

$$\Phi_{[-l/2+l_B,l/2]}(V^2_x) \cap \Phi_{[-l/2-\epsilon_B,-l/2]}(V^2_x) = \varnothing;$$
$$\Phi_{[-l/2,l/2-l_B]}(V^2_x) \cap \Phi_{[l/2,l/2+\epsilon_B]}(V^2_x) = \varnothing.$$

Put
$$V_x = \bigcap_{g \in G_x} g \cdot \left(V^1_x \cap V^2_x\right).$$

Then $V_x \subseteq X$ is a closed G_x–invariant neighborhood of x with the properties

- $\Phi_{[-l/2,l/2]}(V_x)$ is a $\mathcal{F}$in–subset of the G–space X;
- $G_{\Phi_{[-l/2,l/2]}(V_x)} = G_x$;
- $\Phi_{[-l/2+l_B,l/2]}(V_x) \cap \Phi_{[-l/2-\epsilon_B,-l/2]}(V_x) = \varnothing$;
- $\Phi_{[-l/2,l/2-l_B]}(V_x) \cap \Phi_{[l/2,l/2+\epsilon_B]}(V_x) = \varnothing$.

Let $V^\circ_x \subseteq X - X^{\mathbb{R}}$ be the interior of V_x. Let T be the component of $S_B \cap B^\circ \cap V^\circ_x$ that contains x. Since $S_B \cap B^\circ$ is locally connected by Lemma 2.6 (iv) and $S_B \cap B^\circ \cap V^\circ_x$ is an open subset of $S_B \cap B^\circ$, the component T is an open subset of $S_B \cap B^\circ \cap V^\circ_x$ and hence of S_B. Let $\overline{T}$ be the closure of T in S_B. This is a closed connected G_x–invariant neighborhood of $x \in S_B$ which is contained in V_x. Since S_B is a $\mathcal{F}$in–subset of X with $G_{S_B} = G_x$, $\overline{T}$ is a $\mathcal{F}$in–subset of S_B and we can consider the restriction $B(\overline{T})$. We can assume without loss of generality that the central slice S_B is a G_x–invariant connected subset of V_x, otherwise replace B by the restriction $B(\overline{T})$.

We define $C := \Phi_{[-l/2,l/2]}(S_B)$.

Next we show that C is a box of length l. Since $C \subseteq \Phi_{[-l/2,l/2]}(V_x)$ and C is G_x–invariant, C is a compact $\mathcal{F}$in–subset of the G–space X. Consider $y \in C$. We can write it as $y = \Phi_{\tau_y}(s)$ for $\tau_y \in [-l/2, l/2]$ and $s \in S_B$. Put $a_-(y) = -l/2 - \tau_y$ and $a_+(y) = l/2 - \tau_y$. Obviously $l = a_+(y) - a_-(y)$ and $\Phi_\tau(y) \in C$ for $\tau \in [a_-(y), a_+(y)]$. It remains to show that $\Phi_{\tau'}(y) \notin C$ holds for $\tau' \in (a_-(y) - \epsilon_B, a_-(y)) \cup (a_+(y), a_+(y) + \epsilon_B)$. This is equivalent to showing that $\Phi_{\tau'}(s) \notin C$ holds for $\tau' \in (-l/2 - \epsilon_B, -l/2) \cup (l/2, l/2 + \epsilon_B)$. Since $s \in S_B \subseteq V_x$, we have

$$\Phi_{[-l/2+l_B,l/2]}(S_B) \cap \Phi_{[-l/2-\epsilon_B,-l/2]}(s) = \varnothing;$$
$$\Phi_{[-l/2,l/2-l_B]}(S_B) \cap \Phi_{[l/2,l/2+\epsilon_B]}(s) = \varnothing.$$

The main property of ϵ_B is

$$\Phi_{[-l_B/2,l_B/2]}(S_B) \cap \Phi_{(-l_B/2-\epsilon_B,-l_B/2)\cup(l_B/2,l_B/2+\epsilon_B)}(s) = \varnothing.$$

Applying $\Phi_{(l_B-l)/2}$ respectively $\Phi_{(l-l_B)/2}$ we obtain

$$\Phi_{[-l/2,-l/2+l_B]}(S_B) \cap \Phi_{(-l/2-\epsilon_B,-l/2)}(s) = \varnothing;$$
$$\Phi_{[l/2-l_B,l/2]}(S_B) \cap \Phi_{(l/2,l/2+\epsilon_B)}(s) = \varnothing.$$

We conclude

$$\Phi_{[-l/2,l/2]}(S_B) \cap \Phi_{(-l/2-\epsilon_B,-l/2)\cup(l/2,l/2+\epsilon_B)}(s) \;=\; \varnothing.$$

Hence C is a box of length l. By construction $S_C = S_B$ is connected, $G_C = G_{S_C} = G_x$ and $x \in S_B \cap B^\circ$ and hence $x \in S_C \cap C^\circ$. □

Lemma 2.17 *Consider real numbers $a, b, c > 0$ satisfying $c > a + 2b$. Let K be a cocompact G–invariant subset of $X_{>a+2b+2c}$.*

Then there exist a G–set Λ and for every $\lambda \in \Lambda$ boxes $A_\lambda \subseteq B_\lambda \subseteq C_\lambda$ such that:

(i) *Λ is G–cofinite;*

(ii) *We have*

$$l_{A_\lambda} = a;$$
$$l_{B_\lambda} = a + 2b;$$
$$l_{C_\lambda} = a + 2b + 2c;$$

(iii) *S_{C_λ} is connected;*

(iv) *We have $S_{A_\lambda} \subseteq S_{B_\lambda} \subseteq S_{C_\lambda}$;*

(v) *$A_\lambda \subseteq B_\lambda^\circ$ and $B_\lambda \subseteq C_\lambda^\circ$;*

(vi) *$K \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda^\circ$;*

(vii) $gA_\lambda = A_{g\lambda}$, $gB_\lambda = B_{g\lambda}$ and $gC_\lambda = C_{g\lambda}$ for $g \in G$;

(viii) If $B_\lambda \cap B_{\lambda'} \neq \varnothing$, then $B_\lambda \subseteq C^\circ_{\lambda'}$ and S_{B_λ} is transversal to the flow with respect to $C_{\lambda'}$.

Proof Lemma 2.16 implies that we can find for every $x \in X_{>a+2b+2c}$ a box C_x of length $a+2b+2c$ such that $x \in S_{C_x} \cap C^\circ_x$ and $G_{C_x} = G_x$ holds and S_{C_x} is connected. Since G_x is finite, we can find a G_x–invariant closed neighborhood T_x of x in S_{C_x} such that $T_x \subseteq S_{C_x} \cap C^\circ_x$ holds. We can arrange that $gC_x = C_{gx}$ and $gT_x = T_{gx}$ holds for $g \in G$ and $x \in X_{>a+2b+2c}$. Obviously $K \subseteq \bigcup_{x\in K} C_x(T_x; a)^\circ$, where we use the notation from Definition 2.10. Since K is cocompact and G–invariant, we can find a cofinite G–subset $I \subseteq K$ satisfying

$$K \subseteq \bigcup_{x\in I} C_x(T_x; a)^\circ. \tag{2.18}$$

Fix $x \in I$. Consider $y \in T_x$. Since the G–action is proper, $g \cdot C_z(T_z) = C_{gz}(T_{gz})$ holds for $z \in I$ and $g \in G$ and $C_x(\{y\}; a+2b)$ and $C_z(T_z; a+2b)$ are compact, we can find a closed G_y–invariant neighborhood $V_y \subseteq T_x$ of y such that for all $z \in I$

$$C_x(\{y\}; a+2b) \cap C_z(T_z; a+2b) = \varnothing \;\Rightarrow\; C_x(V_y; a+2b) \cap C_z(T_z; a+2b) = \varnothing.$$

For $y \in T_x$ we define

$$I_y = \{z \in I \mid C_x(\{y\}; a+2b) \cap C^\circ_z \neq \varnothing\}.$$

Since the set G–set I is cofinite, $C_x(\{y\}; a+2b)$ and C_z are compact and the G–action on X is proper, the set I_y is finite. From $a+2b < c$ and $l_{C_z} = a+2b+2c$ we conclude for $z \in I_y$ that $C_x(\{y\}; a+2b) = \Phi_{[-a/2-b,a/2+b]}(y)$ is contained in C°_z. Since $C_x(\{y\}; a+2b)$ is compact, we can find for $z \in I_y$ a closed G_y–invariant neighborhood $U_y(z) \subseteq T_x$ of y such that $C_x(U_y(z); a+2b) = \Phi_{[-a/2-b,a/2+b]}(U_y(z))$ is contained in C°_z. Because of Lemma 2.13 applied to $C_x(U_y(z); a+2b) \subseteq C_z$ we can assume without loss of generality that $U_y(z)$ is transversal to the flow with respect to C_z for every $z \in I_y$.

Put $U_y := V_y \cap \bigcap_{z\in I_y} U_y(z)$. Then $U_y \subseteq T_x$ is a G_y–invariant closed neighborhood of y such that

$$C_x(U_y; a+2b) \subseteq C^\circ_z \quad \text{if } z \in I_y; \tag{2.19}$$

$$U_y \text{ is transversal to the flow with respect to } C_z \text{ for } z \in I_y; \tag{2.20}$$

$$C_x(U_y; a+2b) \cap C_z(T_z; a+2b) = \varnothing \quad \text{if } z \in I \text{ and } C_x(\{y\}; a+2b) \cap C_z(T_z; a+2b) = \varnothing. \tag{2.21}$$

Choose a G_y–invariant closed neighborhood $W_y \subseteq T_x$ of y such that $W_y \subseteq U_y^\circ$. Obviously $T_x = \bigcup_{y \in T_x} W_y^\circ$. Since T_x is compact, we can find $y(x)_1, y(x)_2, \ldots, y(x)_{n(x)}$ in T_x such that $T_x = \bigcup_{i=1}^{n(x)} W_{y(x)_i}$. We can arrange $W_{gy} = gW_y$, $n(x) = n(gx)$ and $y(gx)_i = gy(x)_i$ for $g \in G$ since C_x is a $\mathcal{F}$in–subset of X with $G_{C_x} = G_x$. Obviously

$$C_x(T_x; a)^\circ = \bigcup_{i=1}^{n(x)} C_x(W_{y(x)_i}; a)^\circ. \tag{2.22}$$

Define $$\Lambda = \{y_i(x) \mid x \in I, i \in \{1, 2, \ldots, n(x)\}\}.$$

This is a cofinite G–set. Define for $\lambda = y_i(x)$ in Λ

$$\begin{aligned} C_\lambda &= C_x; \\ B_\lambda &= C_x(U_{y_i(x)}; a + 2b); \\ A_\lambda &= C_x(W_{y_i(x)}; a). \end{aligned}$$

It remains to check that this collection of boxes has the desired properties. This is obvious for assertions (i), (ii), (iii), (iv) and (vii). Assertion (v) follows from $W_y \subseteq U_y^\circ$ and $U_y \subseteq S_{C_x} \cap C_x^\circ$. Assertion (vi) follows from (2.18) and (2.22).

Finally we prove assertion (viii). Suppose that $B_\lambda \cap B_{\lambda'} \neq \varnothing$. Write $\lambda = y_i(x)$ and $\lambda' = y_{i'}(x')$. Since $B_\lambda = C_x(U_{y_i(x)}; a+2b)$ and $B_{\lambda'} \subseteq C_{x'}(T_{x'}; a+2b)$, we conclude that $C_x(U_{y_i(x)}; a + 2b) \cap C_{x'}(T_{x'}; a + 2b) \neq \varnothing$. By (2.21) we have $x' \in I_{y_i(x)}$. We conclude from (2.19)

$$B_\lambda = C_x(U_{y_i(x)}; a + 2b) \subseteq C_{x'}^\circ = C_{\lambda'}^\circ.$$

The central slice $S_{B_\lambda} = U_{y_i(x)}$ is transversal to the flow with respect to $C_{\lambda'}$ by (2.20). This finishes the proof of Lemma 2.17. □

3 General position in metric spaces

Definition 3.1 Let Z be a metric space, $A \subseteq Z$ and $\delta > 0$. Then we define the sets

$$\begin{aligned} A^\delta &= \{x \in Z \mid \exists\, a \in A \text{ such that } d(x, a) < \delta\}, \\ A^{-\delta} &= \{x \in A \mid d(x, Z - A) > \delta\}. \end{aligned}$$

For $x \in Z$, we will abbreviate $x^\delta = \{x\}^\delta$.

Notice that A^δ and $A^{-\delta}$ are open. The following Propositions 3.2 and 3.3 are the main results of this section which is entirely devoted to their proof.

Proposition 3.2 *Let Z be a compact metrizable space of covering dimension n with an action of a finite group F. Let $\mathcal{U}$ be a finite collection of open $\mathcal{F}$in–subsets such that $gU \in \mathcal{U}$ for $g \in F$, $U \in \mathcal{U}$. Suppose that we are given for each $U \in \mathcal{U}$ an open subset $U'' \subseteq U$ satisfying $\overline{U''} \subseteq U$. Put $m = (n+1)\cdot|F|$.*

Then for each $U \in \mathcal{U}$ we can find an open subset $U' \subseteq Z$ such that:

(i) $U'' \subseteq \overline{U''} \subseteq U' \subseteq \overline{U'} \subseteq U$;

(ii) *If $\mathcal{U}_0 \subseteq \mathcal{U}$ has more than m elements, then*

$$\bigcap_{U \in \mathcal{U}_0} \partial U' = \varnothing;$$

(iii) $(gU)' = g(U')$ *for $g \in F$, $U \in \mathcal{U}$.*

Proposition 3.3 *Let Z be a compact metric space of covering dimension n with an action of a finite group F by isometries. Let $Y \subseteq Z$ be an open locally connected F–invariant subset. Let $\mathcal{U}$ be a finite collection of open subsets such that $gU \in \mathcal{U}$ for $g \in F$, $U \in \mathcal{U}$ and $\bar{U} \subseteq Y$ for $U \in \mathcal{U}$ holds. Assume that there is k such that for every subset $\mathcal{U}_0 \subseteq \mathcal{U}$ with more than k elements we have*

$$\bigcap_{U \in \mathcal{U}_0} \partial U = \varnothing.$$

Let $\delta > 0$. Put $m = (n+1)\cdot|F|$, $l = k|F|$.

Then there are finite collections $\mathcal{V}^j$, $j = 0, \ldots, m$ of open subsets of Z, such that:

(i) $\mathcal{V} = \mathcal{V}^0 \cup \cdots \cup \mathcal{V}^m$ *is an open cover of Z;*

(ii) $\operatorname{diam}(V) < \delta$ *for every $V \in \mathcal{V}$;*

(iii) *For $V \in \mathcal{V}$ there are at most l different sets $U \in \mathcal{U}$ such that U and V intersect, but U does not contain V;*

(iv) *For fixed j and $V_0 \in \mathcal{V}^j$ we have $V_0 \cap V \neq 0$ for at most $2^{j+1} - 2$ different sets $V \in \mathcal{V}^0 \cup \cdots \cup \mathcal{V}^j$;*

(v) *For fixed j and $V_0, V_1 \in \mathcal{V}^j$ we have either $V_0 = V_1$ or $\overline{V_0} \cap \overline{V_1} = \varnothing$;*

(vi) *Each $\mathcal{V}^i$ is F–invariant, ie, $gV \in \mathcal{V}^i$ for $g \in F$, $V \in \mathcal{V}^i$;*

(vii) $V = \bar{V}^\circ$ *for $V \in \mathcal{V}$.*

In order to prove these two propositions we will compare the metric space Z to the nerve of a suitable open cover of Z. The results will be first proven for simplicial complexes and then be pulled back to Z using the map from the next remark.

Remark 3.4 Let Z be a metric space and $\mathcal{U}$ be a locally finite open cover of Z. Denote by $\mathcal{N}(\mathcal{U})$ the simplicial complex given by the nerve of $\mathcal{U}$. Then

$$z \mapsto \sum_{U \in \mathcal{U}} \left(\frac{d(z, Z-U)}{\sum_{V \in \mathcal{U}} d(z, Z-V)} \right) [U] \tag{3.5}$$

defines a map $f_{\mathcal{U}}\colon Z \to \mathcal{N}(\mathcal{U})$, where $[U]$ denotes the vertex of $\mathcal{N}(\mathcal{U})$ that corresponds to $U \in \mathcal{U}$. If a group G acts by isometries on Z and on the cover $\mathcal{U}$ (ie, $gU \in \mathcal{U}$ for $g \in G$, $U \in \mathcal{U}$) then $f_{\mathcal{U}}$ is equivariant for the induced action on $\mathcal{N}(\mathcal{U})$.

Lemma 3.6 *Let $f\colon Y \to Z$ be a continuous map between metric spaces. Assume that Y is compact. Let U be an open subset of Z and $\beta > 0$. Then there exists $\alpha > 0$ such that*

$$(f^{-1}(U))^{-\beta} \subseteq f^{-1}(U^{-\alpha}).$$

Proof We proceed by contradiction and assume that for every $n \in \mathbb{N}$ there is $x_n \in Y$ such that

$$x_n \in f^{-1}(U)^{-\beta}, \tag{3.7}$$

$$x_n \notin f^{-1}(U^{-1/n}). \tag{3.8}$$

By compactness of Y, we may assume $x_n \to x$ as $n \to \infty$. We derive from (3.8) that there is $z_n \in Z - U$ such that $d(z_n, f(x_n)) \leq 2/n$. Then $z_n \to f(x)$ as $n \to \infty$ and thus $f(x) \notin U$. On the other hand (3.7) implies that $f((x_n)^{\beta}) \subseteq U$. Since $x \in (x_n)^{\beta}$ for sufficiently large n we have $f(x) \in U$. This is the desired contradiction. □

In the sequel interior of a simplex means simplicial interior, ie, the simplex with all its proper faces removed.

Lemma 3.9 *Let Z be a simplicial complex and let $Z^{(n)}$ be the n–th barycentric subdivision of Z, $n \geq 1$. Let Δ be a simplex of $Z^{(n)}$. Let σ, τ be simplices of Z such that both the interior of σ and the interior of τ intersect Δ. Then $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$.*

Proof Let Δ' be a simplex of the first barycentric subdivision $Z' = Z^{(1)}$ of Z which contains Δ. Then both the interior of σ and the interior τ intersect Δ'. Thus it suffices to prove the claim for the first barycentric subdivision.

Now a d–simplex Δ' of the first barycentric subdivision Z' of Z is given by a sequence $\sigma_0, \sigma_1, \ldots, \sigma_d$ such that each σ_i is a simplex of Z and σ_{i-1} is a proper face of σ_i for $i = 1, 2, \ldots, d$. This is the simplex in the barycentric subdivision whose vertices are the barycenters of $\sigma_0, \sigma_1, \ldots, \sigma_d$. Then the simplices of Z whose interior intersect Δ' are precisely the σ_i. □

Recall that the *open star* of a vertex e of a simplicial complex Z is defined to be the set of all points $z \in Z$ whose carrier simplex has v as an vertex. Equivalently, one obtains the star of v by taking the union of all simplices σ which have v as vertex and then deleting those faces of these simplices σ which do not have v as a vertex. We will denote it by $\operatorname{star}(v)$. The set of open stars of vertices of Z is an open covering of Z.

Let σ be a simplex of Z. Define $\operatorname{star}'(\sigma)$ to be the star in the first barycentric subdivision Z' of the vertex in Z' given by σ.

Proof of Proposition 3.3 Since $\mathcal{U}$ is finite and ∂U closed for $U \in \mathcal{U}$, the assumptions on $\mathcal{U}$ imply that we can find for every $z \in Z$ an open neighborhood W_z of z such that for all $z \in Z$ the following holds.

- $\operatorname{diam}(W_z) < \delta/|F|$;
- W_z intersects the boundary of at most k sets in $\mathcal{U}$;
- If W_z intersects $\bar{U}$ for some $U \in \mathcal{U}$, then $W_z \subseteq Y$.

Since the covering dimension of Z is n by assumption and Z is compact, we can choose a finite open refinement $\mathcal{W}$ of the open cover $\{W_z \mid z \in Z\}$ such that $\dim(\mathcal{W}) \leq n$. Let $\mathcal{W}_F$ be the collection of subsets of Z which consists of the components of subsets of the form $\bigcup_{g \in F} gW$ for $W \in \mathcal{W}$ with $W \subseteq Y$ and subsets of the form gW for $g \in F$ and $W \in \mathcal{W}$ with $W \not\subseteq Y$. Since Y is a locally connected open subset of Z, the above components are open subsets of Z. Thus the elements in the finite set $\mathcal{W}_F$ are open subsets. Hence $\mathcal{W}_F$ is a finite open covering of Z. Since $\dim(\mathcal{W}) \leq n$ every orbit of F in Z meets at most $(n+1) \cdot |F|$ members of $\mathcal{W}$. We conclude $\dim(\mathcal{W}_F) \leq m = (n+1) \cdot |F|$. Obviously $gV \in \mathcal{W}_F$ holds if $V \in \mathcal{W}_F$. If V is a component of $\bigcup_{g \in F} gW$ for some $W \in \mathcal{W}$, we can find elements $g_1, g_2, \ldots, g_r$ in F such that V is contained in $\bigcup_{i=1}^{r} g_i W$ and $(\bigcup_{j=1}^{i} g_j W) \cap g_{i+1} W \neq \varnothing$ holds for $i = 1, 2, \ldots, (r-1)$. One shows by induction over $i = 1, 2 \ldots, r$ that the diameter of $\bigcup_{j=1}^{i} g_j W$ is less or equal to the sum of the diameters of the sets $g_1 W$, $g_2 W, \ldots,$ $g_i \cdot W$. Hence the diameter of any element V of $\mathcal{W}_F$ is bounded by δ.

Consider $U \in \mathcal{U}$ and an element $V \in \mathcal{W}_F$ such that V intersects U but is not contained in U. By construction V is a component of $\bigcup_{g \in F} gW$ for some $W \in \mathcal{W}$ with $W \subseteq Y$. Since V is connected we must have $V \cap \partial U \neq \varnothing$. Hence there exists $g \in F$ with $gW \cap \partial U \neq \varnothing$, or, equivalently, with $W \cap g^{-1} \partial U \neq \varnothing$. Since each set W_z intersects the boundary of at most k sets in $\mathcal{U}$ and $g^{-1}U \in \mathcal{U}$, there are at most $l = |F| \cdot k$ elements $U \in \mathcal{U}$ satisfying $W \cap g^{-1} \partial U \neq \varnothing$ for some $g \in F$. Hence for every $V \in \mathcal{W}_F$ there are at most l elements $U \in \mathcal{U}$ such that V intersects U but is not contained in U.

Consider the map $f = f_{\mathcal{W}_F}\colon Z \to \mathcal{N}(\mathcal{W}_F)$ from Remark 3.4. Put for $j = 0, 1, \ldots, m$

$$\mathcal{X}^j \;=\; \{\mathrm{star}'(\sigma) \mid \sigma \in \mathcal{N}(\mathcal{W}_F), \dim(\sigma) = j\}.$$

Then $\mathcal{X} = \mathcal{X}^0 \cup \mathcal{X}^1 \cup \ldots \cup \mathcal{X}^m$ is an open cover of $\mathcal{N}(\mathcal{W}_F)$.

Let σ and τ be simplices in Z with $\mathrm{star}'(\sigma) \cap \mathrm{star}'(\tau) \neq \varnothing$. Let Δ be the carrier simplex in Z' of some point in $\mathrm{star}'(\sigma) \cap \mathrm{star}'(\tau)$. Then the two vertices of Z' given by σ and τ belong to Δ. Hence the barycenters of σ and τ belong to Δ. In particular the interior of σ and the interior of τ intersect Δ. We conclude from Lemma 3.9 that $\sigma \subseteq \tau$ or $\tau \subseteq \sigma$ holds. This implies that the elements in $\mathcal{X}^j$ are pairwise disjoint and every $V \in \mathcal{X}^j$ intersects at most $2^{j+1} - 2$ elements in $\{\mathcal{X}^0 \cup \mathcal{X}^1 \cup \ldots \cup \mathcal{X}^j\}$ since a j–simplex has $2^{j+1} - 2$ proper faces.

If the simplex σ in $\mathcal{N}(\mathcal{W}_F)$ is given by the subset $\{V_0, V_1, \ldots, V_D\} \subseteq \mathcal{W}_F$, then $f^{-1}(\mathrm{star}'(\sigma))$ is contained in $V_0 \cap V_1 \cap \ldots \cap V_D$, since $\mathrm{star}'(\sigma)$ is contained in the intersection of the stars in $\mathcal{N}(\mathcal{W}_F)$ of the vertices of σ and for $V \in \mathcal{N}(\mathcal{W}_F)$ we have $f^{-1}(\mathrm{star}(V)) \subseteq V$. Hence for every $X \in \mathcal{X}$ there exists $W \in \mathcal{W}_F$ with $f^{-1}(X) \subseteq W$. Put

$$\widehat{\mathcal{V}}^j = \{f_{\mathcal{W}_F}^{-1}(X) \mid X \in \mathcal{X}^j\}.$$

Then $\widehat{\mathcal{V}}^j$ has the following properties:

- $\widehat{\mathcal{V}} = \widehat{\mathcal{V}}^0 \cup \cdots \cup \widehat{\mathcal{V}}^m$ is an open cover of Z consisting of finitely many elements.
- $\mathrm{diam}\, \widehat{V} < \delta$ for every $\widehat{V} \in \widehat{\mathcal{V}}$;
- For $\widehat{V} \in \widehat{\mathcal{V}}$ there are at most l different sets $U \in \mathcal{U}$ such that U and $\widehat{V}$ intersect, but U does not contain $\widehat{V}$;
- For fixed j, every $\widehat{V}_0 \in \widehat{\mathcal{V}}^j$ intersects at most $2^{j+1} - 2$ different sets $\widehat{V} \in \widehat{\mathcal{V}}^0 \cup \cdots \cup \widehat{\mathcal{V}}^j$;
- For fixed j and $\widehat{V}_0, \widehat{V}_1 \in \widehat{\mathcal{V}}^j$ we have either $\widehat{V}_0 = \widehat{V}_1$ or $\widehat{V}_0 \cap \widehat{V}_1 = \varnothing$;
- Each $\widehat{\mathcal{V}}^i$ is F–invariant, ie, $g\widehat{V} \in \widehat{\mathcal{V}}^i$ for $g \in F$, $\widehat{V} \in \widehat{\mathcal{V}}^i$.

For $\epsilon > 0$ define

$$\mathcal{V}^j = \left\{\left(\overline{(\widehat{V})^{-\epsilon}}\right)^{\circ} \mid \widehat{V} \in \widehat{\mathcal{V}}^j\right\}.$$

Then $\mathcal{V}^j$ has properties (ii), (iii), (iv), (v), (vi) and (vii) since $\left(\overline{(\widehat{V})^{-\epsilon}}\right)^{\circ} \subseteq \widehat{V}$ for $\widehat{V} \in \mathcal{V}^{j'}$ and we have $\overline{T} = \overline{T^{\circ}}$ for any open subset T of a topological space. Since Z is compact, we can choose ϵ so small that also property (i) is satisfied. This finishes the proof of Proposition 3.3. □

Proposition 3.10 *Let Z be a simplicial complex and let $Z^{(m)}$ be the m–th barycentric subdivision of Z, $m \geq 1$. Let B be a subcomplex of Z. Let B_0 be the union of all simplices of $Z^{(m)}$ that are contained in the interior of B. Let β be a simplex of Z and let σ be a simplex of $Z^{(m)}$ that is contained in the intersection of the boundary of B_0 with β. Then $\dim(\sigma) < \dim(\beta)$.*

We recall that the topological interior and boundary of a subcomplex of a simplicial complex can be described combinatorial as follows: the interior is the union of all open simplices (simplices with all proper faces removed) that are contained in the subcomplex; its boundary is the union of all simplices that are contained in the subcomplex and are in a addition a face of simplex not contained in the subcomplex.

