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ON THE MAXIMAL RANK PROBLEM FOR THE COMPLEX HOMOGENEOUS MONGE–AMPÈRE EQUATION

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We give examples of regular boundary data for the Dirichlet problem for the complex homogeneous Monge–Ampère equation over the unit disc, whose solution is completely degenerate on a nonempty open set and thus fails to have maximal rank.

1. Introduction

Let (X, ω) be a compact Kähler manifold of dimension n and B be a Riemann surface with boundary ∂B. Suppose $(\phi_\tau)_{\tau\in\partial B}$ is a smooth family of Kähler potentials on X; so each ϕ_τ is a smooth function on X, varying smoothly in τ, that satisfies

$$\omega + dd^c\phi_\tau > 0.$$

Then let Φ be the solution to the Dirichlet problem for the complex homogeneous Monge–Ampère equation (HMAE) with this boundary data, so Φ is a function on $X \times B$ that satisfies

$$\begin{aligned}\Phi(\,\cdot\,,\tau) &= \phi_\tau(\,\cdot\,) \quad \text{for } \tau \in \partial B,\\ \pi_X^*\omega + dd^c\Phi &\geq 0,\\ (\pi_X^*\omega + dd^c\Phi)^{n+1} &= 0,\end{aligned} \tag{1}$$

where $\pi_X : X \times B \to X$ is the projection. From standard pluripotential theory we know there exists a unique weak solution Φ to this equation. The *maximal rank problem* in this setting asks whether the current

$$\pi_X^*\omega + dd^c\Phi$$

has maximal rank in the fibre directions, that is, whether the current $\omega + dd^c\Phi(\,\cdot\,,\tau)$ on X is strictly positive for each $\tau \in B$. Said another way, this asks if the rank of $\pi_X^*\omega + dd^c\Phi$ is precisely n at every point in $X \times B$, which is the maximum possible since $(\pi_X^*\omega + dd^c\Phi)^{n+1} = 0$. Similarly one has the *constant-rank problem* in which one asks if the rank of $\pi_X^*\omega + dd^c\Phi$ is the same at every point. The purpose of this note is to answer this question negatively, giving an explicit example in which the rank fails to be maximal.

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Theorem 1.1. *Let $B = \overline{\mathbb{D}} \subset \mathbb{C}$ be the closed unit disc and $(X, \omega) = (\mathbb{P}^1, \omega_{\mathrm{FS}})$, where ω_{FS} denotes the Fubini–Study form. Then there exists a smooth family of Kähler potentials $(\phi_\tau)_{\tau\in\partial\mathbb{D}}$ on $\mathbb{P}^1$ such that the solution Φ to the HMAE (1) is completely degenerate on some nonempty open subset $S \subset \mathbb{P}^1 \times \mathbb{D}$, i.e.,*

$$\pi_{\mathbb{P}^1}^* \omega_{\mathrm{FS}} + dd^c\Phi|_S = 0.$$

A more precise version of this statement is provided in Theorem 2.1. The motivation and ideas build on previous work of the authors [Ross and Witt Nyström 2015a; 2015b; 2015c] in which we understand the solution to the HMAE of a certain kind through a free boundary problem in the plane called the Hele-Shaw flow. But rather than expecting the reader to be an expert in this topic we have chosen to give a direct proof, which can be found in Section 2, that is both self-contained and rather simple. Then in Section 3 we explain the motivation behind our construction, as well as give a second (but essentially equivalent) proof that relies on more machinery. We then end with some questions and possible extensions.

Of course in the above theorem, $\pi_{\mathbb{P}^1}^* \omega_{\mathrm{FS}} + dd^c\Phi$ is not identically zero, and so does not have constant rank. In fact we can say more, and it is possible to arrange so that there is a nonempty open set in $\mathbb{P}^1 \times \mathbb{D}$ on which $\pi_{\mathbb{P}^1}^* \omega_{\mathrm{FS}} + dd^c\Phi$ is regular (i.e., smooth and of maximal rank). It is worth commenting from the outset that we do not expect the solution we have here to be everywhere smooth, but it should be possible to describe precisely where it is regular and where it is degenerate. All of this will be discussed in more detail in Section 3.

