

ANALYSIS & PDE

Volume 9

No. 6

2016

RYAN HYND AND ERIK LINDGREN

**HÖLDER ESTIMATES AND LARGE TIME BEHAVIOR
FOR A NONLOCAL DOUBLY NONLINEAR EVOLUTION**

HÖLDER ESTIMATES AND LARGE TIME BEHAVIOR FOR A NONLOCAL DOUBLY NONLINEAR EVOLUTION

RYAN HYND AND ERIK LINDGREN

The nonlinear and nonlocal PDE

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v = 0,$$

where

$$(-\Delta_p)^s v(x, t) = 2 \text{ P.V. } \int_{\mathbb{R}^n} \frac{|v(x, t) - v(x + y, t)|^{p-2}(v(x, t) - v(x + y, t))}{|y|^{n+sp}} dy,$$

has the interesting feature that an associated Rayleigh quotient is nonincreasing in time along solutions. We prove the existence of a weak solution of the corresponding initial value problem which is also unique as a viscosity solution. Moreover, we provide Hölder estimates for viscosity solutions and relate the asymptotic behavior of solutions to the eigenvalue problem for the fractional p -Laplacian.

1. Introduction

We study the nonlinear and nonlocal PDE

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v = 0, \tag{1-1}$$

where $p \in (1, \infty)$, $s \in (0, 1)$ and $(-\Delta_p)^s$ is the fractional p -Laplacian

$$(-\Delta_p)^s u(x) := 2 \text{ P.V. } \int_{\mathbb{R}^n} \frac{|u(x) - u(x + y)|^{p-2}(u(x) - u(x + y))}{|y|^{n+sp}} dy. \tag{1-2}$$

Here and throughout P.V. denotes principal value. The main reason of our interest in solutions of (1-1) is the connection with ground states for $(-\Delta_p)^s$, i.e., extremals of the nonlocal Rayleigh quotient

$$\lambda_{s,p} = \inf_{u \in W_0^{s,p}(\Omega) \setminus \{0\}} \frac{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy}{\int_{\Omega} |u(x)|^p dx}. \tag{1-3}$$

Here and throughout $\Omega \subset \mathbb{R}^n$ is a bounded domain. Clearly, $1/\lambda_{s,p}$ is the optimal constant in the Poincaré inequality in the fractional Sobolev space $W_0^{s,p}(\Omega)$.

In recent years there has been a surge of interest around this nonlinear and nonlocal eigenvalue problem; see [Lindgren and Lindqvist 2014; Brasco and Franzina 2014; Brasco and Parini 2015; Brasco et al. 2016; Del Pezzo and Salort 2015; Franzina and Palatucci 2014; Iannizzotto and Squassina 2014]. In particular, it is known that ground states (or first eigenfunctions) are unique up to a multiplicative constant and have

MSC2010: 35J60, 47J35, 35J70, 35R09.

Keywords: doubly nonlinear evolution, Hölder estimates, eigenvalue problem, fractional p -Laplacian, nonlocal equation.

a definite sign (see Theorem 14 in [Lindgren and Lindqvist 2014] together with Corollary 3.14 in [Brasco and Parini 2015]). The corresponding local problem (formally $s = 1$), i.e., the eigenvalue problem for the p -Laplacian, has been extensively studied throughout the years. See, for instance, [Lieb 1983; Lindqvist 1990; 2008].

The first of our main results is a local Hölder estimate for *viscosity solutions* of (1-1). This is one of the first continuity estimates for parabolic equations involving the fractional p -Laplacian.

Theorem 1.1. *Let $p \geq 2$, $s \in (0, 1)$ and $v \in L^\infty(\mathbb{R}^n \times (-2, 0])$ be a viscosity solution of*

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v = 0 \quad \text{in } B_2 \times (-2, 0].$$

Then v is Hölder continuous in $B_1 \times (-1, 0]$ and in particular there exist α and C depending on p and s such that

$$\|v\|_{C^\alpha(B_1 \times (-1, 0])} \leq C \|v\|_{L^\infty(\mathbb{R}^n \times (-2, 0])}.$$

We also study the initial value problem

$$\begin{cases} |v_t|^{p-2}v_t + (-\Delta_p)^s v = 0 & \text{in } \Omega \times (0, \infty), \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega \times [0, \infty), \\ v = g & \text{in } \Omega \times \{0\}, \end{cases} \tag{1-4}$$

and show that (1-4) has a *weak solution* in the sense of a doubly nonlinear evolution and a unique *viscosity solution*. In addition, we relate the long time behavior of solutions to the eigenvalue problem for the fractional p -Laplacian. These results are presented in the two theorems below.

Theorem 1.2. *Let $p \in (1, \infty)$ and $s \in (0, 1)$. Assume $g \in W_0^{s,p}(\Omega)$ and define*

$$\mu_{s,p} := \lambda_{s,p}^{\frac{1}{p-1}}.$$

Then for any weak solution v of (1-4),

$$w(x) := \lim_{t \rightarrow \infty} e^{\mu_{s,p}t} v(x, t) \tag{1-5}$$

exists in $W^{s,p}(\mathbb{R}^n)$ and is a ground state for $(-\Delta_p)^s$, provided it is not identically zero. In this case, $v(\cdot, t) \neq 0$ for $t \geq 0$ and

$$\lambda_{s,p} = \lim_{t \rightarrow \infty} \frac{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|v(x,t) - v(y,t)|^p}{|x-y|^{n+sp}} dx dy}{\int_{\Omega} |v(x, t)|^p dx}.$$

Theorem 1.3. *Let $p \geq 2$, $s \in (0, 1)$, Ω be a $C^{1,1}$ domain and assume that $g \in W_0^{s,p}(\Omega) \cap C(\bar{\Omega})$ satisfies $|g| \leq \Psi$, where Ψ is a ground state for $(-\Delta_p)^s$. Then there is a unique viscosity solution of (1-4) that is also a weak solution. In addition, the convergence in (1-5) is uniform in $\bar{\Omega}$.*

In our previous work [Hynd and Lindgren 2016], we studied the large time behavior of the doubly nonlinear, local equation

$$|v_t|^{p-2}v_t = \Delta_p v. \tag{1-6}$$

One of the novelties of the present paper in comparison with the above-mentioned work is that we obtain *uniform* convergence to a ground state and a uniform Hölder estimate for the doubly nonlinear, nonlocal equation (1-1). No such results are known for (1-6). Related to this is also the work for more general systems in [Hynd 2016]. The method in these papers, as the method in the present paper, differs substantially from most of the other methods used in the literature to study asymptotic behavior of nonlinear and possibly degenerate flows, as in [Agueh et al. 2010; Aronson and Peletier 1981; Armstrong and Trokhimtchouk 2010; Kamin and Vázquez 1988; Kim and Lee 2013; Stan and Vázquez 2013]. Our methods are based on energy and compactness in Sobolev spaces, while most of the earlier work is based on comparison principles. This allows us, in contrast to most earlier work, to treat initial data without any assumption on the sign.

In the case of a linear equation, i.e., when $p = 2$, the large time behavior of solutions is especially well understood. Due to the theory of eigenfunctions in Hilbert spaces, one can then recover our result (and more) using the eigenfunction expansion. When $p \neq 2$, this expansion is not available.

The literature on equations of the type (1-1) is very limited. Equations of type (1-6) appear in [Kilpeläinen and Lindqvist 1996] and in the theory of doubly nonlinear flows. In the case of linear nonlocal equations, i.e., when $p = 2$, the literature on regularity is vast. We mention only a fraction; see [Silvestre 2010; 2012; Caffarelli and Vasseur 2010; Lara and Dávila 2014]. Neither of these results apply to our setting. However, our proof of the Hölder regularity is very much inspired by the work of Luis Silvestre [2010; 2012]. We also seize the opportunity to mention the recent papers [Puhst 2015; Mazón et al. 2016; Vázquez 2016; Warma 2016] where the corresponding heat flow is studied, i.e., the equation

$$v_t + (-\Delta_p)^s v = 0.$$

The stationary equation, i.e.,

$$(-\Delta_p)^s v = 0,$$

has in recent years attracted a lot of attention; see [Ishii and Nakamura 2010; Brasco and Lindgren 2017; Chasseigne and Jakobsen 2015; Di Castro et al. 2014; 2016; Chambolle et al. 2012; Korvenpää et al. 2016; Iannizzotto et al. 2016; Kuusi et al. 2015a; 2015b; Gal and Warma 2016; Lindgren 2016]. In [Bjorland et al. 2012], a different nonlocal version of the p -Laplacian is studied.

The plan of the paper is as follows. In Section 2, we introduce the fractional Sobolev spaces $W^{s,p}$, the fractional p -Laplacian $(-\Delta_p)^s$ and additional notation used in this paper. In Section 3, we define weak solutions and derive several of their important properties. The section ends with a key compactness result and some brief explanations on how to construct weak solutions. This is followed by Section 4, where we introduce viscosity solutions and prove that the weak solution constructed in Section 3 is also the unique viscosity solution. In Section 5, we verify Hölder estimates for viscosity solutions. Finally, in Section 6, we prove Theorem 1.2 and Theorem 1.3, which involves the large time behavior of weak solutions.

2. Notation and prerequisites

The fractional Rayleigh quotient (1-3) naturally relates to the so-called fractional Sobolev spaces $W^{s,p}(\mathbb{R}^n)$. If $1 < p < \infty$ and $s \in (0, 1)$, the norm is given by

$$\|u\|_{W^{s,p}(\mathbb{R}^n)}^p = [u]_{W^{s,p}(\mathbb{R}^n)}^p + \|u\|_{L^p(\mathbb{R}^n)}^p,$$

where the Gagliardo seminorm is

$$[u]_{W^{s,p}(\mathbb{R}^n)}^p = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(y) - u(x)|^p}{|y - x|^{sp+n}} dx dy = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(y) - u(x)|^p d\mu(x, y).$$

Here and throughout, we will use the notation

$$d\mu(x, y) := |x - y|^{-n-sp} dx dy.$$

The space $W_0^{s,p}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{W^{s,p}(\mathbb{R}^n)}$. Many properties that are known for the more common Sobolev spaces $W^{1,p}$, also hold for $W^{s,p}$ and can be found in [Di Nezza et al. 2012]. In particular, we have the compact embedding of $W_0^{s,p}(\Omega)$ in $L^q(\Omega)$ for $q \in [1, p]$. This result can be found in Theorem 2.7 in [Brasco et al. 2014] (see also Theorem 7.1 on page 33 in [Di Nezza et al. 2012]).

The operator $(-\Delta_p)^s$ arises as the first variation of the functional

$$[u]_{W^{s,p}(\mathbb{R}^n)}^p.$$

More specifically, minimizers satisfy

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^{p-2} (u(x) - u(y)) (\phi(x) - \phi(y)) d\mu(x, y) = 0$$

for each $\phi \in W_0^{s,p}(\Omega)$. If the solution is regular enough, one can split this into two equal terms, make a change of variables and write the equation in the sense of the principal value, as in (1-2). Note that the notation $(-\Delta_p)^s$ is slightly abusive; this operator is not the s -th power of $-\Delta_p$ unless $p = 2$. See Section 3 in [Di Nezza et al. 2012].

Ground states of $(-\Delta_p)^s$ are minimizers of the Rayleigh quotient

$$\lambda_{s,p} = \inf_{u \in W_0^{s,p}(\Omega)} \frac{\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |u(x) - u(y)|^p d\mu(x, y)}{\int_{\Omega} |u(x)|^p dx},$$

and they are signed solutions of

$$\begin{cases} (-\Delta_p)^s u = \lambda_{s,p} |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

The notation

$$\mathcal{J}_p(t) = |t|^{p-2} t$$

will also come in handy. With this notation, equation (1-1) can be written as

$$\mathcal{J}_p(v_t) + (-\Delta_p)^s v = 0$$

and the operator $(-\Delta_p)^s$ can be written as

$$(-\Delta_p)^s u(x) = 2 \text{ P.V. } \int_{\mathbb{R}^n} \frac{\mathcal{J}_p(u(x) - u(x+y))}{|y|^{n+sp}} dy.$$

Due to the scaling of the equation, we introduce the following notation for parabolic cylinders

$$Q_r(x_0, t_0) = B_r(x_0) \times (t_0 - r^{\frac{sp}{p-1}}, t_0 + r^{\frac{sp}{p-1}}), \quad Q_r^-(x_0, t_0) = B_r(x_0) \times (t_0 - r^{\frac{sp}{p-1}}, t_0),$$

where $B_r(x_0)$ is the ball of radius r centered at x_0 . When $x_0 = 0$ and $t_0 = 0$, we will simply write B_r , Q_r and Q_r^- .

3. Weak solutions

In this section, we present our theory of weak solutions of (1-4). The main results are that the Rayleigh quotient is monotone along the flow (Proposition 3.6) and that “bounded” sequences of weak solutions are compact (Theorem 3.8). The interested reader could also consult [Hynd and Lindgren 2016], where a similar theory is built for equation (1-6).