Proof of Proposition 3.10 By the definition of B_0 there exists a simplex Δ of $Z^{(m)}$ such that $\sigma \subseteq \Delta$, but Δ is not contained in the interior of B. Thus there exists a simplex α of Z that is contained in the boundary of B and intersects Δ. By passing to a face of α, we can arrange that α is contained in the boundary of B and the interior of α intersects Δ. Since σ is contained in β, we can find a face β' of β such that σ intersects the interior of β'. Hence the interior of the simplex $\beta' \subseteq Z$ intersects Δ. Lemma 3.9 implies $\alpha \subseteq \beta'$ or $\beta' \subseteq \alpha$. In the second case, $\sigma \subseteq \beta' \subseteq \alpha$. But σ has to be disjoint from α, because α lies on the boundary of B, while σ is contained in the interior of B. We conclude $\alpha \subseteq \beta'$ and hence $\alpha \subseteq \beta$. Therefore, $\tau = \beta \cap \Delta$ is a simplex of $Z^{(m)}$, that contains the simplex σ as a face and intersects α. Since σ and α are disjoint, σ is a proper face of τ. This implies $\dim(\sigma) < \dim(\tau) \leq \dim(\beta)$. □

Proof of Proposition 3.2 The strategy is first to prove a simplicial version and then use the map appearing in Remark 3.4 to handle the general case of a metric space.

Since we assume that Z is metrizable and F is finite, we can choose a metric d_Z on Z which is F–invariant. Since Z and hence each $\overline{U''}$ is compact and the collection $\mathcal{U}$ is finite, we can find $\delta > 0$ such that $\overline{U''} \subseteq U^{-\delta}$ holds for $U \in \mathcal{U}$. Hence we can assume in the sequel without loss of generality that $U'' = U^{-\delta}$.

So we start with the special case where Z is in addition a simplicial complex, each $U \in \mathcal{U}$ is the interior of a subcomplex of Z and F acts simplicially on Z. Let $\{U_1, \ldots, U_r\} \subseteq \mathcal{U}$ contain exactly one element from every F–orbit in the F–set $\mathcal{U}$. Pick $m \geq 0$ such that the simplices of the m–th barycentric subdivision $Z^{(m)}$ of Z have diameter $< \delta$. For $i = 1, 2, \ldots, r$ let $Z^{(m+i)}$ be the $(m+i)$–th barycentric subdivision of Z and let A_i be the union of all those simplices of $Z^{(m+i)}$ which are contained in U_i. Then A_i is the largest subcomplex of $Z^{(m+i)}$ that is contained in U_i. Since each simplex of $Z^{(m+i)}$ has diameter $< \delta$, we get $U_i^{-\delta} \subseteq A_i \subseteq U_i$.

Define U_i' to be the interior of A_i. For $U \in \mathcal{U}$ define $U' = gU_i'$ for any choice of $g \in F$ and $i \in \{1, 2, \ldots, r\}$ satisfying $U = gU_i$. One easily checks that i is uniquely determined by U and the choice of $g \in F$ does not matter in the definition of U' since each U is by assumption a $\mathcal{F}$in–subset. Obviously (i) and (iii) are satisfied for $\mathcal{U}' = \{U' \mid U \in \mathcal{U}\}$.

Next we prove (ii) but with m replaced by $n = \dim(Z)$. Consider a subset $\mathcal{U}_0 \subseteq \mathcal{U}$ with $|\mathcal{U}_0| = k$. Notice that $\bigcap_{U \in \mathcal{U}_0} \partial U'$ is a subcomplex of $Z^{(m+a)}$ for some a since the intersection of a subcomplex of $Z^{(n+a)}$ with a subcomplex of $Z^{(n+b)}$ is a subcomplex of $Z^{(n+b)}$ if $a \leq b$. It suffices to show $\dim\left(\bigcap_{U \in \mathcal{U}_0} \partial U'\right) \leq n - k$ since this implies

$$\textbf{(3.11)} \qquad \bigcap_{U \in \mathcal{U}_0} \partial U' = \varnothing \quad \text{if } \mathcal{U}_0 \text{ contains more than } n \text{ elements.}$$

Choose $i_1, i_2, \ldots, i_k$ in $\{1, 2, \ldots, r\}$ and $g_1, g_2, \ldots, g_k$ in F with the property that $\mathcal{U}_0$ consists of the mutually different elements $g_1U_{i_1}$, $g_2U_{i_2}$, $\ldots$, $g_kU_{i_k}$ and $i_1 \leq i_2 \leq \ldots \leq i_k$ holds. If for some $j \in \{1, 2, \ldots, (r-1)\}$ we have $i_j = i_{j+1}$, then $g_jU_{i_j} \neq g_{j+1}U_{i_{j+1}}$ implies already $g_jU_{i_j} \cap g_{j+1}U_{i_{j+1}} = \varnothing$ and the claim is obviously true. Hence we can assume without loss of generality $i_1 < i_2 < \ldots < i_k$. Next we show by induction for $j = 1, 2, \ldots, k$ that

$$\dim\left(\bigcap_{l=1}^{j} \partial g_{i_l} U_{i_l}'\right) \leq n - j.$$

The induction beginning is obvious since the dimension of the boundary of a simplicial complex is smaller than the dimension of the simplicial complex itself. The induction step from $j-1$ to j is done as follows. By induction hypothesis the dimension of the simplicial subcomplex $\bigcap_{l=1}^{j-1} \partial g_{i_l} U_{i_l}'$ of $Z^{(n+i_{j-1})}$ is less or equal to $(n - j + 1)$. Let σ be a simplex of the subcomplex $\bigcap_{l=1}^{j} \partial g_{i_l} U_{i_l}'$ of $Z^{(n+i_j)}$. We can find a simplex β in $\bigcap_{l=1}^{j-1} \partial g_{i_l} U_{i_l}'$ such that σ is contained in β. Recall that by assumption U_j is the interior of a subcomplex B_j of $Z^{(n+i_{j-1})}$. Proposition 3.10 applied to the case $m = i_j - i_{j-1}$ and $B = g_jB_j$ and σ and β as above implies

$$\dim(\sigma) < \dim(\beta) \leq n - j + 1$$

since in the notation of Proposition 3.10 we have $B_0 = g_jU_{i_j}'$. Hence $\dim(\sigma) \leq n - j$. This finishes the proof of Proposition 3.2 in the special case where Z is a simplicial complex, each $U \in \mathcal{U}$ is the interior of a subcomplex of Z and F acts simplicially on Z.

In the general case we start with an open cover $\mathcal{V}$ of Z such that each $V \in \mathcal{V}$ has diameter $< \epsilon = \delta/3$ and $\dim(\mathcal{V}) \leq n$. Let $\mathcal{V}_F$ be the cover of Z whose members are

the open sets of the form gV with $V \in \mathcal{V}$ and $g \in F$. Then $\mathcal{V}_F$ is an open cover of Z whose members have diameter $< \epsilon$ and we have $gV \in \mathcal{V}_F$ for $V \in \mathcal{V}_F$ and $g \in F$. Analogously as in Proposition 3.3 one shows that $\dim(\mathcal{V}_F) \leq m$.

For $U \in \mathcal{U}$ let $\mathcal{V}_U = \{V \in \mathcal{V}_F \mid V \subseteq U\}$ and consider $\mathcal{N}(\mathcal{V}_U)$ as a subcomplex of $\mathcal{N}(\mathcal{V}_F)$ (this is the subcomplex spanned by the vertices $[V]$ with $V \in \mathcal{V}_U$) and denote by $\widehat{U}$ the interior of $\mathcal{N}(\mathcal{V}_U)$. Consider the map $f = f_{\mathcal{V}_F} \colon Z \to \mathcal{N}(\mathcal{V}_F)$ from Remark 3.4. If $x \in f^{-1}(\mathcal{N}(\mathcal{V}_U))$ and $x \in V$, $V \in \mathcal{V}_F$ then by the construction of f, $V \in \mathcal{V}_U$. Therefore

$$f^{-1}(\widehat{U}) \subseteq f^{-1}(\mathcal{N}(\mathcal{V}_U)) \subseteq U.$$

Let $x \in U^{-\epsilon}$. If $x \in V$ with $V \in \mathcal{V}_F$ then $V \in \mathcal{V}_U$, because the diameter of V is $< \epsilon$. Therefore $f(x) \in \mathcal{N}(\mathcal{V}_U)$. If $f(x)$ lies on the boundary of $\mathcal{N}(\mathcal{V}_U)$, then there are $V \in \mathcal{V}_U$, $V' \in \mathcal{V}_F - \mathcal{V}_U$ such that $x \in V$ and $V \cap V' \neq \varnothing$. In particular this implies $x \notin U^{-2\epsilon}$. We have thus shown that

$$U^{-2\epsilon} \subseteq f^{-1}(\widehat{U}) \subseteq U. \tag{3.12}$$

Equip $\mathcal{N}(\mathcal{V}_F)$ with a metric such that the action of F is by isometries. By Lemma 3.6 there is α such that

$$(f^{-1}(\widehat{U}))^{-\epsilon} \subseteq f^{-1}(\widehat{U}^{-\alpha}) \tag{3.13}$$

for all $U \in \mathcal{U}$. By the special case treated in the first part of this proof for each $\widehat{U}$ there is $\widehat{U}'$ such that

$$\widehat{U}^{-\alpha} \subseteq \overline{\widehat{U}^{-\alpha}} \subseteq \widehat{U}' \subseteq \overline{\widehat{U}'} \subseteq \widehat{U},$$

$(g\widehat{U})' = g(\widehat{U}')$ for $g \in F$ and

$$\bigcap_{U \in \mathcal{U}_0} \partial \widehat{U}' = \varnothing \quad \text{if } \mathcal{U}_0 \text{ contains more than } m \text{ elements.}$$

(See (3.11) and recall that $\dim(\mathcal{V}_F) \leq m$.) Finally set $U' = f^{-1}(\widehat{U}')$ for $U \in \mathcal{U}$. Since $\partial f^{-1}(U) \subseteq f^{-1}(\partial U)$ and taking preimages preserves inclusions and intersections (ii) and (iii) are satisfied. Moreover, by (3.12) and (3.13)

$$U^{-\delta} = U^{-3\epsilon} \subseteq (f^{-1}(\widehat{U}))^{-\epsilon} \subseteq f^{-1}(\widehat{U}^{-\alpha}) \subseteq f^{-1}(\widehat{U}') \subseteq f^{-1}(\widehat{U}) \subseteq U$$

and therefore (i) is also satisfied. □

4 Covering $X_{>\gamma}$ by long boxes

Throughout this Section we will assume that we are in the situation of Convention 1.3. In particular k_G is the maximum over the orders of finite subgroups of G and

d_X is the dimension of $X - X^{\mathbb{R}}$. Both k_G and d_X are finite. Also recall the notation $X_{>r}$ introduced in Definition 2.15. This section is entirely devoted to the proof of the following

Proposition 4.1 *There exists a natural number $M = M(k_G, d_X)$ depending only on k_G and d_X which has the following property:*

For every $\alpha, \epsilon \in \mathbb{R}$ with $0 < \epsilon < \alpha$ there exists $\gamma = \gamma(\alpha, \epsilon, M) > 0$ such that for every cocompact G–invariant subset K of $X_{>\gamma}$, there is a collection $\mathcal{D}$ of boxes satisfying:

(4.2) $K \subseteq \bigcup_{D\in\mathcal{D}} \Phi_{(-\epsilon,\epsilon)}(D^\circ)$;

(4.3) *For every $x \in X$ which lies on the open bottom or open top of a box in $\mathcal{D}$, the set $\Phi_{[-\alpha,-\epsilon]\cup[\epsilon,\alpha]}(x)$ does not intersect the open bottom or the open top of a box in $\mathcal{D}$;*

(4.4) *For any $x \in X$ there is no box $D \in \mathcal{D}$ such that $\Phi_{[0,\alpha]}(x)$ intersects both the open bottom and open top of D;*

(4.5) *The dimension of the collection $\{D^\circ \mid D \in \mathcal{D}\}$ is less or equal to M, ie the intersection of $(M+2)$ pairwise distinct elements is always empty;*

(4.6) *For $g \in G$, $D \in \mathcal{D}$ we have $gD \in \mathcal{D}$;*

(4.7) *There is a finite subset $\mathcal{D}_0 \subseteq \mathcal{D}$ such that for every $D \in \mathcal{D}$ there exists $g \in G$ with $gD \in \mathcal{D}_0$;*

(4.8) *$\Phi_{[-\alpha-\epsilon,\alpha+\epsilon]}(D)$ is a $\mathcal{F}in$–subset of X for all $D \in \mathcal{D}$.*

The idea of the proof is very roughly as follows. Conditions (4.2) and (4.3) require the boxes to be very long, but we have still the freedom to make the boxes very thin. Proposition 3.2 applied to the transversal directions will be important to arrange the boxes during the construction to be in general position. This will allow the application of Proposition 3.3, where property (iii) will be crucial in order control how many boxes from previous steps of the construction interfere at each step in the construction.

4.1 Preliminaries and the basic induction structure

We begin with fixing some numbers and collection of boxes. Define numbers

$$\begin{aligned} m &:= k_G \cdot (d_X + 1);\\ M &:= (k_G)^2 \cdot (d_X + 1) + 2^{m+1};\\ a &:= \epsilon/2;\\ b &:= 4M \cdot (\alpha + 2\epsilon) + 3(\alpha + \epsilon);\\ c &:= a + 2(b + \epsilon) + 1 + \epsilon;\\ \gamma &:= a + 2b + 2c. \end{aligned}$$

Notice that m and M depend only on k_G and d_X and all the other numbers depend only on α, ϵ and M. (The reader may wonder why we picked a small, we are after all looking for long boxes. But for our construction it is only important that b and c are large and in the proof of Lemma 4.49 (iii) our choice of a small a will be convenient.)

Let $A_\lambda \subseteq B_\lambda \subseteq C_\lambda$ for $\lambda \in \Lambda$ be three collection of boxes as in the assertion of Lemma 2.17, where we use a as defined above and replace b by $b+\epsilon$, and c by $c-\epsilon$. Then we replace B_λ by the restriction $B_\lambda(a+2b)$. The resulting collections satisfy:

- Λ is G–cofinite;
- $l_A = a$, $l_B = a+2b$ and $l_C = a+2b+2c$;
- S_{C_λ} is connected;
- $S_{A_\lambda} \subseteq S_{B_\lambda} \subseteq S_{C_\lambda}$;
- $A_\lambda \subseteq B_\lambda^\circ$ and $B_\lambda \subseteq C_\lambda^\circ$;
- $K \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda^\circ$;
- $gA_\lambda = A_{g\lambda}$, $gB_\lambda = B_{g\lambda}$ and $gC_\lambda = C_{g\lambda}$ for $g \in G$ and $\lambda \in \Lambda$;
- If $B_\lambda \cap B_{\lambda'} \neq \varnothing$, then $\phi_{[-\epsilon,\epsilon]}(B_\lambda) \subseteq C_{\lambda'}^\circ$ and S_{B_λ} is transversal to the flow with respect to $C_{\lambda'}$.

Next we discuss the main strategy of the proof. We will construct the desired collection $\mathcal{D}$ inductively over larger and larger parts of K. The boxes C_λ will be used to control the group action, ie, they will allow us to restrict attention to the finite group G_{C_λ}. The boxes B_λ will be used to control properties **(4.2)** and **(4.3)**. The boxes A_λ will be used to control which part of K is already covered. We will need to sharpen the induction assumptions. We introduce a minor but useful variation of **(4.2)** as follows.

Definition 4.9 A collection of boxes is *δ–overlong* for $0 \le \delta < \epsilon$ if for every $x \in X$ which lies on the open bottom or open top of a box in this collection, the set $\Phi_{[-\alpha-\delta,-\epsilon+\delta]\cup[\epsilon-\delta,\alpha+\delta]}(x)$ does not intersect the open bottom or the open top of any box in this collection.

The assertion **(4.2)** is then that $\mathcal{D}$ is 0–overlong. Clearly, δ–overlong for some $0 \le \delta < \epsilon$ implies 0–overlong.

Definition 4.10 We say that a box D is *not huge* if for every $\lambda \in \Lambda$ such that D intersects B_λ we have $\phi_{[-\epsilon,\epsilon]}(D) \subset C_\lambda^\circ$ and S_D is transversal to the flow with respect to C_λ.

Every box which is obtained from one of the boxes B_λ by restriction is automatically not huge.

Definition 4.11 We say a collection $\mathcal{E}$ of boxes is a *δ–good box cover* of a subset S of X if it has the following properties:

- $\mathcal{E}$ is δ–overlong;
- Every box in $\mathcal{E}$ is not huge;
- Assertions (4.3), (4.4), (4.5), (4.6) and (4.7) hold for $\mathcal{E}$ in place of $\mathcal{D}$;
- $S \subseteq \bigcup_{E\in\mathcal{E}} \Phi_{(-\epsilon,\epsilon)}(E^\circ)$.

To prove Proposition 4.1 we will construct a 0–good box cover of K. Let N be the number of G–orbits of Λ.

Put

$$\delta_r = \frac{N-r}{N+1}\cdot\epsilon \quad \text{for} \quad r = 0,1,\dots,N. \tag{4.12}$$

Clearly,

$$\epsilon > \frac{N}{N+1}\epsilon = \delta_0 > \cdots > \delta_r > \delta_{r+1} > \cdots > \delta_N = 0.$$

We will show inductively for $r = 0,1,2,\dots,N$ that for any subset $\Xi \subseteq \Lambda$ consisting of r G–orbits there exists a δ_r–good box cover of $K_\Xi := \bigcup_{\xi\in\Xi} A_\xi$. The induction beginning $r = 0$ is trivial, take $\mathcal{D} = \varnothing$. The induction step from r to $r+1$ is summarized in the next lemma.

Lemma 4.13 (Induction step : r to $r+1$) *Let $\Xi \subseteq \Lambda$ consist of r G–orbits and assume that $\mathcal{D}$ is a δ_r–good box cover of $K_\Xi = \bigcup_{\xi\in\Xi} A_\xi$. Let $\lambda \in \Lambda - \Xi$ and $\Xi' := G\lambda \cup \Xi$. Then there is a δ_{r+1}–good box cover $\mathcal{D}'$ of $K_{\Xi'} = \bigcup_{\xi\in\Xi'} A_\xi$.*

Clearly Lemma 4.13 implies Proposition 4.1. The proof of Lemma 4.13 will occupy the remainder of Section 4.

4.2 Boxes in C_λ°

Before we explain the construction of $\mathcal{D}'$, we have to introduce some notation and to rearrange $\mathcal{D}$ as follows. In the sequel everything will take place in the interior of the box C_λ. Recall for the sequel the G_{C_λ}–homeomorphism

$$\mu_{C_\lambda}\colon S_{C_\lambda}\times[-a/2-b-c,a/2+b+c] \to C_\lambda, \quad (x,\tau) \mapsto \Phi_\tau(x)$$

from Lemma 2.6 (iii). Let $\pi_{C_\lambda}\colon C_\lambda \to S_{C_\lambda}, x \mapsto \Phi_{(a_+(x)+a_-(x))/2}(x)$ be the retraction onto the central slice. Closures and interiors of subsets of S_{C_λ} are always understood with respect to S_{C_λ}.

Definition 4.14 Let $E \subseteq C_\lambda^\circ$ be a box such that S_E is transversal to the flow with respect to C_λ.

Define an open subset of $S_{C_\lambda} \cap C_\lambda^\circ$ by

$$U_E := \pi_{C_\lambda}(S_E \cap E^\circ) = \pi_{C_\lambda}(E^\circ).$$

Define the continuous G_E–invariant map

$$\tau_E \colon \pi_{C_\lambda}(S_E) \to [-l_C/2, l_C/2]$$

to be the composite of the inverse of $\pi_{C_\lambda}|_{S_E} \colon S_E \xrightarrow{\cong} \pi_{C_\lambda}(S_E)$, the inverse of μ_{C_λ} restricted to S_E and the projection $S_{C_\lambda} \times [-l_C/2, l_C/2] \to [-l_C/2, l_C/2]$.

For a subset $T \subseteq S_{C_\lambda}$ define a subset of S_E by

$$\sigma_E(T) := \pi_{C_\lambda}^{-1}(T) \cap S_E.$$

Lemma 4.15 *Let $E, E' \subseteq C_\lambda^\circ$ be boxes such that S_E and $S_{E'}$ are transversal to the flow with respect to C_λ. Then:*

(i) *If $gE \cap E' \neq \varnothing$ for some $g \in G$, then $g \in G_{C_\lambda}$. In particular G_E is a subgroup of G_{C_λ};*

(ii) *The map τ_E is uniquely characterized by*

$$\mu_{C_\lambda}(x) = (\pi_{C_\lambda}(x), \tau_E \circ \pi_{C_\lambda}(x))$$

for $x \in S_E$;

(iii) *If $T \subseteq S_{C_\lambda}$ is a closed $\mathcal{F}in$–subset of the G_E–space S_{C_λ}, then $\sigma_E(T)$ is a closed $\mathcal{F}in$–subset of the G_E–space S_E and the restriction $E(\sigma_E(T))$ is defined;*

(iv) *Let $U \subseteq S_{C_\lambda}$ be an open $\mathcal{F}in$–subset of the G_E–space S_{C_λ} with $\bar{U} \subseteq \pi_{C_\lambda}(S_E)$ and $U = \bar{U}^\circ$. Then*

$$\begin{aligned} E(\sigma_E(\bar{U})) &= \Phi_{[-l_E/2, l_E/2]}(\sigma_E(\bar{U})); \\ E(\sigma_E(\bar{U}))^\circ &= \Phi_{(-l_E/2, l_E/2)}(\sigma_E(U)); \\ U_{E(\sigma_E(\bar{U}))} &= U. \end{aligned}$$

Proof (i) Suppose $gE \cap E' \neq \varnothing$. Since then $gC_\lambda \cap C_\lambda \neq \varnothing$, we get $g \in G_{C_\lambda}$.

(ii) This follows from the definitions.

(iii) This is obvious since π_{C_λ} is G_E–equivariant.

(iv) We conclude $E(\sigma_E(\bar{U})) = \Phi_{[-l_E/2,l_E/2]}(\sigma_E(\bar{U}))$ from the definition of the restriction. The set $\Phi_{(-l_E/2,l_E/2)}(\sigma_E(U))$ is mapped under the homeomorphism μ_{C_λ} to the set $\{(u,t) \mid u \in U, |t-\tau_E(u)| < l_C/2\}$. Since τ_E is continuous and U is open in S_{C_λ}, this set and hence $\Phi_{(-l_E/2,l_E/2)}(\sigma_E(U)) \subseteq C_\lambda^\circ$ are open. This implies

$$\Phi_{(-l_E/2,l_E/2)}(\sigma_E(U)) \subseteq E(\sigma_E(\bar{U}))^\circ;$$
$$\sigma_E(U) \subseteq \sigma_E(\bar{U}) \cap E(\sigma_E(\bar{U}))^\circ.$$

If we apply π_{C_λ} to the latter inclusion, we conclude

$$U \subseteq \bar{U} \cap \pi_{C_\lambda}(E(\sigma_E(\bar{U}))^\circ) \subseteq \bar{U}^\circ = U.$$

This implies $\sigma_E(U) = \sigma_E(\bar{U}) \cap E(\sigma_E(\bar{U}))^\circ$ and $U_{E(\sigma_E(\bar{U}))} = U$. Lemma 2.6 (iii) implies $\Phi_{(-l_E/2,l_E/2)}(\sigma_E(U)) = E(\sigma_E(\bar{U}))^\circ$. □

4.3 Rearranging the data of the induction beginning

Let $\mathcal{D}$ be the collection of boxes, from the hypothesis of Lemma 4.13 (with respect to $\Xi \subset \Lambda$ and $\lambda \in \Lambda - \Xi$).

Definition 4.16 Put

$$\mathcal{D}_\lambda := \{D \in \mathcal{D} \mid D \cap B_\lambda \neq \varnothing\};$$
$$\mathcal{U}(D_\lambda) := \{U_D \mid D \in \mathcal{D}_\lambda\}.$$

Since the G–action on X is proper and B_λ is compact, property **(4.6)** implies that $\mathcal{D}_\lambda$ is finite. We will consider the G_{C_λ} sets

$$G_{C_\lambda} \cdot \mathcal{D}_\lambda = \{gD \mid D \in \mathcal{D}_\lambda, g \in G_{C_\lambda}\};$$
$$G_{C_\lambda} \cdot \mathcal{U}(\mathcal{D}_\lambda) = \{gU_D \mid D \in \mathcal{D}_\lambda, g \in G_{C_\lambda}\} = \{U_E \mid E \in G_{C_\lambda} \cdot \mathcal{D}_\lambda\}.$$

Since every $D \in \mathcal{D}_\lambda$ is not huge, the set D is contained in C_λ° and S_D is transversal to the flow with respect to C_λ. Both these properties also hold for every $D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda$.

We use Proposition 3.2 to diminish the elements in $\mathcal{D}$ slightly in order to obtain a general position property for $\mathcal{D}$. At this point it is important, that we arranged the central slice S_{C_λ} to be connected.

We main goal of this subsection will be the proof of the following lemma.

Lemma 4.17 *We can assume without loss of generality that $\mathcal{D}$ has the following general position properties:*

(4.18) $\bigcap_{U\in\mathcal{U}_0} \partial U = \varnothing$ *when* $\mathcal{U}_0 \subseteq G_{C_\lambda}\cdot\mathcal{U}(\mathcal{D}_\lambda)$ *fulfills* $|\mathcal{U}_0| > m = k_G\cdot(d_X+1)$;

(4.19) *If* $D, D' \in G_{C_\lambda}\cdot\mathcal{D}_\lambda$ *and* $U_D = U_{D'}$ *then* $D = D'$.

The proof of Lemma 4.17 will use the following lemma, that we will prove first.

Lemma 4.20 *There exists collections* $\{V_D \mid D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ *and* $\{W_D \mid D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ *of open subsets of* S_{C_λ} *satisfying:*

(i) *For* $D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$ *the sets* W_D , $\overline{W_D}$, V_D *and* $\overline{V_D}$ *are* G_D*–invariant subsets of the* G_D*–space* U_D;

(ii) *We have* $W_D \subseteq \overline{W_D} \subseteq V_D \subseteq \overline{V_D} \subseteq U_D$ *for* $D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$;

(iii) *We have* $W_D = (\overline{W_D})^\circ$ *and* $V_D = (\overline{V_D})^\circ$ *for* $D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$;

(iv) $W_{gD} = gW_D$ *and* $V_{gD} = gV_D$ *holds for* $g\in G_{C_\lambda}$ *and* $D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$;

(v) $K_\Xi \subseteq \bigcup_{\substack{D\in\mathcal{D}\\ D\notin G\cdot\mathcal{D}_\lambda}} \Phi_{(-\epsilon,\epsilon)}(D^\circ) \cup \bigcup_{g\in G}\bigcup_{D\in\mathcal{D}_\lambda} g\cdot\Phi_{(-\epsilon,\epsilon)}\big(D(\sigma_D(\overline{W_D}))^\circ\big)$;

(vi) *If* $V_D = V_E$ *for* $D, E\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$, *then* $D = E$;

(vii) *If for* $D, E\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$ *the intersection* $V_D\cap V_E$ *contains both* W_D *and* W_E, *then* $V_D = V_E$.