1A. ***Comparison with other work.*** It is known that convex solutions to elliptic partial differential equations have a constant-rank property. Early works of this include [Caffarelli and Friedman 1985; Singer, Wong, Yau, and Yau 1985]. These have since been built upon by many others, and it is now known that the constant-rank property holds for a wide class of elliptic equations; see, for instance, [Korevaar and Lewis 1987; Bian and Guan 2009; 2010; Caffarelli, Guan and Ma 2007; Székelyhidi and Weinkove 2016]. In this paper we are interested in the complex degenerate situation, about which much less has been written. The most famous result along these lines, and in the positive direction, is that of Lempert [1981] who proved that on a convex domain in $\mathbb{C}^n$ the solution to the complex HMAE with prescribed singularity at an interior point (the pluricomplex Green function) is smooth and of maximal rank. The maximal rank problem for other partial differential equations in the complex case has also been taken up by Guan, Li and Zhang [2009] and by Li [2009].

The closest previous work to that of this paper is probably that of Guan and Phong [2012], who studied the problem of finding uniform lower bounds for the eigenvalues of the solution to the (nondegenerate) Monge–Ampère equation in the limit as the equation becomes degenerate. Moreover, they asked whether solutions to the complex HMAE have maximal rank [Guan and Phong 2012, discussion after Theorem 4]. The idea of maximal rank for the complex HMAE also appears in the ideas of Chen and Tian [2008] through the concept of an almost-regular solution to the HMAE, which fails to have maximal rank only on a set which is small in a precise sense. The kinds of envelopes that we use in our proof also can be defined more generally, and even in higher dimensions, which is the topic of previous work of the authors [Ross and Witt Nyström 2017b], in which we prove a constant-rank theorem, Theorem 6.2 of that paper, that we call "optimal regularity".

Questions concerning the regularity of the solution to the Dirichlet problem for the kind of complex HMAE we consider here go back at least as far as [Semmes 1992; Donaldson 2002], and this HMAE has been the focus of much interest due to it being the geodesic equation in the space of Kähler metrics. By [Chen 2000] with complements by Błocki [2012] we know such a solution always has bounded Laplacian (so in particular is $C^{1,\alpha}$ for any $\alpha < 1$). In fact in our case, since we are working on $\mathbb{P}^1$, the results of [Błocki 2012] imply that Φ is $C^{1,1}$. (We observe that we do not actually need to know this regularity for the direct proof of our main theorem). Donaldson [2002] gives examples of boundary data for which the solution is not regular, but the nature of the irregularity there is left unknown (for instance Donaldson's example may have maximal rank but fail to be everywhere smooth).

2. Main theorem

2A. *Notation.* We let $\mathbb{D}_r$ be the open disc of radius r in the complex plane about the origin, $\mathbb{D} = \mathbb{D}_1$ and $\mathbb{D}^\times = \mathbb{D} \setminus \{0\}$. Throughout we consider the standard cover of $\mathbb{P}^1$ by two charts equal to the complex plane with coordinates z and $w = 1/z$. We shall denote these two charts by $\mathbb{C}_z$ and $\mathbb{C}_w$ respectively. We use the convention $d^c = \frac{1}{2\pi}(\bar\partial - \partial)$ so $dd^c \log|z|^2 = \delta_0$, and normalise the Fubini–Study form ω_{FS} so $\int_{\mathbb{P}^1} \omega_{\mathrm{FS}} = 1$. Thus $\omega_{\mathrm{FS}} = dd^c \log(1 + |z|^2)$ locally on $\mathbb{C}_z$.

2B. *Statement of the main theorem.* The following is a precise version of our main theorem. By an *arc* in $\mathbb{C}$ we mean the image γ of a smooth map $[0, 1] \to \mathbb{C}$ that does not intersect itself. From now on $B = \overline{\mathbb{D}}$ is the closed unit disc and $(X, \omega) = (\mathbb{P}^1, \omega_{\mathrm{FS}})$.

Theorem 2.1. *Suppose that $\phi \in C^\infty(\mathbb{P}^1)$ satisfies*:

(1) $\omega_{\mathrm{FS}} + dd^c\phi > 0$.

(2) *On $\mathbb{C}_w \subset \mathbb{P}^1$ it holds that*

$$\phi(w) \geq -\ln(1 + |w|^2)$$

with equality precisely on an arc in $\mathbb{C}_w$.