Definition 3.1. Let $g \in W_0^{s,p}(\Omega)$. We say that v is a *weak solution* of (1-4) if

$$v \in L^\infty([0, \infty); W_0^{s,p}(\Omega)), \quad v_t \in L^p(\Omega \times [0, \infty)), \tag{3-1}$$

and

$$\int_\Omega |v_t(x, t)|^{p-2} v_t(x, t) \phi(x) dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{J}_p(v(x, t) - v(y, t)) (\phi(x) - \phi(y)) d\mu(x, y) = 0 \tag{3-2}$$

for each $\phi \in W_0^{s,p}(\Omega)$ and for a.e. $t > 0$, and

$$v(x, 0) = g(x). \tag{3-3}$$

Remark 3.2. We note that if v satisfies (3-1), the L^p norm of v is absolutely continuous in time (one can for instance adapt the proof of Theorem 3 on page 287 of [Evans 2010]), so that it makes sense to assign values in $L^p(\Omega)$ at $t = 0$, as in (3-3).

In the rest of this section, we derive various identities and estimates for weak solutions.

Lemma 3.3. Assume v is a weak solution of (1-4). Then

$$[v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p$$

is absolutely continuous in t and

$$\frac{d}{dt} \frac{1}{p} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p = - \int_\Omega |v_t(x, t)|^p dx \tag{3-4}$$

holds for almost every $t > 0$.

Proof. Define

$$\Phi(w) := \begin{cases} \frac{1}{p} [w]_{W^{s,p}(\mathbb{R}^n)}^p, & w \in W_0^{s,p}(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

for each $w \in L^p(\Omega)$. Then Φ is convex, proper, and lower-semicontinuous. In addition, (3-2) implies

$$\partial\Phi(v(\cdot, t)) = \{-|v_t(\cdot, t)|^{p-2} v_t(\cdot, t)\}$$

for almost every $t > 0$. Since $t \mapsto v(\cdot, t)$ is absolutely continuous in $L^p(\Omega)$ (see Remark 3.2) and since $|\partial\Phi(v)||v_t| \in L^1(\Omega \times [0, T])$ for any $T > 0$, Remark 1.4.6 in [Ambrosio et al. 2008] implies that $t \mapsto \Phi(v(\cdot, t))$ is absolutely continuous and that identity (3-4) holds for a.e. $t > 0$. \square

Lemma 3.4. *Assume v is a weak solution of (1-4). Then*

$$\frac{d}{dt} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \leq -p\mu_{s,p} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \tag{3-5}$$

for a.e. $t > 0$, and

$$[v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \leq e^{-(p\mu_{s,p})t} [g]_{W^{s,p}(\mathbb{R}^n)} \tag{3-6}$$

for each $t > 0$.

Proof. By Lemma 3.3, $v(\cdot, t)$ is an admissible test function in (3-2), which yields

$$\begin{aligned} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{J}_p(v(x, t) - v(y, t))(v(x, t) - v(y, t)) \, d\mu(x, y) \\ &= - \int_{\Omega} |v_t(x, t)|^{p-2} v_t(x, t) \cdot v(x, t) \, dx \\ &\leq \left(\int_{\Omega} |v_t(x, t)|^p \, dx \right)^{1-\frac{1}{p}} \left(\int_{\Omega} |v(x, t)|^p \, dx \right)^{\frac{1}{p}} \\ &\leq \lambda_{s,p}^{-\frac{1}{p}} \left(\int_{\Omega} |v_t(x, t)|^p \, dx \right)^{1-\frac{1}{p}} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}. \end{aligned} \tag{3-7}$$

Hence,

$$[v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \leq \frac{1}{\mu_{s,p}} \int_{\Omega} |v_t(x, t)|^p \, dx. \tag{3-8}$$

Identity (3-4) together with (3-8) implies (3-5). From Grönwall’s inequality, we can now deduce inequality (3-6). \square

Corollary 3.5. *Let v be a weak solution of (1-4). Then the function*

$$e^{(\mu_{s,p})t} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p$$

is nonincreasing in t and

$$\frac{1}{p} \frac{d}{dt} (e^{(\mu_{s,p})t} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p) = e^{(\mu_{s,p})t} \left(\mu_{s,p} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p - \int_{\Omega} |v_t(x, t)|^p \, dx \right) \leq 0 \tag{3-9}$$

for a.e. $t \geq 0$.

Proof. The monotonicity is a consequence of (3-5). The identity (3-9) follows from (3-4). \square

Proposition 3.6. *Assume that v is a weak solution of (1-4) such that $v(\cdot, t) \neq 0 \in L^p(\Omega)$ for each $t \geq 0$. Then the Rayleigh quotient*

$$\frac{[v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v(x, t)|^p \, dx}$$

is nonincreasing in t .

Proof. By (3-1),

$$\frac{d}{dt} \int_{\Omega} \frac{1}{p} |v(x, t)|^p dx = \int_{\Omega} |v(x, t)|^{p-2} v(x, t) v_t(x, t) dx \tag{3-10}$$

for a.e. $t > 0$. Suppressing the (x, t) -dependence, we compute, using (3-4) in Lemma 3.3 and (3-10), to find

$$\begin{aligned} \frac{d}{dt} \frac{[v]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v|^p dx} &= -p \frac{\int_{\Omega} |v_t|^p dx}{\int_{\Omega} |v|^p dx} - p \frac{[v]_{W^{s,p}(\mathbb{R}^n)}^p}{(\int_{\Omega} |v|^p dx)^2} \int_{\Omega} |v|^{p-2} v v_t dx \\ &= \frac{p}{(\int_{\Omega} |v|^p dx)^2} \left([v]_{W^{s,p}(\mathbb{R}^n)}^p \int_{\Omega} |v|^{p-2} v(-v_t) dx - \int_{\Omega} |v|^p dx \int_{\Omega} |v_t|^p dx \right) \end{aligned} \tag{3-11}$$

for a.e. $t > 0$. By Hölder’s inequality,

$$\int_{\Omega} |v|^{p-2} v(-v_t) dx \leq \left(\int_{\Omega} |v|^p dx \right)^{1-\frac{1}{p}} \left(\int_{\Omega} |v_t|^p dx \right)^{\frac{1}{p}},$$

which together with (3-7) gives

$$[v]_{W^{s,p}(\mathbb{R}^n)}^p \int_{\Omega} |v|^{p-2} v(-v_t) dx \leq \int_{\Omega} |v|^p dx \int_{\Omega} |v_t|^p dx.$$

Inserted into (3-11), this yields

$$\frac{d}{dt} \frac{[v]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v|^p dx} \leq 0. \tag{3-12}$$

As a corollary, we obtain that any weak solution with a ground state as initial data can be written explicitly. Since the proof is exactly the same as the proof of the corresponding result in [Hynd and Lindgren 2016], Corollary 2.5, we have chosen to omit it.

Corollary 3.7. *Suppose that v is a weak solution of (1-4) and that g is a ground state of $(-\Delta_p)^s$. Then*

$$v(x, t) = e^{-\mu_{s,p} t} g(x).$$

The following compactness result is the key both to the long time behavior and to the construction of weak solutions, as we will see. The proof is based on the compact embedding of $W_0^{s,p}(\Omega)$ into $L^p(\Omega)$ and it is fairly similar to the proof of Theorem 2.6 in [Hynd and Lindgren 2016].

Theorem 3.8. *Assume $\{g^k\}_{k \in \mathbb{N}} \in W_0^{s,p}(\Omega)$ is uniformly bounded in $W_0^{s,p}(\Omega)$ and that v^k is a weak solution of (1-4) with $v^k(x, 0) = g^k(x)$. Then there is a subsequence $\{v^{k_j}\}_{j \in \mathbb{N}} \subset \{v^k\}_{k \in \mathbb{N}}$ and v satisfying (3-1) such that*

$$v^{k_j} \rightarrow v \quad \text{in} \quad \begin{cases} C([0, T]; L^p(\Omega)) & \text{for all } T > 0, \\ L^r_{\text{loc}}([0, \infty); W^{s,p}(\mathbb{R}^n)) & \text{for all } 1 \leq r < \infty \end{cases} \tag{3-13}$$

and

$$v_t^{k_j} \rightarrow v_t \quad \text{in} \quad L^p_{\text{loc}}([0, \infty); L^p(\Omega)) \tag{3-14}$$

as $j \rightarrow \infty$. Moreover, v is a weak solution of (1-4), where g is a weak limit of $\{g^{k_j}\}_{j \in \mathbb{N}}$ in $W_0^{s,p}(\Omega)$.

Proof. As in (3-4),

$$\frac{d}{dt} \frac{1}{p} [v^k(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p = - \int_{\Omega} |v_t^k(x, t)|^p dx \tag{3-14}$$

for almost every $t > 0$. After integration, we obtain

$$p \int_0^\infty \int_{\Omega} |v_t^k(x, t)|^p dx dt + \sup_{t \geq 0} [v_k(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)} \leq 2[g^k]_{W^{s,p}(\mathbb{R}^n)}^p. \tag{3-15}$$

By assumption, the right-hand side above is uniformly bounded. It follows that the sequence $\{v^k\}_{k \in \mathbb{N}} \in C_{\text{loc}}([0, \infty), L^p(\Omega))$ is equicontinuous, and $\{v^k(\cdot, t)\}_{k \in \mathbb{N}} \in W_0^{s,p}(\Omega)$ is uniformly bounded for each $t \geq 0$. By Theorem 1 in [Simon 1987], we can conclude that there is a subsequence $\{v^{k_j}\}_{j \in \mathbb{N}} \subset \{v^k\}_{k \in \mathbb{N}}$ converging in $C_{\text{loc}}([0, \infty), L^p(\Omega))$ to some v satisfying (3-1). Passing to a further subsequence, we may also assume that $v^{k_j} \rightharpoonup v$ in $L_{\text{loc}}^p([0, \infty); W^{s,p}(\mathbb{R}^n))$.

Since v_t^k is bounded in $L^p(\Omega \times [0, \infty))$, we may also assume

$$\begin{cases} v_t^{k_j} \rightharpoonup v_t & \text{in } L^p(\Omega \times [0, \infty)), \\ \mathcal{J}_p(v_t^{k_j}) \rightharpoonup \xi & \text{in } L^q(\Omega \times [0, \infty)), \end{cases}$$

where $1/p + 1/q = 1$. We will prove below that

$$\xi = \mathcal{J}_p(v_t). \tag{3-16}$$

Let us assume for the moment that (3-16) holds. Note that since the function $|z|^p$ is convex,

$$\frac{1}{p} [w]_{W^{s,p}(\mathbb{R}^n)}^p \geq \frac{1}{p} [v^{k_j}(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p - \int_{\Omega} \mathcal{J}_p(v_t^{k_j}(x, t))(w(x) - v^{k_j}(x, t)) dx$$

for any $w \in W_0^{s,p}(\Omega)$. Integrating over the interval $[t_0, t_1]$ and passing to the limit, we obtain

$$\int_{t_0}^{t_1} \frac{1}{p} [w]_{W^{s,p}(\mathbb{R}^n)}^p dt \geq \int_{t_0}^{t_1} \left(\frac{1}{p} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p - \int_{\Omega} \xi(x, t)(w(x) - v(x, t)) dx \right) dt.$$

Here we made use of Fatou’s lemma, the weak convergence of $\mathcal{J}_p(v_t^{k_j})$ and the strong convergence of v^{k_j} .

Therefore,

$$\frac{1}{p} [w]_{W^{s,p}(\mathbb{R}^n)}^p \geq \frac{1}{p} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p - \int_{\Omega} \xi(x, t)(w(x) - v(x, t)) dx$$

for a.e. $t \geq 0$. In particular, for each $\phi \in W_0^{s,p}(\Omega)$,

$$\int_{\Omega} \xi(x, t)\phi(x) dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{J}_p(v(x, t) - v(y, t))(\phi(x) - \phi(y)) d\mu(x, y) = 0 \tag{3-17}$$

for a.e. $t \geq 0$. Thus, once we verify (3-16), v is then a weak solution of (1-4).