Proof Choose a metric d on S_{C_λ} which is G_{C_λ}–invariant. Consider $D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$. Put

$$V_D(n) = \big(\overline{U_D^{-1/n}}\big)^\circ.$$

Notice for the sequel that for an open subset Y of a topological space we have $\overline{Y} = \overline{\overline{Y}^\circ}$ and $Y\subseteq\overline{Y}^\circ$ but in general $Y\neq\overline{Y}^\circ$. Hence $V_D(n)$ is a G_D–invariant open subset of S_{C_λ} with $V_D(n) = \big(\overline{V_D(n)}\big)^\circ$. We get for $D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$ and $g\in G_{C_\lambda}$ that $\overline{V_D(n)}\subseteq V_D(n+1)\subseteq U_D$, $U_D = \bigcup_{n\geq 1} V_D(n)$ and $gV_D(n) = V_{gD}(n)$ holds. Denote for $D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$ and $n\geq 1$ the restriction by

$$D_n \;=\; D\left(\sigma_D(\overline{V_D(n)}), l_D - 1/n\right).$$

Lemma 4.15 (iv) and Lemma 2.6 (iii) imply $D^\circ = \bigcup_{n\geq 1} D_n^\circ$ and $D_n\subseteq D_{n+1}^\circ$ for $n\geq 1$.

Put $$K'_\Xi := K_\Xi - \left(K_\Xi \cap \bigcup_{\substack{D\in\mathcal{D}\\ D\notin G\cdot\mathcal{D}_\lambda}} \Phi_{(-\epsilon,\epsilon)}(D^\circ)\right).$$

Since $K_\Xi \subseteq \bigcup_{D\in\mathcal{D}} \Phi_{(-\epsilon,\epsilon)}(D^\circ)$ by assumption, we have

$$K'_\Xi \subseteq \bigcup_{D\in G\cdot\mathcal{D}_\lambda} \Phi_{(-\epsilon,\epsilon)}(D^\circ); \tag{4.21}$$

$$K_\Xi \subseteq K'_\Xi \cup \bigcup_{\substack{D\in\mathcal{D}\\ D\notin G\cdot\mathcal{D}_\lambda}} \Phi_{(-\epsilon,\epsilon)}(D^\circ). \tag{4.22}$$

Since each $D\in\mathcal{D}$ is not huge, $\Phi_{[-\epsilon,\epsilon]}(D)$ is contained in C_λ° for $D\in\mathcal{D}_\lambda$. Since $gC_\lambda\cap C_\lambda\neq\varnothing\Rightarrow g\in G_{C_\lambda}$, we get from (4.21)

$$K'_\Xi\cap C_\lambda \subseteq \bigcup_{D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda} \Phi_{(-\epsilon,\epsilon)}(D^\circ); \tag{4.23}$$

$$K'_\Xi = \bigcup_{g\in G} g\cdot\left(K'_\Xi\cap C_\lambda\right). \tag{4.24}$$

Since K'_Ξ is closed, $K'_\Xi\cap C_\lambda$ is compact. Because $D^\circ=\bigcup_{n\geq 1} D_n^\circ$ and $D_n^\circ\subset D_{n+1}^\circ$, (4.23) implies that there exists a natural number N with

$$K'_\Xi\cap C_\lambda \subseteq \bigcup_{D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda} \Phi_{(-\epsilon,\epsilon)}(D_N^\circ).$$

Since $D_N\subseteq D\big(\sigma_D(\overline{V_D(N)})\big)$ we conclude

$$K'_\Xi\cap C_\lambda \subseteq \bigcup_{D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda} \Phi_{(-\epsilon,\epsilon)}\big(D\big(\sigma_D(\overline{V_D(N)})\big)^\circ\big).$$

We conclude from (4.24)

$$K'_\Xi \subseteq \bigcup_{g\in G}\bigcup_{D\in\mathcal{D}_\lambda} g\cdot\Phi_{(-\epsilon,\epsilon)}\big(D\big(\sigma_D(\overline{V_D(N)})\big)^\circ\big).$$

We conclude from (4.22)

$$K_\Xi \subseteq \bigcup_{\substack{D\in\mathcal{D}\\ D\notin G\cdot\mathcal{D}_\lambda}} \Phi_{(-\epsilon,\epsilon)}(D^\circ) \cup \bigcup_{g\in G}\bigcup_{D\in\mathcal{D}_\lambda} g\cdot\Phi_{(-\epsilon,\epsilon)}\big(D(\sigma_D(\overline{V_D(N)}))^\circ\big).$$

So for every choice of natural numbers $\{n_D \mid D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ satisfying $n_D\geq N$ and $n_{gD}=n_D$ for $g\in G_{C_\lambda}$ and $D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$ the collection $\{V_D(n_D)\mid D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ has the following properties:

- $V_D(n_D)$ is a G_D–invariant subset of the G_D–space V_D;
- $V_D(n_D)=\big(\overline{V_D(n_D)}\big)^\circ$;
- We have $V_D(n_D)\subseteq\overline{V_D(n_D)}\subseteq U_D$ for $D\in G_{C_\lambda}\cdot\mathcal{D}_\lambda$;

- $V_{gD}(n_{gD}) = gV_D(n_D)$ holds for $g \in G_{C_\lambda}$ and $D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda$;
- $K_\Xi \subseteq \bigcup_{\substack{D \in \mathcal{D} \\ D \notin G \cdot \mathcal{D}_\lambda}} \Phi_{(-\epsilon,\epsilon)}(D^\circ) \cup \bigcup_{g \in G} \bigcup_{D \in \mathcal{D}_\lambda} g \cdot \Phi_{(-\epsilon,\epsilon)}\big(D(\sigma_D(\overline{V_D(n_D)}))^\circ\big)$.

Next we show that for some choice of numbers $\{n_D \mid D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda\}$ satisfying $n_D \geq N$ and $n_{gD} = n_D$ for $g \in G_{C_\lambda}$ and $D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda$ the collection $\{V_D(n_D) \mid D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda\}$ also has property (vi). We can write $G_{C_\lambda} \cdot \mathcal{D}_\lambda$ as the disjoint union

$$G_{C_\lambda} \cdot \mathcal{D}_\lambda = \mathcal{C}_1 \amalg \mathcal{C}_2 \amalg \ldots \amalg \mathcal{C}_r$$

of its G_{C_λ}–orbits. We show by induction that we can find numbers $n_1, n_2, \ldots, n_r$ with $n_k \geq N$ such that if we set $n_D = n_k$ for $D \in \mathcal{C}_k$ the collection $\{V_D = V_D(n_D) \mid D \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_k\}$ satisfies property (vi). The induction beginning $k = 1$ is trivial, the induction step from $k-1$ to k is done as follows. For given $D \in \mathcal{C}_k$ choose n_0 with $\overline{V_D(n_0)} \neq \varnothing$. Since S_{C_λ} is connected, the nonempty closed subset $\overline{V_D(n)}$ and the open subset $V_D(n+1) \neq S_{C_\lambda}$ cannot agree for $n \geq n_0$. In particular $V_D(n) \subset V_D(n+1)$ for all $n \geq n_0$ and since $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_{k-1}$ is a finite set we can find a number n_k such that $V_D(n_k) \neq V_E(n_E)$ for $E \in \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_{k-1}$. By invariance under the G_{C_λ}–action the same statement holds for all $D \in \mathcal{C}_k$ with this n_k. If we have $D, gD \in \mathcal{C}_k$ with $g \in G_{C_\lambda}$ then since μ_{C_λ} from Lemma 2.6 (iii) is G_{C_λ}–invariant and the action on the interval trivial and D is a $\mathcal{F}$in–set, $U_D = gU_D$ already implies $D = gD$. Therefore property (vi) holds for $\mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_k$. We have therefore verified property (vi) for the collection $V_D = V_D(n_D)$ with $D \in G_{C_\lambda}\mathcal{D}_\lambda$.

In order to achieve property (vii) we repeat this construction replacing the collection of boxes $G_{C_\lambda}\mathcal{D}_\lambda$ with the collection of boxes $\big\{D\big(\sigma_D(\overline{V_D(n_0)})\big) \mid D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda\big\}$. Namely, put

$$W_D(n) \;=\; \big(\overline{V_D^{-1/n}}\big)^\circ.$$

Lemma 4.15 (iv) implies $U_{D(\sigma_D(\overline{V_D(n)}))} = V_D(n)$. Thus we get open subsets $W_D(n) \subseteq V_D$ and a natural number N' such that for every choice of natural numbers $\{n_D \mid D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda\}$ satisfying $n_D \geq N'$ and $n_{gD} = n_D$ for $g \in G_{C_\lambda}$ and $D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda$ the collection $\{W_D(n_D) \mid D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda\}$ has the following properties:

- $W_D(n_D)$ is a G_D–invariant subset of the G_D–space V_D;
- $W_D(n_D) = \big(\overline{W_D(n_D)}\big)^\circ$;
- We have $W_D(n_D) \subseteq \overline{W_D(n_D)} \subseteq V_D$ for $D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda$;
- $W_{gD}(n_{gD}) = gW_D(n_D)$ holds for $g \in G_\lambda$ and $D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda$;
- $K_\Xi \subseteq \bigcup_{\substack{D \in \mathcal{D} \\ D \notin G \cdot \mathcal{D}_\lambda}} \Phi_{(-\epsilon,\epsilon)}(D^\circ) \cup \bigcup_{g \in G} \bigcup_{D \in \mathcal{D}_\lambda} g \cdot \Phi_{(-\epsilon,\epsilon)}\big(D\big(\sigma_D(\overline{W_D(n_D)})\big)^\circ\big)$.

Consider $D, E \in \mathcal{C}$ with $V_D \neq V_E$. Since $V_D = \bigcup_{n\geq 1} W_D(n)$ and $V_E = \bigcup_{n\geq 1} W_E(n)$ holds, we can find $N'(D,E)$ such that $V_D \cap V_E$ does not contain both $W_D(n)$ and $W_E(n)$ for $n \geq N'(D,E)$. Define N'' to be the maximum over the numbers $N'(D,E)$ for $D, E \in G_{C_\lambda}\cdot\mathcal{D}_\lambda$ with $V_D \neq V_E$ and N'. Put $W_D = W_D(N'')$ for $D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda$. Then the collections $\{V_D \mid D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ and $\{W_D \mid D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ have all the desired properties. This finishes the proof of Lemma 4.20. □

Now we can prove Lemma 4.17.

Proof In the sequel we will use the collections $\{V_D \mid D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ and $\{W_D \mid D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ appearing in Lemma 4.20. We apply Proposition 3.2 in the case, where the space Z is S_{C_λ}, the finite group F is G_{C_λ}, the collection $\mathcal{U}$ is $\{V_D \mid D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ and we use the subsets $W_D \subseteq V_D$. Since $S_{C_\lambda} \subseteq X$ is closed, we have $\dim(S_{C_\lambda}) \leq d_X$. So from Proposition 3.2 we obtain for every $D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda$ an open subset $V''_D \subseteq S_{C_\lambda}$ such that the following holds:

- $W_D \subseteq V''_D \subseteq V_D$;
- If $\mathcal{U}_0 \subseteq \{V''_D \mid D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda\}$ has more than $m = k_G\cdot(d_X+1)$ elements, then
$$\bigcap_{U''\in\mathcal{U}_0} \partial U'' = \varnothing;$$
- $(V_{gD})'' = (gV_D)'' = g(V''_D)$ for $g \in G_{C_\lambda}$ and $D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda$.

Now define
$$V'_D = \overline{V''_D}^{\circ}.$$
Since V''_D is open, we conclude $V'_D = \overline{V'_D}^{\circ}$. Recall that $V_D = \overline{V_D}^{\circ}$ and $W_D = \overline{W_D}^{\circ}$. Notice that V''_D is not necessarily V'_D. Since V''_D is open, we get $V''_D \subseteq V'_D$ and hence $V''_D \cap \partial V'_D = \varnothing$. We have
$$\partial V'_D \subseteq \overline{V'_D} = \overline{\overline{V''_D}^{\circ}} = \overline{V''_D}.$$
Hence $\partial V'_D \subseteq \partial V''_D$. Thus we have constructed for every $D \in G_{C_\lambda}\cdot\mathcal{D}_\lambda$ an open subset $V'_D \subseteq S_{C_\lambda}$ such that the following holds

- $W_D \subseteq V'_D \subseteq V_D$;
- If $\mathcal{V}_0 \subseteq \{V'_D \mid D \in G_{C_\lambda}\cdot\mathcal{U}(\mathcal{D}_\lambda)\}$ has more than $m = k_G\cdot(d_X+1)$ elements, then
$$\bigcap_{V'\in\mathcal{U}_0} \partial V' = \varnothing;$$
- $(V_{gD})' = (gV_D)' = g(V'_D)$ for $g \in G_{C_\lambda}$ and $D \in G_{C_\lambda}\cdot\mathcal{U}(\mathcal{D}_\lambda)$;

- $\overline{V'_D}^{\circ} = V'_D$.

We use next restriction of boxes to diminish some of the boxes in $\mathcal{D}$ as follows. Consider $D \in \mathcal{D}$. Suppose there exists $g_0 \in G$ and $D_0 \in G_{C_\lambda} \cdot \mathcal{D}_\lambda$ with $D = g_0 D_0$. Then define $\widehat{D} = g_0 D_0\big(\sigma_{D_0}(\overline{V'_{D_0}})\big)$. We have to check that this is well-defined. Suppose we have $g_i \in G$ and $D_i \in G_{C_\lambda} \cdot \mathcal{D}_\lambda$ with $D = g_i D_i$ for $i = 0, 1$. We have $D_1 = g_1^{-1} g_0 D_0$. This implies $g_1^{-1} g_0 \in G_{C_\lambda}$ (see Lemma 4.15 (i)) and hence $g_1^{-1} g_0 U_{D_0} = U_{D_1}$. We conclude $g_1^{-1} g_0 (V_{D_0})' = (V_{D_1})'$ and hence $g_1^{-1} g_0 D_0\big(\sigma_{D_0}(\overline{V'_{D_0}})\big) = D_1\big(\sigma_{D_1}(\overline{V'_{D_1}})\big)$. If there does not exist $g_0 \in G$ and $D_0 \in \mathcal{D}_\lambda$ with $D = g_0 D_0$, we put $\widehat{D} = D$. Define a new collection of boxes

$$\widehat{\mathcal{D}} = \{\widehat{D} \mid D \in \mathcal{D}\}.$$

Next we want to show that $\widehat{\mathcal{D}}$ is a δ_r–good box cover of $K_\Xi = \bigcup_{\xi \in \Xi} A_\xi$. Since $W_D \subseteq V'_D$ for $D \in \mathcal{D}_\lambda$, we conclude from property (v) appearing in Lemma 4.20

$$K_\Xi \subseteq \bigcup_{\widehat{D} \in \widehat{\mathcal{D}}} \Phi_{(-\epsilon,\epsilon)}(\widehat{D}^{\circ}).$$

One easily checks that the other required properties of a δ_r–good box cover do pass from $\mathcal{D}$ to $\widehat{\mathcal{D}}$ since elements in $\widehat{\mathcal{D}}$ are obtained from those in $\mathcal{D}$ by restriction in a G–equivariant way.

We conclude $G_{C_\lambda} \cdot \mathcal{U}(\widehat{\mathcal{D}}_\lambda) \subseteq \{V'_D \mid D \in G_{C_\lambda} \cdot \mathcal{D}_\lambda\}$ from Lemma 4.15 (iv). (We remark that $D \cap B_\lambda \neq \varnothing$ does not necessarily imply $\widehat{D} \cap B_\lambda \neq \varnothing$ and the above inclusion may be a strict inclusion.) By construction $\widehat{\mathcal{D}}$ satisfies **(4.18)**.

Suppose for $\widehat{D}, \widehat{D}' \in G_{C_\lambda} \cdot \widehat{\mathcal{D}}_\lambda$ that $U_{\widehat{D}} = U_{\widehat{D}'}$. By construction $U_{\widehat{D}} = V'_D$ and $U_{\widehat{D}} = V'_{D'}$. We conclude $V'_D = V'_{D'}$ and hence both W_D and $W_{D'}$ are contained in $V_D \cap V_{D'}$. Properties (vi) and (vii) appearing in Lemma 4.20 imply $D = D'$. We conclude that $\widehat{\mathcal{D}}$ satisfies **(4.19)**. This finishes the proof of Lemma 4.17. □

Now we have finished our arrangement of $\mathcal{D}$ and can now construct the desired new collection $\mathcal{D}'$ out of $\mathcal{D}$ as demanded in Lemma 4.13.

4.4 Carrying out the induction step

Recall that we defined numbers m, M, a, b and c in the beginning of Section 4.1. Recall also that N is the number of G–orbits of Λ and that ϵ is given in Proposition 4.1.

In the sequel we will abbreviate

$$a_\pm = \pm l_{A_\lambda}/2;$$
$$b_\pm = \pm l_{B_\lambda}/2;$$
$$c_\pm = \pm l_{C_\lambda}/2.$$

We have
$$a_+ - a_- = a;$$
$$b_+ - a_+ = a_- - b_- = b;$$
$$c_+ - b_+ = b_- - c_- = c.$$

Put

(4.25) $$\mu := \frac{\epsilon}{(N+1)(m+1)} \quad \text{and} \quad \eta := \frac{\mu}{5}.$$

We will use the G_{C_λ}–homeomorphism

(4.26) $$\mu_{C_\lambda}\colon S_{C_\lambda} \times [c_-, c_+] \to C_\lambda, \quad (x,\tau) \mapsto \Phi_\tau(x)$$

from Lemma 2.6 (iii) as an identification. Note that $A_\lambda = S_{A_\lambda} \times [a_-, a_+]$, $B_\lambda = S_{B_\lambda} \times [b_-, b_+]$ and $B_\lambda^\circ = \big(S_{B_\lambda} \cap B^\circ\big) \times (b_-, b_+)$ under this identification. Note that for $g \in G_{C_\lambda}$ we have $g \cdot (x,t) = (gx, t)$, by Lemma 2.6 (iii).

Choose a G_{C_λ}–invariant metric $d_{S_{C_\lambda}}$ on S_{C_λ}. Consider $D \in \mathcal{D}_\lambda$. Recall that $D \subseteq C_\lambda$ and that the retraction $\pi_{C_\lambda}\colon C_\lambda \to S_{C_\lambda}$ induces a G_D–homeomorphism $S_D \to \pi_{C_\lambda}(S_D)$. We have introduced the continuous G_D–invariant map

$$\tau_D\colon \pi_{C_\lambda}(S_D) \to [c_-, c_+]$$

in Definition 4.14. It is uniquely characterized by $\mu_{C_\lambda}(x) = (\pi_{C_\lambda}(x), \tau_D \circ \pi_{C_\lambda}(x))$ for $x \in S_D$. Since τ_D is continuous and $\pi_{C_\lambda}(S_D)$ is compact, we can find $\delta_D > 0$ such that

(4.27) $$|\tau_D(x) - \tau_D(y)| \le \eta \quad \text{for } x, y \in \pi_{C_\lambda}(S_D) \text{ with } d_{S_{C_\lambda}}(x,y) < \delta_D.$$

Because $\mathcal{D}_\lambda$ is finite we can set

(4.28) $$\delta := \min\{\delta_D \mid D \in \mathcal{D}_\lambda\}.$$

Then $\delta > 0$.

In the sequel interiors and closures of subsets of S_{C_λ} are to be understood with respect to S_{C_λ}. One easily checks with this convention that Lemma 2.6 (iii) implies $S_{B_\lambda}^\circ = B^\circ \cap S_{B_\lambda}$ since $B_\lambda \subseteq C_\lambda^\circ$.

Next we want to apply Proposition 3.3 to the locally connected compact metric space $Z := S_{C_\lambda}$ with the obvious isometric $F := G_{B_\lambda}$–action (note that $F \subseteq G_{C_\lambda}$ by

Lemma 4.15 (i)), the $F = G_{B_\lambda}$–invariant open subset $Y := S_{C_\lambda} \cap C_\lambda^\circ$ which is locally connected by Lemma 2.6 (iv), the collection of sets $\mathcal{U} := \mathcal{U}(\mathcal{D}_\lambda)$ and the number $\delta > 0$ in **(4.28)**. Note that for $D \in \mathcal{D}_\lambda$ we have by definition $D \cap B_\lambda \neq \varnothing$ and therefore $D \subseteq C_\lambda^\circ$ since D is not huge. Therefore $\overline{U_D} \subseteq \pi_{C_\lambda}(D) \subseteq S_{C_\lambda} \cap C_\lambda^\circ = Y$ for $D \in \mathcal{D}_\lambda$. In the notation just introduced this means $\bar{U} \subset Y$ for $U \in \mathcal{U}$. Thus we can indeed apply Proposition 3.3 in this situation. By intersecting the resulting open covering with $S^\circ_{B_\lambda} = S_{B_\lambda} \cap B^\circ$, we obtain a collection $\mathcal{V} = \mathcal{V}^0 \cup \mathcal{V}^1 \cup \cdots \cup \mathcal{V}^m$ of open subsets of $S^\circ_{B_\lambda}$ which has the following properties:

(4.29) $\mathcal{V}$ is a open covering of $S^\circ_{B_\lambda}$ consisting of finitely many elements;

(4.30) For every $V \in \mathcal{V}$ there are at most $k_G^2 \cdot (d_X + 1)$ different $U \in \mathcal{U}(\mathcal{D}_\lambda)$ such that $U \cap V \neq \varnothing$ and $V \not\subset U$;

(4.31) For fixed j and $V_0 \in \mathcal{V}^j$ we have $V_0 \cap V \neq \varnothing$ for at most $2^{j+1} - 2 < 2^{m+1}$ different subsets $V \in \mathcal{V}^0 \cup \cdots \cup \mathcal{V}^{j-1}$;

(4.32) For fixed j and $V_0, V_1 \in \mathcal{V}^j$ we have either $V_0 = V_1$ or $\overline{V_0} \cap \overline{V_1} = \varnothing$;

(4.33) Each $\mathcal{V}^i$ is G_{B_λ}–invariant, ie, $gV \in \mathcal{V}^i$ if $g \in G_{B_\lambda}$, $V \in \mathcal{V}^i$;

(4.34) For $V \in \mathcal{V}$ its closure $\bar{V}$ is a $\mathcal{F}$in–subset of S_{B_λ} with respect to the G_{B_λ}–action;

(4.35) We have $\bar{V}^\circ = V$ for $V \in \mathcal{V}$;

(4.36) For every $V \in \mathcal{V}$ the diameter of V is bounded by δ;

(4.37) $\mathcal{V}^j \cap \mathcal{V}^k = \varnothing$ if $j \neq k$.

Properties **(4.29)**, **(4.30)**, **(4.31)**, **(4.32)**, **(4.33)** and **(4.36)** are direct consequences of Proposition 3.3. Property **(4.34)** follows from properties **(4.32)** and **(4.33)**. Property **(4.37)** can be achieved by replacing $\mathcal{V}^j$ by a subset of $\mathcal{V}^j$ if necessary.

Since for every subset $Y \subseteq S_{C_\lambda}$ with $\bar{Y}^\circ = Y$ we have

$$\big(\overline{Y \cap S^\circ_{B_\lambda}}\big)^\circ \subseteq \big(\bar{Y} \cap S_{B_\lambda}\big)^\circ = \bar{Y}^\circ \cap S^\circ_{B_\lambda} = Y \cap S^\circ_{B_\lambda} \subseteq \big(\overline{Y \cap S^\circ_{B_\lambda}}\big)^\circ$$

and hence $\big(\overline{Y \cap S^\circ_{B_\lambda}}\big)^\circ = Y \cap S^\circ_{B_\lambda}$, property **(4.35)** holds. We mention that because of **(4.34)** we can consider for $V \in \mathcal{V}$ the restriction $B_\lambda(\bar{V})$ and property **(4.35)** ensures

$$\begin{aligned} S_{B_\lambda(\bar{V})} &= \bar{V}; \\ S^\circ_{B_\lambda(\bar{V})} &= S_{B_\lambda(\bar{V})} \cap B^\circ_\lambda = V. \end{aligned}$$

The collection $\mathcal{D}'$ we are seeking will be of the form $\mathcal{D} \cup \{gB_\lambda(\overline{W}; a_-^W, a_+^W) \mid W \in \mathcal{W}, g \in G\}$, where $\mathcal{W} \subseteq \mathcal{V}$. In order to find suitable $\mathcal{W} \subseteq \mathcal{V}$ and $a_\pm^W$ we proceed by a

subinduction over $n=-1,\dots,m$. Using (4.12) and (4.25) we set

$$\delta_{r,n} := \delta_r-(n+1)\cdot\mu \quad \text{for } n=-1,0,1,\dots,m.$$

Clearly $\delta_r=\delta_{r,-1}>\delta_{r,0}>\cdots>\delta_{r,n-1}>\delta_{r,n}>\cdots>\delta_{r,m}=\delta_{r+1}$

and $\delta_{r,n-1}-\delta_{r,n}=\mu$. For $j=0,\dots,m$ let

$$K_\lambda^{(j)} := \bigcup_{V\in\mathcal{V}_0\cup\cdots\cup\mathcal{V}_j} V\times[a_-,a_+].$$

(Recall that we use (4.26) to identify $K_\lambda^{(j)}$ with a subset of C_λ.) The induction step from $(n-1)$ to n is formulated in the following lemma.

Lemma 4.38 (Induction step : $n-1$ to n) *Assume that we have for $j=0,\dots,n-1$ subsets $\mathcal{W}^j\subseteq\mathcal{V}^j$ and numbers $\{a_\pm^W \mid W\in\mathcal{W}^j\}$ satisfying $b_-\le a_-^W<a_+^W\le b_+$ such that the collection of boxes $\mathcal{D}^{n-1}=\mathcal{D}\cup\{gD^W \mid W\in\mathcal{W}^0\cup\cdots\cup\mathcal{W}^{n-1}, g\in G\}$ is a $\delta_{r,n-1}$–good box cover of $K_\Xi\cup GK_\lambda^{(n-1)}$, where $D^W:=B_\lambda(\overline{W};a_-^W,a_+^W)$ for $W\in\mathcal{W}^0\cup\cdots\cup\mathcal{W}^{n-1}$.*

Then there is a subset $\mathcal{W}^n\subseteq\mathcal{V}^n$ and numbers $a_\pm^W\in\mathbb{R}$ with $b_-\le a_-^W<a_+^W\le b_+$ for $W\in\mathcal{W}^n$ such that $\mathcal{D}^n=\mathcal{D}^{n-1}\cup\{gD^W \mid W\in\mathcal{W}^n, g\in G\}$ is a $\delta_{r,n}$–good box cover of $K_\Xi\cup GK_\lambda^{(n)}$, where $D^W=B_\lambda(\overline{W};a_-^W,a_+^W)$ for $W\in\mathcal{W}^n$.

Since (4.29) implies $S_{A_\lambda}\subseteq S^\circ_{B_\lambda}=\bigcup_{V\in\mathcal{V}}V$ we get $A_\lambda\subseteq\bigcup_{V\in\mathcal{V}}V\times[a_-,a_+]$. We conclude $K_{\Xi'}\subseteq K_\Xi\cup GK_\lambda^{(m)}$. ($K_{\Xi'}$ was defined in Lemma 4.13.) Hence $\mathcal{D}'=\mathcal{D}^m$ is the desired δ_{r+1}–good box cover of $K_{\Xi'}$. Therefore Lemma 4.13 follows from Lemma 4.38.

The proof of Lemma 4.38 will occupy the remainder of Section 4.