Then setting

$$\phi_\tau(z) := \phi(\tau z) \quad \text{for } \tau \in \partial\mathbb{D},$$

the solution Φ to the HMAE (1) does not have maximal rank. In fact there is a nonempty open subset $S \subset \mathbb{P}^1 \times \mathbb{D}$ such that

$$\pi^*_{\mathbb{P}^1}\omega_{\mathrm{FS}} + dd^c\Phi|_S = 0.$$

2C. *Envelopes.* For the proof we need some background concerning envelopes of subharmonic functions. Fix a potential $\phi \in C^\infty(\mathbb{P}^1)$ so $\omega_{\mathrm{FS}} + dd^c\phi > 0$. For a topological space X let

$$\mathrm{USC}(X) = \{\psi : X \to \mathbb{R} \cup \{-\infty\} \text{ such that } \psi \text{ is upper semicontinuous}\}.$$

Definition 2.2. For $t \in (0, 1]$ set

$$\psi_t := \sup\{\psi \in \mathrm{USC}(\mathbb{P}^1) : \psi \leq \phi \text{ and } \omega_{\mathrm{FS}} + dd^c\psi \geq 0 \text{ and } \nu_{z=0}(\psi) \geq t\}.$$

Here $\nu_{z=0}$ denotes the Lelong number at the point $z=0$, so $\nu_{z=0}(\psi)\geq t$ means $\psi(z)\leq t\ln|z|^2+O(1)$ near $z=0$. As the upper-semicontinuous regularisation of ψ_t is itself a candidate for the envelope defining ψ_t, we see that ψ_t is itself upper-semicontinuous.

Definition 2.3. For $t\in(0,1]$ set

$$\Omega_t:=\Omega_t(\phi):=\{z\in\mathbb{P}^1:\psi_t(z)<\phi(z)\}. \tag{2}$$

Clearly if $t\leq t'$ then $\psi_{t'}\leq\psi_t$ and so $\Omega_t\subset\Omega_{t'}$. Now, unless one assumes some additional symmetry of ϕ, it is generally quite hard to describe the sets Ω_t. However, as the next lemma shows, it is possible, under a suitable hypothesis, to describe the largest one Ω_1 by looking at the level set on which ϕ takes its minimum value.

Lemma 2.4. *Let $\phi\in C^\infty(\mathbb{P}^1)$ be such that $\omega_{\mathrm{FS}}+dd^c\phi>0$ and $\phi(w)\geq-\ln(1+|w|^2)$ on $\mathbb{C}_w$ with equality precisely on some nonempty subset $\gamma\subset\mathbb{C}_w$ containing $w=0$. Then*

$$\psi_1(z)=\ln\left(\frac{|z|^2}{1+|z|^2}\right)$$

and

$$\Omega_1(\phi)=\mathbb{P}^1\setminus\gamma.$$

Proof. Observe first that the only upper-semicontinuous $\psi:\mathbb{P}^1\to\mathbb{R}\cup\{-\infty\}$ with $\omega_{\mathrm{FS}}+dd^c\psi\geq0$ and $\nu_{z=0}(\psi)\geq1$ is, up to an additive constant, equal to

$$\zeta(z):=\ln\left(\frac{|z|^2}{1+|z|^2}\right)\quad\text{on }\mathbb{C}_z.$$

To see this observe first that we certainly cannot have $\nu_{z=0}(\psi)>1$ since we have normalised so $\int_{\mathbb{P}^1}\omega_{\mathrm{FS}}=1$. Thus we may assume $\nu_{z=0}\psi=1$. Then observe that ζ is ω_{FS}-harmonic on $\mathbb{C}_z\setminus\{0\}$, and that the difference $\psi-\zeta$ is bounded near 0. Thus $\psi-\zeta$ extends to a bounded subharmonic function on all of $\mathbb{C}_z$, and hence is constant by the Liouville property. Thus the envelope ψ_1 from Definition 2.2 must be

$$\psi_1=\zeta+C,$$

where C is the largest constant one can choose so that $\psi_1\leq\phi$. Now on $\mathbb{C}_w$ we have

$$\psi_1(w)=-\ln(1+|w|^2)+C$$

and so as γ is nontrivial our hypothesis forces $C=0$. Thus

$$\Omega_1=\{-\ln(1+|w|^2)<\phi(w)\}=\mathbb{P}^1\setminus\gamma. \qquad\square$$

2D. ***Weak solutions to the HMAE.*** We now discuss the weak solution to two versions of the Dirichlet problem for the complex HMAE, first over the disc and second over the punctured disc; this follows the discussion in [Ross and Witt Nyström 2015b]. Again we let $\phi\in C^\infty(\mathbb{P}^1)$ be such that $\omega_{\mathrm{FS}}+dd^c\phi>0$.