For each interval $[t_0, t_1]$,

$$\begin{aligned} \lim_{j \rightarrow \infty} \int_{t_0}^{t_1} [v^{k_j}(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p dt &= \lim_{j \rightarrow \infty} \int_{t_0}^{t_1} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{J}_p(v^{k_j}(x, t) - v^{k_j}(y, t))(v^{k_j}(x, t) - v^{k_j}(y, t)) d\mu(x, y) \\ &= - \lim_{j \rightarrow \infty} \int_{t_0}^{t_1} \int_{\Omega} \mathcal{J}_p(v_t^{k_j}(x, t))v^{k_j}(x, t) dx dt \\ &= - \int_{t_0}^{t_1} \int_{\Omega} \xi(x, t)v(x, t) dx dt \\ &= \int_{t_0}^{t_1} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p dt, \end{aligned}$$

where the last equality is a consequence of (3-17). Since weak convergence together with convergence of the norm implies strong convergence, we have

$$v^{k_j} \rightarrow v \quad \text{in } L^p_{\text{loc}}([0, \infty); W^{s,p}(\mathbb{R}^n)).$$

It is now routine to combine the interpolation of L^p spaces with the uniform bound (3-15) to obtain the stronger convergence $v^{k_j} \rightarrow v$ in $L^r_{\text{loc}}([0, \infty); W^{s,p}(\mathbb{R}^n))$ for each $1 \leq r < \infty$. Further, upon extracting yet another subsequence, we can assume that

$$[v^{k_j}(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \rightarrow [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \tag{3-18}$$

as $j \rightarrow \infty$ for a.e. $t \geq 0$.

We will now verify (3-16). As in the proof of Lemma 3.3, (3-17) implies

$$\frac{d}{dt} \frac{1}{p} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p = - \int_{\Omega} \xi(x, t)v_t(x, t) dx$$

for a.e. $t \geq 0$. Therefore, for each $t_1 > t_0$,

$$\int_{t_0}^{t_1} \int_{\Omega} \xi(x, s)v_t(x, s) dx ds + \frac{1}{p} [v(\cdot, t_1)]_{W^{s,p}(\mathbb{R}^n)}^p = \frac{1}{p} [v(\cdot, t_0)]_{W^{s,p}(\mathbb{R}^n)}^p. \tag{3-19}$$

In addition, integrating (3-14) yields

$$\begin{aligned} \int_{t_0}^{t_1} \int_{\Omega} \frac{1}{p} |v_t^{k_j}(x, s)|^p + \frac{1}{q} |\mathcal{J}_p(v_t^{k_j}(x, s))|^q dx ds + \frac{1}{p} [v^{k_j}(\cdot, t_1)]_{W^{s,p}(\mathbb{R}^n)}^p \\ = \frac{1}{p} [v^{k_j}(\cdot, t_0)]_{W^{s,p}(\mathbb{R}^n)}^p. \end{aligned} \tag{3-20}$$

Let now t_0 and t_1 be times for which (3-18) holds and pass to the limit to obtain

$$\int_{t_0}^{t_1} \int_{\Omega} \frac{1}{p} |v_t(x, s)|^p + \frac{1}{q} |\xi(x, s)|^q dx ds + \frac{1}{p} [v(\cdot, t_1)]_{W^{s,p}(\mathbb{R}^n)}^p \leq \frac{1}{p} [v(\cdot, t_0)]_{W^{s,p}(\mathbb{R}^n)}^p$$

by weak convergence. Together with (3-19) this implies

$$\int_{t_0}^{t_1} \int_{\Omega} \left(\frac{1}{p} |v_t(x, s)|^p + \frac{1}{q} |\xi(x, s)|^q - \xi(x, s)v_t(x, s) \right) dx ds \leq 0.$$

Identity (3-16) now follows from the case of equality in Young’s inequality.

Substituting $\xi = \mathcal{J}_p(v_t)$ into (3-19) and passing to the limit as $j \rightarrow \infty$ in (3-20) also gives

$$\lim_{j \rightarrow \infty} \int_{t_0}^{t_1} \int_{\Omega} |v_t^{kj}(x, s)|^p dx ds = \int_{t_0}^{t_1} \int_{\Omega} |v_t(x, s)|^p dx ds.$$

Again, since weak convergence together with convergence of the norm implies strong convergence, we obtain (3-13). □

Let us now discuss how the ideas above can be used to construct weak solutions. As in [Hynd and Lindgren 2016], we aim to build weak solutions (1-4) by using the implicit time scheme for $\tau > 0$: $v^0 := g$,

$$\begin{cases} \mathcal{J}_p\left(\frac{v^k - v^{k-1}}{\tau}\right) + (-\Delta_p)^s v^k = 0 & \text{for } x \in \Omega, \\ v^k = 0 & \text{for } x \in \mathbb{R}^n \setminus \Omega, \end{cases} \quad \text{where } k = 1, 2, \dots, N. \tag{3-21}$$

The direct methods in the calculus of variations can be used to show that this scheme has a unique weak solution sequence $\{v^1, \dots, v^N\} \subset W_0^{s,p}(\Omega)$ for each $\tau > 0$ and N , in the sense that

$$\int_{\Omega} \mathcal{J}_p\left(\frac{v^k(x) - v^{k-1}(x)}{\tau}\right) \phi(x) dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{J}_p(v^k(x) - v^k(y))(\phi(x) - \phi(y)) d\mu(x, y) = 0$$

for any $\phi \in W_0^{s,p}(\Omega)$. Our candidate for a solution $v(x, t)$ of (1-4) is the limit of $v^N(x)$, when N tends to infinity with $\tau = t/N$.

Choosing $\phi = v^k - v^{k-1}$ as test function, we obtain

$$\int_{\Omega} \frac{|v^k - v^{k-1}|^p}{\tau^{p-1}} dx + \frac{1}{p} [v^k]_{W^{s,p}(\mathbb{R}^n)}^p \leq \frac{1}{p} [v^{k-1}]_{W^{s,p}(\mathbb{R}^n)}^p, \quad \text{where } k = 1, \dots, N.$$

Summing over $k = 1, \dots, j \leq N$ yields

$$\sum_{k=1}^j \int_{\Omega} \frac{|v^k - v^{k-1}|^p}{\tau^{p-1}} dx + \frac{1}{p} [v^j]_{W^{s,p}(\mathbb{R}^n)}^p \leq \frac{1}{p} [g]_{W^{s,p}(\mathbb{R}^n)}^p, \tag{3-22}$$

which is a discrete analogue of the energy identity (3-4).

Let $\tau = T/N$ and $\tau_k = k\tau$, and define the “linear interpolation” of the solution sequence as

$$w_N(x, t) := v^{k-1}(x) + \left(\frac{t - \tau_{k-1}}{\tau}\right)(v^k(x) - v^{k-1}(x)), \quad \text{where } \tau_{k-1} \leq t \leq \tau_k, \quad k = 1, \dots, N. \tag{3-23}$$

From (3-22) we conclude

$$p \int_0^T \int_{\Omega} |\partial_t w_N(x, t)|^p dx dt + \sup_{0 \leq t \leq T} [w_N(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \leq 2[g]_{W^{s,p}(\mathbb{R}^n)}^p. \tag{3-24}$$

Arguing as in the proof of [Theorem 3.8](#), we can obtain a subsequence w_{N_j} and a weak solution w of (1-1) on $\Omega \times (0, T)$ such that

$$w_{N_j} \rightarrow w \quad \text{in} \quad \begin{cases} C([0, T]; L^p(\Omega)), \\ L^p([0, T]; W^{s,p}(\mathbb{R}^n)) \end{cases}$$

and

$$\begin{cases} \partial_t w_{N_j} \rightarrow w_t & \text{in } L^p(\Omega \times [0, T]), \\ \mathcal{J}_p(\partial_t w_{N_j}) \rightharpoonup \mathcal{J}_p(w_t) & \text{in } L^q(\Omega \times [0, T]). \end{cases}$$

It remains to construct a global-in-time solution. This can be accomplished as follows: Let $k \in \mathbb{N}$ and let w^k be the weak solution of (1-1) above for $T = k$. Define

$$z^k(\cdot, t) = \begin{cases} w^k(\cdot, t), & t \in [0, k], \\ w^k(\cdot, k), & t \in [k, \infty). \end{cases}$$

One readily verifies that z^k satisfies (3-1). In addition, the proof of [Theorem 3.8](#) can easily be adapted to give that z^k has a subsequence converging as in (3-12) and (3-13) to a global weak solution of (1-4). We omit the details.

Remark 3.9. At this point, we seize the opportunity to mention that the “step function” approximation

$$v_N(\cdot, t) := \begin{cases} g, & t = 0, \\ v^k, & t \in (\tau_{k-1}, \tau_k], \quad k = 1, \dots, N, \end{cases} \tag{3-25}$$

converges in $C([0, T]; L^p(\Omega))$ to the same weak solution v as the linear interpolating sequence (3-23). Indeed, by (3-22),

$$\begin{aligned} \int_{\Omega} |w_N(x, t) - v_N(x, t)|^p dx &\leq \max_{1 \leq k \leq N} \int_{\Omega} |v^k(x) - v^{k-1}(x)|^p dx \\ &\leq \frac{1}{p} \tau^{p-1} [g]_{W^{s,p}(\mathbb{R}^n)}^p \\ &= \frac{1}{p} \left(\frac{T}{N}\right)^{p-1} [g]_{W^{s,p}(\mathbb{R}^n)}^p. \end{aligned}$$

This fact will be used in [Section 4](#), where we verify that the viscosity solution we construct is also a weak solution.

4. Viscosity solutions

Throughout this section we assume that $\partial\Omega$ is $C^{1,1}$, $p \geq 2$, $g \in W_0^{s,p}(\Omega) \cap C(\bar{\Omega})$ and that there is a ground state Ψ such that

$$|g| \leq \Psi.$$

Our main result in this section is:

Proposition 4.1. *There is a unique viscosity solution v of the initial value problem (1-4) which is also a weak solution.*

It is not known whether or not uniqueness holds for weak solutions of (1-4), even in the local case. However, quite standard methods for viscosity solutions apply to (1-4). The key here is that the term $|v_t|^{p-2}v_t$ is strictly monotone with respect to v_t . In what follows, we will prove that the discrete scheme (3-21) converges both to the unique viscosity solution and to a weak solution.

We first define *viscosity solutions* of the relevant equations.

Definition 4.2. Let Ω be an open set in \mathbb{R}^n and $f(x, u)$ a continuous function. A function $u \in L^\infty(\mathbb{R}^n)$ which is upper semicontinuous in Ω is a *subsolution* of

$$(-\Delta_p)^s u \leq f(x, u) \quad \text{in } \Omega$$

if the following holds: whenever $x_0 \in \Omega$ and $\phi \in C^2(B_r(x_0))$ for some $r > 0$ are such that

$$\phi(x_0) = u(x_0), \quad \phi(x) \geq u(x) \quad \text{for } x \in B_r(x_0) \subset \Omega,$$

then we have

$$(-\Delta_p)^s \phi^r(x_0) \leq f(x_0, \phi(x_0)),$$

where

$$\phi^r = \begin{cases} \phi & \text{in } B_r(x_0), \\ u & \text{in } \mathbb{R}^n \setminus B_r(x_0). \end{cases}$$

A *supersolution* is defined similarly and a *solution* is a function which is both a sub- and a supersolution.

Remark 4.3. For a bounded function f which is C^2 in a neighborhood of x_0 , we know $(-\Delta_p)^s f(x_0)$ is well defined. Indeed, we may split the operator into one integral over $B_1(x_0)$ and another over $\mathbb{R}^n \setminus B_1(x_0)$. The latter is well-defined since f is bounded. For the former, we may write, for $\varepsilon > 0$,

$$\begin{aligned} 2P.V. \int_{B_1(x_0)} \mathcal{J}_p(f(x_0) - f(y))|x_0 - y|^{-n-sp} dy \\ = 2 \lim_{\varepsilon \rightarrow 0} \int_{B_1 \setminus B_\varepsilon} \mathcal{J}_p(f(x_0) - f(x_0 + y))|y|^{-n-sp} dy \\ = \lim_{\varepsilon \rightarrow 0} \int_{B_1 \setminus B_\varepsilon} (\mathcal{J}_p(f(x_0) - f(x_0 + y)) + \mathcal{J}_p(f(x_0) - f(x_0 - y)))|y|^{-n-sp} dy. \end{aligned}$$

Since f is a C^2 function and $p \geq 2$, we have the estimate

$$|f(x) - f(x + y)|^{p-2}(f(x) - f(x + y)) + |f(x) - f(x - y)|^{p-2}(f(x) - f(x - y)) \leq C|y|^p.$$

This is due to the elementary inequality

$$||a + b|^{p-2}(a + b) - |a|^{p-2}a| \leq C|b|(|a| + |b|)^{p-2}.$$

Therefore, the integral

$$\int_{B_1} (\mathcal{J}_p(f(x_0) - f(x_0 + y)) + \mathcal{J}_p(f(x_0) - f(x_0 - y)))|y|^{-n-sp} dy$$

is absolutely convergent, so that the principal value exists.

We define viscosity solutions of the evolutionary equation (1-1). We introduce the class of test functions

$$C_{x,t}^{2,1}(D \times I), \quad \text{where } D \times I \subset \mathbb{R}^n \times \mathbb{R},$$

consisting of functions that are C^2 in the spatial variables and C^1 in t , in the set $D \times I$.