Definition 4.39 Let $\mathcal{W}^n$ be the set of all $W\in\mathcal{V}^n$ for which

$$W\times[a_-,a_+]\not\subset\bigcup_{D\in\mathcal{D}^{n-1}}\Phi_{(-\epsilon,\epsilon)}(D^\circ).$$

Lemma 4.40 *Let $V\in\mathcal{V}$ and let $\partial_\pm D$ be the top or bottom of a box $D\in\mathcal{D}_\lambda$. Consider $t\in(c_-,c_+)$. Suppose that $\partial_\pm D\cap(V\times\{t\})\ne\varnothing$. Then*

$$\partial_\pm D\cap(V\times[c_-,c_+])\subseteq V\times(t-\eta,t+\eta).$$

Proof Consider $v\in V$ with $(v,t)\in\partial_\pm D$. Then $(v,t\mp l_D/2)\in S_D$. Hence $\tau_D(v)=t\mp l_D/2$, where τ_D is the function introduced in Definition 4.14. Consider $w\in V$ and $s\in[c_-,c_+]$ with $(w,s)\in\partial_\pm D$. Then $s\mp l_D/2=\tau_D(w)$. From (4.27), (4.28)

and (4.36) we conclude $|\tau_D(v)-\tau_D(w)|<\eta$ and hence $|t-s|<\eta$. This implies $\partial_\pm D\cap(V\times[c_-,c_+])\subseteq V\times(t-\eta,t+\eta)$. □

Recall that for $D\in\mathcal{D}_\lambda$ we have $D\subseteq C_\lambda^\circ$ and we have associated to such D an open subset $U_D=\pi_{C_\lambda}(S_D\cap D^\circ)=\pi_{C_\lambda}(D^\circ)$ of S_{C_λ}.

Definition 4.41 For $W\in\mathcal{W}^n$ define

$$\begin{aligned}
\mathcal{D}_W &:= \{D\in\mathcal{D}^{n-1}\mid D^\circ\cap W\times(b_-,b_+)\neq\varnothing\};\\
\mathcal{D}_W^{\mathrm{good}} &:= \{D\in\mathcal{D}_W\mid W\subseteq U_D\};\\
\mathcal{D}_W^{\mathrm{bad}} &:= \{D\in\mathcal{D}_W\mid W\cap U_D\neq\varnothing, W\not\subseteq U_D\};\\
J_W^{\mathrm{good},\pm} &:= \{t\in(b_-,b_+)\mid \exists D\in\mathcal{D}_W^{\mathrm{good}}\mid \partial_\pm D^\circ\cap W\times\{t\}\neq\varnothing\};\\
J_W^{\mathrm{bad},\pm} &:= \{t\in(b_-,b_+)\mid \exists D\in\mathcal{D}_W^{\mathrm{bad}}\mid \partial_\pm D^\circ\cap W\times\{t\}\neq\varnothing\};\\
J_W^{\mathrm{good}} &:= \{t\in(b_-,b_+)\mid \exists D\in\mathcal{D}_W^{\mathrm{good}}\mid D^\circ\cap W\times\{t\}\neq\varnothing\};\\
J_W^{\partial} &:= J_W^{\mathrm{good},-}\cup J_W^{\mathrm{good},+}\cup J_W^{\mathrm{bad},-}\cup J_W^{\mathrm{bad},+}.
\end{aligned}$$

Since $D\in\mathcal{D}_W$ implies $W\cap U_D\neq\varnothing$, we have $\mathcal{D}_W=\mathcal{D}_W^{\mathrm{good}}\cup\mathcal{D}_W^{\mathrm{bad}}$. The reason for the names of $\mathcal{D}_W^{\mathrm{good}}$ and $\mathcal{D}_W^{\mathrm{bad}}$ is this. In the construction of D^W for $W\in\mathcal{W}^n$ we will be able to allow top and bottom of D^W to be very close to top or bottom of a box in $\mathcal{D}_W^{\mathrm{good}}$ (compare Lemma 4.43) but will have to make sure that top and bottom of D^W will be far away from top and bottom of every box in $\mathcal{D}_W^{\mathrm{bad}}$. Thus, both for choosing a_-^W and a_+^W there will be two cases: either we find a suitable top (for a_-^W) respectively bottom (for a_+^W) of a box in $\mathcal{D}_W^{\mathrm{good}}$ to put $W\times\{a_\mp^W\}$ close by, or all boxes from $\mathcal{D}_W^{\mathrm{good}}$ are far away and we will only have to worry about the boxes from $\mathcal{D}_W^{\mathrm{bad}}$. The crucial point will then be, that the number of members of $\mathcal{D}_W^{\mathrm{bad}}$ is uniformly bounded, see Lemma 4.42.

Note that for $g\in G_{B_\lambda}$ we have $J_W^{\mathrm{good},\pm}=J_{gW}^{\mathrm{good},\pm}$, $J_W^{\mathrm{bad},\pm}=J_{gW}^{\mathrm{bad},\pm}$, $J_W^{\mathrm{good}}=J_{gW}^{\mathrm{good}}$ and $J_W^\partial=J_{gW}^\partial$, because g acts trivially on the second factor of $C_\lambda=S_{C_\lambda}\times[c_-,c_+]$.

Lemma 4.42 *We have*

$$|\mathcal{D}_W^{\mathrm{bad}}|\leq M=(k_G)^2\cdot(d_X+1)+2^{m+1}.$$

Proof We conclude from the definitions, Lemma 4.15 (iv) and (4.35)

$$\begin{aligned}
\mathcal{D}^{n-1} &= \mathcal{D}\cup\{gD^W\mid W\in\mathcal{W}^0\cup\cdots\cup\mathcal{W}^{n-1}, g\in G\};\\
D^W &= B_\lambda(\overline{W};a_-^W,a_+^W) = \overline{W}\times[a_-^W,a_+^W] \quad \text{for } W\in\mathcal{W}^0\cup\cdots\cup\mathcal{W}^n,\\
U_{D^W} &= W \quad \text{for } W\in\mathcal{W}^0\cup\cdots\cup\mathcal{W}^n.
\end{aligned}$$

Recall that $gB_\lambda \cap B_\lambda \neq \varnothing \Rightarrow g \in G_{B_\lambda}$ holds. Hence we get

$$\begin{aligned}
\mathcal{D}_W^{\mathrm{bad}} &= \left\{D \in \mathcal{D} \mid D^\circ \cap W \times (b_-, b_+) \neq \varnothing, W \cap U_D \neq \varnothing, W \not\subset U_D\right\} \\
&\qquad \cup \left\{gD^{W'} \mid W' \in \mathcal{W}^0 \cup \cdots \cup \mathcal{W}^{n-1}, g \in G, \right. \\
&\qquad\qquad \left. \left(gD^{W'}\right)^\circ \cap W \times (b_-, b_+) \neq \varnothing, W \cap U_{gD^{W'}} \neq \varnothing, W \not\subset U_{gD^{W'}}\right\} \\
&= \left\{D \in \mathcal{D} \mid D^\circ \cap W \times (b_-, b_+) \neq \varnothing, W \cap U_D \neq \varnothing, W \not\subset U_D\right\} \\
&\quad \cup \left\{gD^{W'} \mid W' \in \mathcal{W}^0 \cup \cdots \cup \mathcal{W}^{n-1}, g \in G_{B_\lambda}, W \cap gW' \neq \varnothing, W \not\subset gW'\right\} \\
&\subseteq \left\{D \in \mathcal{D}_\lambda \mid W \cap U_D \neq \varnothing, W \not\subset U_D\right\} \\
&\quad \cup \left\{gD^{W'} \mid W' \in \mathcal{W}^0 \cup \cdots \cup \mathcal{W}^{n-1}, g \in G_{B_\lambda}, W \cap gW' \neq \varnothing, W \not\subset gW'\right\}.
\end{aligned}$$

We have $gD^{W'} = D^{gW'}$ and $D^{W'} = D^{W''} \Leftrightarrow W' = W''$. Hence we conclude using **(4.19)**, **(4.30)**, **(4.31)** and **(4.37)** that

$$\begin{aligned}
|\mathcal{D}_W^{\mathrm{bad}}| &\leq |\{D \in \mathcal{D}_\lambda \mid W \cap U_D \neq \varnothing, W \not\subset U_D\}| \\
&\quad + \left|\left\{gD^{W'} \mid W' \in \mathcal{W}^0 \cup \cdots \cup \mathcal{W}^{n-1}, g \in G_{B_\lambda}, W \cap gW' \neq \varnothing, W \not\subset gW'\right\}\right| \\
&= |\{U_D \in \mathcal{U}(\mathcal{D}_\lambda) \mid W \cap U_D \neq \varnothing, W \not\subset U_D\}| \\
&\quad + \left|\left\{gW' \mid W' \in \mathcal{W}^0 \cup \cdots \cup \mathcal{W}^{n-1}, g \in G_{B_\lambda}, W \cap gW' \neq \varnothing, W \not\subset gW'\right\}\right| \\
&\leq |\{U_D \in \mathcal{U}(\mathcal{D}_\lambda) \mid W \cap U_D \neq \varnothing, W \not\subset U_D\}| \\
&\quad + \left|\left\{V \mid V \in \mathcal{V}^0 \cup \cdots \cup \mathcal{V}^{n-1}, W \cap V \neq \varnothing, W \not\subset V\right\}\right| \\
&\leq k_G^2 \cdot (d_X + 1) + 2^{m+1} \\
&= M.
\end{aligned}$$

This finishes the proof of Lemma 4.42. □

Lemma 4.43 *Let $W \in \mathcal{W}^n$. If $t_0 \in J_W^{\mathrm{good},-} \cup J_W^{\mathrm{good},+}$ and $t_1 \in J_W^{\partial}$, then*

$$|t_0 - t_1| \notin [\epsilon - \delta_{r,n-1} + \eta, \alpha + \delta_{r,n-1} - \eta].$$

Proof There are $D_0 \in \mathcal{D}_W^{\mathrm{good}}$, $D_1 \in \mathcal{D}_W$, $\sigma_0, \sigma_1 \in \{-, +\}$ and $w_0, w_1 \in W$ such that $(w_0, t_0) \in \partial_{\sigma_0} D_0^\circ$ and $(w_1, t_1) \in \partial_{\sigma_1} D_1^\circ$. By definition of $\mathcal{D}_W^{\mathrm{good}}$ we have $W \subseteq U_{D_0}$ and $D_0 \subseteq C_\lambda^\circ$. Therefore there is $\tau \in \mathbb{R}$ such that $\Phi_\tau(w_1, t_1) = (w_1, t_1 + \tau) \in \partial_{\sigma_0} D_0^\circ$ and $t_1 + \tau \in (c_-, c_+)$. We conclude from Lemma 4.40 that $|t_0 - (t_1 + \tau)| < \eta$. Because

$\mathcal{D}^{n-1}$ satisfies the induction assumption, we know that $\mathcal{D}_W \subseteq \mathcal{D}^{n-1}$ is $\delta_{r,n-1}$–overlong. Therefore

$$|\tau| \notin [\epsilon - \delta_{r,n-1}, \alpha + \delta_{r,n-1}].$$

This implies our result, because $||t_0 - t_1| - |\tau|| \leq |(t_0 - t_1) - \tau| < \eta$. □

Definition 4.44 For $W \in \mathcal{W}^n$ let

$$R_-^W := \sup\left((b_- + \alpha + \epsilon, a_-) \cap J_W^{\text{good},+}\right) \cup \{b_- + \alpha + \epsilon\};$$
$$R_+^W := \inf\left((a_+, b_+ - \alpha - \epsilon) \cap J_W^{\text{good},-}\right) \cup \{b_+ - \alpha - \epsilon\}.$$

Lemma 4.45 *Let $W \in \mathcal{W}^n$. We have:*

(i) $(R_-^W, R_+^W) \cap J_W^{\text{good}} = \varnothing$;

(ii) $R_-^W + \alpha + \delta_{r,n} + 2\eta < R_+^W$.

Proof We first show that

$$[a_- - \epsilon/2 + \eta, a_+ + \epsilon/2 - \eta] \cap J_W^{\text{good}} = \varnothing. \tag{4.46}$$

We proceed by contradiction. If (4.46) fails then there are $t_0 \in [a_- - \epsilon/2 + \eta, a_+ + \epsilon/2 - \eta]$, $w_0 \in W$ and a box $D \in \mathcal{D}_W^{\text{good}}$ such that $(w_0, t_0) \in D^\circ$. We have $W \subseteq U_D$. For every $w \in W$ there exist unique real numbers $\tau_\pm(w)$ such that $(w, \tau_\pm(w)) \in \partial_\pm D^\circ$. From Lemma 4.40 we conclude

$$\tau_\pm(w) \in (\tau_\pm(w_0) - \eta, \tau_\pm(w_0) + \eta) \quad \text{for } w \in W.$$

We have $\tau_-(w_0) \leq t_0 \leq \tau_+(w_0)$. From $a_+ - a_- = \epsilon/2$ we conclude $t_0 \leq a_+ + \epsilon/2 - \eta = a_- + \epsilon - \eta$ and $t_0 \geq a_- - \epsilon/2 + \eta = a_+ - \epsilon + \eta$. We estimate

$$\tau_-(w) - \epsilon < \tau_-(w_0) + \eta - \epsilon \leq t_0 + \eta - \epsilon \leq a_-;$$
$$\tau_+(w) + \epsilon > \tau_+(w_0) - \eta + \epsilon \geq t_0 - \eta + \epsilon \geq a_+.$$

This implies $W \times [a_-, a_+] \subset \Phi_{(-\epsilon,\epsilon)} D^\circ$ which contradicts the definition of $\mathcal{W}^n$ in Definition 4.39. This proves (4.46).

We give now the proof of (i). Assume that there is $D \in \mathcal{D}_W^{\text{good}}$, $t_0 \in (R_-^W, R_+^W)$ and $w_0 \in W$ such that $(w_0, t_0) \in D^\circ$. Because $\eta = \mu/5 < \epsilon/5 < \epsilon/2$ we conclude from (4.46) that either $t_0 < a_-$ or $t_0 > a_+$. We treat the first case, in the second case there is an analogous argument. There is $\tau_+ \geq 0$ such that $(w_0, t_0 + \tau_+) \in \partial_+ D^\circ$. If $t_0 + \tau_+ > a_-$, then $(w_0, a_-) \in D^\circ$, that is, $a_- \in J_W^{\text{good}}$. Since this contradicts (4.46) we conclude $t_0 + \tau_+ \leq a_-$. Clearly $t_0 + \tau_+ \in J_W^{\text{good},+}$ and $b_- + \alpha + \epsilon \leq R_-^W < t_0 \leq t_0 + \tau_+$. But this contradicts the construction of R_-^W in Definition 4.44. This proves (i).

Next we prove (ii). First we treat the case $R^W_- = b_- + \alpha + \epsilon$. Since $2\eta = 2\mu/5 < 2\epsilon/5 < \epsilon$, $\delta_{r,n} < \epsilon$ and $2\alpha + 3\epsilon < b = a_- - b_-$ we conclude

$$R^W_- + \alpha + \delta_{r,n} + 2\eta = b_- + 2\alpha + \epsilon + \delta_{r,n} + 2\eta < b_- + 2\alpha + 3\epsilon < b_- + b = a_- \le R^W_+.$$

The case $R^W_+ = b_+ - \alpha - \epsilon$ can be treated similarly. Therefore we may assume now $R^W_- \neq b_- + \alpha + \epsilon$ and $R^W_+ \neq b_+ - \alpha - \epsilon$. From the construction of $R^W_\pm$ we conclude then that there are $t_\pm \in J^{\mathrm{good},\mp}_W$, such that $R^W_- - \eta < t_- \le R^W_-$ and $R^W_+ \le t_+ < R^W_+ + \eta$. Clearly $t_- \le R^W_- \le a_- < a_+ \le R^W_+ \le t_+$. Thus $t_+ - t_- > 0$. By Lemma 4.43

$$t_+ - t_- \notin [\epsilon - \delta_{r,n-1} + \eta, \alpha + \delta_{r,n-1} - \eta].$$

On the other hand **(4.46)** implies $t_- < a_- - \epsilon/2 + \eta$ and $t_+ > a_+ + \epsilon/2 - \eta$. Using $a_+ - a_- = \epsilon/2$, $2\eta = 2\mu/5 < 2\epsilon/5 < \epsilon/2$ and $\delta_{r,n-1} - \eta = \delta_{r,n} + \mu - \eta > \delta_{r,n} \ge 0$ we estimate

$$t_+ - t_- \;>\; (a_+ + \epsilon/2 - \eta) - (a_- - \epsilon/2 + \eta) \;=\; 3\epsilon/2 - 2\eta \;>\; \epsilon \;>\; \epsilon - \delta_{r,n-1} + \eta.$$

Therefore $t_+ - t_- > \alpha + \delta_{r,n-1} - \eta = \alpha + \delta_{r,n} + \mu - \eta$. This implies $t_- + \alpha + \delta_{r,n} < t_+ + \eta - \mu$. Using this and $5\eta = \mu$ we compute

$$\begin{aligned} R^W_- + \alpha + \delta_{r,n} + 2\eta \;&<\; t_- + \alpha + \delta_{r,n} + 3\eta \\ &<\; t_+ - \mu + 4\eta \;<\; R^W_+ - \mu + 5\eta \;=\; R^W_+. \end{aligned}$$ □

We can now give the construction of $a^W_\pm$ for $W \in \mathcal{W}^n$. If $R^W_- > b_- + \alpha + \epsilon$ then we set $a^W_- := R^W_- + \eta$. Otherwise $R^W_- = b_- + \alpha + \epsilon$ and we will use the fact that we arranged $b = a_- - b_-$ to be very large. It follows from Lemmas 4.42 and 4.45 (i) and $a_- \le R^W_+$ that $J^\partial_W \cap [R^W_-, a_-]$ is contained in the union of $2M$ intervals of length 2η. Using $\delta_{r,n} + \eta < \delta_{r,n} + \mu = \delta_{r,n-1} < \epsilon$ we estimate

$$\begin{aligned} a_- - (b_- + \alpha + \epsilon) \;&=\; b - (\alpha + \epsilon) \;=\; 4M(\alpha + 2\epsilon) + 2(\alpha + \epsilon) \\ &>\; (2M+1)(2\alpha + 2\epsilon) \;>\; (2M+1)(2\alpha + 2\eta + 2\delta_{r,n}). \end{aligned}$$

If from an interval I of length strictly larger than L, we take out $2M$ or less intervals, each of which has length less than or equal to l, then the remaining set contains an interval of length $\widetilde{l} := (L - 2Ml)/(2M+1)$. The center of such an interval, will have distance $\widetilde{l}/2$ from all points in the $2M$ intervals and from the boundary of I. Therefore we find $a^W_- \in [(b_- + \alpha + \epsilon) + (\alpha + \delta_{r,n}), a_- - (\alpha + \delta_{r,n})]$ such that

$$|a^W_- - t| > \alpha + \delta_{r,n} \text{ for all } t \in J^\partial_W. \tag{4.47}$$

This finishes the construction of a^W_-. To construct a^W_+ we proceed similarly. If $R^W_+ < b_+ - \alpha - \epsilon$ then we set $a^W_+ := R^W_+ - \eta$. Otherwise $R^W_+ = b_+ - \alpha - \epsilon$ and there

is $a_+^W \in [a_+ + (\alpha+\delta_{r,n}), (b_+ - \alpha + \epsilon) - (\alpha+\delta_{r,n})]$ such that

$$|a_+^W - t| > \alpha + \delta_{r,n} \text{ for all } t \in J_W^\partial. \tag{4.48}$$

This finishes the construction of a_+^W. We can arrange that $a_\pm^W = a_\pm^{gW}$ for $g \in G_{B_\lambda}$. For $W \in \mathcal{W}^n$ let now $D^W := B_\lambda(\overline{W}; a_-^W, a_+^W) = \overline{W} \times [a_-^W, a_+^W]$.

Lemma 4.49 *Let $W \in \mathcal{W}^n$.*

(i) $\Phi_{(-\alpha-\epsilon,\alpha+\epsilon)}(D^W) \subset B_\lambda$;

(ii) *If x lies in the open bottom or open top of D^W then*

$$\Phi_{[-\alpha-\delta_{r,n},-\epsilon+\delta_{r,n}]\cup[\epsilon-\delta_{r,n},\alpha+\delta_{r,n}]}(x)$$

does not intersect the open bottom or top of a box $D \in \mathcal{D}^{n-1}$;

(iii) $l_{D^W} = a_+^W - a_-^W > \alpha + \delta_{r,n}$;

(iv) $W \times [a_-, a_+] \subseteq \Phi_{(-\epsilon,\epsilon)}(D^W)^\circ$;

(v) $|\{D \in \mathcal{D}^{n-1} \mid D^\circ \cap (D^W)^\circ \neq \varnothing\}| \leq M$.

Proof (i) By construction $b_- + \alpha + \epsilon < a_-^W < a_+^W < b_+ - \alpha - \epsilon$ and (4.34) implies $\overline{W} \subseteq S_{B_\lambda}$.

(ii) We consider the open bottom first. Let $w \in W$ and $x = (w, a_-^W)$. By (i) $\Phi_{[-\alpha-\delta_{r,n},-\epsilon+\delta_{r,n}]\cup[\epsilon-\delta_{r,n},\alpha+\delta_{r,n}]}(x)$ is contained in B_λ and can therefore only intersect boxes from $\mathcal{D}_W$. The claim follows thus if $|a_-^W - t| \notin [\epsilon - \delta_{r,n}, \alpha + \delta_{r,n}]$ for all $t \in J_W^\partial$. If $R_-^W = b_- + \alpha + \epsilon$ then (4.47) holds and implies our claim. Otherwise $a_-^W = R_-^W + \eta$ and there is $t_0 \in J_W^{\text{good},+} \cap [R_-^W - \eta, R_-^W]$ by the construction of R_-^W in Definition 4.44. Now Lemma 4.43 implies our claim since $\delta_{r,n-1} - \delta_{r,n} = \mu > 3\mu/5 = 3\eta$. The open top can be treated completely analogously.

(iii) Clearly $a_+^W - a_-^W$ is the length of D^W. By construction $a_-^W \leq a_- + \eta < a_+$ and $a_+^W \geq a_+ - \eta > a_-$ since $a_+ - a_- = a = \epsilon/2 > \mu/5 = \eta$. If $R_-^W = b_- + \alpha + \epsilon$, then by construction $a_-^W \leq a_- - (\alpha + \delta_{r,n})$ and our claim follows. Similarly the claim follows if $R_+^W = b_+ - \alpha - \epsilon$. Thus we are left with the case $a_\pm^W = R_\pm^W \mp \eta$ and the claim follows from Lemma 4.45 (ii).

(iv) As noted above the construction of $a_\pm^W$ implies that $a_-^W - \eta \leq a_-$ and $a_+^W + \eta \geq a_+$. The claim follows therefore from $\eta < \epsilon$.

(v) Because $(D^W)^\circ \subset W \times (b_-, b_+)$,

$$\{D \in \mathcal{D}^{n-1} \mid D^\circ \cap (D^W)^\circ \neq \varnothing\} = \{D \in \mathcal{D}_W \mid D^\circ \cap (D^W)^\circ \neq \varnothing\}.$$

By construction $R_-^W < a_-^W < a_+^W < R_+^W$. Thus Lemma 4.45 (i) and Lemma 4.42 imply

$$|\{D \in \mathcal{D}_W \mid D^\circ \cap (D^W)^\circ \neq \varnothing\}| = |\{D \in \mathcal{D}_W^{\mathrm{bad}} \mid D^\circ \cap (D^W)^\circ \neq \varnothing\}| \leq M. \qquad \square$$

We now define

$$\mathcal{D}^n := \mathcal{D}^{n-1} \cup \{gD^W \mid W \in \mathcal{W}^n, g \in G\}.$$

It remains to check that $\mathcal{D}^n$ is a $\delta_{r,n}$–good box cover of $K_\Xi \cup GK_\lambda^{(n)}$. Recall the induction hypothesis that $\mathcal{D}^{n-1}$ is a $\delta_{r,n-1}$–good box cover of $K_\Xi \cup GK_\lambda^{(n-1)}$.

We begin with showing that $\mathcal{D}^n$ is $\delta_{r,n}$–overlong. So we have to show for every $x \in X$ which lies on the open bottom or open top of a box D_1 in $\mathcal{D}^n$, that the set $\Phi_{[-\alpha-\delta_{r,n},-\epsilon+\delta_{r,n}]\cup[\epsilon-\delta_{r,n},\alpha+\delta_{r,n}]}(x)$ does not intersect the open bottom or the open top of any box D_2 in $\mathcal{D}^n$.

If D_1 and D_2 lie in $\mathcal{D}^{n-1}$, this follows from the induction hypothesis.

Suppose that $D_1 \notin \mathcal{D}^{n-1}$ and $D_2 \in \mathcal{D}^{n-1}$. Then we can assume without loss of generality that $D_1 = D^W$ for some $W \in \mathcal{W}^n$ since $\mathcal{D}^{n-1}$ is G–invariant. The claim follows then from Lemma 4.49 (ii).

The case $D_1 \in \mathcal{D}^{n-1}$ and $D_2 \notin \mathcal{D}^{n-1}$ is treated analogously.

If $D_1 = D_2$ and $D_1 \notin \mathcal{D}^{n-1}$, then the claim follows from Lemma 4.49 (i) and (iii).

If $D_1 \neq D_2$ and $D_1, D_2 \notin \mathcal{D}^{n-1}$, the claim follows from **(4.32)** and Lemma 4.49 (i) since B_λ is a $\mathcal{F}$in–subset of X. Hence $\mathcal{D}^n$ is $\delta_{r,n}$–overlong.

We conclude from Lemma 4.49 (i) and (iii) that $\mathcal{D}^n$ satisfies **(4.3)**.

We derive the inclusion $K_\Xi \cup GK_\lambda^{(n)} \subset \bigcup_{D\in\mathcal{D}^n} \Phi_{(-\epsilon,\epsilon)}(D^\circ)$ from Definition 4.39 and Lemma 4.49 (iv). (The set $K_\lambda^{(n)}$ was defined before Lemma 4.38.)

By **(4.32)** the D^W are mutually disjoint. Therefore Lemma 4.49 (v) implies that **(4.4)** holds for $\mathcal{D}^n$.

It is clear that **(4.5)** and **(4.6)** hold for $\mathcal{D}^n$.

Next we prove property **(4.7)**. Because of the induction hypothesis it suffices to prove the assertion for the boxes gD^W for $g \in G$ and $W \in \mathcal{W}^n$, where $D^W = B_\lambda(\overline{W}; a_-^W, a_+^W)$. From Lemma 4.49 (i) we conclude $\Phi_{[-\alpha-\epsilon,\alpha+\epsilon]}(D^W) \subseteq B_\lambda(W)$. Since $\overline{W}$ is a $\mathcal{F}$in–subset of S_{B_λ} with respect to the G_{B_λ}–action by **(4.34)** and B is a $\mathcal{F}$in–subset of the G–space X, $\Phi_{[-\alpha-\epsilon,\alpha+\epsilon]}(D^W)$ is a $\mathcal{F}$in–subset of the G–space X.

Finally we show that elements in $\mathcal{D}^{(n)}$ are not huge. For $g \in G$ and $W \in \mathcal{W}^n$ the box gD^W can be obtained by restriction from $B_{g\lambda}$ and is therefore not huge; compare Definition 4.10 and the subsequent comment.

We have shown that $\mathcal{D}^n$ is the required $\delta_{r,n}$–good box cover of $K_\Xi \cup GK_\lambda^{(n)}$. This finishes the proof of Lemma 4.38.

As was noted before, Proposition 4.1 follows from Lemma 4.13 which follows from Lemma 4.38. The proof of Proposition 4.1 is therefore now completed.

5 Construction of long $\mathcal{VC}$yc–covers of X

At the end of this section we will give the proof of Theorem 1.4. Throughout this section we will work in the situation of Convention 1.3. In order to construct the long and thin cover of X we need to discuss covers of $X^{\mathbb{R}}$ and $X_{\leq\gamma} - X^{\mathbb{R}}$.