Definition 2.5. Let

$$\Phi := \sup\bigl\{\psi \in \mathrm{USC}(\mathbb{P}^1 \times \overline{\mathbb{D}}) : \pi_{\mathbb{P}^1}^*\omega_{\mathrm{FS}} + dd^c\psi \geq 0 \text{ and } \psi(z,\tau) \leq \phi(\tau z) \text{ for } (z,\tau) \in \mathbb{P}^1 \times \partial\mathbb{D}\bigr\}.$$

and

$$\widetilde{\Phi} := \sup\bigl\{\psi \in \mathrm{USC}(\mathbb{P}^1 \times \overline{\mathbb{D}}) : \pi_{\mathbb{P}^1}^*\omega_{\mathrm{FS}} + dd^c\psi \geq 0 \text{ and } \psi(z,\tau) \leq \phi(z) \text{ for } (z,\tau) \in \mathbb{P}^1 \times \partial\mathbb{D} \text{ and } \nu_{(z=0,\tau=0)}(\psi) \geq 1\bigr\}. \tag{3}$$

The function Φ is the weak solution to the complex HMAE with boundary data $\phi(\tau z)$ for $\tau \in \partial\mathbb{D}$, that is, the solution to (1). Similarly $\widetilde{\Phi}$ is the weak solution to the Dirichlet problem with boundary data $\phi(z)$, but with the additional requirement of having a prescribed singularity at the point $p := (0,0) \subset \mathbb{C}_z \times \mathbb{D} \subset \mathbb{P}^1 \times \mathbb{D}$. That is, $\widetilde{\Phi}$ is upper-semicontinuous, $\pi_{\mathbb{P}^1}^*\omega_{\mathrm{FS}} + dd^c\widetilde{\Phi} \geq 0$ and $(\pi_{\mathbb{P}^1}^*\omega_{\mathrm{FS}} + dd^c\widetilde{\Phi})^2 = 0$ away from p and $\widetilde{\Phi}(z,\tau) = \phi(z)$ for $\tau \in \partial\mathbb{D}$. Moreover it is not hard to show that $\widetilde{\Phi}$ is locally bounded away from p and $\nu_p\widetilde{\Phi} = 1$. These two quantities carry the same information, as given by:

Proposition 2.6. *We have that*

$$\Phi(z,\tau) + \ln|\tau|^2 + \ln(1+|z|^2) = \widetilde{\Phi}(\tau z,\tau) + \ln(1+|\tau z|^2) \quad \textit{for } (z,\tau) \in \mathbb{P}^1 \times \overline{\mathbb{D}}^\times.$$

Proof. It is easily seen from the definition that $\Phi(z,\tau) + \ln|\tau|^2 + \ln(1+|z|^2) - \ln(1+|\tau z|^2)$ is a candidate for the envelope defining $\widetilde{\Phi}(\tau z,\tau)$, giving one inequality and the other inequality is proved similarly. □

2E. *Proof of Theorem 2.1*. Without loss of generality we assume the arc γ goes through the point $w = 0$. By Lemma 2.4

$$\psi_1(z) = \ln\left(\frac{|z|^2}{1+|z|^2}\right)$$

and

$$\Omega_1 = \mathbb{P}^1 \setminus \gamma.$$

Looking at the other coordinate patch $\mathbb{C}_z$, we have that γ is a curve passing through infinity, and so $\mathbb{C}_z \setminus \gamma$ is an open, simply connected proper subset of $\mathbb{C}_z$. Hence by the Riemann mapping theorem there is a biholomorphism

$$f : \mathbb{D} \to \mathbb{C}_z \setminus \gamma \quad \text{with } f(0) = 0.$$

For $\tau \in \mathbb{D}^\times$ set

$$A_\tau := f(\mathbb{D}_{|\tau|}) \subset \mathbb{C}_z \subset \mathbb{P}^1.$$

Clearly each A_τ is a proper subset of $\mathbb{C}_z$ containing the origin, whose complement has nonempty interior.