Definition 4.4. Let $\Omega \in \mathbb{R}^n$ be an open set, $I \in \mathbb{R}$ be an open interval. A function $v \in L^\infty(\mathbb{R}^n \times I)$ which is upper semicontinuous in $\Omega \times I$ is a *subsolution* of

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v \leq C \quad \text{in } \Omega \times I$$

if the following holds: whenever $(x_0, t_0) \in \Omega \times I$ and $\phi \in C_{x,t}^{2,1}(B_r(x_0) \times (t_0 - r, t_0 + r))$ for some $r > 0$ are such that

$$\phi(x_0, t_0) = v(x_0, t_0), \quad \phi(x, t) \geq v(x, t) \quad \text{for } (x, t) \in B_r(x_0) \times (t_0 - r, t_0 + r)$$

then

$$|\phi_t(x_0, t_0)|^{p-2}\phi_t(x_0, t_0) + (-\Delta_p)^s \phi^r(x_0, t_0) \leq C,$$

where

$$\phi^r(x, t) = \begin{cases} \phi & \text{in } B_r(x_0) \times (t_0 - r, t_0 + r), \\ v & \text{in } \mathbb{R}^n \setminus B_r(x_0) \times (t_0 - r, t_0 + r). \end{cases}$$

A *supersolution* is defined similarly and a *solution* is a function which is both a sub- and a supersolution.

Remark 4.5. In both of the definitions above, it is obvious that we can replace the condition that ϕ touches v from above at a point with the condition that $v - \phi$ has a maximum at a point. In addition, as is standard when dealing with viscosity solutions, it is enough to ask that ϕ touches v strictly at a point or equivalently that $v - \phi$ has a strict maximum at a point.

Now we are ready to treat the implicit scheme (3-21). We first construct viscosity solutions v^1, \dots, v^N .

Lemma 4.6. For each N and τ , the implicit scheme (3-21) generates viscosity solutions $v^k \in C(\bar{\Omega})$ for $k = 1, \dots, N$. Moreover,

$$\max_{1 \leq k \leq N} \sup_{\mathbb{R}^n} |v^k| \leq \sup_{\mathbb{R}^n} |g|.$$

Proof. Consider the implicit scheme (3-21) for $k = 1$:

$$\mathcal{J}_p\left(\frac{v^1 - g}{\tau}\right) + (-\Delta_p)^s v^1 = 0, \quad \text{where } x \in \Omega.$$

This means that

$$\int_{\Omega} \mathcal{J}_p\left(\frac{v^1 - g}{\tau}\right) \phi \, dx + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mathcal{J}_p(v^1(x) - v^1(y))(\phi(x) - \phi(y)) \, d\mu(x, y) = 0$$

for any $\phi \in W_0^{s,p}(\Omega)$. The existence of such a weak solution follows from the direct methods of calculus of variations. Since \mathcal{J}_p is strictly increasing, it is standard to prove a comparison principle for weak sub- and supersolutions; see, for instance, Lemma 9 in [Lindgren and Lindqvist 2014] for a proof. Clearly, the constant function $\sup_{\mathbb{R}^n} |g|$ is a supersolution; hence $v^1 \leq \sup_{\mathbb{R}^n} |g|$. Similarly, $v^1 \geq -\sup_{\mathbb{R}^n} |g|$, and thus $|v^1| \leq \sup_{\mathbb{R}^n} |g|$. By induction, $|v^k| \leq \sup_{\mathbb{R}^n} |g|$ for $k = 2, \dots, N$. As the left-hand side of the PDE

(3-21) is bounded it follows by Theorem 1.1 in [Iannizzotto et al. 2016], that v^k is continuous in $\bar{\Omega}$ for $k = 1, \dots, N$.

That each v^k is a viscosity solution can be verified by following the proof of Proposition 11 in [Lindgren and Lindqvist 2014] line by line. We omit the details. □

The natural candidate for a viscosity solution of (1-4) is $\lim_{N \rightarrow \infty} v_N$, where v_N is defined in (3-25). Before proving this, we present some technical lemmas.

Lemma 4.7. *Let $N \in \mathbb{N}$. Further assume $\{\psi^0, \psi^1, \dots, \psi^N\} \subset C^2(\Omega)$ and $(x_0, k_0) \in \Omega \times \{1, \dots, N\}$ are such that*

$$v^k(x) - \psi^k(x) \leq v^{k_0}(x_0) - \psi^{k_0}(x_0) \tag{4-1}$$

for x in $B_r(x_0)$ and $k \in \{k_0 - 1, k_0\}$. Then

$$\mathcal{J}_p \left(\frac{\psi^{k_0}(x_0) - \psi^{k_0-1}(x_0)}{\tau} \right) + (-\Delta)_p^s(\psi^{k_0})^r(x_0) \leq 0.$$

Proof. Evaluating the left-hand side of (4-1) at $k = k_0$ gives

$$\mathcal{J}_p \left(\frac{v^{k_0}(x_0) - v^{k_0-1}(x_0)}{\tau} \right) + (-\Delta)_p^s(\psi^{k_0})^r(x_0) \leq 0,$$

as v^k is a viscosity solution of (3-21). Evaluating the left-hand side of (4-1) at $x = x_0$ and $k = k_0 - 1$ gives $\psi^{k_0}(x_0) - \psi^{k_0-1}(x_0) \leq v^{k_0}(x_0) - v^{k_0-1}(x_0)$. The claim follows from the above inequality and the monotonicity of \mathcal{J}_p . □

Let \bar{v} and \underline{v} denote the weak upper and lower limits respectively of v_N defined in (3-25), i.e.,

$$\bar{v}(x, t) := \limsup_{\substack{N \rightarrow \infty \\ (y,s) \rightarrow (x,t)}} v_N(y, s), \quad \underline{v}(x, t) := \liminf_{\substack{N \rightarrow \infty \\ (v,s) \rightarrow (x,t)}} v_N(y, s).$$

By Lemma 4.6, the sequence $\{v_N\}_{N \in \mathbb{N}}$ is bounded independently of $N \in \mathbb{N}$. As a result, \bar{v} and \underline{v} are well defined and finite. In addition, \bar{v} and $-\underline{v}$ are upper semicontinuous. We recall the following result, which is Lemma 4.4 in [Hynd and Lindgren 2016]. Its statement there is for smooth ϕ , but the proof holds also for ϕ as below.

Lemma 4.8. *Assume $\phi \in C_{x,t}^{2,1}(\Omega \times (0, T))$. For $N \in \mathbb{N}$, define*

$$\phi_N(x, t) := \begin{cases} \phi(x, 0), & t = 0, \\ \phi(x, \tau_k), & t \in (\tau_{k-1}, \tau_k], \quad k = 1, \dots, N. \end{cases}$$

Suppose $\bar{v} - \phi$ has a strict local maximum at $(x_0, t_0) \in \Omega \times (0, T)$. Then there exist $(x_j, t_j) \rightarrow (x_0, t_0)$ and $N_j \rightarrow \infty$, as $j \rightarrow \infty$, such that $v_{N_j} - \phi_{N_j}$ has local maximum at (x_j, t_j) . A corresponding result holds in the case of a strict local minimum.

Before proving the uniqueness of viscosity solutions, we need the following result, which verifies that whenever we can touch a subsolution from above with a $C_{x,t}^{2,1}$ function, we can treat the subsolution as a classical subsolution in space. The proof is almost identical to the one of Theorem 2.2 in [Caffarelli and Silvestre 2009] or the one of Proposition 1 in [Lindgren 2016].

Proposition 4.9. *Suppose*

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v \leq C \quad \text{in } B_1 \times I$$

in the viscosity sense. Further assume that $(x_0, t_0) \in B_1 \times I$ and $\phi \in C_{x,t}^{2,1}(B_r(x_0) \times (t_0 - r, t_0 + r))$ are such that

$$\phi(x_0, t_0) = v(x_0, t_0), \quad \phi(x, t) \geq v(x, t) \quad \text{for } (x, t) \in B_r(x_0) \times (t_0 - r, t_0 + r)$$

for some $r > 0$. Then $(-\Delta_p)^s v$ is defined pointwise at (x_0, t_0) and

$$|\phi_t(x_0, t_0)|^{p-2}\phi_t(x_0, t_0) + (-\Delta_p)^s v(x_0, t_0) \leq C.$$

Proof. For $0 < \rho \leq r$, let

$$\phi^\rho = \begin{cases} \phi & \text{in } B_\rho(x_0) \times (t_0 - r, t_0 + r), \\ v & \text{in } \mathbb{R}^n \setminus B_\rho(x_0) \times (t_0 - r, t_0 + r). \end{cases}$$

Since v is a viscosity subsolution,

$$|\phi_t(x_0, t_0)|^{p-2}\phi_t(x_0, t_0) + (-\Delta_p)^s \phi^\rho(x_0, t_0) \leq C.$$

Now introduce the notation

$$\begin{aligned} \delta(\phi^\rho, x, y, t) &:= \frac{1}{2}|\phi^\rho(x, t) - \phi^\rho(x + y, t)|^{p-2}(\phi^\rho(x, t) - \phi^\rho(x + y, t)) \\ &\quad + \frac{1}{2}|\phi^\rho(x, t) - \phi^\rho(x - y, t)|^{p-2}(\phi^\rho(x, t) - \phi^\rho(x - y, t)), \end{aligned}$$

$$\delta^\pm(\phi^\rho, x, y, t) = \max(\pm\delta(\phi^\rho, x, y, t), 0).$$

Since ϕ^ρ is C^2 in space near x_0 , we can substitute $-y$ for y in the integral and obtain the convergent integral

$$2 \int_{\mathbb{R}^n} \delta(\phi^\rho, x_0, y, t_0) |y|^{-n-sp} dy \leq C - |\phi_t(x_0, t_0)|^{p-2}\phi_t(x_0, t_0) := D. \tag{4-2}$$

See [Remark 4.3](#) for more details.

We note that

$$\delta(\phi^{\rho_2}, x_0, y, t_0) \leq \delta(\phi^{\rho_1}, x_0, y, t_0) \leq \delta(v, x_0, y, t_0) \quad \text{for } \rho_1 < \rho_2 < r,$$

so that

$$\delta^-(\phi^{\rho_2}, x_0, y, t_0) \geq \delta^-(\phi^{\rho_1}, x_0, y, t_0) \geq \delta^-(v, x_0, y, t_0) \quad \text{for } \rho_1 < \rho_2 < r. \tag{4-3}$$

In particular,

$$\delta^-(v, x_0, y, t_0) \leq |\delta(\phi^r, x_0, y, t_0)|.$$

Since $|\delta(\phi^r, x_0, y, t_0)| |y|^{-n-sp}$ is integrable, so is $\delta^-(v, x_0, y, t_0) |y|^{-n-sp}$. In addition, by [\(4-2\)](#),

$$2 \int_{\mathbb{R}^n} \delta^+(\phi^\rho, x_0, y, t_0) |y|^{-n-sp} dy \leq 2 \int_{\mathbb{R}^n} \delta^-(\phi^\rho, x_0, y, t_0) |y|^{-n-sp} dy + D.$$

Thus, for $\rho_1 < \rho_2$,

$$\begin{aligned} 2 \int_{\mathbb{R}^n} \delta^+(\phi^{\rho_1}, x_0, y, t_0) |y|^{-n-sp} dy &\leq 2 \int_{\mathbb{R}^n} \delta^-(\phi^{\rho_1}, x_0, y, t_0) |y|^{-n-sp} dy + D \\ &\leq 2 \int_{\mathbb{R}^n} \delta^-(\phi^{\rho_2}, x_0, y, t_0) |y|^{-n-sp} dy + D < \infty, \end{aligned} \tag{4-4}$$

where we have used (4-3).

Since $\delta^+(\phi^\rho, x_0, y, t_0) \nearrow \delta^+(v, x_0, y, t_0)$, the monotone convergence theorem implies

$$\int_{\mathbb{R}^n} \delta^+(\phi^\rho, x_0, y, t_0) |y|^{-n-sp} dy \rightarrow \int_{\mathbb{R}^n} \delta^+(v, x_0, y, t_0) |y|^{-n-sp} dy.$$

By (4-4),

$$2 \int_{\mathbb{R}^n} \delta^+(v, x_0, y, t_0) |y|^{-n-sp} dy \leq 2 \int_{\mathbb{R}^n} \delta^-(\phi^\rho, x_0, y, t_0) |y|^{-n-sp} dy + D < \infty \tag{4-5}$$

for any $0 < \rho < r$. We conclude that $\delta^+(v, x_0, y, t_0) |y|^{-n-sp}$ is integrable. By the dominated convergence theorem, we can pass to the limit in the right-hand side of (4-5) and obtain

$$2 \int_{\mathbb{R}^n} \delta^+(v, x_0, y, t_0) |y|^{-n-sp} dy \leq 2 \int_{\mathbb{R}^n} \delta^-(v, x_0, y, t_0) |y|^{-n-sp} dy + D < \infty.$$

This is simply another way of writing

$$2 \int_{\mathbb{R}^n} \delta(v, x_0, y, t_0) |y|^{-n-sp} dy \leq D.$$

Therefore $(-\Delta_p)^s v(x_0, t_0)$ exists in the pointwise sense and $(-\Delta_p)^s v(x_0, t_0) \leq D$, which concludes the proof as $D = C - |\phi_t(x_0, t_0)|^{p-2} \phi_t(x_0, t_0)$. □

Proposition 4.10. *Assume that u is a viscosity subsolution and that v is a viscosity supersolution of*

$$|v_t|^{p-2} v_t + (-\Delta_p)^s v = 0 \quad \text{in } \Omega \times (0, T).$$

Suppose $u, v \in L^\infty(\mathbb{R}^n \times [0, T])$, $u \leq v$ in $\mathbb{R}^n \setminus \Omega \times [0, T]$ and

$$\limsup_{(x,t) \rightarrow (x_0,t_0)} u(x_0, t_0) \leq \liminf_{(x,t) \rightarrow (x_0,t_0)} v(x_0, t_0) \quad \text{for } (x_0, t_0) \in \partial\Omega \times (0, T) \cup \Omega \times \{0\}.$$

Then $u \leq v$.