Lemma 5.1 *There exists a collection $\mathcal{U}_{X^{\mathbb{R}}}$ of open $\mathcal{F}$in–subsets of X such that $G\backslash\mathcal{U}_{X^{\mathbb{R}}}$ is finite, $X^{\mathbb{R}} \subset \bigcup_{U\in\mathcal{U}_{X^{\mathbb{R}}}} U$ and $\dim(\mathcal{U}_{X^{\mathbb{R}}}) < \infty$.*

Proof Because the action of G on X is proper there is for $x \in X^{\mathbb{R}}$ an open $\mathcal{F}$in–neighborhood W_x of x. Because the action of G on X is cocompact and $X^{\mathbb{R}}$ is closed, there is a finite subset $\Lambda \subset X^{\mathbb{R}}$ such that $X^{\mathbb{R}} \subset \bigcup_{g\in G}\bigcup_{\lambda\in\Lambda} gW_\lambda$. Let $\mathcal{U}_{X^{\mathbb{R}}} = \{gW_\lambda \mid g \in G, \lambda \in \Lambda\}$. Because the W_λ are $\mathcal{F}$in–sets we have $gW_\lambda \neq W_\lambda \Rightarrow gW_\lambda \cap W_\lambda = \varnothing$. Therefore $\dim(\mathcal{U}_{X^{\mathbb{R}}}) \leq |\Lambda| - 1$. □

Lemma 5.2 *Fix $\gamma > 0$. Let $\mathcal{L}_{\leq\gamma}$ be the set of orbits $L = \Phi_{\mathbb{R}}(x)$ in X whose G–period satisfies $0 < \mathrm{per}_\Phi^G(L) \leq \gamma$. Then there exists a collection $\mathcal{U}_\gamma = \{U_L \mid L \in \mathcal{L}_{\leq\gamma}\}$ of open $\mathcal{VC}$yc–subsets U_L of the G–space X such that $L \subseteq U_L$ for $L \in \mathcal{L}_\gamma$ and $\dim\mathcal{U} = 0$.*

Proof By assumption we can find finitely many pairwise distinct elements L_1, L_2, …, L_r in $\mathcal{L}_{\leq\gamma}$ such that $\mathcal{L}_{\leq\gamma} = G\cdot\{L_1, L_2, \ldots, L_r\}$. We can arrange that $L_j = g\cdot L_k$ for some $g \in G$ implies $j = k$. Since the G_{L_j}–action on L_j is proper and cocompact and L_j is homeomorphic to $\mathbb{R}$ or S^1, the group G_{L_j} is virtually cyclic. (A group that acts cocompact and properly on $\mathbb{R}$ has two ends and is therefore virtually cyclic [5, Theorem I.8.32(2)].) We can choose compact subsets $K_j \subseteq L_j$ with $L_j = G_{L_j}\cdot K_j$ and $G\backslash GL_j$ closed.

Since $G\backslash X$ is compact and $G\backslash GL_j \cap G\backslash GL_k \neq \varnothing \Rightarrow j = k$ holds, we can find open subsets $V_1', V_2', \ldots, V_r'$ in $G\backslash X$ such that $G\backslash GL_j \subseteq V_j'$ and $V_j' \cap V_k' \neq \varnothing \Rightarrow j = k$

holds. Let V_j be the preimage of V_j' under the projection $X \to G\backslash X$. Then V_j is a G–invariant open neighborhood of L_j and $V_j \cap V_k \neq \varnothing \Rightarrow j = k$.

Fix $j \in \{1, 2, \dots r\}$. Since the G–action on X is proper, we can find an open neighborhood W_j' of K_j and a finite subset $S \subseteq G$ such that $W_j' \cap gW_j' \neq \varnothing \Rightarrow g \in S$. Let $S_0 \subseteq S$ be the subset consisting of those elements $g \in S$ for which $K_j \cap gK_j = \varnothing$. Since K_j is compact, we can find for $s \in S_0$ an open neighborhood W_s'' of K_j such that $W_s'' \cap sW_s'' = \varnothing$. Put $U_j' := W_j' \cap \bigcap_{s \in S_0} W_s''$. Then U_j' is an open neighborhood of K_j such that $U_j' \cap gU_j' \neq \varnothing$ implies $K_j \cap gK_j \neq \varnothing$. Put $U_j = G_{L_j} \cdot U_j'$. Then U_j is a G_{L_j}–invariant open subset containing $L_j = G_{L_j} K_j$. Next we prove $gU_j \cap U_j \neq \varnothing \Rightarrow g \in G_{L_j}$. Suppose for $g \in G$ that $gU_j \cap U_j \neq \varnothing$. Then we can find $g_0, g_1 \in G_{L_j}$ such that $gg_0U_j' \cap g_1U_j \neq \varnothing$. This implies $g_1^{-1}gg_0K_j \cap K_j \neq \varnothing$. We conclude $g_1^{-1}gg_0L_j \cap L_j \neq \varnothing$ and hence $g_1^{-1}gg_2L_j = L_j$. This shows $g_1^{-1}gg_0 \in G_{L_j}$ and thus $g \in G_{L_j}$. Hence U_j is an open $\mathcal{VC}$yc–subset of the G–space X such that $G_{U_j} = G_{L_j}$ and $L_j \subseteq U_j$.

Define for any element $L \in \mathcal{L}_{\leq \gamma}$

$$U_L = g \cdot (V_j \cap U_j) \quad \text{for } g \in G \text{ with } L = gL_j.$$

This is independent of the choice of g and j and U_L is a $\mathcal{VC}$yc–subset of the G–space X with $G_{U_L} = G_L$ since $(V_j \cap U_j)$ is a $\mathcal{VC}$yc–subset of the G–space X with $G_{V_j \cap U_j} = G_{L_j}$. We have by construction

$$U_{L_1} \cap U_{L_2} \neq \varnothing \Rightarrow L_1 = L_2. \qquad \square$$

Finally we can give the proof of Theorem 1.4.

Proof Let $\alpha > 0$ be given. Choose ϵ such that $0 < \epsilon < \alpha$. Let $M = M(k_G, d_X)$ and $\gamma = \gamma(4\alpha, \epsilon, M) > 0$ be as in Proposition 4.1.

Let $\mathcal{U}_{X^{\mathbb{R}}}$ be the collection of open $\mathcal{F}$in–sets from Lemma 5.1 and $\mathcal{U}_\gamma$ be the collection of open $\mathcal{VC}$yc–sets from Lemma 5.2. Note that $\dim(\mathcal{U}_{X^{\mathbb{R}}} \cup \mathcal{U}_\gamma) = \dim(\mathcal{U}_{X^{\mathbb{R}}}) + 1$ is finite and does not depend on α, but only on an arbitrarily small neighborhood of $X^{\mathbb{R}}$ as a G–space.

Put $$S = \{x \in X \mid \exists\, U \in \mathcal{U}_\gamma \cup \mathcal{U}_{X^{\mathbb{R}}} \text{ such that } \Phi_{[-\alpha,\alpha]}(x) \subset U\}.$$

Note that S is G–invariant, because $\mathcal{U}_\gamma$ and $\mathcal{U}_{X^{\mathbb{R}}}$ are. Consider $x \in S$. Choose $U_x \in \mathcal{U}_\gamma \cup \mathcal{U}_{X^{\mathbb{R}}}$ with $\Phi_{[-\alpha,\alpha]}(x) \subseteq U$. Since $\{x\} \times [-\alpha, \alpha]$ is compact and contained in $\Phi^{-1}(U_x)$, we can find an open neighborhood V_x of x such that $V_x \times [-\alpha, \alpha] \subseteq \Phi^{-1}(U_x)$. This implies $V_x \subseteq S$. Hence S is an open G–invariant subset of X which contains $X_{\leq \gamma}$.

Let K be the closure of the complement of S in X. Since $G\backslash X$ is compact $K \subseteq X$ is a cocompact G–invariant subset which does not meet $X_{\leq\gamma}$. Hence we can apply Proposition 4.1 to K with respect to 4α instead of α and ϵ with $0 < \epsilon < \alpha$. Recall that $M = M(k_G, d_X)$ and $\gamma = \gamma(M, 4\alpha, \epsilon)$ are the numbers appearing in Proposition 4.1. So we get a collection of boxes $\mathcal{D}$ with the properties described in Proposition 4.1. Put

$$\mathcal{U}_K = \{\Phi_{(-\alpha-\epsilon,+\alpha+\epsilon)} D^\circ \mid D \in \mathcal{D}\}.$$

Then for $x \in K$ there is $U \in \mathcal{U}_K$ such that $\Phi_{[-\alpha,\alpha]}(x) \subset U$ and every element in $\mathcal{U}_K$ is an open $\mathcal{F}$in–subset of X.

Next we show $\dim(\mathcal{U}_K) \leq 2M + 1$. Consider pairwise disjoint elements D_1, D_2, $\ldots$, D_{2M+3} of $\mathcal{D}$. We have to show that $\bigcap_{k=1}^{2M+3} \Phi_{(-\alpha-\epsilon,\alpha+\epsilon)}(D_k^\circ) = \varnothing$. Suppose the contrary, ie, there exist $x \in X$ such that $x \in \Phi_{(-\alpha-\epsilon,\alpha+\epsilon)}(D_k^\circ)$ holds for $k = 1, 2, \ldots, (2M + 3)$. Obviously $x \in \Phi_{(-\alpha-\epsilon,\alpha+\epsilon)}(D_k^\circ)$ implies that $\Phi_{2\alpha}(x) \in D_k^\circ$ or $\Phi_{-2\alpha}(x) \in D_k^\circ$ since $\epsilon < \alpha$ and for every $y \in X$ the set $\Phi_{[0,4\alpha]}(y)$ can not intersect both the open bottom and the open top of D_k. Hence we can find $(M + 2)$ pairwise distinct elements $k_1, k_2, \ldots, k_{M+2} \subseteq \{1, 2, \ldots, 2M + 3\}$ such that $\Phi_{2\alpha}(x) \in D_{k_j}^\circ$ holds for $j = 1, 2, \ldots, M+2$ or that $\Phi_{-2\alpha}(x) \in D_{k_j}^\circ$ holds for $j = 1, 2, \ldots, M+2$. In both cases we get a contradiction to $\dim(\{D^\circ \mid D \in \mathcal{D}\}) \leq M$. This shows $\dim(\mathcal{U}_K) \leq 2M + 1$. Note that this bound depends only on k_G and d_X and is independent of α. Because

$$\dim(\mathcal{U}_K \cup \mathcal{U}_\gamma \cup \mathcal{U}_{X^{\mathbb{R}}}) \;\leq\; \dim(\mathcal{U}_K) + \dim(\mathcal{U}_\gamma \cup \mathcal{U}_{X^{\mathbb{R}}}) + 1$$

this implies that the dimension of

$$\mathcal{U} \;:=\; \mathcal{U}_K \cup \mathcal{U}_\gamma \cup \mathcal{U}_{X^{\mathbb{R}}}$$

is bounded by a number that depends only on k_G, d_X and the G–action on an arbitrary small neighborhood of $X^{\mathbb{R}}$. Thus $\mathcal{U}$ is the required $\mathcal{VC}$yc–cover of X. This finishes the proof of Theorem 1.4. □

Remark 5.3 In Convention 1.3 we assumed that the number of closed orbits, which are not stationary and whose period is $\leq C$, of the flow induced on $G\backslash X$ is finite for every $C > 0$. This assumption can be replace with the following less restrictive assumption.

> There is a number N such that for every $\gamma > 0$ there is a G–invariant collection $\mathcal{U}$ of open $\mathcal{VC}$yc–subsets of X such that for each $x \in X_{\leq\gamma}$ there is $U \in \mathcal{U}$ such that $\Phi_{[-\gamma,\gamma]} \subset U$, $\dim(\mathcal{U}) \leq N$ and $G\backslash\mathcal{U}$ is finite.

The proof of Theorem 1.4 given above clearly also works under this less restrictive assumption. This might be useful in nonpositively curved situations.

6 Mineyev's flow space

6.1 Hyperbolic complexes, double difference and Gromov product

We collect some basic concepts such as hyperbolic complexes, double difference and Gromov product which are all taken from the paper by Mineyev [14] and which we will need for our purposes.

A simplicial complex is called *uniformly locally finite* if there exists a number N such that any element in the 0–skeleton X_0 occurs as vertex of at most N simplices.

Let X be a simplicial complex. Given any metric d on its 0–skeleton, one can extend it to a metric $\widetilde{d}$ on X as follows. Given points u_k for $k = 1, 2$ in X, we can find vertices $x_k[0]$, $x_k[1]$, $x_k[n_k]$ such that u_k belongs to the simplex with vertices $x_k[0]$, $x_k[1]$, …, $x_k[n_k]$. There are unique numbers $\alpha_k[0]$, $\alpha_k[1]$, …, $\alpha_k[n_k]$ in $[0,1]$ with $\sum_{i_k=0}^{n_k}\alpha_k[i_k]=1$ such that u_k is given by $\sum_{i_k=0}^{n_k}\alpha_k[i_k]\cdot x_k[n_k]$. Then define

$$\widetilde{d}(u_1,u_2) \;=\; \sum_{i_1=0}^{n_1}\sum_{i_2=0}^{n_2}\alpha_1[i_1]\cdot\alpha_2[i_2]\cdot d(x_1[i_1],x_2[i_2]).$$

This is a well-defined metric extending d such that each simplex with the metric induced by $\widetilde{d}$ is homeomorphic to the standard simplex.

Given a connected simplicial complex, define a metric d_0 on its 0–skeleton by defining $d_0(x,y)$ as the minimum of the numbers $n \geq 0$ such that there is a sequence of vertices $x = x_0, x_1, \ldots, x_n = y$ with the property that x_i and x_{i+1} are joint by an edge for $i = 0, 1, \ldots, n-1$. The *word metric* d_{word} on a connected simplicial complex X is the metric $\widetilde{d_0}$.

A *metric complex* (X,d) is a connected uniformly locally finite simplicial complex X equipped with its word metric $d = d_{\text{word}}$. A *hyperbolic complex* X is a metric complex (X,d) such that (X,d) is δ–hyperbolic in the sense of Gromov for some $\delta > 0$ (see Gromov [10] and Bridson, Haefliger and Buch [5, Definition III.H.1.1]). Let ∂X be the boundary and $\overline{X} = X \cup \partial X$ be the compactification of the hyperbolic complex X in the sense of Gromov [10; 5, III.H.3].

Mineyev [14, 6.1] constructs for a hyperbolic metric complex (X,d) a new metric $\widehat{D}$ with certain properties (see [14, Lemma 2.7 on page 449 and Theorem 32 on page 446]). For instance $\widehat{D}$ is quasi-isometric to the word metric d_{word}. For a simplicial map $f\colon X \to X$ the following conditions are equivalent: (i) f is a simplicial automorphism, (ii) f is a simplicial automorphism preserving the word metric d_{word}, (iii) f is a

simplicial automorphism preserving the metric $\hat{D}$. Define for $a, a', b, b' \in X$ the *double difference* to be the real number

$$\langle a, a' | b, b' \rangle := \frac{1}{2} \cdot \Big(\hat{D}(a,b) + \hat{D}(a',b') - \hat{D}(a',b) - \hat{D}(a,b') \Big). \tag{6.1}$$

Recall that the *Gromov product* for $a, b, c \in X$ is defined to be the positive real number

$$\langle a | b \rangle_c := \frac{1}{2} \cdot \Big(\hat{D}(a,c) + \hat{D}(b,c) - \hat{D}(a,b) \Big). \tag{6.2}$$

Define the subset

$$S(\bar{X}) \subseteq \big\{ (a, a', b, b') \in \bar{X} \times \bar{X} \times \bar{X} \times \bar{X} \big\} = \bar{X}^4$$

to consist of those quadruples (a, a', b, b') satisfying

$$\begin{aligned} a, b \in \partial X &\Rightarrow a \neq b; \\ a, b' \in \partial X &\Rightarrow a' \neq b; \\ a', b \in \partial X &\Rightarrow a \neq b'; \\ a', b' \in \partial X &\Rightarrow a' \neq b'. \end{aligned}$$

Let $\quad T(\bar{X}) := \{(a,b,c) \mid \bar{X} \times \bar{X} \times \bar{X} \mid c \in \partial X \Rightarrow (a \neq c \text{ and } b \neq c)\}.$

We equip $S(\bar{X}) \subseteq \bar{X}^4$ and $T(\bar{X}) \subseteq \bar{X}^3$ with the subspace topology. The following result is a special case of [14, Theorem 35 on page 448 and Theorem 36 on page 452]. (We only need and want to consider the case where the double difference takes values in $\mathbb{R}$.)

Theorem 6.3 (Mineyev) *Let (X, d) be a hyperbolic complex. Then the double difference of (6.1) extends to a continuous function invariant under simplicial automorphisms of X*

$$\langle -, - | -, - \rangle \colon S(\bar{X}) \to \mathbb{R}$$

satisfying

(i) $\langle a, a' | b, b' \rangle = \langle b, b' | a, a' \rangle$;

(ii) $\langle a, a' | b, b' \rangle = -\langle a', a | b, b' \rangle = -\langle a, a' | b', b \rangle$;

(iii) $\langle a, a | b, b' \rangle = \langle a, a' | b, b \rangle = 0$;

(iv) $\langle a, a' | b, b' \rangle + \langle a', a'' | b, b' \rangle = \langle a, a'' | b, b' \rangle$;

(v) $\langle a, b | c, x \rangle + \langle c, a | b, x \rangle + \langle b, c | a, x \rangle = 0$;

The Gromov product of (6.2) extends to a continuous function invariant under simplicial automorphisms of X

$$\langle -|-\rangle_-\colon T(\bar{X}) \to [0,\infty]$$

satisfying

(i) $\langle a|b\rangle_c = \infty \Leftrightarrow (c \in \partial X)$ *or* $(a,b \in \partial X$ *and* $a = b)$;

(ii) $\langle a,b|x,y\rangle = \langle b|x\rangle_a - \langle b|y\rangle_a$ *for* $a \in X$ *and* $(a,b,x,y) \in S(\bar{X})$.

We will often use the rules appearing in Theorem 6.3 tacitly.

Another important ingredient will be the following result due to Mineyev [14, Proposition 38 on page 453].

Proposition 6.4 (Mineyev) *Let (X,d) be a hyperbolic complex. Then there exist constants $\lambda \in (e^{-1},1)$ and $T \in [0,\infty)$ depending only on X such that for all $a,b,c,u \in \bar{X}$ satisfying*

$$\begin{gathered}(a,c,u,b) \in S(\bar{X});\\ (b,c,u,a) \in S(\bar{X});\\ \max\{\langle a,c|u,b\rangle, \langle b,c|u,a\rangle\} \geq T,\end{gathered}$$

we have

$$\begin{gathered}(u,c,a,b) \in S(\bar{X});\\ |\langle u,c|a,b\rangle| \leq \lambda^{\max\{\langle a,c|u,b\rangle,\langle b,c|u,a\rangle\}}.\end{gathered}$$

6.2 Two auxiliary functions

In the sequel we will use the following two functions for $\alpha,\beta \in \bar{\mathbb{R}} := \mathbb{R} \coprod \{-\infty,\infty\}$ with $\alpha \leq \beta$.

(6.5) $$\theta_{[\alpha,\beta]}\colon \bar{\mathbb{R}} \to [\alpha,\beta]$$

(6.6) $$\Theta_{[\alpha,\beta]}\colon \bar{\mathbb{R}} \to [\alpha,\beta]$$

which are defined by

$$\theta_{[\alpha,\beta]}(t) := \begin{cases} \alpha & \text{if } -\infty \leq t \leq \alpha;\\ t & \text{if } \alpha \leq t \leq \beta;\\ \beta & \text{if } \beta \leq t \leq \infty,\end{cases}$$

$$\Theta_{[\alpha,\beta]}(t) := \begin{cases} -\infty & \text{if } -\infty = t = \alpha,\\ \alpha + e^{t-\alpha}/2 - e^{t-\beta}/2 & \text{if } -\infty \leq t \leq \alpha, -\infty < \alpha < \infty;\\ t + e^{\alpha-t}/2 - e^{t-\beta}/2 & \text{if } \alpha \leq t \leq \beta, -\infty < t < \infty;\\ \beta + e^{\alpha-t}/2 - e^{\beta-t}/2 & \text{if } \beta \leq t \leq \infty, -\infty < \beta < \infty;\\ \infty & \text{if } t = \beta = \infty.\end{cases}$$

Here and in the sequel we use the convention that for $r, s \in \mathbb{R}$ the expressions $r + s$, $r \cdot s$ and e^r are defined as usual and furthermore

$$\begin{aligned} &r + \infty = \infty + r = \infty && \text{for } r \in \mathbb{R};\\ &r - \infty = -\infty + r = -\infty && \text{for } r \in \mathbb{R};\\ &e^{-\infty} = 0;\\ &e^{\infty} = \infty;\\ &|\pm\infty| = \infty. \end{aligned}$$

The function $\Theta_{[\alpha,\beta]}$ agrees with the function denoted by $\theta'[\alpha, \beta; \cdot]$ in Mineyev [14, Section 1.6]. We equip $\overline{\mathbb{R}}$ with the topology uniquely determined by the properties that $\mathbb{R} \subseteq \overline{\mathbb{R}}$ is an open subset, the subspace topology on $\mathbb{R} \subseteq \overline{\mathbb{R}}$ is the standard topology and a fundamental system for open neighborhoods of ∞ is $\{(R, \infty) \cup \{\infty\} \mid R \in \mathbb{R}\}$ and of $-\infty$ is $\{(-\infty, R) \cup \{-\infty\} \mid R \in \mathbb{R}\}$.

The elementary proof of the following basic properties of $\Theta_{[\alpha,\beta]}$ is left to the reader.

Lemma 6.7 *Suppose that $\alpha < \beta$. Then:*

(i) *We have for $t \in \mathbb{R}$*

$$\Theta_{[\alpha,\beta]}(t) \;=\; \int_{-\infty}^{\infty} \frac{\theta_{[\alpha,\beta]}(t+s)}{2 \cdot e^{|s|}}\, ds \;=\; \theta_{[\alpha,\beta]}(t) + e^{-|\alpha - t|}/2 - e^{-|\beta - t|}/2;$$

(ii) *The restriction of $\Theta_{[\alpha,\beta]}$ to $\mathbb{R}$ is a homeomorphism $\mathbb{R} \xrightarrow{\cong} (\alpha, \beta)$ which is a C^1–function. Its first derivative is the continuous function*

$$t \in \mathbb{R} \;\mapsto\; \begin{cases} e^{t-\alpha}/2 - e^{t-\beta}/2 & \text{if } -\infty < t \le \alpha;\\ 1 - e^{\alpha - t}/2 - e^{t-\beta}/2 & \text{if } \alpha \le t \le \beta;\\ -e^{\alpha - t}/2 + e^{\beta - t}/2 & \text{if } \beta \le t < \infty; \end{cases}$$

(iii) *The function $\Theta_{[\alpha,\beta]}$ is strictly monotone increasing. The function $\theta_{[\alpha,\beta]}$ is monotone increasing;*

(iv) *The function $\Theta_{[\alpha,\beta]}$ is nonexpanding, ie, $|\Theta_{[\alpha,\beta]}(t) - \Theta_{[\alpha,\beta]}(s)| \le |t - s|$ for $t, s \in \overline{\mathbb{R}}$. The same is true for $\theta_{[\alpha,\beta]}$;*

(v) *The map $\Theta_{[\alpha,\beta]} \colon \overline{\mathbb{R}} \to [\alpha, \beta]$ is a homeomorphism;*

(vi) *We have for $t \in \overline{\mathbb{R}}$ and $s \in \mathbb{R}$*

$$\begin{aligned} \theta_{[\alpha+s,\beta+s]}(t+s) &= \theta_{[\alpha,\beta]}(t) + s;\\ \Theta_{[\alpha+s,\beta+s]}(t+s) &= \Theta_{[\alpha,\beta]}(t) + s; \end{aligned}$$

(vii) *Consider $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \overline{\mathbb{R}}$ such that $\alpha_i \le \beta_i$ for $i = 0, 1$, $(\alpha_0 = -\infty \Leftrightarrow \alpha_1 = -\infty)$ and $(\beta_0 = \infty \Leftrightarrow \beta_1 = \infty)$ holds. Put*

$$C := \begin{cases} \max\{|\alpha_1 - \alpha_0|, |\beta_1 - \beta_0|\} & \text{if } \alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbb{R}; \\ |\alpha_1 - \alpha_0| & \text{if } \alpha_0, \alpha_1, \in \mathbb{R}, \beta_0 = \beta_1 = \infty; \\ |\beta_1 - \beta_0| & \text{if } \alpha_0 = \alpha_1 = -\infty, \beta_0, \beta_1, \in \mathbb{R}; \\ 0 & \text{if } \alpha_0 = \alpha_1 = -\infty, \beta_0 = \beta_1 = \infty. \end{cases}$$

Then we get for all $t \in \mathbb{R}$ that

$$\left|\theta_{[\alpha_1,\beta_1]}(t) - \theta_{[\alpha_0,\beta_0]}(t)\right| \le C;$$
$$\left|\Theta_{[\alpha_1,\beta_1]}(t) - \Theta_{[\alpha_0,\beta_0]}(t)\right| \le C;$$

(viii) *If $\alpha \le t$, then*

$$\Theta_{[\alpha,\beta]}(t) - \Theta_{[-\infty,\beta]}(t) = e^{\alpha - t}/2.$$

If $\beta \ge t$, then

$$\Theta_{[\alpha,\infty]}(t) - \Theta_{[\alpha,\beta]}(t) = e^{t-\beta}/2;$$

(ix) *Consider $\alpha, \beta \in \overline{\mathbb{R}}$ with $-\infty < \alpha < \beta$ and $t \in \mathbb{R}$.*
If $\Theta_{[\alpha,\beta]}(t) \le (\alpha + \beta)/2$, then

$$t \le \min\{\beta, \Theta_{[\alpha,\beta]}(t) + 1/2\}.$$

If $\Theta_{[\alpha,\beta]}(t) \ge (\alpha + \beta)/2$, then

$$t \ge \max\{\alpha, \Theta_{[\alpha,\beta]}(t) - 1/2\}.$$

6.3 The construction of the flow space

Let (X, d) be a hyperbolic complex. We want to define the associated *flow space*, ie a metric space $FS(X)$ together with a flow, following Mineyev [14]. (It is the same as the half open symmetric join $*\overline{X}$ constructed by Mineyev [14, Section 8.3].) The underlying set is

$$\textbf{(6.8)}\quad FS(X) := \big\{(a, b, t) \in \overline{X} \times \overline{X} \times \overline{\mathbb{R}} \mid (a \in \partial X \Rightarrow t \ne -\infty) \text{ and } (b \in \partial X \Rightarrow t \ne \infty) \text{ and } (a, b \in \partial X \Rightarrow a \ne b)\big\} / \sim,$$

where we identify $(a, b, -\infty) \sim (a, b', -\infty)$, $(a, b, \infty) \sim (a', b, \infty)$, and $(a, a, t) \sim (a, a, t')$. In the sequel we will denote for $(a, b, t) \in \overline{X} \times \overline{X} \times \overline{\mathbb{R}}$ which satisfies $a \in \partial X \Rightarrow t \ne -\infty$, $b \in \partial X \Rightarrow t \ne \infty$ and $a, b \in \partial X \Rightarrow a \ne b$ its class in FSX again by (a, b, t).

From now on we fix a base point $x_0 \in X$. The metric on $FS(X)$ will depend on this choice.