Proposition 2.7. *We have*

$$\widetilde{\Phi}(z,\tau) = \psi_1(z) \quad \textit{for all } \tau \in \mathbb{D}^\times \textit{ and } z \in \mathbb{P}^1 \setminus A_\tau.$$

Proof. By abuse of notation we write ψ_1 also for the pullback of ψ_1 to $\mathbb{P}^1 \times \overline{\mathbb{D}}$. Then

$$\widetilde{\Phi}(z,\tau) \geq \psi_1(z) \quad \text{for } (z,\tau) \in \mathbb{P}^1 \times \overline{\mathbb{D}} \tag{4}$$

since ψ_1 is a candidate for the envelope (3) defining $\widetilde{\Phi}$.

We next claim that

$$\widetilde{\Phi}(f(\tau),\tau)=\psi_1(f(\tau)) \quad \text{for all } \tau\in\mathbb{D}. \tag{5}$$

To see this, observe that $\tau\mapsto\widetilde{\Phi}(f(\tau),\tau)$ is $f^*\omega_{\mathrm{FS}}$-subharmonic and has Lelong number 1 at $\tau=0$. On the other hand $\psi_1(f(\tau))$ is $f^*\omega_{\mathrm{FS}}$-harmonic except at $\tau=0$ where it has Lelong number 1. But $\widetilde{\Phi}(f(\tau),\tau)$ tends to $\psi_1(f(\tau))$ as $|\tau|$ tends to 1, and hence from the maximum principle along with (4), we get (5).

Now fix some $\tau\in\mathbb{D}^\times$ and set

$$\phi_\tau(z):=\widetilde{\Phi}(z,\tau).$$

Then the above says that $\phi_\tau=\psi_1$ on ∂A_τ. On the other hand by (4) we have $\phi_\tau\geq\psi_1$ everywhere. Moreover ϕ_τ is ω_{FS}-subharmonic on A_τ^c, whereas ψ_1 is bounded and ω_{FS}-harmonic on A_τ^c. Thus by the maximum principle we deduce $\phi_\tau=\psi_1$ on A_τ^c as required. □

Proof of Theorem 2.1. Set

$$S:=\{(z,\tau)\in\mathbb{P}^1\times\mathbb{D}^\times:\tau z\in(A_\tau^c)^\circ\},$$

which is nonempty and open in $\mathbb{P}^1\times\mathbb{D}^\times$. Then by Proposition 2.6 and then Proposition 2.7 if $(z,\tau)\in S$ we have

$$\Phi(z,\tau)=\widetilde{\Phi}(\tau z,\tau)+\ln\left(\frac{1+|\tau z|^2}{|\tau|^2(1+|z|^2)}\right)=\psi_1(\tau z)+\ln\left(\frac{1+|\tau z|^2}{|\tau|^2(1+|z|^2)}\right).$$

Thus on S we have

$$\pi_{\mathbb{P}^1}\omega_{\mathrm{FS}}+dd^c\Phi=\pi_{\mathbb{P}^1}\omega_{\mathrm{FS}}+dd^c\psi_1(\tau z)=0$$

as ψ_1 is ω_{FS}-harmonic away from $z=0$. □

2F. *A specific example.* We now construct a specific potential ϕ that satisfies the hypotheses of Theorem 2.1. Fix γ to be the interval $[-1,1]\subset\mathbb{R}\subset\mathbb{C}_w$. Our goal is to find a $\phi\in C^\infty(\mathbb{P}^1)$ such that $\omega_{\mathrm{FS}}+dd^c\phi>0$ and $\phi\geq-\ln(1+|w|^2)$ with equality precisely on γ.

To do so, let $\alpha:\mathbb{R}\to\mathbb{R}$ be a nonnegative smooth nondecreasing convex function with $\alpha(t)=0$ for $t\leq 1$ and $\alpha(t)>0$ for $t>1$. Let

$$u(w):=\alpha(|w|^2)+\mathrm{Im}(w)^2.$$

Thus u is a smooth strictly subharmonic function on $\mathbb{C}_w$ that vanishes precisely on γ. Then $\epsilon u-\ln(1+|w|^2)$ for some small constant $\epsilon>0$ is essentially the function that we want; we simply need to adjust it to have the correct behaviour far away from γ.