Proof. We employ the usual trick of adding a term $\delta/(t - T)$; let $\tilde{u} = u + \delta/(t - T)$. Then v is a supersolution, \tilde{u} is a subsolution of

$$|v_t|^{p-2} v_t + (-\Delta_p)^s v = -\frac{\delta}{(t - T)^2},$$

$\tilde{u} < v$ in $\mathbb{R}^n \setminus \Omega \times [0, T]$ and

$$\limsup_{(x,t) \rightarrow (x_0,t_0)} \tilde{u}(x_0, t_0) < \liminf_{(x,t) \rightarrow (x_0,t_0)} v(x_0, t_0) \tag{4-6}$$

for $(x_0, t_0) \in \partial\Omega \times (0, T) \cup \Omega \times \{0\}$. Moreover, $\tilde{u}(x, t) - v(x, t) \rightarrow -\infty$ as $t \rightarrow T$. It is now sufficient to prove that $\tilde{u} \leq v$ for any $\delta > 0$ since we can then let $\delta \rightarrow 0$. We argue by contradiction and assume that

$$\sup_{\mathbb{R}^n \times [0, T]} (\tilde{u} - v) > 0.$$

Fix $\varepsilon > 0$ and define

$$M_\varepsilon := \sup_{\mathbb{R}^n \times [0, T] \times \mathbb{R}^n \times [0, T]} \left(\tilde{u}(x, t) - v(y, \tau) - \frac{|x - y|^2 + |t - \tau|^2}{\varepsilon} \right).$$

Note $M_\varepsilon \geq \sup_{\mathbb{R}^n \times [0, T]} (\tilde{u} - v) > 0$ and select $x_\varepsilon, y_\varepsilon \in \mathbb{R}^n$ and $t_\varepsilon, \tau_\varepsilon \in [0, T]$ for which

$$M_\varepsilon < \tilde{u}(x_\varepsilon, t_\varepsilon) - v(y_\varepsilon, \tau_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - \tau_\varepsilon|^2}{\varepsilon} + \varepsilon.$$

By Proposition 3.7 in [Crandall et al. 1992], $(x_\varepsilon, t_\varepsilon)$ and $(y_\varepsilon, \tau_\varepsilon)$ each have subsequences converging to $(\hat{x}, \hat{t}) \in \Omega \times (0, T)$ as $\varepsilon \rightarrow 0$ for which

$$\sup_{\mathbb{R}^n \times [0, T]} (\tilde{u} - v) = (\tilde{u} - v)(\hat{x}, \hat{t}).$$

As a result, there is ε small enough such that $x_\varepsilon, y_\varepsilon \in \Omega$ and $t_\varepsilon, \tau_\varepsilon \in (0, T)$. For this ε , it also follows that the maximum M_ε is attained in $\Omega \times (0, T) \times \Omega \times (0, T)$. For convenience, we will again call this point $(x_\varepsilon, t_\varepsilon, y_\varepsilon, \tau_\varepsilon)$.

Observe that the function

$$\frac{|x - y_\varepsilon|^2 + |t - \tau_\varepsilon|^2}{\varepsilon} + \tilde{u}(x_\varepsilon, t_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - \tau_\varepsilon|^2}{\varepsilon}$$

touches \tilde{u} from above at $(x_\varepsilon, t_\varepsilon)$ and

$$\frac{|x_\varepsilon - y|^2 + |t_\varepsilon - \tau|^2}{\varepsilon} - v(y_\varepsilon, \tau_\varepsilon) - \frac{|x_\varepsilon - y_\varepsilon|^2 + |t_\varepsilon - \tau_\varepsilon|^2}{\varepsilon}$$

touches $-v$ from above at $(y_\varepsilon, \tau_\varepsilon)$. From Proposition 4.9, we can conclude that $(-\Delta_p)^s \tilde{u}(x_\varepsilon, t_\varepsilon)$ and $(-\Delta_p)^s v(y_\varepsilon, \tau_\varepsilon)$ exist pointwise and satisfy

$$(-\Delta_p)^s \tilde{u}(x_\varepsilon, t_\varepsilon) < (\Delta_p)^s v(y_\varepsilon, \tau_\varepsilon).$$

In addition, since the function

$$\tilde{u}(x, t) - v(y, \tau) - \frac{|x - y|^2 + |t - \tau|^2}{\varepsilon}$$

is larger at $(x_\varepsilon, y_\varepsilon, t_\varepsilon, \tau_\varepsilon)$ than at $(x_\varepsilon + y, y_\varepsilon + y, t_\varepsilon, \tau_\varepsilon)$ for any y , we obtain

$$\tilde{u}(x_\varepsilon, t_\varepsilon) - \tilde{u}(x_\varepsilon + y, t_\varepsilon) \geq v(y_\varepsilon, \tau_\varepsilon) - v(y_\varepsilon + y, \tau_\varepsilon).$$

This implies

$$(-\Delta_p)^s \tilde{u}(x_\varepsilon, t_\varepsilon) \geq (-\Delta_p)^s v(y_\varepsilon, \tau_\varepsilon),$$

which is a contradiction. Therefore, we must have $\tilde{u} \leq v$. □

Now we present a general result for nonlocal parabolic equations, inspired by previous work of Petri Juutinen [2001, Theorem 1]. This fact will be important in the proof of Hölder regularity of solutions of (1-1).

Proposition 4.11. *Suppose that v is a viscosity subsolution of*

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v \leq 0$$

in $B_r(x_0) \times (t_0 - r, t_0 + r)$ and $\phi \in C_{x,t}^{2,1}(B_r(x_0) \times (t_0 - r, t_0 + r))$. If

$$\phi(x_0, t_0) = v(x_0, t_0), \quad \phi(x, t) \geq v(x, t) \quad \text{for } (x, t) \in B_r(x_0) \times (t_0 - r, t_0], \tag{4-7}$$

then

$$|\phi_t(x_0, t_0)|^{p-2}\phi_t(x_0, t_0) + (-\Delta_p)^s \phi^r(x_0, t_0) \leq 0.$$

Proof. We argue by contradiction. If the assertion is not true then

$$|\phi_t(x_0, t_0)|^{p-2}\phi_t(x_0, t_0) + (-\Delta_p)^s \phi^r(x_0, t_0) \geq \varepsilon > 0$$

for some ε . Recall ϕ^r is defined in Definition 4.4. By continuity, we have

$$|\phi_t|^{p-2}\phi_t + (-\Delta_p)^s \phi^r \geq \frac{1}{2}\varepsilon > 0$$

in $B_\rho(x_0) \times (t_0 - \rho, t_0)$ for ρ small enough.

Let $\eta : \mathbb{R}^{n+1} \rightarrow \mathbb{R}$ be a smooth function satisfying

$$\begin{cases} 0 \leq \eta \leq 1, \\ \eta(x_0, t_0) = 0, \\ \eta(x, t) > 0 \quad \text{if } (x, t) \neq (x_0, t_0), \\ \eta(x, t) \geq 1 \quad \text{if } (x, t) \notin B_\rho(x_0) \times (t_0 - \rho, t_0). \end{cases}$$

Also define

$$\phi_\delta(x, t) = \phi(x, t) + \delta\eta(x, t) - \delta,$$

where $\delta > 0$ is considered small. By continuity,

$$|(\phi_\delta)_t|^{p-2}(\phi_\delta)_t + (-\Delta_p)^s (\phi_\delta)^r \geq \frac{1}{4}\varepsilon > 0$$

in $B_\rho(x_0) \times (t_0 - \rho, t_0)$, provided δ is small enough.

This means that $(\phi_\delta)^r$ is a supersolution in the pointwise classical sense in $B_\rho(x_0) \times (t_0 - \rho, t_0)$, and in particular it means that $(\phi_\delta)^r$ is a viscosity supersolution in this region. Moreover, $(\phi_\delta)^r \geq \phi^r \geq v$ in the complement of

$$\mathbb{R}^n \setminus B_\rho(x_0) \times (t_0 - \rho, t_0) \cup B_\rho(x_0) \times \{t_0 - \rho\}.$$

By Proposition 4.10, $(\phi_\delta)^r \geq v$ in $\mathbb{R}^n \times [t_0 - \rho, t_0]$. Furthermore, $(\phi_\delta)^r(x_0, t_0) \geq v(x_0, t_0)$ which is a contradiction since $(\phi_\delta)^r(x_0, t_0) = \phi(x_0, t_0) - \delta = v(x_0, t_0) - \delta$. □

Let us now return to our study of the implicit time scheme. We are now in position to construct barriers that assure that \bar{v} and \underline{v} satisfy the correct boundary and initial conditions.

Lemma 4.12. *Assume that $-\Psi \leq g \leq \Psi$, where Ψ is a nonnegative ground state of $(-\Delta_p)^s$. Then \bar{v} and \underline{v} satisfy the boundary condition in the classical sense; i.e.,*

$$\lim_{y \rightarrow x} \bar{v}(y, t) = \lim_{y \rightarrow x} \underline{v}(y, t) = 0 \quad \text{for any } x \in \partial\Omega \text{ and any } t \geq 0.$$

Proof. We observe that

$$-(-\Delta_p)^s \Psi - \frac{|\Psi - g|^{p-2}(\Psi - g)}{\tau^{p-1}} = -\lambda_{s,p} \Psi^{p-1} - \frac{(\Psi - g)^{p-1}}{\tau^{p-1}} \leq 0.$$

Hence Ψ is a supersolution of (3-21). Since $\Psi = v^1 = 0$ in $\mathbb{R}^n \setminus \Omega$, the comparison principle implies

$$v^1 \leq \Psi.$$

We can argue similarly to obtain

$$|v^1| \leq \Psi.$$

Iterating this method for each v^k yields $|v^k| \leq \Psi$ for any $k = 1, \dots, N$. By the definition of v_N in (3-25),

$$|v_N| \leq \Psi. \tag{4-8}$$

By inequality (4-8), the assertion would follow as long as Ψ is continuous up to the boundary. To establish this continuity, we first note that Ψ is globally bounded. This fact is due to Theorem 3.2 in [Franzina and Palatucci 2014], Theorem 3.3 in [Brasco et al. 2014] or Theorem 3.1 together with Remark 3.2 in [Brasco and Parini 2015]. Theorem 1.1 in [Iannizzotto et al. 2016] can now be used to establish the desired continuity of Ψ . □

Lemma 4.13. *Assume g is continuous. Then \bar{v} and \underline{v} satisfy the initial condition in the classical sense; i.e.,*

$$\lim_{t \rightarrow 0} \bar{v}(x, t) = \lim_{t \rightarrow 0} \underline{v}(x, t) = g(x) \quad \text{for any } x \in \Omega.$$

Proof. Take η to be a bounded, smooth and strictly increasing radial function such that $\eta(0) = 0$. Let $d = \text{diam } \Omega$ and

$$\alpha^{p-1} = \sup_{x \in B_d} |(-\Delta_p)^s \eta(x)|.$$

Clearly α is finite. Now we fix $x_0 \in \Omega$. We first prove that given $\varepsilon > 0$ there is $C = C(x_0, \varepsilon)$ such that

$$u(x) = g(x_0) + \varepsilon + C(\alpha\tau + \eta(x - x_0))$$

lies above v^1 .