Define the map

$$\textbf{(6.9)} \qquad \begin{aligned} l_{x_0}\colon X \times FS(X) &\to \mathbb{R} \\ (u,(a,b,t)) &\mapsto \langle a|b\rangle_u + \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \langle a,b|u,x_0\rangle\right|. \end{aligned}$$

It is easy to check that it is compatible with the equivalence relation appearing in the definition of $FS(X)$.

Definition 6.10 Define a pseudometric on $FS(X)$

$$\begin{aligned} d^{\times}_{FS,x_0} = d^{\times}_{FS(X),x_0}\colon FS(X) \times FS(X) &\to \mathbb{R}, \\ (a,b,t),(c,d,s) &\mapsto \sup_{u\in X} \left|l_{x_0}(u,(a,b,t)) - l_{x_0}(u,(c,d,s))\right|. \end{aligned}$$

Recall that a pseudometric satisfies the same axioms as a metric except that the condition $d(x,y) = 0 \Rightarrow x = y$ is dropped. The proof that this definition makes sense and yields a pseudometric $d^{\times}_{FS,x_0}$ is given in Mineyev [14, Theorem 44 on page 459].

Lemma 6.11 *The inclusion $X \to FS(X), x \mapsto (x,x,0)$ is an isometric embedding with respect to the metric $\widehat{D}$ on X and the pseudometric $d^{\times}_{FS,x_0}$ on $FS(X)$.*

Proof We compute for $u \in X$ and $x \in X$ and $t \in \overline{\mathbb{R}}$

$$\begin{aligned} l_{x_0}(u,(x,x,t)) &= \langle x|x\rangle_u + \left|\theta_{[-\langle x|x_0\rangle_x, \langle x,x_0\rangle_x]}(t) - \langle x,x|u,x_0\rangle\right| \\ &= \widehat{D}(x,u) + \left|\theta_{[0,0]}(t) - 0\right| \\ &= \widehat{D}(u,x). \end{aligned}$$

Consider $a,b \in X$. Since by the triangle inequality $|\widehat{D}(u,a) - \widehat{D}(u,b)| \le \widehat{D}(a,b)$ and $|\widehat{D}(b,a) - \widehat{D}(b,b)| = \widehat{D}(a,b)$ holds, we conclude

$$\begin{aligned} d^{\times}_{FS,x_0}((a,a,t),(b,b,s)) &= \sup_{u\in X} \left|l_{x_0}(u,(a,a,t)) - l_{x_0}(u,(b,b,s))\right| \\ &= \sup_{u\in X} |\widehat{D}(u,a) - \widehat{D}(u,b)| \\ &= \widehat{D}(a,b). \end{aligned}$$

□

The canonical $\mathbb{R}$–action on $\overline{\mathbb{R}}$

$$\textbf{(6.12)} \qquad \phi\colon \mathbb{R} \times \overline{\mathbb{R}}, \quad (\tau,t) \mapsto \phi_\tau(t)$$

is defined by $\phi_\tau(t) = t + \tau$, if $t \in \mathbb{R}$, $\phi_\tau(-\infty) = -\infty$ and $\phi_\tau(\infty) = \infty$. This $\mathbb{R}$–action on $\overline{\mathbb{R}}$ together with the trivial $\mathbb{R}$–action on $\overline{X} \times \overline{X}$ yields an action of $\mathbb{R}$ on $\overline{X} \times \overline{X} \times \overline{\mathbb{R}}$ which in turn induces an $\mathbb{R}$–action

$$\phi\colon \mathbb{R} \times FS(X) \to FS(X), \quad (\tau, (a,b,t)) \mapsto (a,b,\phi_\tau(t)). \tag{6.13}$$

For $a, b \in \overline{X}$ we define the *line* $(a,b)_{FS(X)}$ to be the set of points $\{(a,b,t) \mid t \in \mathbb{R}\}$. Obviously $(a,b)_{FS(X)}$ is a transitive free $\mathbb{R}$–set if $a \neq b$.

Next we construct the desired metric from the pseudometric above.

Definition 6.14 Define a metric on $FS(X)$

$$d_{FS(X),x_0}\colon FS(X) \times FS(X) \to \mathbb{R},$$
$$(a,b,t),(c,d,s) \mapsto \int_{\mathbb{R}} \frac{d^{\times}_{FS,x_0}(\phi_\tau(a,b,t),\phi_\tau(c,d,s))}{2 \cdot e^{|\tau|}} d\tau.$$

Obviously d_{FS,x_0} inherits from $d^{\times}_{FS,x_0}$ the properties of a pseudometric. The proof that d_{FS,x_0} is a metric can be found in [14, Theorem 14 on page 426 and Theorem 45 on page 459].

Lemma 6.15 *The inclusion $X \to FS(X)$ is an isometric embedding with respect to the metric $\widehat{D}$ on X and the metric d_{FS,x_0} on $FS(X)$.*

Proof We compute for $u \in X$ and $x \in X$ and $t \in \overline{\mathbb{R}}$ using Lemma 6.11.

$$\begin{aligned}
d_{FS,x_0}((a,a,t),(b,b,s)) &= \int_{\mathbb{R}} \frac{d^{\times}_{FS,x_0}(\phi_\tau(a,a,t),\phi_\tau(b,b,s))}{2 \cdot e^{|\tau|}}\, d\tau \\
&= \int_{\mathbb{R}} \frac{d^{\times}_{FS,x_0}((a,a,t+\tau),(b,b,s+\tau))}{2 \cdot e^{|\tau|}}\, d\tau \\
&= \int_{\mathbb{R}} \frac{\widehat{D}(a,b)}{2 \cdot e^{|\tau|}}\, d\tau \\
&= \widehat{D}(a,b) \cdot \int_{\mathbb{R}} \frac{1}{2 \cdot e^{|\tau|}}\, d\tau \\
&= \widehat{D}(a,b).
\end{aligned}$$

□

In the sequel we will consider X as a subspace of $FS(X)$ by the isometric embedding appearing in Lemma 6.15.

Let $f\colon X \to X$ be an *isometry*, ie a bijection respecting the word metric d on X. It extends uniquely to a homeomorphism $\overline{f}\colon \overline{X} \to \overline{X}$ and induces an isometry with respect to the metric $d_{FS(X),x_0}$

$$FS(f)\colon FS(X) \to FS(X), \quad (a,b,t) \mapsto (\overline{f}(a), \overline{f}(b), t + \langle a,b|x_0, f^{-1}(x_0)\rangle).$$

(Already $d^{\times}_{FS,x_0}$ is invariant under this map by a straight-forward calculation that uses Lemma 6.7 (vi).) We have $FS(g \circ f) = FS(g) \circ FS(f)$ and $FS(\mathrm{id}) = \mathrm{id}$. In particular a G–action on X by isometries with respect to the word metric extends to a G–action on $FS(X)$ by isometries with respect to the metric $d_{FS(X),x_0}$.

Next we compute the pseudometric $d^{\times}_{FS}$ and the metric d_{FS} on a line.

Lemma 6.16 *We get for $(a,b,t),(a,b,s) \in FS(X)$ and a given base point $x_0 \in X$*

$$d^{\times}_{FS,x_0}((a,b,t),(a,b,s)) = \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s)\right|;$$
$$d_{FS,x_0}((a,b,t),(a,b,s)) = \left|\Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s)\right|.$$

Proof We have

$$\begin{aligned}
&d^{\times}_{FS,x_0}((a,b,t),(a,b,s))\\
&= \sup_{u\in X} \left|l_{x_0}(u,(a,b,t)) - l_{x_0}(u,(a,b,s))\right|\\
&= \sup_{u\in X} \Big|\big(\langle a|b\rangle_u + \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \langle a,b|u,x_0\rangle\right|\big)\\
&\qquad\qquad - \big(\langle a|b\rangle_u + \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s) - \langle a,b|u,x_0\rangle\right|\big)\Big|\\
&= \sup_{u\in X} \left|\left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \langle a,b|u,x_0\rangle\right| - \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s) - \langle a,b|u,x_0\rangle\right|\right|.
\end{aligned}$$

We conclude from the triangle inequality

$$\begin{aligned}
&\left|\left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \langle a,b|u,x_0\rangle\right| - \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s) - \langle a,b|u,x_0\rangle\right|\right|\\
&\le \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \langle a,b|u,x_0\rangle - \big(\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s) - \langle a,b|u,x_0\rangle\big)\right|\\
&\le \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s)\right|,
\end{aligned}$$

and we get for $u = a$

$$\begin{aligned}
&\left|\left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \langle a,b|a,x_0\rangle\right| - \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s) - \langle a,b|a,x_0\rangle\right|\right|\\
&= \left|\left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - (-\langle b|x_0\rangle_a)\right| - \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s) - (-\langle b|x_0\rangle_a)\right|\right|\\
&= \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s)\right|.
\end{aligned}$$

This implies

$$d^{\times}_{FS,x_0}((a,b,t),(a,b,s)) = \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s)\right|.$$

We prove the claim for d_{FS,x_0} only in the case $t \geq s$, the case $t \leq s$ is analogous. Then $t+\tau \geq s+\tau$ holds for all $\tau \in \mathbb{R}$. Since both $\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}$ and $\Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}$ are monotone increasing, we conclude for all $\tau \in \mathbb{R}$,

$$\begin{aligned}
&\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t+\tau) - \theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s+\tau) \\
&\qquad = \left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t+\tau) - \theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s+\tau)\right|; \\
&\Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t+\tau) - \Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s+\tau) \\
&\qquad = \left|\Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t+\tau) - \Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s+\tau)\right|.
\end{aligned}$$

Now we get

$$\begin{aligned}
&d_{FS,x_0}((a,b,t),(a,b,s)) \\
&= \int_{\mathbb{R}} \frac{d^{\times}_{FS,x_0}(\phi_\tau((a,b,t)), \phi_\tau(a,b,s))}{2\cdot e^{|\tau|}} d\tau \\
&= \int_{\mathbb{R}} \frac{d^{\times}_{FS,x_0}((a,b,t+\tau)),(a,b,s+\tau))}{2\cdot e^{|\tau|}} d\tau \\
&= \int_{\mathbb{R}} \frac{\left|\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t+\tau) - \theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s+\tau)\right|}{2\cdot e^{|\tau|}} d\tau \\
&= \int_{\mathbb{R}} \frac{\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t+\tau) - \theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s+\tau)}{2\cdot e^{|\tau|}} d\tau \\
&= \int_{\mathbb{R}} \frac{\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t+\tau)}{2\cdot e^{|\tau|}} d\tau - \int_{\mathbb{R}} \frac{\theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s+\tau)}{2\cdot e^{|\tau|}} d\tau \\
&= \Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s) \\
&= \left|\Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(t) - \Theta_{[-\langle b|x_0\rangle_a, \langle a|x_0\rangle_b]}(s)\right|.
\end{aligned}$$

This finishes the proof of Lemma 6.16. □

Remark 6.17 We have fixed a base point $x_0 \in X$. For a different base point $x_1 \in X$ there is a canonical isometry $(FS(X), d_{FS,x_0}) \to (FS(X), d_{FS,x_1})$ defined by $(a,b,t) \mapsto (a,b,t+\langle a,b|x_0,x_1\rangle)$. (Of course this isometry appeared already when we defined $FS(f)$ for an isometry $f\colon X \to X$.) Using these isometries and a colimit over all choices of base points it is possible to give a canonical construction of the metric space $FS(X)$ without choosing a base point. However, then we do no longer have canonical coordinates in $FS(X)$, ie to make sense out of $(a,b,t) \in FS(X)$ we would still need to pick a base point. Since the base-point free formulation is not directly relevant for our applications, we do not give any details here.

Remark 6.18 As pointed out before, $FS(X)$ and $*\bar{X}$ agree as topological spaces. But it should be noted that the construction of the metric d_{FS,x_0} on $FS(X)$ differs slightly

from the metric d_* constructed by Mineyev on $*\bar{X}$. The definitions of l_{x_0} in (6.9) and of $l(u,x)$ in [14, Definition 10 on page 422] do not quite agree. There Mineyev uses a different parametrization to the effect that his formula translates to replacing θ by Θ in (6.9). (The point $[\![a,b;s]\!]_{x_0} \in *\bar{X}$ corresponds to $(a,b,\Theta_{[-\langle b|x_0\rangle_a,\langle a|x_0\rangle_b]}(s))$ in our parametrization, because $[\![a,b;s]\!]'_{x_0} = [\![a,b;\Theta_{[-\langle b|x_0\rangle_a,\langle a|x_0\rangle_b]}(s)]\!]_{x_0}$ is used in [14, Section 2.3] to identify the models $*X$ and $\bar{X}$.) However, this difference is not important. All results of [14] that we will use are also valid with this minor variation. Moreover we remark that since $|\theta_{[-\langle b|x_0\rangle_a,\langle a|x_0\rangle_b]}(t) - \Theta_{[-\langle b|x_0\rangle_a,\langle a|x_0\rangle_b]}(t)| \le 1$ for all $t \in \mathbb{R}$, it is easy to check that the identification of $FS(X)$ and $*\bar{X}$ is a quasi-isometry with respect to d_{FS,x_0} and d_*.

7 Flow estimates

In this section we prove the main exponential flow estimate for $FS(X)$. Recall that we have fixed a base point $x_0 \in X$.

Theorem 7.1 (Exponential flow estimate) *Let $\lambda \in (e^{-1},1)$ and $T \in [0,\infty)$ be the constants depending only on X which appear in Proposition 6.4. Consider $a,b,c \in \bar{X}$ such that $a,c \in \partial X \Rightarrow a \neq c$ and $b,c \in \partial X \Rightarrow b \neq c$ holds. Let $t,s,\tau \in \mathbb{R}$. Put*

$$\tau_0 = t - s - \langle a,b|c,x_0\rangle;$$
$$N = 2 + \frac{2}{\lambda^T \cdot (-\ln(\lambda))}.$$

Then we get

$$d_{FS,x_0}(\phi_\tau(a,c,t),\phi_{\tau+\tau_0}(b,c,s)) \le \frac{N}{1-\ln(\lambda)^2} \cdot \lambda^{(t-\langle a,c|b,x_0\rangle)} \cdot \lambda^\tau.$$

For the sphere bundle of the universal cover of a strictly negatively curved manifold estimates as above are classical results and have been used in algebraic K–theory by Farrell and Jones [7]. Compare also the work of Bartels, Farrell, Jones and Reich [2, Proposition 14.2]. There only $c \in \partial X$ is considered and τ_0 is chosen to ensure that $\phi_\tau(a,c,t)$ and $\phi_{\tau+\tau_0}(b,c,s)$ both lie on the same horosphere around c.

As mentioned in the introduction, the proof of Theorem 7.1 is strongly based on ideas due to Mineyev [14, Theorem 57 on page 468].

We also will use the following basic flow estimate.

Lemma 7.2 *We get for $(a,b,t),(c,d,s) \in FS(X)$ and $\tau \in \mathbb{R}$,*

$$d_{FS,x_0}(\phi_\tau(a,b,t),\phi_\tau(c,d,s)) \le e^{|\tau|} \cdot d_{FS,x_0}((a,b,t),(c,d,s)).$$

Proof We compute

$$\begin{aligned}
&d_{FS,x_0}(\phi_\tau(a,b,t),\phi_\tau(c,d,s))\\
&=\int_{\mathbb{R}}\frac{d^{\times}_{FS,x_0}(\phi_\sigma(\phi_\tau(a,b,t)),\phi_\sigma(\phi_\tau(c,d,s)))}{2\cdot e^{|\sigma|}}\,d\sigma\\
&=\int_{\mathbb{R}}\frac{d^{\times}_{FS,x_0}(\phi_{\sigma+\tau}(a,b,t),\phi_{\sigma+\tau}(c,d,s))}{2\cdot e^{|\sigma|}}\,d\sigma\\
&=\int_{\mathbb{R}}\frac{d^{\times}_{FS,x_0}(\phi_\sigma(a,b,t),\phi_\sigma(c,d,s))}{2\cdot e^{|\sigma-\tau|}}\,d\sigma\\
&\le\int_{\mathbb{R}}\frac{d^{\times}_{FS,x_0}(\phi_\sigma(a,b,t),\phi_\sigma(c,d,s))}{2\cdot e^{|\sigma|-|\tau|}}\,d\sigma\\
&\le e^{|\tau|}\cdot\int_{\mathbb{R}}\frac{d^{\times}_{FS,x_0}(\phi_\sigma(a,b,t),\phi_\sigma(c,d,s))}{2\cdot e^{|\sigma|}}\,d\sigma\\
&=e^{|\tau|}\cdot d_{FS,x_0}((a,b,t),(c,d,s)).
\end{aligned}$$

□

We record the following result due to Mineyev[14, Proposition 48 on page 460].

Theorem 7.3 *The map*

$$\big(\bar{X}\times\bar{X}-\Delta(\bar{X})\big)\times\mathbb{R}\ \xrightarrow{\cong}\ FS(X)-X,\quad((a,b),t)\mapsto(a,b,t)$$

is a homeomorphism, where $\Delta(\bar{X})\subseteq\bar{X}\times\bar{X}$ is the diagonal.

7.1 Flow estimates for the pseudo metric

The goal of this subsection is to prove the version of Theorem 7.1 for the pseudometric $d^{\times}_{FS,x_0}$.

Theorem 7.4 *Let $\lambda\in(e^{-1},1)$ and $T\in[0,\infty)$ be the constants depending only on X which appear in Proposition 6.4. Consider $a,b,c\in\bar{X}$ such that $a,c\in\partial X\Rightarrow a\neq c$ and $b,c\in\partial X\Rightarrow b\neq c$ holds. Let $t,s,\tau\in\mathbb{R}$. Put*

$$\begin{aligned}
\tau_0&=t-s-\langle a,b|c,x_0\rangle;\\
N&=2+\frac{2}{\lambda^T\cdot(-\ln(\lambda))}.
\end{aligned}$$

Then we get

$$\textbf{(7.5)}\qquad d^{\times}_{FS,x_0}(\phi_\tau(a,c,t),\phi_{\tau+\tau_0}(b,c,s))\ \le\ N\cdot\lambda^{t+\tau-\langle a,c|b,x_0\rangle}.$$

Its proof needs some preparations and is then done in several steps.

We begin with the trivial case $a = b$.

Lemma 7.6 *Consider the situation appearing in Theorem 7.4. If $a = b$ then (7.5) holds.*

Proof Since

$$\tau_0 \;=\; t - s - \langle a, b | c, x_0\rangle \;=\; t - s - \langle a, a | c, x_0\rangle \;=\; t - s,$$

we get for all $\tau \in \mathbb{R}$

$$\phi_\tau(a, c, t) \;=\; (a, c, t + \tau) = \; (b, c, s + \tau + \tau_0) \;=\; \phi_{\tau+\tau_0}(b, c, s)$$

and hence

$$d^{\times}_{FS,x_0}(\phi_\tau(a, c, t), \phi_{\tau+\tau_0}(b, c, s)) \;=\; 0. \qquad \square$$

So we can make in the sequel the additional assumption that $a \neq b$. This has the advantage that the expressions $\langle a, c | b, x_0\rangle$, $\langle b, c | a, x_0\rangle$, $\langle a, c | u, b\rangle$ and $\langle b, c | u, a\rangle$ for $u \in X$ which will appear below are well defined elements in $\mathbb{R}$.

Lemma 7.7 *Define $\tau, \tau_0, a, b, c, x_0$ as in Theorem 7.4. Suppose that $a, b \in \partial X \Rightarrow a \neq b$. Then:*

(i) *We have*

$$\begin{aligned}
\langle a|x_0\rangle_c - \langle a, c|b, x_0\rangle &= \langle b|x_0\rangle_c - \langle b, c|a, x_0\rangle \;=\; \langle a|b\rangle_c;\\
t + \tau - \langle a, c|b, x_0\rangle &= s + \tau + \tau_0 - \langle b, c|a, x_0\rangle;\\
\langle a|x_0\rangle_c - (t + \tau) &= \langle b|x_0\rangle_c - (s + \tau + \tau_0);\\
-\langle c|x_0\rangle_a - \langle a, c|b, x_0\rangle &= -\langle c|b\rangle_a;\\
-\langle c|x_0\rangle_b - \langle b, c|a, x_0\rangle &= -\langle c|a\rangle_b;\\
\max\{\langle a, c|u, b\rangle, \langle b, c|u, a\rangle\} \le \langle a|x_0\rangle_c - \langle a, c|b, x_0\rangle &\quad \text{for } u \in X;
\end{aligned}$$

(ii) *If*

$$\begin{aligned}
t + \tau &\ge -\langle c|x_0\rangle_a;\\
s + \tau + \tau_0 &\ge -\langle c|x_0\rangle_b,
\end{aligned}$$

then

$$\begin{aligned}
&\theta_{[-\langle c|x_0\rangle_a, \langle a|x_0\rangle_c]}(t + \tau) - \langle a, c|b, x_0\rangle\\
&\qquad = \theta_{[-\langle c|x_0\rangle_b, \langle b|x_0\rangle_c]}(s + \tau_0 + \tau) - \langle b, c|a, x_0\rangle;
\end{aligned}$$

(iii) *If we assume*

$$t+\tau \geq -\langle c|x_0\rangle_a;$$
$$\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|b,x_0\rangle < \max\{\langle a,c|u,b\rangle,\langle b,c|u,a\rangle\},$$

then $$\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)=t+\tau.$$

Proof (i) Notice for the sequel that $\langle a|a'\rangle_c=\infty$ if $c\in\partial X$. Hence the first equality, the third equality and the last inequality are true for trivial reasons if $c\in\partial X$.

We compute for $c\in X$

$$\begin{aligned}\langle a|x_0\rangle_c-\langle a,c|b,x_0\rangle &= \langle a|x_0\rangle_c+\langle c,a|b,x_0\rangle\\ &= \langle a|x_0\rangle_c+(\langle a|b\rangle_c-\langle a|x_0\rangle_c)\\ &= \langle a|b\rangle_c,\end{aligned}$$

and analogously

$$\langle b|x_0\rangle_c-\langle b,c|a,x_0\rangle \;=\; \langle b|a\rangle_c \;=\; \langle a|b\rangle_c.$$

This proves the first equation.

The second follows from

$$\begin{aligned}s+\tau+\tau_0-&\langle b,c|a,x_0\rangle\\ &= s+\tau+t-s-\langle a,b|c,x_0\rangle-\langle b,c|a,x_0\rangle\\ &= \tau+t-\langle a,b|c,x_0\rangle-\langle b,c|a,x_0\rangle-\langle c,a|b,x_0\rangle+\langle c,a|b,x_0\rangle\\ &= \tau+t-\langle a,c|b,x_0\rangle.\end{aligned}$$

The third equation is a direct consequence of the first two if $c\in X$, and hence true for all $c\in\bar{X}$.

The proof of the fourth and fifth equation is analogous to the one of the first one.

Since for $c\in X$

$$\begin{aligned}&\langle a,c|u,b\rangle \;=\; \langle c,a|b,u\rangle \;=\; \langle a|b\rangle_c-\langle a|u\rangle_c \;\leq\; \langle a|b\rangle_c\\ \text{and}\quad &\langle b,c|u,a\rangle \;=\; \langle c,b|a,u\rangle \;=\; \langle b|a\rangle_c-\langle b|u\rangle_c \;\leq\; \langle b|a\rangle_c \;=\; \langle a|b\rangle_c\end{aligned}$$

holds, the last inequality follows from the first equality.

(ii) We begin with the case $t+\tau\geq\langle a|x_0\rangle_c$. Then we get from the third equation of assertion (i) that also $s+\tau+\tau_0\geq\langle b|x_0\rangle_c$ holds. We conclude from the definitions

and the first equation of assertion (i)

$$\begin{aligned}\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|b,x_0\rangle &= \langle a|x_0\rangle_c-\langle a,c|b,x_0\rangle\\ &= \langle b|x_0\rangle_c-\langle b,c|a,x_0\rangle\\ &= \theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau_0+\tau)-\langle b,c|a,x_0\rangle.\end{aligned}$$

Next we treat the case $t+\tau \le \langle a|x_0\rangle_c$. Then we get from the third equation of assertion (i) that also $s+\tau+\tau_0 \le \langle b|x_0\rangle_c$ holds. Since by assumption $t+\tau \ge -\langle c|x_0\rangle_a$ and $s+\tau+\tau_0 \ge -\langle c|x_0\rangle_b$ holds, we conclude from the definitions and the second equation of assertion (i)

$$\begin{aligned}\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|b,x_0\rangle &= t+\tau-\langle a,c|b,x_0\rangle\\ &= s+\tau+\tau_0-\langle b,c|a,x_0\rangle\\ &= \theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau_0+\tau)-\langle b,c|a,x_0\rangle.\end{aligned}$$

(iii) Since the inequality in assertion (i) implies

$$\begin{aligned}\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) &< \max\{\langle a,c|u,b\rangle,\langle b,c|u,a\rangle\}+\langle a,c|b,x_0\rangle\\ &\le \langle a|x_0\rangle_c,\end{aligned}$$

we get $t+\tau<\langle a|x_0\rangle_c$. Since we assume $-\langle c|x_0\rangle_a \le t+\tau$, we conclude

$$\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)=t+\tau.$$

This finishes the proof of Lemma 7.7. □

The elementary proof of the next lemma is left to the reader.