To do so we shall use a regularised version of the maximum function, which can be explicitly given as follows: Let $|\cdot|_{\mathrm{reg}}$ be a smooth convex function on $\mathbb{R}$ so that $|t|_{\mathrm{reg}}=|t|$ for $|t|\geq 1$. Set $\max_{\mathrm{reg}}(a,b):=\frac{1}{2}(|a-b|_{\mathrm{reg}}+a+b)$ and for $\delta>0$ put

$$\max_\delta(a,b):=\delta\max_{\mathrm{reg}}(\delta^{-1}a,\delta^{-1}b). \tag{6}$$

Then $\max_\delta(\,\cdot\,,\cdot\,)$ is smooth, and satisfies

$$\max_\delta(a,b)=\begin{cases} a & \text{if } a>b+\delta,\\ b & \text{if } b>a+\delta.\end{cases}$$

Returning to the construction of ϕ, fix a sufficiently large constant C and a sufficiently small positive constant ϵ so that

$$\epsilon u \geq \ln(1+|w|^2)-C+1 \quad \text{on } \mathbb{D}_2,$$
$$\epsilon u \leq \ln(1+|w|^2)-C-1 \quad \text{on } \mathbb{D}_4\setminus\mathbb{D}_3.$$

Then for $0<\delta\ll 1$ set

$$v:=\max_\delta(\epsilon u, \ln(1+|w|^2)-C).$$

So v is smooth, nonnegative, strictly subharmonic, equal to $\ln(1+|w|^2)-C$ on $\mathbb{D}_4\setminus\mathbb{D}_3$ and vanishes precisely on γ. We then put

$$\phi:=v-\ln(1+|w|^2)$$

and extend ϕ to take the constant value C in $\mathbb{C}_w\setminus\mathbb{D}_4$. So ϕ extends to a smooth function over $\mathbb{P}^1$ with the desired properties.

3. Discussion

3A. *Context.* Fix a $\phi\in C^\infty(\mathbb{P}^1)$ such that $\omega_{\mathrm{FS}}+dd^c\phi>0$. Then associated to ϕ we have two constructions:

(1) The solution $\widetilde{\Phi}$ to the complex HMAE on $\mathbb{P}^1\times\overline{\mathbb{D}}$ with boundary data given by $\phi_\tau=\phi$ for all $\tau\in\partial\mathbb{D}$ and the requirement of having Lelong number 1 at the point $(z,\tau)=(0,0)\in\mathbb{C}_z\times\mathbb{D}\subset\mathbb{P}^1\times\overline{\mathbb{D}}$.

(2) The envelopes ψ_t for $t\in(0,1]$ and the associated sets $\Omega_t(\phi)=\{\psi_t<\phi\}$.

In previous work we showed that these sets of data are intimately connected. First, $\widetilde{\Phi}$ and ψ_t are Legendre dual to each other [Ross and Witt Nyström 2015b, Theorem 2.7] in that

$$\psi_t(z)=\inf_{|\tau|>0}\{\widetilde{\Phi}(z,\tau)-(1-t)\ln|\tau|^2\} \tag{7}$$

and

$$\widetilde{\Phi}(z,\tau)=\sup_t\{\psi_t(z)+(1-t)\ln|\tau|^2\}. \tag{8}$$

Second, the collection of sets $\Omega_t(\phi)$ that are biholomorphic to a disc describes the harmonic discs of $\widetilde{\Phi}$. That is, if t is such that $\Omega_t(\phi)$ is a proper simply connected subset of $\mathbb{C}_z$ and $f:\mathbb{D}\to\Omega_t$ is a Riemann map with $f(0)=0$ then the restriction of $\widetilde{\Phi}$ to the graph $\{(f(\tau),\tau)\in\mathbb{P}^1\times\mathbb{D}\}$ is ω_{FS}-harmonic. Furthermore it is shown in [Ross and Witt Nyström 2015b, Theorem 3.1] these are (essentially) the only harmonic discs that occur.

We can say more. For $\tau\in\mathbb{D}^\times$ set

$$\phi_\tau(z):=\widetilde{\Phi}(z,\tau).$$

If $\widetilde{\Phi}$ is regular then each ϕ_τ will be a smooth Kähler potential, but in general this will not be the case. Nevertheless, by [Błocki 2012] we know ϕ_τ is $C^{1,1}$ and since $\pi_{\mathbb{P}^1}^*\omega_{\mathrm{FS}} + dd^c\widetilde{\Phi} \geq 0$, we know $\omega_{\mathrm{FS}} + dd^c\phi_\tau$ is semipositive. We can then define the associated sets $\Omega_t(\phi_\tau)$ in exactly the same way as before.