As g is continuous, for each $\varepsilon > 0$ there is $\delta > 0$ and $C > 0$ so that

$$|g(x) - g(x_0)| < \varepsilon \quad \text{if } |x - x_0| < \delta$$

and

$$\sup |g| \leq C\eta(x - x_0) \quad \text{if } |x - x_0| \geq \delta.$$

Upon choosing C even larger, we may also assume that $u \geq 0$ in $\mathbb{R}^n \setminus \Omega$. In addition

$$\begin{aligned} -(-\Delta_p)^s u - \frac{|u - g|^{p-2}(u - g)}{\tau^{p-1}} &\leq C^{p-1}\alpha^{p-1} - \frac{(g(x_0) + \varepsilon + C\eta(\cdot - x_0) - g + C\alpha\tau)^{p-1}}{\tau^{p-1}} \\ &\leq C^{p-1}\alpha^{p-1} - C^{p-1}\alpha^{p-1} = 0, \end{aligned}$$

since $g(x_0) + \varepsilon + C\eta(\cdot - x_0) - g(\cdot) \geq 0$ by construction. Now it follows from the comparison principle that

$$v^1(x) \leq u(x) = g(x_0) + \varepsilon + C(\alpha\tau + \eta(x - x_0)).$$

Arguing in the same fashion, we have

$$v^1(x) \geq g(x_0) - \varepsilon - C(\alpha\tau + \eta(x - x_0)).$$

Similarly we can obtain the bounds

$$g(x_0) - \varepsilon - C(k\alpha\tau + \eta(x - x_0)) \leq v^k(x) \leq g(x_0) + \varepsilon + C(k\alpha\tau + \eta(x - x_0))$$

for each $k = 1, \dots, N$. Using the definition (3-25) of v_N , we also have

$$v_N(x, t) \leq v^k(x) \leq g(x_0) + \varepsilon + C(\alpha k\tau + \eta(x - x_0)) \leq g(x_0) + \varepsilon + C\left(\alpha t + \alpha \frac{T}{N} + \eta(x - x_0)\right)$$

for $t \in ((k - 1)\tau, k\tau)$ as $\tau = T/N$. A similar estimate from below holds as well. In total,

$$g(x_0) - \varepsilon - C\left(\alpha t + \alpha \frac{T}{N} + \eta(x - x_0)\right) \leq v_N(x, t) \leq g(x_0) + \varepsilon + C\left(\alpha t + \alpha \frac{T}{N} + \eta(x - x_0)\right).$$

Passing to the liminf and limsup in the above inequalities, we find

$$g(x_0) - \varepsilon - C(\alpha t + \eta(x - x_0)) \leq \underline{v}(x, t) \leq \bar{v}(x, t) \leq g(x_0) + \varepsilon + C(\alpha t + \eta(x - x_0)).$$

And after letting $x = x_0$ and $t \rightarrow 0$,

$$g(x_0) - \varepsilon \leq \liminf_{t \rightarrow 0} \underline{v}(x_0, t) \leq \limsup_{t \rightarrow 0} \bar{v}(x_0, t) \leq g(x_0) + \varepsilon.$$

Since both ε and $x_0 \in \Omega$ are arbitrary, the desired result follows. □

Proof of Proposition 4.1. It is enough to show that \bar{v} is a viscosity subsolution of (1-1). The same argument (applied to $-\bar{v}$) yields that \underline{v} is a supersolution. Combining Lemma 4.12, Lemma 4.13, and Proposition 4.10, would then imply $\bar{v} \leq \underline{v}$. Hence, $v := \bar{v} = \underline{v}$ is continuous and v_N converges to v locally uniformly. The claim would then follow as v_N has a subsequence converging to a weak solution of (1-1) in $C([0, T], L^p(\Omega))$; see Remark 3.9.

We now prove that \bar{v} is a viscosity subsolution of (1-1). Assume that $\phi \in C_{x,t}^{2,1}(B_r(x_0) \times (t_0 - r, t_0 + r))$ and $\bar{v} - \phi$ has a strict maximum in $B_r(x_0) \times (t_0 - r, t_0 + r)$ at $(x_0, t_0) \in \Omega \times (0, T)$. By Lemma 4.8, there are points (x_j, t_j) converging to (x_0, t_0) and $N_j \in \mathbb{N}$ tending to $+\infty$, as $j \rightarrow \infty$, such that $v_{N_j} - \phi_{N_j}$ has a maximum in $B_r(x_0) \times (t_0 - r, t_0 + r)$ at (x_j, t_j) . Observe that for each $j \in \mathbb{N}$, we have $t_j \in (\tau_{k_j-1}, \tau_{k_j}]$ for some $k_j \in \{0, 1, \dots, N_j\}$. Hence, by the definition of v_{N_j} and ϕ_{N_j} ,

$$\Omega \times \{0, 1, \dots, N_j\} \ni (x, k) \mapsto v^k(x) - \phi(x, \tau_k)$$

has a local maximum in $B_r(x_0) \times \{0, 1, \dots, N_j\}$ at $(x, k) = (x_j, k_j)$. By Lemma 4.7,

$$\mathcal{J}_p \left(\frac{\phi(x_j, \tau_{k_j}) - \phi(x_j, \tau_{k_j-1})}{T/N_j} \right) + (-\Delta_p)^s \phi^r(x_j, \tau_{k_j}) \leq 0.$$

As $\tau_{k_j-1} = \tau_{k_j} - T/N_j$ and $|t_j - \tau_{k_j}| \leq T/N_j$ for $j \in \mathbb{N}$, we can send $j \rightarrow \infty$ above by appealing to the smoothness of ϕ and arrive at

$$\mathcal{J}_p(\phi_t(x_0, t_0)) + (-\Delta_p)^s \phi^r(x_0, t_0) \leq 0.$$

It follows that \bar{v} is a viscosity subsolution. □

5. Hölder estimates for viscosity solutions

In this section we prove Theorem 1.1. The proof of this regularity result is based on Lemma 5.1 below. We start by noting an elementary inequality that will come in handy:

$$|a + b|^{p-2}(a + b) \leq 2^{p-2}(|a|^{p-2}a + |b|^{p-2}b), \quad a + b \geq 0, \quad p \geq 2. \tag{5-1}$$

Lemma 5.1. Fix $\delta > 0$. Suppose v is continuous in Q_1^- and satisfies (in the viscosity sense)

$$|v_t|^{p-2}v_t + (-\Delta_p)^s v \leq 0 \quad \text{in } Q_1^-,$$

$$v \leq 1 \quad \text{in } Q_1^-,$$

$$v(x, t) \leq 2|2x|^\eta - 1 \quad \text{in } \mathbb{R}^n \setminus B_1 \times (-1, 0),$$

$$\left| \left\{ B_1 \times \left[-1, -\frac{1}{2^{\frac{sp}{p-1}}} \right] \right\} \cap \{v \leq 0\} \right| > \delta.$$

Then for η small enough, $v \leq 1 - \theta < 1$ in $Q_{1/2}^-$, where $\theta = \theta(\delta, p, s) > 0$.

Recall that the parabolic cylinders Q_1^- and $Q_{\frac{1}{2}}^-$ have been defined on page 1. Before proving this lemma, we will first need to gain control of a certain function.

Lemma 5.2. Fix $\delta > 0$, let $\varepsilon > 0$ and assume the following:

$$m(t) = e^{-c_1 t} \int_{-1}^t c_0 e^{c_1 s} |G(s)| ds,$$

$$G(t) = \{x \in B_1 : v(x, t) \leq 0\},$$

$$b(x, t) = 1 + \varepsilon - m(t)\rho(x),$$

$$0 \leq \rho \leq 1, \quad \rho = 1 \quad \text{in } B_{\frac{1}{2}}, \quad \rho \in C_0^\infty(B_{\frac{3}{4}}),$$

$$\left| \left\{ B_1 \times \left[-1, -\frac{1}{2^{\frac{sp}{p-1}}} \right] \right\} \cap \{v \leq 0\} \right| > \delta.$$

If $c_0|B_1| \leq \frac{1}{2}c_1$ then

$$b(x, t) \geq \frac{1}{2},$$

and for $0 \geq t \geq -1/2^{\frac{sp}{p-1}}$,

$$m(t) \geq c_0 e^{-c_1} \delta.$$

Remark 5.3. Note that m solves the equation

$$m'(t) = c_0|G(t)| - c_1m(t)$$

for a.e. $t \in [-1, 0]$.

Proof. As $|G(t)| \leq |B_1|$, it follows that $m(t) \leq c_0/c_1|B_1|$. And since $c_0|B_1| \leq \frac{1}{2}c_1$,

$$b(x, t) \geq 1 + \varepsilon - \frac{c_0}{c_1}|B_1|\rho(x) \geq 1 + \varepsilon - \frac{c_0}{c_1}|B_1| \geq \frac{1}{2}.$$

Moreover,

$$m(t) = e^{-c_1t} \int_{-1}^t c_0e^{c_1s}|G(s)| ds \geq e^{c_1(-1-t)} \int_{-1}^t c_0|G(s)| ds.$$

From our hypotheses,

$$\int_{-1}^{-1/2^{\frac{sp}{p-1}}} |G(s)| ds \geq \delta.$$

Therefore,

$$m(t) \geq c_0e^{-c_1}\delta$$

for $0 \geq t \geq -1/2^{\frac{sp}{p-1}}$. □

Proof of Lemma 5.1. Assume the hypotheses of Lemma 5.2. Choose c_0, c_1 and ε so that

$$c_0^{p-1} < 1/(2^{2p-4+n+sp}|B_1|^{p-2}), \quad c_0|B_1| \leq \frac{1}{2}c_1, \quad 2^{2-p}(c_1)^{p-1} > 2 \sup_{x_0 \in B_{\frac{3}{4}}} (-\Delta_p)^s \left(\frac{\rho}{\rho(x_0)} \right) (x_0),$$

and

$$2\varepsilon < e^{-c_1}c_0\delta.$$

Note that the quantity

$$2 \sup_{x_0 \in B_{\frac{3}{4}}} (-\Delta_p)^s \left(\frac{\rho}{\rho(x_0)} \right) (x_0)$$

is finite, since the only way it could be infinite is if there is maximizing sequence of points x_j where $\rho(x_j) \rightarrow 0$. But then

$$(-\Delta_p)^s \left(\frac{\rho}{\rho(x_j)} \right) (x_j)$$

would be negative for j large enough.

We claim that $v \leq b$ in Q_1^- . Let us describe how the lemma follows once this claim is proved. By the lower bound on m in Lemma 5.2, we have

$$b(x, t) \leq 1 + \varepsilon - e^{-c_1}c_0\delta$$

for $0 \geq t \geq -1/2^{\frac{sp}{p-1}}$. Since $2\varepsilon < e^{-c_1}c_0\delta$,

$$b(x, t) \leq 1 - \frac{1}{2}e^{-c_1}c_0\delta.$$

Therefore,

$$v \leq b \leq 1 - \theta$$

in $Q_{\frac{1}{2}}^-$ as long as we choose

$$\theta = \frac{1}{2}e^{-c_1}c_0\delta.$$

Let us now prove that $v \leq b$ in Q_1^- . We argue by contradiction. Assume that, starting from $t = -1$, the first time v touches b at some point in Q_1^- is at the point (x_0, t_0) . Since $\rho = 0$ outside $B_{\frac{3}{4}}$ and $v \leq 1$ in Q_1^- , we know that $x_0 \in B_{\frac{3}{4}}$. In addition, since $m(-1) = 0$, we know $t_0 > -1$. It is not difficult to see b touches v from above at (x_0, t_0) in the sense of (4-7). In order to simplify the presentation, we first assume that b is C^1 at (x_0, t_0) and explain in the last paragraph of this proof how to relax this assumption.

By Proposition 4.9, $(-\Delta_p)^s v(x_0, t_0)$ is well defined and

$$|b_t(x_0, t_0)|^{p-2}b_t(x_0, t_0) + (-\Delta_p)^s v(x_0, t_0) \leq 0. \tag{5-2}$$

Note that $b_t(x_0, t_0) = -m'(t_0)\rho(x_0)$. We will now estimate $(-\Delta_p)^s(b - v)(x_0, t_0)$ from above and from below and arrive at a contradiction. This part of the proof will be divided into four steps. Along the way, we will use the notation

$$L_D w(x, t) := 2 \text{P.V.} \int_{y \in D} \frac{\mathcal{J}_p(w(y, t) - w(x, t))}{|x - y|^{n+ps}} dy$$

for a measurable function w and an open or closed set $D \subset \mathbb{R}^n$. Notice that

$$(-\Delta_p)^s w = -L_{\mathbb{R}^n} w.$$

Step 1: Estimate L_{B_1} . Since $b(\cdot, t_0) \geq v(\cdot, t_0)$ in B_1 , (5-1) implies

$$\begin{aligned} L_{B_1}(b - v)(x_0, t_0) &= 2 \text{P.V.} \int_{B_1} \frac{\mathcal{J}_p((b - v)(y, t_0) - (b - v)(x_0, t_0))}{|x_0 - y|^{n+ps}} dy \\ &\leq 2^{p-1} \text{P.V.} \int_{B_1} \frac{\mathcal{J}_p(b(y, t_0) - b(x_0, t_0)) - \mathcal{J}_p(v(y, t_0) - v(x_0, t_0))}{|x_0 - y|^{n+ps}} dy \\ &= 2^{p-2}(L_{B_1}b(x_0, t_0) - L_{B_1}v(x_0, t_0)). \end{aligned}$$

In addition, since $v(x_0, t_0) = b(x_0, t_0)$,

$$\begin{aligned} L_{B_1}(b - v)(x_0, t_0) &= 2 \text{P.V.} \int_{B_1} \frac{\mathcal{J}_p((b - v)(y, t_0))}{|x_0 - y|^{n+ps}} dy \\ &\geq 2 \int_{G(t_0)} \frac{|b(y, t_0)|^{p-2}b(y, t_0)}{|x_0 - y|^{n+ps}} dy \\ &\geq 2\left(\frac{1}{2}\right)^{n+sp} \inf_{y \in B_1} |b(y, t_0)|^{p-1} |G(t_0)| \\ &\geq \left(\frac{1}{2}\right)^{p-2+n+sp} |G(t_0)|, \end{aligned}$$

from Lemma 5.2.