Lemma 7.8 *Consider numbers $\lambda \in (e^{-1},1)$ and $T \in [0,\infty)$. Put*

$$N := 2+\frac{2}{\lambda^T\cdot(-\ln(\lambda))}.$$

Then we get

$$\begin{array}{lcll} 2 & \le & N; & \\ 2\cdot(T-v+\lambda^v) & \le & N\cdot\lambda^v & \text{for all } v\le T;\\ v+N\cdot\lambda^{v-w} & \le & N\cdot\lambda^{-w} & \text{for } 0\le v\le w.\end{array}$$

Lemma 7.9 *Consider the situation appearing in Theorem 7.4. Suppose that $a,b\in\partial X \Rightarrow a\ne b$. Suppose that*

$$\begin{aligned}t+\tau &\ge -\langle c|x_0\rangle_a;\\ s+\tau+\tau_0 &\ge -\langle c|x_0\rangle_b.\end{aligned}$$

Then we get for all $u \in X$

$$\begin{aligned}&\big|l_{x_0}(u,\phi_\tau(a,c,t)) - l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))\big| \\ &\qquad = \big|\langle u,c|a,b\rangle + \big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle a,c|u,b\rangle\big| \\ &\qquad\qquad - \big|\theta_{[-\langle c|x_0\rangle_a,\langle a,x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle b,c|u,a\rangle\big|\big|\,.\end{aligned}$$

Proof We compute

$$\begin{aligned}&\big|l_{x_0}(u,\phi_\tau(a,c,t)) - l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))\big| \\ &= \big|l_{x_0}(u,(a,c,t+\tau)) - l_{x_0}(u,(b,c,s+\tau+\tau_0))\big| \\ &= \big|\langle a|c\rangle_u + \big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|u,x_0\rangle\big| - \langle b|c\rangle_u \\ &\qquad - \big|\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau_0+\tau) - \langle b,c|u,x_0\rangle\big|\big| \\ &= |\langle c|a\rangle_u - \langle c|b\rangle_u \\ &\qquad + \big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle + \langle a,c|b,x_0\rangle - \langle a,c|u,x_0\rangle\big| \\ &\qquad - \big|\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau_0+\tau) - \langle b,c|a,x_0\rangle + \langle b,c|a,x_0\rangle - \langle b,c|u,x_0\rangle\big|\big| \\ &= \big|\langle u,c|a,b\rangle + \big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle a,c|u,b\rangle\big| \\ &\qquad - \big|\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau_0+\tau) - \langle b,c|a,x_0\rangle - \langle b,c|u,a\rangle\big|\big|\,.\end{aligned}$$

Since we get from Lemma 7.7 (ii)

$$\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle \;=\; \theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau_0+\tau) - \langle b,c|a,x_0\rangle.$$

Lemma 7.9 follows. □

Lemma 7.10 *Consider the situation appearing in Theorem 7.4. Suppose that $a,b \in \partial X \Rightarrow a \neq b$. Suppose that*

$$\begin{aligned}t+\tau &\geq -\langle c|x_0\rangle_a;\\ s+\tau+\tau_0 &\geq -\langle c|x_0\rangle_b,\end{aligned}$$

and $\quad \theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle \;\geq\; \max\{\langle a,c|u,b\rangle, \langle b,c|u,a\rangle\}.$

Then we get for all $u \in X$

$$\big|l_{x_0}(u,\phi_\tau(a,c,t)) - l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))\big| \;=\; 0.$$

Proof We compute

$$\begin{aligned}
&\big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle a,c|u,b\rangle\big| \\
&\qquad\qquad - \big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle b,c|u,a\rangle\big| \\
&= \big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle a,c|u,b\rangle\big) \\
&\qquad\qquad - \big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle b,c|u,a\rangle\big) \\
&= -\langle a,c|u,b\rangle + \langle b,c|u,a\rangle.
\end{aligned}$$

This implies together with Lemma 7.9

$$\begin{aligned}
&|l_{x_0}(u,\phi_\tau(a,c,t)) - l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))| \\
&\qquad = |\langle u,c|a,b\rangle - \langle a,c|u,b\rangle + \langle b,c|u,a\rangle| \\
&\qquad = |\langle b,a|c,u\rangle + \langle a,c|b,u\rangle + \langle c,b|a,u\rangle| \\
&\qquad = 0. \qquad \square
\end{aligned}$$

Lemma 7.11 *Consider the situation appearing in Theorem 7.4. Suppose that $a,b \in \partial X \Rightarrow a \neq b$. Suppose that*

$$\begin{aligned}
t+\tau &\geq -\langle c|x_0\rangle_a; \\
s+\tau+\tau_0 &\geq -\langle c|x_0\rangle_b,
\end{aligned}$$

and that

$$\begin{aligned}
&\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle < \max\{\langle a,c|u,b\rangle, \langle b,c|u,a\rangle\}; \\
&\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle \geq T.
\end{aligned}$$

Then we get for all $u \in X$

$$\big|l_{x_0}(u,\phi_\tau(a,c,t)) - l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))\big| \leq 2\cdot\lambda^{t+\tau-\langle a,c|b,x_0\rangle}.$$

Proof We get from Lemma 7.9

$$\begin{aligned}
&|l_{x_0}(u,\phi_\tau(a,c,t)) - l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))| \\
&\quad = \big|\langle u,c|a,b\rangle + \big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle a,c|u,b\rangle\big| \\
&\qquad - \big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle b,c|u,a\rangle\big|\big| \\
&\quad \leq |\langle u,c|a,b\rangle| + \big|\big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle a,c|u,b\rangle\big) \\
&\qquad - \big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) - \langle a,c|b,x_0\rangle - \langle b,c|u,a\rangle\big)\big| \\
&\quad = |\langle u,c|a,b\rangle| + |-\langle a,c|u,b\rangle + \langle b,c|u,a\rangle| \\
&\quad = |\langle u,c|a,b\rangle| + |\langle c,a|u,b\rangle + \langle a,u|c,b\rangle|
\end{aligned}$$

$$\begin{aligned}&= |\langle u, c|a, b\rangle| + |\langle u, c|a, b\rangle| \\ &= 2 \cdot |\langle u, c|a, b\rangle| .\end{aligned}$$

Our assumptions imply $\max\{\langle a, c|u, b\rangle, \langle b, c|u, a\} \geq T$. We conclude from Proposition 6.4

$$|\langle u, c|a, b\rangle| \leq \lambda^{\max\{\langle a,c|u,b\rangle,\langle b,c|u,a\rangle\}}.$$

Thus we get using Lemma 7.7 (iii)

$$\begin{aligned}\left|l_{x_0}(u, \phi_\tau(a, c, t)) - l_{x_0}(u, \phi_{\tau+\tau_0}(b, c, s))\right| &\leq 2 \cdot \lambda^{\max\{\langle a,c|u,b\rangle,\langle b,c|u,a\rangle\}} \\ &\leq 2 \cdot \lambda^{\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|b,x_0\rangle} \\ &= 2 \cdot \lambda^{t+\tau-\langle a,c|b,x_0\rangle}.\end{aligned}$$

□

Lemma 7.12 *Consider the situation appearing in Theorem 7.4. Suppose that $a, b \in \partial X \Rightarrow a \neq b$. Suppose that*

$$\begin{aligned}t + \tau &\geq -\langle c|x_0\rangle_a; \\ s + \tau + \tau_0 &\geq -\langle c|x_0\rangle_b,\end{aligned}$$

and that

$$\begin{aligned}&\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t + \tau) - \langle a, c|b, x_0\rangle < \max\{\langle a, c|u, b\rangle, \langle b, c|u, a\rangle\}; \\ &\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t + \tau) - \langle a, c|b, x_0\rangle < T.\end{aligned}$$

Then we get for all $u \in X$

$$\left|l_{x_0}(u, \phi_\tau(a, c, t)) - l_{x_0}(u, \phi_{\tau+\tau_0}(b, c, s))\right| \leq N \cdot \lambda^{t+\tau-\langle a,c|b,x_0\rangle}.$$

Proof Since $\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}$ is monotone increasing and by Lemma 7.7 (i) and by assumption

$$\begin{aligned}\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t + \tau) - \langle a, c|b, x_0\rangle &< \min\{\max\{\langle a, c|u, b\rangle, \langle b, c|u, a\rangle\}, T\}; \\ \max\{\langle a, c|u, b\rangle, \langle b, c|u, a\rangle\} &\leq \langle a|x_0\rangle_c - \langle a, c|b, x_0\rangle,\end{aligned}$$

holds, we can choose $\tau' \in \mathbb{R}$ satisfying

$$\begin{aligned}\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t + \tau') - \langle a, c|b, x_0\rangle &= \min\{\max\{\langle a, c|u, b\rangle, \langle b, c|u, a\rangle\}, T\}; \\ \tau &\leq \tau'.\end{aligned}$$

In particular we have

$$\begin{aligned}t + \tau' &\geq -\langle c|x_0\rangle_a; \\ s + \tau' + \tau_0 &\geq -\langle c|x_0\rangle_b.\end{aligned}$$

Hence Lemma 7.10 and Lemma 7.11 imply

$$\begin{aligned}(7.13)\qquad \left|l_{x_0}(u,\phi_{\tau'}(a,c,t))-l_{x_0}(u,\phi_{\tau'+\tau_0}(b,c,s))\right| &\le 2\cdot\lambda^{t+\tau'-\langle a,c|b,x_0\rangle}\\ &\le 2\cdot\lambda^{t+\tau-\langle a,c|b,x_0\rangle}.\end{aligned}$$

We compute

$$\begin{aligned}(7.14)\qquad &|l_{x_0}(u,\phi_{\tau}(a,c,t))-l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))|\\ &\quad=\big|l_{x_0}(u,\phi_{\tau}(a,c,t))-l_{x_0}(u,\phi_{\tau'}(a,c,t))\\ &\qquad\qquad+l_{x_0}(u,\phi_{\tau'}(a,c,t))-l_{x_0}(u,\phi_{\tau'+\tau_0}(b,c,s))\\ &\qquad\qquad+l_{x_0}(u,\phi_{\tau'+\tau_0}(b,c,s))-l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))\big|\\ &\quad\le\big|l_{x_0}(u,\phi_{\tau}(a,c,t))-l_{x_0}(u,\phi_{\tau'}(a,c,t))\big|\\ &\qquad\qquad+\big|l_{x_0}(u,\phi_{\tau'}(a,c,t))-l_{x_0}(u,\phi_{\tau'+\tau_0}(b,c,s))\big|\\ &\qquad\qquad+\big|l_{x_0}(u,\phi_{\tau'+\tau_0}(b,c,s))-l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))\big|.\end{aligned}$$

We estimate using Lemma 7.7 (iii)

$$\begin{aligned}(7.15)\qquad &|l_{x_0}(u,\phi_{\tau}(a,c,t))-l_{x_0}(u,\phi_{\tau'}(a,c,t))|\\ &\quad=\big|\big(\langle a|c\rangle_u+\big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|u,x_0\rangle\big|\big)\\ &\qquad\qquad-\big(\langle a|c\rangle_u+\big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau')-\langle a,c|u,x_0\rangle\big|\big)\big|\\ &\quad=\big|\,\big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|u,x_0\rangle\big|\\ &\qquad\qquad-\big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau')-\langle a,c|u,x_0\rangle\big|\,\big|\\ &\quad\le\big|\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau')\big|\\ &\quad=\big|\,\big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|b,x_0\rangle\big)\\ &\qquad\qquad-\big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau')-\langle a,c|b,x_0\rangle\big)\,\big|\\ &\quad=\big|\,\big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|b,x_0\rangle\big)\\ &\qquad\qquad-\min\{\max\{\langle a,c|u,b\rangle,\langle b,c|u,a\rangle\},T\}\,\big|\\ &\quad=\min\{\max\{\langle a,c|u,b\rangle,\langle b,c|u,a\rangle\},T\}\\ &\qquad\qquad-\big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|b,x_0\rangle\big)\\ &\quad=\min\{\max\{\langle a,c|u,b\rangle,\langle b,c|u,a\rangle\},T\}-(t+\tau-\langle a,c|b,x_0\rangle)\\ &\quad\le T-(t+\tau-\langle a,c|b,x_0\rangle).\end{aligned}$$

Analogously we get using Lemma 7.7 (ii)

$$\begin{aligned}
\textbf{(7.16)}\quad & \big|l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))-l_{x_0}(u,\phi_{\tau'+\tau_0}(b,c,s))\big| \\
&= \big|\big(\langle b|c\rangle_u+\big|\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau+\tau_0)-\langle b,c|u,x_0\rangle\big|\big) \\
&\qquad -\big(\langle b|c\rangle_u+\big|\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau'+\tau_0)-\langle b,c|u,x_0\rangle\big|\big)\big| \\
&= \big|\big|\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau+\tau_0)-\langle b,c|u,x_0\rangle\big| \\
&\qquad -\big|\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau'+\tau_0)-\langle b,c|u,x_0\rangle\big|\big| \\
&\le \big|\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau+\tau_0)-\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau'+\tau_0)\big| \\
&= \big|\big(\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau+\tau_0)-\langle b,c|a,x_0\rangle\big) \\
&\qquad -\big(\theta_{[-\langle c|x_0\rangle_b,\langle b,x_0\rangle_c]}(s+\tau'+\tau_0)-\langle b,c|a,x_0\rangle\big)\big| \\
&= \big|\big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|b,x_0\rangle\big) \\
&\qquad -\big(\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau')-\langle a,c|b,x_0\rangle\big)\big| \\
&\le T-(t+\tau-\langle a,c|b,x_0\rangle).
\end{aligned}$$

Lemma 7.7 (iii) implies

$$t+\tau-\langle a,c|b,x_0\rangle \;=\; \theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau)-\langle a,c|b,x_0\rangle \;\le\; T.$$

Hence we conclude from Lemma 7.8 for $v=t+\tau-\langle a,c|b,x_0\rangle$

$$\textbf{(7.17)}\qquad 2\cdot\Big(T-(t+\tau-\langle a,c|b,x_0\rangle)+\lambda^{t+\tau-\langle a,c|b,x_0\rangle}\Big)\le N\cdot\lambda^{t+\tau-\langle a,c|b,x_0\rangle}.$$

Combining **(7.13)**, **(7.14)**, **(7.15)**, **(7.16)** and **(7.17)** yields

$$\begin{aligned}
&\big|l_{x_0}(u,\phi_\tau(a,c,t))-l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s))\big| \\
&\quad\le 2\cdot\Big(T-(t+\tau-\langle a,c|b,x_0\rangle)+\lambda^{t+\tau-\langle a,c|b,x_0\rangle}\Big) \\
&\quad\le N\cdot\lambda^{t+\tau-\langle a,c|b,x_0\rangle}.
\end{aligned}$$

□

Lemma 7.18 *Consider the situation appearing in Theorem 7.4. Suppose that $a,b\in\partial X\Rightarrow a\neq b$. Suppose that*

$$\begin{aligned}
t+\tau &\ge -\langle c|x_0\rangle_a;\\
s+\tau+\tau_0 &\ge -\langle c|x_0\rangle_b.
\end{aligned}$$

Then (7.5) holds.

Proof This follows from Lemma 7.10, Lemma 7.11 (note that $2 \le N$ by Lemma 7.8) and Lemma 7.12 since by definition

$$d^{\times}_{FS,x_0}(\phi_\tau(a,c,t),\phi_{\tau+\tau_0}(b,c,s)) = \sup_{u\in X} \left| l_{x_0}(u,\phi_\tau(a,c,t)) - l_{x_0}(u,\phi_{\tau+\tau_0}(b,c,s)) \right|. \qquad \square$$

Lemma 7.19 *Consider the situation appearing in Theorem 7.4. Suppose that $a,b \in \partial X \Rightarrow a \neq b$. Suppose that at least one of the following inequalities is true:*

$$t+\tau \le -\langle c|x_0\rangle_a;$$
$$s+\tau+\tau_0 \le -\langle c|x_0\rangle_b.$$

Then (7.5) holds.

Proof Put $\tau'' := \max\{-\langle c|x_0\rangle_a - t, -\langle c|x_0\rangle_b - s - \tau_0\}$.

Since by assumption $\tau \le -\langle c|x_0\rangle_a - t$ or $\tau \le -\langle c|x_0\rangle_b - s - \tau_0$ holds, we must have

$$\tau \le \tau''.$$

We estimate

$$\begin{aligned} \textbf{(7.20)} \quad & d^{\times}_{FS,x_0}(\phi_\tau(a,c,t),\phi_{\tau+\tau_0}(b,c,s)) \\ &= d^{\times}_{FS,x_0}((a,c,t+\tau),(b,c,s+\tau+\tau_0)) \\ &\le d^{\times}_{FS,x_0}((a,c,t+\tau),(a,c,t+\tau'')) \\ &\qquad + d^{\times}_{FS,x_0}((a,c,t+\tau''),(b,c,s+\tau''+\tau_0)) \\ &\qquad + d^{\times}_{FS,x_0}((b,c,s+\tau''+\tau_0),(b,c,s+\tau+\tau_0)). \end{aligned}$$

Since

$$t+\tau'' \ge -\langle c|x_0\rangle_a;$$
$$s+\tau''+\tau_0 \ge -\langle c|x_0\rangle_b,$$

holds by definition of τ'', we get from Lemma 7.18

$$\textbf{(7.21)} \quad d^{\times}_{FS,x_0}((a,c,t+\tau''),(b,c,s+\tau''+\tau_0)) \le N\cdot\lambda^{t+\tau''-\langle a,c|b,x_0\rangle}.$$

Next we want to show

$$\begin{aligned} \textbf{(7.22)} \quad & d^{\times}_{FS,x_0}((a,c,t+\tau),(a,c,t+\tau'')) \\ &\quad + d^{\times}_{FS,x_0}((b,c,s+\tau''+\tau_0),(b,c,s+\tau+\tau_0)) \le \tau''-\tau. \end{aligned}$$

Inspecting the definition of τ'' we see that we have to consider two cases, namely,

$$t+\tau'' = -\langle c|x_0\rangle_a \text{ and } s+\tau_0+\tau'' \geq -\langle c|x_0\rangle_b,$$

and

$$t+\tau'' \geq -\langle c|x_0\rangle_a \text{ and } s+\tau_0+\tau'' = -\langle c|x_0\rangle_b.$$

We only treat the first one, the second is completely analogous. From $t+\tau \leq t+\tau'' = -\langle c|x_0\rangle_a$ we conclude $\theta_{[-\langle c|x_0\rangle_a,\langle a|x_0\rangle_c]}(t+\tau) = \theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(t+\tau'')$. Lemma 6.16 implies

$$d^{\times}_{FS,x_0}((a,c,t+\tau),(a,c,t+\tau'')) = 0.$$

We conclude from Lemma 6.16

$$\begin{aligned} &d^{\times}_{FS,x_0}((b,c,s+\tau''+\tau_0),(b,c,s+\tau+\tau_0)) \\ &\quad = \left|\theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau''+\tau_0) - \theta_{[-\langle c|x_0\rangle_b,\langle b|x_0\rangle_c]}(s+\tau+\tau_0)\right| \\ &\quad \leq \left|(s+\tau''+\tau_0)-(s+\tau+\tau_0)\right| \\ &\quad = \tau''-\tau. \end{aligned}$$

This finishes the proof of **(7.22)**.

If we combine **(7.20)**, **(7.21)** and **(7.22)**, we get

(7.23) $$d^{\times}_{FS,x_0}(\phi_\tau(a,c,t),\phi_{\tau+\tau_0}(b,c,s)) \leq \tau''-\tau+N\cdot\lambda^{t+\tau''-\langle a,c|b,x_0\rangle}.$$

We estimate

$$-\langle c|x_0\rangle_a - t \;\leq\; \langle c|b\rangle_a - \langle c|x_0\rangle_a - t \;=\; \langle a,c|b,x_0\rangle - t,$$

and for $b\in X$

$$\begin{aligned} &-\langle c|x_0\rangle_b - s - \tau_0 \\ &= -\langle c|x_0\rangle_b - t + \langle a,b|c,x_0\rangle \\ &= \langle a,c|b,x_0\rangle - t - \langle a,c|b,x_0\rangle + \langle a,b|c,x_0\rangle - \langle c|x_0\rangle_b \\ &= \langle a,c|b,x_0\rangle - t + \langle c,a|b,x_0\rangle + \langle a,b|c,x_0\rangle + \langle b,c|a,x_0\rangle - \langle b,c|a,x_0\rangle - \langle c|x_0\rangle_b \\ &= \langle a,c|b,x_0\rangle - t - \langle b,c|a,x_0\rangle - \langle c|x_0\rangle_b \\ &= \langle a,c|b,x_0\rangle - t - \langle c|a\rangle_b \\ &\leq \langle a,c|b,x_0\rangle - t. \end{aligned}$$

This inequality holds for $b\in\partial X$ for trivial reasons. The last two inequalities imply

$$\tau'' \;\leq\; \langle a,c|b,x_0\rangle - t$$

and hence

$$0 \;\leq\; \tau''-\tau \;\leq\; -\left(t+\tau-\langle a,c|b,x_0\rangle\right).$$

Lemma 7.8 applied to $v = \tau'' - \tau$ and $w = -(t + \tau - \langle a, c|b, x_0\rangle)$ yields

$$\tau'' - \tau + N \cdot \lambda^{t+\tau''-\langle a,c|b,x_0\rangle} \leq N \cdot \lambda^{t+\tau-\langle a,c|b,x_0\rangle}. \tag{7.24}$$

If we combine (7.23), (7.24), we get the desired inequality

$$d^{\times}_{FS,x_0}(\phi_\tau(a,c,t), \phi_{\tau+\tau_0}(b,c,s)) \leq N \cdot \lambda^{t+\tau-\langle a,c|b,x_0\rangle}. \qquad \square$$

Now Theorem 7.4 follows from Lemma 7.6, Lemma 7.18 and Lemma 7.19.

7.2 Flow estimates for the metric

Next we prove Theorem 7.1.

Proof We estimate using $e \cdot \lambda > 1$, $0 < e^{-1} \cdot \lambda < 1$ and Theorem 7.4

$$\begin{aligned}
&d_{FS,x_0}(\phi_\tau(a,c,t), \phi_{\tau+\tau_0}(b,c,s)) \\
&\quad = \int_{-\infty}^{\infty} \frac{d^{\times}_{FS,x_0}\left(\phi_\sigma(\phi_\tau(a,c,t)), \phi_\sigma(\phi_{\tau+\tau_0}(b,c,s))\right)}{2 \cdot e^{|\sigma|}}\, d\sigma \\
&\quad = \int_{-\infty}^{\infty} \frac{d^{\times}_{FS}\left(\phi_{\sigma+\tau}((a,c,t)), \phi_{\sigma+\tau+\tau_0}((b,c,s))\right)}{2 \cdot e^{|\sigma|}}\, d\sigma \\
&\quad \leq \int_{-\infty}^{\infty} \frac{N \cdot \lambda^{t+\sigma+\tau-\langle a,c|b,x_0\rangle}}{2 \cdot e^{|\sigma|}}\, d\sigma \\
&\quad = \int_{-\infty}^{0} \frac{N \cdot \lambda^{t+\sigma+\tau-\langle a,c|b,x_0\rangle}}{2 \cdot e^{|\sigma|}}\, d\sigma + \int_{0}^{\infty} \frac{N \cdot \lambda^{t+\sigma+\tau-\langle a,c|b,x_0\rangle}}{2 \cdot e^{|\sigma|}}\, d\sigma. \\
&\quad = \int_{-\infty}^{0} \frac{N \cdot \lambda^{t+\sigma+\tau-\langle a,c|b,x_0\rangle}}{2 \cdot e^{-\sigma}}\, d\sigma + \int_{0}^{\infty} \frac{N \cdot \lambda^{t+\sigma+\tau-\langle a,c|b,x_0\rangle}}{2 \cdot e^{\sigma}}\, d\sigma. \\
&\quad = \frac{N}{2} \cdot \lambda^{t+\tau-\langle a,c|b,x_0\rangle} \cdot \left(\int_{-\infty}^{0} (e \cdot \lambda)^{\sigma}\, d\sigma + \int_{0}^{\infty} (e^{-1} \cdot \lambda)^{\sigma}\, d\sigma \right) \\
&\quad = \frac{N}{2} \cdot \lambda^{t+\tau-\langle a,c|b,x_0\rangle} \cdot \left(\left[\frac{(e \cdot \lambda)^{\sigma}}{\ln(e \cdot \lambda)} \right]_{-\infty}^{0} + \left[\frac{(e^{-1} \cdot \lambda)^{\sigma}}{\ln(e^{-1} \cdot \lambda)} \right]_{0}^{\infty} \right) \\
&\quad = \frac{N}{2} \cdot \lambda^{t+\tau-\langle a,c|b,x_0\rangle} \cdot \left(\frac{1}{\ln(e \cdot \lambda)} + \frac{1}{-\ln(e^{-1} \cdot \lambda)} \right) \\
&\quad = \frac{N}{2} \cdot \lambda^{t+\tau-\langle a,c|b,x_0\rangle} \cdot \left(\frac{1}{1 + \ln(\lambda)} + \frac{1}{1 - \ln(\lambda)} \right) \\
&\quad = \frac{N}{1 - \ln(\lambda)^2} \cdot \lambda^{(t-\langle a,c|b,x_0\rangle)} \cdot \lambda^{\tau}. \qquad \square
\end{aligned}$$

8 The flow estimates for the map ι

Let X be a hyperbolic complex and $x_0 \in X$ be a base point. We define a map

$$\iota_{x_0}\colon X \times \overline{X} \to FS(X) \tag{8.1}$$

by $\iota_{x_0}(a,c) :=$

$$\begin{cases} \left(a, c, \Theta^{-1}_{[-\langle c|x_0\rangle_a, \langle a|x_0\rangle_c]}\left(\min\{2, \hat{D}(a,c)/2\} - \langle c|x_0\rangle_a\right)\right) & c \in X \text{ and } a \neq c; \\ \left(a, c, \Theta^{-1}_{[-\langle c|x_0\rangle_a, \infty]}(2 - \langle c|x_0\rangle_a)\right) & c \in \partial X; \\ c = (c, c, 0) & \text{if } a = c. \end{cases}$$

We remind the reader that for $a, c \in X$ we have $a = (a,a,0) = (a,a,-\infty) = (a,c,-\infty)$ in $FS(X)$.

Remark 8.2 Because of Lemma 6.16 the point $\iota_{x_0}(a,c)$ is c if $a = c$, is the unique point on the line $(a,c)_{FS(X)}$ whose distance with respect to d_{FS,x_0} from a is $\min\{2, \hat{D}(a,c)/2\}$ if $c \in X$ and $a \neq c$, and is the unique point on the line $(a,c)_{FS(X)}$ whose distance with respect to d_{FS,x_0} from a is 2 if $c \in \partial X$.

Lemma 8.3 *Consider $a, b \in X$ and $c \in \overline{X}$ with $a \neq c$ and $t \in \mathbb{R}$. Suppose*

$$d_{FS,x_0}((a,c,t),(a,a,0)) = \begin{cases} \min\{2, \hat{D}(a,c)/2\} & c \in X; \\ 2 & c \in \partial X. \end{cases}$$

Then

$$-\hat{D}(a,b) \leq t - \langle a, c|b, x_0\rangle \leq 5/2.$$

Proof Note that $(a,a,0) = (a,c,-\infty) \in FS(X)$. We conclude from Lemma 6.16, Lemma 7.7 (i) and Lemma 6.7 (vi)

$$\begin{aligned} &\Theta_{[-\langle c|b\rangle_a, \langle a|b\rangle_c]}(t - \langle a, c|b, x_0\rangle) \\ &= \Theta_{[-\langle c|x_0\rangle_a - \langle a,c|b,x_0\rangle, \langle a|x_0\rangle_c - \langle a,c|b,x_0\rangle]}(t - \langle a, c|b, x_0\rangle) \\ &= \Theta_{[-\langle c|x_0\rangle_a, \langle a|x_0\rangle_c]}(t) - \langle a, c|b, x_0\rangle \\ &= \Theta_{[-\langle c|x_0\rangle_a, \langle a|x_0\rangle_c]}(t) - (-\langle c|x_0\rangle_a) + (-\langle c|x_0\rangle_a) - \langle a, c|b, x_0\rangle \\ &= \Theta_{[-\langle c|x_0\rangle_a, \langle a|x_0\rangle_c]}(t) - (-\langle c|x_0\rangle_a) - \langle c|b\rangle_a \\ &= d_{FS,x_0}((a,c,t),(a,a,0)) - \langle c|b\rangle_a. \end{aligned} \tag{8.4}$$

If $c \in X$, we get by assumption

$$d_{FS,x_0}((a,c,t),(a,a,0)) \leq \frac{\hat{D}(a,c)}{2} = \frac{\langle a|b\rangle_c - (-\langle c|b\rangle_a)}{2},$$

and hence

$$d_{FS,x_0}((a,c,t),(a,a,0)) - \langle c|b\rangle_a \;\leq\; \frac{\langle a|b\rangle_c + (-\langle c|b\rangle_a)}{2}.$$

This inequality is true for $c \in \partial X$ for trivial reasons. Hence we get from **(8.4)**

$$\Theta_{[-\langle c|b\rangle_a,\langle a|b\rangle_c]}(t - \langle a,c|b,x_0\rangle) \;\leq\; \frac{-\langle c|b\rangle_a + \langle a|b\rangle_c}{2}.$$

Lemma 6.7 (ix) together with **(8.4)** implies

$$\begin{aligned} t - \langle a,c|b,x_0\rangle &\leq \Theta_{[-\langle c|b\rangle_a,\langle a|b\rangle_c]}(t - \langle a,c|b,x_0\rangle) + 1/2 \\ &= d_{FS,x_0}((a,c,t),(a,a,0)) - \langle c|b\rangle_a + 1/2 \\ &\leq d_{FS,x_0}((a,c,t),(a,a,0)) + 1/2 \\ &\leq 2 + 1/2 \\ &= 5/2. \end{aligned}$$

Thus we have proven the upper bound $t - \langle a,c|b,x_0\rangle \leq 5/2$. It remains to show the lower bound $-\widehat{D}(a,b) \;\leq\; t - \langle a,c|b,x_0\rangle$.