Proposition 3.1. *Suppose t is such that $\Omega_t(\phi) \subset \mathbb{C}_z$ is proper and simply connected and let $f_t : \mathbb{D} \to \Omega_t(\phi)$ be a Riemann map with $f(0) = 0$. Then for each $\tau \in \mathbb{D}^\times$ we have*

$$f_t(\mathbb{D}_{|\tau|}) = \Omega_t(\phi_\tau).$$

We shall give a proof of this fact below, but assuming it for now we can give an alternative proof that, under the hypotheses of Theorem 2.1, for each $\tau \in \mathbb{D}^\times$ the current $\omega_{\mathrm{FS}} + dd^c\widetilde{\Phi}(\,\cdot\,, \tau)$ is degenerate on some nonempty open subset of $\mathbb{P}^1$. First Lemma 2.4 gives

$$\Omega_1(\phi) = \mathbb{P}^1 \setminus \gamma,$$

which is a simply connected proper subset of $\mathbb{C}_z$. We then take our Riemann map $f : \mathbb{D} \to \Omega_1(\phi)$ and consider the image

$$A_\tau := f(\mathbb{D}_{|\tau|}) = \Omega_1(\phi_\tau) \quad \text{for } \tau \in \mathbb{D}^\times.$$

As observed before, A_τ is a proper subset of $\mathbb{C}_z$ whose complement has nonempty interior.

On the other hand, it is a general fact that for each t the set $\Omega_t(\phi_\tau)$ has measure t with respect to the current $\omega_{\mathrm{FS}} + dd^c\phi_\tau$. (If ϕ_τ is smooth and $\omega_{\mathrm{FS}} + dd^c\phi_\tau$ is strictly positive, this is a standard piece of potential theory and is discussed in [Ross and Witt Nyström 2015b, Proposition 1.1]; when ϕ_τ is merely C^2 and $t < 1$ then this is proved in [Ross and Witt Nyström 2017a, Theorem 1.2] and the case $t = 1$ follows from this by continuity as $\Omega_1(\phi_\tau) = \bigcup_{t<1}\Omega_t(\phi_\tau)$; finally when ϕ_τ is merely $C^{1,1}$ this is given in [Berman and Demailly 2012, Remark 1.19, Corollary 2.5].)

Therefore

$$\int_{A_\tau}(\omega_{\mathrm{FS}} + dd^c\phi_\tau) = \int_{\Omega_1(\phi_\tau)}(\omega_{\mathrm{FS}} + dd^c\phi_\tau) = 1.$$

But our normalisation is that $\int_{\mathbb{P}^1}(\omega_{\mathrm{FS}} + dd^c\phi_\tau) = \int_{\mathbb{P}^1}\omega_{\mathrm{FS}} = 1$ as well, and so the current $\omega_{\mathrm{FS}} + dd^c\phi_\tau$ gives zero measure to the complement of A_τ, which is precisely what we were aiming to prove.

Proof of Proposition 3.1. Fix $\sigma \in \mathbb{D}^\times$ and set $r := |\sigma|$. Our aim is to show

$$f_t(\mathbb{D}_r) = \Omega_t(\phi_\sigma).$$

Consider the S^1-action on $\mathbb{P}^1 \times \overline{\mathbb{D}}$ given by $e^{i\theta}\cdot(z, \tau) = (z, e^{i\theta}\tau)$, and observe that the boundary data used to define $\widetilde{\Phi}$ from (3) is S^1-invariant, which implies $\widetilde{\Phi}$ is S^1-invariant as well. Thus we may as well assume that σ is real, so $\phi_\sigma = \phi_r$.

For a function F on $\mathbb{P}^1 \times \overline{\mathbb{D}}$ and $D \subset \overline{\mathbb{D}}$ we write $F|_D$ for the restriction of F to $\mathbb{P}^1 \times D$. Then $\widetilde{\Phi}|_{\overline{\mathbb{D}}_r}$ is the solution to the Dirichlet problem for the HMAE with boundary data $(\phi_\tau)_{\tau\in\partial\mathbb{D}_r} = \phi_r$ and the requirement that $\widetilde{\Phi}|_{\overline{\mathbb{D}}_r}$ has Lelong number 1 at the point $(0, 0) \in \mathbb{C}_z \times \mathbb{D}_r \subset \mathbb{P}^1 \times \mathbb{D}_r$.