Step 2: Estimate $L_{\mathbb{R}^n \setminus B_1}$. By our hypotheses,

$$v(x, t_0) \leq 2|2y|^\eta - 1, \quad b = 1 + \varepsilon > 1$$

whenever $|y| > 1$. Hence, $b(y, t_0) - v(y, t_0) \geq 2(1 - |2y|^\eta)$ so that

$$b(y, t_0) - v(y, t_0) \leq b(y, t_0) - v(y, t_0) + 2(|2y|^\eta - 1) \tag{5-3}$$

and

$$b(y, t_0) - v(y, t_0) + 2(|2y|^\eta - 1) \geq 0. \tag{5-4}$$

By (5-1), (5-3) and (5-4),

$$\begin{aligned} L_{\mathbb{R}^n \setminus B_1}(b-v)(x_0, t_0) &\leq 2 \int_{\mathbb{R}^n \setminus B_1} \frac{\mathcal{J}_p(b(y, t_0) - v(y, t_0) + 2(|2y|^\eta - 1) - (b(x_0, t_0) - v(x_0, t_0)))}{|x_0 - y|^{n+ps}} dy \\ &\leq 2^{p-2} \left(-L_{\mathbb{R}^n \setminus B_1} v(x_0, t_0) + 2 \int_{\mathbb{R}^n \setminus B_1} \frac{\mathcal{J}_p(b(y, t_0) + 2(|2y|^\eta - 1) - b(x_0, t_0))}{|x_0 - y|^{n+ps}} dy \right). \end{aligned}$$

Using (5-4), we obtain the estimate from below

$$\begin{aligned} L_{\mathbb{R}^n \setminus B_1}(b-v)(x_0, t_0) &= 2 \int_{\mathbb{R}^n \setminus B_1} \frac{\mathcal{J}_p(b(y, t_0) - v(y, t_0))}{|x_0 - y|^{n+ps}} dy \\ &\geq -2 \int_{\mathbb{R}^n \setminus B_1} (2(|2y|^\eta - 1))^{p-1} \frac{dy}{|x_0 - y|^{n+sp}} \\ &:= -c_\eta. \end{aligned}$$

We note that $\lim_{\eta \rightarrow 0^+} c_\eta = 0$ by an application of the dominated convergence theorem.

Step 3: Use the equation. The two steps above together imply

$$\begin{aligned} \left(\frac{1}{2}\right)^{p-2+n+sp} |G(t_0)| - c_\eta &\leq -(-\Delta_p)^s (b-v)(x_0, t_0) \\ &\leq 2^{p-2} (-\Delta_p)^s v(x_0, t_0) + 2^{p-2} L_{B_1} b(x_0, t_0) \\ &\quad + 2^{p-1} \int_{\mathbb{R}^n \setminus B_1} \frac{\mathcal{J}_p(b(y, t_0) + 2(|2y|^\eta - 1) - b(x_0, t_0))}{|x_0 - y|^{n+ps}} dy. \end{aligned} \tag{5-5}$$

From inequality (5-2), it follows that

$$\begin{aligned} (-\Delta_p)^s v(x_0, t_0) &\leq -|b_t|^{p-2} b_t(x_0, t_0) \\ &= |m'(t_0)\rho(x_0)|^{p-2} m'(t_0)\rho(x_0) \\ &= |\rho(x_0)c_0|G(t_0)| - \rho(x_0)c_1m(t_0)|^{p-2} (\rho(x_0)c_0|G(t_0)| - \rho(x_0)c_1m(t_0)) \\ &\leq |c_0|G(t_0)| - \rho(x_0)c_1m(t_0)|^{p-2} (c_0|G(t_0)| - \rho(x_0)c_1m(t_0)). \end{aligned} \tag{5-6}$$

Using (5-1), with $a = c_1\rho(x_0)m(t_0) - c_0|G(t_0)|$ and $b = c_0|G(t_0)|$, we then obtain

$$\begin{aligned} (c_1\rho(x_0)m(t_0))^{p-1} &\leq 2^{p-2} |\rho(x_0)c_1m(t_0) - c_0|G(t_0)||^{p-2} (\rho(x_0)c_1m(t_0) - c_0|G(t_0)|) + 2^{p-2} (c_0|G(t_0)|)^{p-1}. \end{aligned}$$

After rearranging

$$2^{p-2} |c_0|G(t_0)| - \rho(x_0)c_1m(t_0)|^{p-2} (c_0|G(t_0)| - \rho(x_0)c_1m(t_0)) \leq 2^{p-2} (c_0|G(t_0)|)^{p-1} - (c_1\rho(x_0)m(t_0))^{p-1}. \tag{5-7}$$

Combining (5-5), (5-6) and (5-7) yields

$$(c_1\rho(x_0)m(t_0))^{p-1} - 2^{p-2} (c_0|G(t_0)|)^{p-1} + \left(\frac{1}{2}\right)^{p-2+n+sp} |G(t_0)| - c_\eta \leq 2^{p-2} L_{B_1} b(x_0, t_0) + 2^{p-1} \int_{\mathbb{R}^n \setminus B_1} \frac{\mathcal{J}_p(b(y, t_0) + 2(|2y|^\eta - 1) - b(x_0, t_0))}{|x_0 - y|^{n+ps}} dy.$$

Since we assumed at the outset that $c_0^{p-1} < 1/(2^{2p-4+n+sp}|G(t_0)|^{p-2})$, we have, by the definition of b and L_{B_1} ,

$$2^{2-p} (c_1\rho(x_0)m(t_0))^{p-1} \leq 2^{2-p} c_\eta + 2P.V. \int_{B_1} \frac{\mathcal{J}_p(m(t_0)(\rho(x_0) - \rho(y)))}{|x_0 - y|^{n+ps}} dy + 2 \int_{\mathbb{R}^n \setminus B_1} \frac{\mathcal{J}_p(m(t_0)\rho(x_0) + 2(|2y|^\eta - 1))}{|x_0 - y|^{n+ps}} dy. \tag{5-8}$$

Here we also used that $\rho(y) = 0$ whenever $y \notin B_1$.

Step 4: Arrive at a contradiction. It follows from the proof of Lemma 5.2 that m is uniformly bounded with respect to η . Consequently, the second integral on the right-hand side of (5-8) is uniformly bounded for all small η . We can again apply the dominated convergence theorem to show that the right-hand side of (5-8) converges to the quantity

$$-(-\Delta_p)^s b(x_0, t_0) = (m(t_0))^{p-1} (-\Delta_p)^s \rho(x_0)$$

as $\eta \rightarrow 0$. As m is bounded from below by $\frac{1}{2}$ (by Lemma 5.2), there is $\gamma_\eta \searrow 0$ as $\eta \rightarrow 0$ such that

$$2^{2-p} (c_1\rho(x_0))^{p-1} \leq \gamma_\eta + (-\Delta_p)^s \rho(x_0). \tag{5-9}$$

In general, x_0 will depend on η . Let us now consider two cases depending on the size of $(-\Delta_p)^s \rho(x_0)$ for η small.

For the first case, we suppose $\limsup_{\eta \rightarrow 0^+} (-\Delta_p)^s \rho(x_0) \leq 0$. Then (5-9) forces $\lim_{\eta \rightarrow 0^+} \rho(x_0) = 0$ as $\eta \rightarrow 0$. It would then follow that $(-\Delta_p)^s \rho(x_0) < -\gamma_\eta$ for all small $\eta > 0$. Together with (5-9), this would in turn force $\rho(x_0) < 0$ for η small enough, which is a contradiction.

Alternatively if $\limsup_{\eta \rightarrow 0^+} (-\Delta_p)^s \rho(x_0) > 0$, then for some sequence of $\eta \rightarrow 0$, we have that $(-\Delta_p)^s \rho(x_0) \geq \gamma_\eta$. By (5-9),

$$2^{2-p} (c_1\rho(x_0))^{p-1} \leq 2(-\Delta_p)^s \rho(x_0)$$

along this sequence. Also note that $(-\Delta_p)^s \rho(x_0) > 0$ implies that $x_0 \in B_{\frac{3}{4}}$. After dividing by $(\rho(x_0))^{p-1}$, we have

$$2^{2-p} (c_1)^{p-1} \leq 2(-\Delta_p)^s \left(\frac{\rho}{\rho(x_0)} \right) (x_0) \leq 2 \sup_{x_0 \in B_{\frac{3}{4}}} (-\Delta_p)^s \left(\frac{\rho}{\rho(x_0)} \right) (x_0).$$

However, by our hypotheses on c_1 ,

$$2^{2-p}(c_1)^{p-1} > 2 \sup_{x_0 \in B_{\frac{3}{4}}} (-\Delta_p)^s \left(\frac{\rho}{\rho(x_0)} \right) (x_0),$$

which is a contradiction.

Step 5: Relax the C^1 assumption on b . As mentioned above, m is not necessarily C^1 since $|G(t)|$ is not necessarily continuous. We have chosen to ignore this fact in the reasoning above in order to make the proof more accessible. This issue can be handled as follows.

First, set

$$\chi_k(x, t) := \int_{-1}^0 \phi_k(t-s) \chi_{\{v \leq 0\}}(x, s) ds$$

for $x \in \mathbb{R}^n$ and $t \in \mathbb{R}$, where ϕ_k is a standard mollifier. Also define

$$g_k(t) = \int_{B_1} \chi_k(x, t) dx.$$

Observe $g_k(t) \rightarrow |G(t)|$ a.e. and in $L^1(\mathbb{R})$ as $k \rightarrow \infty$.

Now set

$$m_k(t) := e^{-c_1 t} \int_{-1}^t c_0 e^{c_1 s} g_k(s) ds$$

for $t \in [-1, 0]$ and

$$b_k(x, t) := 1 + \varepsilon - m_k(t)\rho(x)$$

for $(x, t) \in Q_1^-$. It is evident that $m_k \rightarrow m$ and $b_k \rightarrow b$ uniformly as $k \rightarrow \infty$.

Recall that $b - v \geq \varepsilon$ and $b_k - v \geq \varepsilon$ on $\partial B_1 \times [-1, 0] \cup B_1 \times \{-1\}$. These facts combined with the above uniform convergence imply that v touches b_k from below at some $(x_k, t_k) \rightarrow (x_0, t_0)$, where v touches b from below at (x_0, t_0) . Without loss of generality, we may assume that $t_k < 0$ for all $k \in \mathbb{N}$ large enough. Moreover, as in Step 1 we find

$$\begin{aligned} 2\text{P.V.} \int_{B_1} \frac{\mathcal{J}_p((b_k - v)(y, t_k))}{|x_k - y|^{n+ps}} dy &\geq \left(\frac{1}{2}\right)^{p-2+n+sp} \inf_{y \in B_1} \left(\frac{|b_k(y, t_k)|}{1/2} \right)^{p-1} g_k(t_k) \\ &\quad - 2 \int_{B_1} (\chi_k - \chi_{\{v \leq 0\}})(y, t_k) \frac{|b_k(y, t_k)|^{p-2} b_k(y, t_k)}{|x_k - y|^{n+ps}} dy. \end{aligned} \tag{5-10}$$

Notice that as $k \rightarrow \infty$,

$$\inf_{y \in B_1} \left(\frac{|b_k(y, t_k)|}{1/2} \right)^{p-1} \rightarrow \inf_{y \in B_1} \left(\frac{|b(y, t_0)|}{1/2} \right)^{p-1} \geq 1.$$

Let us now argue that the second term on the right-hand side of (5-10) goes to zero as $k \rightarrow \infty$.