We conclude from the assumptions that

$$d_{FS,x_0}((a,c,t),(a,a,0)) \;=\; \frac{\widehat{D}(a,c)}{2} \;=\; \frac{\langle a|b\rangle_c - (-\langle c|b\rangle_a)}{2} \;\text{ and } c \in X$$

or

$$d_{FS,x_0}((a,c,t),(a,a,0)) \;=\; 2$$

holds. We begin with the first case. Then

$$d_{FS,x_0}((a,c,t),(a,a,0)) - \langle c|b\rangle_a \;\geq\; \frac{\langle a|b\rangle_c + (-\langle c|b\rangle_a)}{2}.$$

Lemma 6.7 (ix) together with **(8.4)** implies

$$t - \langle a,c|b,x_0\rangle \;\geq\; -\langle c|b\rangle_a \;\geq\; -\widehat{D}(a,b).$$

Finally we treat the second case. Then **(8.4)** implies

$$\Theta_{[-\langle c|b\rangle_a,\langle a|b\rangle_c]}(t - \langle a,c|b,x_0\rangle) \;\geq\; 2 - \langle c|b\rangle_a.$$

Since for $u \leq -\langle c|b\rangle_a$ we have

$$\Theta_{[-\langle c|b\rangle_a,\langle a|b\rangle_c]}(u) \;=\; -\langle c|b\rangle_a + e^{u-(-\langle c|b\rangle_a)}/2 - e^{u-\langle a|b\rangle_c}/2 \;<\; 2 - \langle c|b\rangle_a,$$

we conclude

$$t - \langle a,c|b,x_0\rangle \;\geq\; -\langle c|b\rangle_a \;\geq\; -\widehat{D}(a,b).$$

This finishes the proof of Lemma 8.3. □

Lemma 8.5 *The map $\iota_{x_0}\colon X \times \bar{X} \to FS(X)$ from (8.1) is continuous. It is* $\mathrm{Isom}(X)$*–equivariant with respect to the diagonal* $\mathrm{Isom}(X)$*–action on the source. For $x \in X$ the map $\iota_{x_0}(x,-)\colon \bar{X} \to FS(X), \quad y \mapsto \iota_{x_0}(x,y)$ is injective.*

Proof We only prove continuity, the other claims are straight-forward to check using Remark 8.2. Recall that X and $FS(X)$ are metric spaces. By [5, Exercise III.H.3.18(4)] the space $\bar{X}$ is metrizable. Hence it suffices to check continuity for sequences. Consider sequences $(a_n)_{n\geq 0}$ in X and $(c_n)_{n\geq 0}$ in $\bar{X}$ and points $a \in X$ and $c \in \bar{X}$ such that $\lim_{n\to\infty} a_n = a$ in X and $\lim_{n\to\infty} c_n = c$ in $\bar{X}$ hold. We have to show $\lim_{n\to\infty} \iota_{x_0}(a_n, c_n) = \iota_{x_0}(a,c)$ in $FS(X)$.

Suppose that $a = c$. Then we can assume $c_n \in X$ and $\widehat{D}(a_n, c_n)/2 < 2$ for $n \geq 0$ and $\lim_{n\to\infty} \widehat{D}(a_n, c_n) = 0$. This implies by the construction of ι_{x_0} that

$$\lim_{n\to\infty} d_{FS,x_0}(a_n, \iota_{x_0}(a_n, c_n)) = \lim_{n\to\infty} d_{FS,x_0}(a_n, c_n)/2 \;=\; 0$$

and hence

$$\lim_{n\to\infty} \iota_{x_0}(a_n, c_n) = \lim_{n\to\infty} a_n = a = c = \iota_{x_0}(a,c).$$

Hence we can assume without loss of generality that $a \neq c$ and $a_n \neq c_n$ for all $n \geq 0$ holds.

For $n \geq 0$ put $\alpha_n = -\langle c_n | x_0 \rangle_{a_n}$ and $\beta_n = \langle a_n | x_0 \rangle_{c_n}$. Put $\alpha = -\langle c | x_0 \rangle_a$ and $\beta = \langle a | x_0 \rangle_c$. Then the continuity of the Gromov product (see Theorem 6.3) implies

$$\lim_{n\to\infty} \alpha_n = \alpha;$$
$$\lim_{n\to\infty} \beta_n = \beta.$$

Define t_n to be the real number satisfying

$$\textbf{(8.6)} \qquad \begin{aligned} \Theta_{[\alpha_n,\beta_n]}(t_n) &= \min\left\{2, \widehat{D}(a_n, c_n)/2\right\} - \langle c_n | x_0 \rangle_{a_n} && c_n \in X; \\ \Theta_{[\alpha_n,\beta_n]}(t_n) &= 2 - \langle c_n | x_0 \rangle_{a_n} && c_n \in \partial X. \end{aligned}$$

Define t to be the real number satisfying

$$\textbf{(8.7)} \qquad \begin{aligned} \Theta_{[\alpha,\beta]}(t) &= \min\{2, \widehat{D}(a,c)/2\} - \langle c | x_0 \rangle_a && c \in X; \\ \Theta_{[\alpha,\beta]}(t) &= 2 - \langle c | x_0 \rangle_a && c \in \partial X. \end{aligned}$$

Then $\iota_{x_0}(a_n, c_n) = (a_n, c_n, t_n)$ and $\iota_{x_0}(a,c) = (a,c,t)$. Because of Theorem 7.3 it suffices to show that $\lim_{n\to\infty} t_n = t$ holds. From Lemma 8.3 applied in the case $b = x_0$ we conclude $-\widehat{D}(a_n, x_0) \leq t_n \leq 5/2$ and $-\widehat{D}(a, x_0) \leq t \leq 5/2$. Hence we can assume without loss of generality for all $n \geq 0$

$$-\widehat{D}(a, x_0) - 1 \leq t_n, t \leq 5/2.$$

We conclude from the Mean Value Theorem for some number $\rho \in [0,1]$

$$\Theta_{[\alpha,\beta]}(t_n) - \Theta_{[\alpha,\beta]}(t) \;=\; \Theta'_{[\alpha,\beta]}(\rho \cdot t_n + (1-\rho)t) \cdot (t_n - t).$$

Choose a constant $C > 0$ such that $\Theta'_{[\alpha,\beta]}(s) \geq C^{-1}$ holds for $s \in [-\widehat{D}(a, x_0) - 1, 5/2]$. Since $\rho \cdot t_n + (1-\rho)t$ lies in $[-\widehat{D}(a, x_0) - 1, 5/2]$, we conclude for all $n \geq 0$

$$|t_n - t| \leq C \cdot |\Theta_{[\alpha,\beta]}(t_n) - \Theta_{[\alpha,\beta]}(t)|.$$

We get from the triangle inequality

$$|\Theta_{[\alpha,\beta]}(t_n) - \Theta_{[\alpha,\beta]}(t)| \leq |\Theta_{[\alpha_n,\beta_n]}(t_n) - \Theta_{[\alpha,\beta]}(t_n)| + |\Theta_{[\alpha_n,\beta_n]}(t_n) - \Theta_{[\alpha,\beta]}(t)|$$

and hence

$$|t_n - t| \leq C \cdot \left(|\Theta_{[\alpha_n,\beta_n]}(t_n) - \Theta_{[\alpha,\beta]}(t_n)| + |\Theta_{[\alpha_n,\beta_n]}(t_n) - \Theta_{[\alpha,\beta]}(t)|\right). \tag{8.8}$$

Since $a_n, a \in X$, we have $-\infty < \alpha_n$ and $-\infty < \alpha$. We have $\beta < \infty$ if and only if $c \in X$ and $\beta_n < \infty$ if and only if $c_n \in X$. If $\beta = \infty$, we can assume without loss of generality $5/2 \leq \beta_n$ and hence $t_n \leq \beta_n$ for all $n \geq 0$. We conclude

$$\begin{array}{lll}
|\Theta_{[\alpha_n,\beta_n]}(t_n) - \Theta_{[\alpha,\beta]}(t_n)| \;\leq\; \max\{|\alpha_n - \alpha|, |\beta_n - \beta|\} & \beta_n, \beta < \infty; \\
|\Theta_{[\alpha_n,\beta_n]}(t_n) - \Theta_{[\alpha,\beta]}(t_n)| \;\leq\; |\alpha_n - \alpha| & \beta_n = \beta = \infty; \\
|\Theta_{[\alpha_n,\beta_n]}(t_n) - \Theta_{[\alpha,\beta]}(t_n)| \;\leq\; |\alpha_n - \alpha| + e^{t_n - \beta_n} & \beta_n < \infty, \beta = \infty.
\end{array}$$

from Lemma 6.7 (vii) and (viii) and (the triangle inequality in the last case). Since $t_n \leq 5/2$, $\lim_{n\to\infty} \alpha_n = \alpha$ and $\lim_{n\to\infty} \beta_n = \beta$, we conclude

$$\lim_{n\to\infty} |\Theta_{[\alpha_n,\beta_n]}(t_n) - \Theta_{[\alpha,\beta]}(t_n)| \;=\; 0.$$

Since $\widehat{D}$ and the Gromov product are continuous, we get using (8.6) and (8.7)

$$\lim_{n\to\infty} |\Theta_{[\alpha_n,\beta_n]}(t_n) - \Theta_{[\alpha,\beta]}(t)| \;=\; 0.$$

Now (8.8) implies $\lim_{n\to\infty} t_n = t$. This finishes the proof of Lemma 8.5. □

Theorem 8.9 (Flow estimate for ι) *Let $\lambda \in (e^{-1}, 1)$ and $T \in [0, \infty)$ be the constants depending only on X which appear in Proposition 6.4. Consider $a, b \in X$ and $c \in \overline{X}$. Put*

$$N = 2 + \frac{2}{\lambda^T \cdot (-\ln(\lambda))}.$$

Then there exists a real number τ_0 such that

$$|\tau_0| \;\leq 2 \cdot \widehat{D}(a, b) + 5$$

and for all $\tau \in \mathbb{R}$

$$d_{FS,x_0}(\phi_\tau \iota_{x_0}(a,c), \phi_{\tau+\tau_0}\iota_{x_0}(b,c)) \;\leq\; \frac{N}{1-\ln(\lambda)^2} \cdot \lambda^{-\widehat{D}(a,b)} \cdot \lambda^{\tau}.$$

Proof Put

$$t := \begin{cases} \left(\Theta_{[-\langle c|x_0\rangle_a, \langle a|x_0\rangle_c]}\right)^{-1} \left(\min\{2, \widehat{D}(a,c)/2\} - \langle c|x_0\rangle_a\right) & \text{if } c \in X \text{ and } a \neq c; \\ \left(\Theta_{[-\langle c|x_0\rangle_a, \infty]}\right)^{-1} (2 - \langle c|x_0\rangle_a) & \text{if } c \in \partial X; \\ \langle a, c|b, x_0\rangle = 0 & \text{if } a = c; \end{cases}$$

$$s := \begin{cases} \left(\Theta_{[-\langle c|x_0\rangle_b, \langle b|x_0\rangle_c]}\right)^{-1} \left(\min\{2, \widehat{D}(b,c)/2\} - \langle c|x_0\rangle_b\right) & \text{if } c \in X \text{ and } b \neq c; \\ \left(\Theta_{[-\langle c|x_0\rangle_b, \infty]}\right)^{-1} (2 - \langle c|x_0\rangle_b) & \text{if } c \in \partial X; \\ \langle b, c|a, x_0\rangle = 0 & \text{if } b = c. \end{cases}$$

Put $$\tau_0 \;=\; t - s - \langle a, b|c, x_0\rangle.$$

We have by definition

$$\begin{aligned} \iota_{x_0}(a,c) &= (a,c,t); \\ \iota_{x_0}(b,c) &= (b,c,s); \\ \tau_0 = t - s - \langle a,b|c,x_0\rangle \;&=\; (t - \langle a,c|b,x_0\rangle) - (s - \langle b,c|a,x_0\rangle). \end{aligned}$$

We conclude from Theorem 7.1

$$d_{FS,x_0}(\phi_\tau \iota_{x_0}(a,c), \phi_{\tau+\tau_0}\iota_{x_0}(b,c)) \;\leq\; \frac{N}{1-\ln(\lambda)^2} \cdot \lambda^{(t-\langle a,c|b,x_0\rangle)} \cdot \lambda^{\tau}.$$

We have $$d_{FS,x_0}((a,c,t),(a,a,0)) \;=\; \begin{cases} \min\{2, \widehat{D}(a,c)/2\} & c \in X; \\ 2 & c \in \partial X. \end{cases}$$

and $$d_{FS,x_0}((b,c,s),(b,b,0)) \;=\; \begin{cases} \min\{2, \widehat{D}(b,c)/2\} & c \in X; \\ 2 & c \in \partial X. \end{cases}$$

We conclude from Lemma 8.3 and the definition of t and s respectively that

$$\begin{aligned} -\widehat{D}(a,b) \;&\leq\; t - \langle a,c|b,x_0\rangle \;\leq\; 5/2; \\ -\widehat{D}(a,b) \;&\leq\; s - \langle b,c|a,x_0\rangle \;\leq\; 5/2, \end{aligned}$$

holds. This finishes the proof of Theorem 8.9. □

We can now prove the flow estimate from the introduction.

Proof of Theorem 1.5 Define $j\colon G\times\overline{X}\to FS(X)$ by $j(g,c):=\iota_{x_0}(gx_0,c)$. It follows from Lemma 8.5 that j is continuous and G–equivariant with respect to the diagonal G–action on the source. Let $C:=\max\{\widehat{D}(x_0,sx_0)\mid s\in S\}$ where S is the generating set S of G used to define the word metric d_G on G. Then

$$\widehat{D}(gx_0,hx_0)\le Cd_G(g,h)\quad\forall\, g,h\in G.$$

Let $\alpha>0$ be given. Let $\lambda\in(e^{-1},1)$ and $T\in[0,\infty)$ be the constants depending only on X which appear in Proposition 6.4. Let N be the number defined in Theorem 8.9. Define

$$\beta(\alpha):=2C\alpha+5;$$
$$f_\alpha(\tau):=\frac{N}{1-\ln(\lambda)^2}\cdot\lambda^{-C\alpha}\cdot\lambda^{\tau}.$$

It follows from Theorem 8.9 that $\beta(\alpha)$ and f_α satisfy the assertion of Theorem 1.5. □

9 Further properties of the flow space

Theorem 9.1 *Let X be a hyperbolic complex with base point $x_0\in X$. Suppose that G acts on X by simplicial automorphisms such that every isotropy group is finite. Then:*

(i) *The metric space $(FS(X),d_{FS,x_0}(X))$ is proper;*

(ii) *The induced G–action on the flow space $(FS(X),d_{FS,x_0})$ is proper;*

(iii) *If G acts cocompactly on X, then G acts cocompactly on $FS(X)$.*

A G–space Y is called *proper* if for every $y\in Y$ there exists an open neighborhood U such that the set $\{g\in G\mid g\cdot U\cap U\neq\varnothing\}$ is finite. Proper implies that all isotropy groups are finite. The converse is not true in general but is true if Y is a G–CW–complex [13, Theorem 1.23 on page 18]. If (Y,d_Y) is a metric space and G acts by isometries, the G–action is proper if and only if for every $y\in Y$ there exists an ϵ such that the set $\{g\in G\mid g\cdot B_\epsilon(y)\cap B_\epsilon(y)\neq\varnothing\}$ is finite, where $B_\epsilon(y)=\{z\in Y\mid d_Y(y,z)<\epsilon\}$. A metric space (Y,d_Y) is called *proper* if and only if $\overline{B_\epsilon(y)}=\{z\in Y\mid d_Y(y,z)\le\epsilon\}$ is compact for all $y\in Y$ and $\epsilon\ge 0$.

The elementary proof of the next result is left to the reader.

Lemma 9.2 *Let (Y,d_Y) be a proper metric space. Let G act on Y by isometries. Then the G–action on Y is proper if and only if for every $C>0$ and $y\in Y$ the set $\{g\mid d_Y(g\cdot y,y)\le C\}$ is finite.*

Now we are ready to prove Theorem 9.1.

Proof (i) This is proven in [14, Proposition 54, page 464].

(ii) Mineyev [14, page 457] constructs a map

$$\Psi\colon FS(X) \to X$$

such that there exists constants K_1 and K_2 depending only on X such that for all $v, w \in FS(X)$ and $g \in G$ we have

$$\begin{aligned} |d_{FS,x_0}(v,w) - \hat{D}(\Psi(v),\Psi(w))| &\le K_1; \\ \hat{D}(\Psi(g\cdot v), g\cdot\Psi(v)) &\le K_2. \end{aligned}$$

Compare also [14, Proposition 43 on page 458].

We conclude for $v \in FS(X)$ and $g \in G$

$$\begin{aligned} &\hat{D}(g\cdot\Psi(v),\Psi(v)) \\ &\le \hat{D}(\Psi(g\cdot v),\Psi(v)) + \hat{D}(\Psi(g\cdot v), g\cdot\Psi(v)) \\ &= d_{FS,x_0}(g\cdot v, v) - d_{FS,x_0}(g\cdot v, v) + \hat{D}(\Psi(g\cdot v),\Psi(v)) + \hat{D}(\Psi(g\cdot v), g\cdot\Psi(v)) \\ &\le d_{FS,x_0}(g\cdot v, v) + \left| d_{FS,x_0}(g\cdot v, v) - \hat{D}(\Psi(g\cdot v),\Psi(v)) \right| + \hat{D}(\Psi(g\cdot v), g\cdot\Psi(v)) \\ &\le d_{FS,x_0}(g\cdot v, v) + K_1 + K_2. \end{aligned}$$

There exist real numbers $A \ge 1$ and $B \ge 0$ depending only on X such that for all $x_1, x_2 \in X$ we have

$$A^{-1}\hat{D}(x_1,x_2) - B \ \le d(x_1,x_2) \ \le \ A\cdot\hat{D}(x_1,x_2) + B$$

where d is the word metric; compare the beginning of Section 6.1. Hence we get

$$\begin{aligned} d(g\cdot\Psi(v),\Psi(v)) &\le A\cdot\hat{D}(g\cdot\Psi(v),\Psi(v)) + B \\ &\le A\cdot\big(d_{FS,x_0}(g\cdot v, v) + K_1 + K_2\big) + B. \end{aligned}$$

Consider $v \in FS(X)$ and $C \ge 0$. Since G acts properly on X and (X,d) is a proper metric space, Lemma 9.2 implies that the set

$$\{g \in G \mid d(g\cdot\Psi(v),\Psi(v)) \le A\cdot(C + K_1 + K_2) + B\}$$

is finite. Since this set contains $\{g \in G \mid d_{FS,x_0}(g\cdot v, v) \le C\}$, also the latter set is finite. Hence the G–action on $(FS(X), d_{FS,x_0})$ is proper by Lemma 9.2.

(iii) Since G acts simplicially and cocompactly on X, we can find a compact subset $C \subseteq X$ such that $G\cdot C = X$. Consider $D = \{x \in FS(X) \mid \hat{D}(\Psi(x), C) \le K_2\}$. Since

C is compact, its diameter $\operatorname{diam}(C)$ is finite. Since for $y, z \in D$ we get

$$d_{FS,x_0}(y,z) \le \widehat{D}(\Psi(y),\Psi(z)) + K_1 \le \operatorname{diam}(C) + 2K_2 + K_1$$

the set D has finite diameter. Since $FS(X)$ is proper as a metric space by assertion (i), the closure of D is a compact subset of $FS(X)$. Next we show $G \cdot D = FS(X)$. Consider $x \in FS(X)$. Choose $g \in G$ such that $g^{-1}\Psi(x) \in C$. From $\widehat{D}(\Psi(g^{-1}x), g^{-1}\Psi(x)) \le K_2$ we conclude $g^{-1}x \in D$. This implies $x \in g \cdot D \subseteq G \cdot D$. Hence $\overline{D}$ is a compact subset of $FS(X)$ with $G \cdot \overline{D} = FS$. Therefore G acts on FS cocompactly. This finishes the proof of Theorem 9.1. □

The following facts are well-known. We include a proof for the convenience of the reader.

Lemma 9.3 *Let X be a δ–hyperbolic complex in the sense of Section 6.1. Let $\overline{X}$ be the compactification of X in the sense of Gromov. Then*

(i) *$\overline{X}$ is locally connected;*

(ii) *$\overline{X}$ has finite covering dimension.*

Proof We start by reviewing the topology of $\overline{X}$ following [5, p 429]. Recall that $X^{(1)}$ denotes the 1–skeleton of X. A generalized ray $c\colon I \to X^{(1)}$ is a geodesic with respect to the word metric d_{word}, where $I = [0, R]$ for $R \ge 0$ or $I = [0, \infty)$. In the later case c will be called a geodesic ray. If $I = [0, R]$ it is convenient to write $c(t) = c(R)$ for $t \ge r$. Two geodesic rays c, c' are called equivalent if there is $C > 0$ such that $d_{\mathrm{word}}(c(t), c'(t)) < C$ for all $t \in [0, \infty)$. $\partial X = \partial X^{(1)}$ is the set of all equivalence classes of such geodesic rays. For a geodesic ray c we denote by $c(\infty)$ the point in ∂X determined by c. Fix a base point $x_0 \in X^{(0)}$ and $k > 2\delta$. Every point in ∂X can be written as $c(\infty)$ where c is a geodesic ray starting at x_0 [5, Lemma 3.1 on page 427]. For a geodesic ray c starting at x_0 and $n \in \mathbb{N}$ let $V_n^{(1)}(c)$ denote the set of all $c'(\infty)$ where c' is a generalized ray starting at x_0 with $d_{\mathrm{word}}(c(n), c'(n)) < k$. (Such a generalized ray may end in X.) Let $V_n(c)$ be the union of $V_n^{(1)}(c)$ with the smallest subcomplex of X containing $V_n^{(1)}(c) \cap X$. The topology on $\overline{X} = X \cup \partial X$ is now defined as follows: a $U \subset \overline{X}$ is open if and only if the following two conditions hold:

- $U \cap X$ is open in X.
- If c is a geodesic ray starting at x_0 and $c(\infty) \in U$ then there is $n \in \mathbb{N}$ such that $V_n(c) \subset U$.

We now prove (i). Obviously X is locally connected as it is a CW–complex. It suffices to show for every $x \in \partial X = \bar{X} - X$ that there is a (not necessarily open) connected neighborhood (see [15, Exercise 10 on page 163]). But this follows because the $V_n(c)$ from above are connected.

Next we prove (ii). Let $D \in \mathbb{N}$ be the maximal number of points in $X^{(0)}$ that are contained in a ball of radius $k + 2\delta$. This is a finite number because X is uniformly locally finite. Because X has finite covering dimension and ∂X is compact, it suffices to show the following: For every finite collection $\mathcal{U}$ of open subsets of $\bar{X}$ that covers ∂X there is an $(D-1)$–dimensional refinement of $\mathcal{U}$ that still covers ∂X.

We will need the following observation: Let c, c' be two geodesics starting at x_0. Let x be the endpoint of c and x' be the endpoint of c'. If N is such that $d(c(N), x) > d(x, x') + \delta$, then $d(c(N), c'(N)) \leq 2\delta$. This is an easy application of the condition that all geodesic triangles in $X^{(1)}$ are δ–thin.

For $N \in \mathbb{N}$ consider the set S_N of all $c(N)$ where c is a geodesic ray in $X^{(1)}$ starting at x_0. This is a finite subset of $X^{(0)}$ because X is locally finite. For every $x \in S_N$ pick a geodesic ray c_x starting at x_0 such that $c_x(N) = x$. Denote by B the close ball of radius $n + \delta$ around x_0. Let $U_x := (V_N(c_x))^\circ - B$. Then the collection $\mathcal{U}_N := \{U_x \mid x \in S_N\}$ covers ∂X. Let $z \in \bar{X} - B$ and choose $x \in S_N$ such that x lies on a geodesic from x_0 to z. Using the above observation it is not hard to show that for every $x' \in S_N$ with $z \in U_{x'}$ we have $d(x, x') < k + 2\delta$. Therefore the dimension of $\mathcal{U}_N$ is bounded by $D - 1$.

It remains to show that for sufficiently large N, the collection $\mathcal{U}_N$ will be a refinement of the given collection $\mathcal{U}$. Let $R > k + 3\delta$, $R \in \mathbb{N}$. By the definition of the topology of ∂X and because ∂X is compact, there are $M_1, \dots, M_n \in \mathbb{N}$ and geodesic rays $c_1, \dots, c_n$ starting at x_0 such that

- $\partial X \subset V_{M_1+R}(c_1) \cup \cdots \cup V_{M_n+R}(c_n)$;
- for every $i = 1, \dots, n$ there is $U \in \mathcal{U}$ such that $V_{M_i}(c_i) \subset U$.

Let $M := \max\{M_1, \dots, M_n\}$. Two applications of the above observation give the following: If c is a geodesic ray starting at x_0 such that $c(\infty) \in V_{M_i+R}(c_i)$, then $V_{M+2R}(c) \subset V_{M_i}(c_i)$. Thus $\mathcal{U}_{M+2R}$ is a refinement of $\mathcal{U}$. □

We can now check the additional properties stated in Section 1.2.

Proof of Proposition 1.6 (i) This follows from [5, Theorem 3.2 on page 459].

(ii) By Lemma 9.3 $\bar{X}$ has finite covering dimension and is locally connected. It follows therefore from Theorem 7.3 that $FS(X) - FS(X)^{\mathbb{R}} = FS(X) - X$ is locally connected and has finite covering dimension.

(iii) This follows from Theorem 9.1.

(iv) For $g \in G$ its translation length on $(FS(X), d_{FS(X),x_0})$ is defined as

$$l(g) := \lim_{n\to\infty} d_{FS,x_0}(g^n(a,b,t),(a,b,t))/n.$$

By the triangle inequality this definition does not depend on the choice of $(a,b,t) \in FS(X)$. In particular, it depends only on the conjugacy class of g. Since the isometric G–action on $FS(X)$ is cocompact and proper, (G, d_G) is quasi-isometric to (FS, d_{FS,x_0}). Thus, there are constants $A \geq 1$, $B > 0$ such that

$$A \cdot l(g) + B \geq \tau(g) := \lim_{n\to\infty} d_G(g^n, 1_G).$$

($\tau(g)$ is the translation length of g on (G, d_G).) By [5, Proposition 3.15 on page 465] for fixed $C > 0$ the number of conjugacy classes whose translation length on (G, d_G) is no more than C is finite. We conclude that the same holds for the translation length on $(FS(X), d_{FS,x_0})$.

Fix $C > 0$. Let $\mathcal{L}$ be the set of all orbits L of the flow ϕ_τ on $FS(X)$ with $0 < \mathrm{per}_\phi^G(L) \leq C$, see Definition 2.15. Every $L \in \mathcal{L}$ is a line $(a_L, b_L)_{FS(X)}$ with a_L, $b_L \in \bar{X}$. For $L \in \mathcal{L}$ there is $g_L \in G$ such that

$$g_L \cdot (a_L, b_L, t) = (a_L, b_L, t + \mathrm{per}_\phi^G(L))$$

for $(a_L, b_L, t) \in L$. In particular $g_L \cdot a_L = a_L$, $g_L \cdot b_L = b_L$. If $a_L \in X$ or $b_L \in X$, then g_L has finite order because the action of G on X is proper. But this would imply $\mathrm{per}_\phi^G(L) = 0$. Therefore a_L, $b_L \in \partial X$. By Lemma 6.16 $l(g_L) = \mathrm{per}_\phi^G(L) \leq C$. Recall that $\partial X \cong \partial G$, because X is quasi-isometric to G [5, Theorem 3.9 on page 430]. Because every $g \in G$ has at most 2 fixed points on ∂G [9, 20.- Corollaire on page 149] the map $L \mapsto g_L$ is injective. Because there are only finitely many conjugacy classes of translation length $\leq C$ this means that $G\backslash\mathcal{L}$ is finite. This is what we needed to prove. □

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