Letting $s := -\ln|\tau|^2$, consider the function on $\mathbb{P}^1 \times \overline{\mathbb{D}}^\times$ given by

$$H(z, \tau) := \frac{\partial}{\partial s}\widetilde{\Phi}(z, e^{-s/2})$$

(when $|\tau| = 1$ and thus $s = 0$, we take the right derivative). As $\widetilde{\Phi}$ is $C^{1,1}$ on $\mathbb{P}^1 \times \overline{\mathbb{D}}^\times$, the function H is well-defined and Lipschitz. Clearly this is compatible with restriction; i.e.,

$$H|_{\overline{\mathbb{D}}_r^\times}(z, \tau) = \frac{\partial}{\partial s}\widetilde{\Phi}|_{\overline{\mathbb{D}}_r}(z, e^{-s/2}).$$

Now, as discussed above, and proved in [Ross and Witt Nyström 2015b, Theorem 3.1], the graph $\{(f(\tau), \tau) : \tau \in \mathbb{D}\}$ of f is a harmonic disc for $\widetilde{\Phi}$. What is also proved is that H takes the constant value $t - 1$ along this disc so

$$H(f(\tau), \tau) = t - 1 \quad \text{for } \tau \in \mathbb{D}^\times.$$

Now H is also S^1-invariant and so this in particular implies

$$H(f(re^{i\theta}), r) = H(f(re^{i\theta}), re^{i\theta}) = t - 1 \quad \text{for all } \theta \in [0, 2\pi].$$

In other words, the function $H(\,\cdot\,, r)$ takes the value $t - 1$ on the boundary of $f(\mathbb{D}_r)$. On the other hand, we prove in [Ross and Witt Nyström 2015b, Proposition 2.9] that the function $H(\,\cdot\,, r)$ describes the set $\Omega_t(\phi_r)$, in that

$$H(z, r) + 1 = \sup\{s : z \notin \Omega_s(\phi_r)\}$$

(we remark that the proof of that proposition does not require any regularity or strict positivity assumptions on the potential ϕ_σ). Thus $\Omega_t(\phi_r)$ is the interior component of the curve $\theta \mapsto f(re^{i\theta})$ (that is, the component containing $z = 0$), which gives $\Omega_t(\phi_r) = f(\mathbb{D}_r)$ as desired. □

3B. ***Extensions and questions.*** Under the hypotheses of Theorem 2.1 we have shown that the current $\omega_{\mathrm{FS}} + dd^c\Phi(\,\cdot\,, \tau)$ fails to be strictly positive on any interior fibre (that is, for any τ with $0 < |\tau| < 1$). Furthermore we have no reason to expect our solution to be smooth everywhere. Thus the following two questions are natural:

Question 3.2. Does there exist a smooth family of potentials $(\phi_\tau)_{\tau\in\partial B}$ for which the solution to the complex HMAE (1) is everywhere smooth but not of maximal rank?

Question 3.3. Does there exist a smooth family of potentials $(\phi_\tau)_{\tau\in\partial B}$ for which the solution to the complex HMAE (1) such that $\omega + dd^c\Phi(\,\cdot\,, \tau)$ is a Kähler form for some τ with $0 < |\tau| < 1$ but not for others.

We are not currently able to answer these questions. However, we believe that the degenerate solutions we describe in this paper are actually regular in the interior of the complement of the degenerate set S (that is, they are smooth there and of maximal rank). In fact from our previous work in [Ross and Witt Nyström 2015b] we can understand the set on which our solution is regular in terms of the collection of sets $\Omega_t(\phi)$ that are simply connected. Now, our specific potential ϕ (Section 2F) was constructed to have curvature equal to ω_{FS} far away from the arc $\gamma = [-1, 1] \subset \mathbb{C}_w \subset \mathbb{P}^1$, from which one can see that $\Omega_t(\phi)$ is a disc for sufficiently small t. This gives an open set of $\mathbb{P}^1 \times \mathbb{D}$ for which the solution Φ is regular. Furthermore, by construction, $\Omega_1(\phi)$ is simply connected. We think it likely that $\Omega_t(\phi)$ is actually simply connected for all t, which would give rather precise information about the set on which our solution is regular, but it does not seem easy to prove that this is the case.

We furthermore believe that the fibrewise Laplacian of such a solution is uniformly bounded from below on the complement of S, and so has a discontinuity on the boundary ∂S where it jumps to zero. A somewhat bold conjecture would be that any solution to the HMAE is regular away from the set where it fails to have maximal rank, and is smooth away from the boundary of this set.

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