By Lemma 5.2, $b > 0$ and so $b_k > 0$ for all k large enough. Hence, $v(x_k, t_k) = b(x_k, t_k) > 0$. Since v is continuous, $v > 0$ in a neighborhood of (x_k, t_k) for k large. This means that $\chi_k = \chi_{\{v \leq 0\}} = 0$ in

$B_\tau(x_k) \times \{t_k\}$ if τ is small enough and k large enough. Hence,

$$\begin{aligned} \int_{B_1} (\chi_k - \chi_{\{v \leq 0\}})(y, t_k) \frac{|b_k(y, t_k)|^{p-2} b_k(y, t_k)}{|x_k - y|^{n+ps}} dy \\ = \int_{B_1 \setminus B_\tau(x_k)} (\chi_k - \chi_{\{v \leq 0\}})(y, t_k) \frac{|b_k(y, t_k)|^{p-2} b_k(y, t_k)}{|x_k - y|^{n+ps}} dy. \end{aligned}$$

As a result, the integrand is uniformly bounded and converges to zero almost everywhere. By Lebesgue's dominated convergence theorem, we can conclude

$$c_k := 2 \int_{B_1} (\chi_k - \chi_{\{v \leq 0\}})(y, t_k) \frac{|b_k(y, t_k)|^{p-2} b_k(y, t_k)}{|x_k - y|^{n+ps}} dy \rightarrow 0$$

as $k \rightarrow \infty$.

Steps 2 and 3 go through with minor modifications, so that we can obtain the following analog of (5-8)

$$\begin{aligned} & 2^{2-p} (c_1 \rho(x_k) m_k(t_k))^{p-1} \\ & \leq 2^{2-p} (c_\eta + c_k) + 2P.V. \int_{B_1} \frac{\mathcal{J}_p(m_k(t_k)(\rho(x_k) - \rho(y)))}{|x_k - y|^{n+ps}} dy + 2 \int_{\mathbb{R}^n \setminus B_1} \frac{\mathcal{J}_p(m_k(t_k)\rho(x_k) + 2(|2y|^\eta - 1))}{|x_k - y|^{n+ps}} dy \end{aligned}$$

for all k sufficiently large. We can then send $k \rightarrow \infty$ and recover (5-8). At this point, we can repeat Step 4 to complete this proof. \square

We are now in a position to verify [Theorem 1.1](#) and prove that solutions of equation (1-1) are Hölder continuous.

Proof of Theorem 1.1. Upon rescaling v by the factor

$$\frac{1}{2\|v\|_{L^\infty(\mathbb{R}^n \times [-2, 0])}},$$

we may assume that v satisfies

$$|v_t|^{p-2} v_t + (-\Delta_p)^s v = 0 \quad \text{in } Q_2^-, \quad \text{osc}_{\mathbb{R}^n \times [-2, 0]} v \leq 1.$$

We will now show that for any $(x_0, t_0) \in Q_1^-$,

$$\text{osc}_{Q_{2^{-j}}^-(x_0, t_0)} v \leq 2^{-j\alpha}, \quad \text{where } j = 0, 1, \dots$$

Here α is chosen so that

$$\frac{2 - \theta}{2} \leq 2^{-\alpha} \quad \text{and} \quad \alpha \leq \eta, \tag{5-11}$$

where $\theta = \theta(\delta(p, s), p, s)$ and η are from [Lemma 5.1](#) with $\delta(p, s) := \frac{1}{2}(1 - 1/2^{\frac{ps}{p-1}})|B_1|$. This will imply the desired result with $C = 2^\alpha$.

To this end, we will find constants a_j and b_j so that

$$b_j \leq v \leq a_j \quad \text{in } Q_{2^{-j}}^-(x_0, t_0), \quad |a_j - b_j| \leq 2^{-j\alpha} \tag{5-12}$$

for $j \in \mathbb{Z}$. We construct these constants by induction on j . For $j \leq 0$, (5-12) holds true with $b_j = \inf_{\mathbb{R}^n \times [-2,0)} v$ and $a_j = b_j + 1$.

Now assume (5-12) holds for all $j \leq k$. We need to construct a_{k+1} and b_{k+1} . Put $m_k = \frac{1}{2}(a_k + b_k)$. Then

$$|v - m_k| \leq 2^{-k\alpha-1} \quad \text{in } Q_{2^{-k}}^-(x_0, t_0).$$

Let

$$w(x, t) = 2^{\alpha k+1} (v(2^{-k}x + x_0, 2^{-k\gamma}t + t_0) - m_k), \quad \gamma = \frac{sp}{p-1}.$$

Then

$$|w_t|^{p-2} w_t + (-\Delta_p)^s w = 0 \quad \text{in } Q_1^-$$

and

$$|w| \leq 1 \quad \text{in } Q_1^-.$$

It also follows for $|y| > 1$, such that $2^\ell \leq |y| \leq 2^{\ell+1}$, and $t \geq -2^{\gamma(\ell+1)}$ that

$$\begin{aligned} w(y, t) &= 2^{\alpha k+1} (v(2^{-k}y + x_0, 2^{-k\gamma}t + t_0) - m_k) \leq 2^{\alpha k+1} (a_{k-\ell-1} - m_k) \\ &\leq 2^{\alpha k+1} (a_{k-\ell-1} - b_{k-\ell-1} + b_k - m_k) \\ &\leq 2^{\alpha k+1} (2^{-\alpha(k-\ell-1)} - \frac{1}{2}2^{-k\alpha}) \\ &\leq 2^{1+\alpha(\ell+1)} - 1 \leq 2|2y|^\alpha - 1 \\ &\leq 2|2y|^\eta - 1. \end{aligned}$$

Here we used that (5-12) holds for $j \leq k$.

Suppose now that

$$|\{(x, t) : w(x, t) \leq 0\} \cap \{B_1 \times [-1, -\frac{1}{2}]\}| \geq \frac{1}{2}(1 - 1/2^{\frac{ps}{p-1}})|B_1| = \delta(p, s).$$

If not we would apply the same procedure to $-w$. Then w satisfies all the assumptions of Lemma 5.1 with $\delta = \delta(p, s)$, and so

$$w \leq 1 - \theta \quad \text{in } Q_{\frac{1}{2}}^-.$$

Scaling back to v yields

$$\begin{aligned} v(x, t) &\leq 2^{-1-\alpha k} (1 - \theta) + m_k \leq 2^{-1-k\alpha} (1 - \theta) + \frac{1}{2}a_k + b_k \\ &\leq b_k + 2^{-1-\alpha k} (1 - \theta) + 2^{-1-\alpha k} \\ &\leq b_k + 2^{-\alpha(k+1)} \end{aligned}$$

for $(x, t) \in Q_{2^{-k-1}}(x_0, t_0)$, by (5-11). Hence, if we let $b_{k+1} = b_k$ and $a_{k+1} = b_k + 2^{-\alpha(k+1)}$, we obtain (5-12) for the step $j = k + 1$ and the induction is complete. □

6. Large time limit

In this section, we prove [Theorem 1.2](#) and [Theorem 1.3](#). The main tools are the monotonicity of the Rayleigh quotient and the $W^{s,p}$ seminorm (equation (3-11) and [Proposition 3.6](#)), the compactness of weak solutions ([Theorem 3.8](#)) and the Hölder estimates ([Theorem 1.1](#)). In order to control the sign of the limiting ground state, we also need the following lemma.

Lemma 6.1. *Assume that v is a weak solution of (1-4). For any positive ground state w for $(-\Delta_p)^s$ and any constant $C > 0$, there is a $\delta = \delta(w, C) > 0$ such that if*

$$(1) [v(\cdot, 0)]_{W^{s,p}(\mathbb{R}^n)}^p \geq \lambda_{s,p} \int_{\Omega} |w|^p dx,$$

$$(2) \int_{\Omega} |v(x, 0)|^p dx \leq C,$$

$$(3) \frac{[v(\cdot, 0)]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v(x, 0)|^p dx} \leq \lambda_{s,p} + \delta,$$

$$(4) \int_{\Omega} |v^+(x, 0)|^p dx \geq \frac{1}{2} \int_{\Omega} |w|^p dx,$$

then

$$\int_{\Omega} |e^{\mu_{s,p}t} v^+(x, t)|^p dx \geq \frac{1}{2} \int_{\Omega} |w|^p dx \tag{6-1}$$

for $t \in [0, 1]$.

Proof. We argue towards a contradiction. If the result fails, then there are w and C such that for every $\delta > 0$, there is a weak solution v that satisfies (1)–(4) while (6-1) fails. Therefore, associated to $\delta_j := 1/j$ ($j \in \mathbb{N}$), there is a weak solution v_j that satisfies

$$(1) [v_j(\cdot, 0)]_{W^{s,p}(\mathbb{R}^n)}^p \geq \lambda_{s,p} \int_{\Omega} |w|^p dx,$$

$$(2) \int_{\Omega} |v_j(x, 0)|^p dx \leq C,$$

$$(3) \frac{[v_j(\cdot, 0)]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v_j(x, 0)|^p dx} \leq \lambda_{s,p} + \frac{1}{j},$$

$$(4) \int_{\Omega} |v_j^+(x, 0)|^p dx \geq \frac{1}{2} \int_{\Omega} |w|^p dx,$$

while

$$\int_{\Omega} |e^{\mu_{s,p}t_j} v_j^+(x, t_j)|^p dx < \frac{1}{2} \int_{\Omega} |w|^p dx \tag{6-2}$$

for some $t_j \in [0, 1]$.

Consequently, the sequence of initial conditions $(v_j(\cdot, 0))_{j \in \mathbb{N}}$ is bounded in $W_0^{s,p}(\Omega)$ and has a subsequence (not relabeled) that converges to a positive ground state g of $(-\Delta_p)^s$ in $W^{s,p}(\mathbb{R}^n)$. By [Theorem 3.8](#), it also follows that (a subsequence of) the sequence of weak solutions $(v_j)_{j \in \mathbb{N}}$ converges

to a weak solution \tilde{w} in $C([0, 2], L^p(\Omega)) \cap L^p([0, 2]; W^{s,p}(\mathbb{R}^n))$ with $\tilde{w}(\cdot, 0) = g$. By [Corollary 3.7](#), $\tilde{w}(\cdot, t) = e^{-\mu_{s,p}t} g$.

In addition, by (1) and since g is a ground state,

$$\int_{\Omega} |g|^p dx = \frac{1}{\lambda_{s,p}} [g]_{W^{s,p}(\mathbb{R}^n)}^p = \frac{1}{\lambda_{s,p}} \lim_{j \rightarrow \infty} [v_j(x, 0)]_{W^{s,p}(\mathbb{R}^n)}^p \geq \int_{\Omega} |w|^p dx.$$

However, sending $j \rightarrow \infty$ in (6-2) gives

$$\int_{\Omega} |g|^p dx = \int_{\Omega} |g^+|^p dx \leq \frac{1}{2} \int_{\Omega} |w|^p dx.$$

This is a contradiction as $w \not\equiv 0$. □

Corollary 6.2. *Assume that v is a weak solution of (1-4). For any positive ground state w for $(-\Delta_p)^s$ and any constant $C > 0$, there is a $\tilde{\delta} = \tilde{\delta}(w, C) > 0$ such that if*

(1) $e^{\mu_{s,p}pt} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \geq \lambda_{s,p} \int_{\Omega} |w|^p$ for all $t \geq 0$,

(2) $\int_{\Omega} |v(x, 0)|^p dx \leq C$,

(3) $\frac{[v(\cdot, 0)]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v(x, 0)|^p dx} \leq \lambda_{s,p} + \tilde{\delta}$,

(4) $\int_{\Omega} |v^+(x, 0)|^p dx \geq \frac{1}{2} \int_{\Omega} |w|^p dx$,

then

$$\int_{\Omega} |e^{\mu_{s,p}pt} v^+(x, t)|^p dx \geq \frac{1}{2} \int_{\Omega} |w|^p dx$$

for all $t \geq 0$.

Proof. Choose

$$\tilde{\delta} = \min(\lambda_{s,p}, \delta(w, 2C)),$$

where $\delta(w, 2C)$ is from [Lemma 6.1](#). It is clear that v satisfies the assumptions of [Lemma 6.1](#), so that in particular

$$\int_{\Omega} |e^{\mu_{s,p}pt} v^+(x, 1)|^p dx \geq \frac{1}{2} \int_{\Omega} |w|^p dx.$$

By [Proposition 3.6](#) combined with (2) and (3)

$$\int_{\Omega} |e^{\mu_{s,p}pt} v(x, t)|^p dx \leq \frac{1}{\lambda_{s,p}} [e^{\mu_{s,p}pt} v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \leq \frac{1}{\lambda_{s,p}} [v(\cdot, 0)]_{W^{s,p}(\mathbb{R}^n)}^p \leq C \left(1 + \frac{\tilde{\delta}}{\lambda_{s,p}} \right) \leq 2C$$

for any $t > 0$.

The above inequality, together with (1) and the monotonicity of the Rayleigh quotient, implies that $e^{\mu_{s,p}kt} v(x, t + k)$ satisfies properties (1)–(3) in [Lemma 6.1](#) with C replaced by $2C$. In particular,

Lemma 6.1 applied to $e^{\mu_{s,p}} v(x, t + 1)$ yields

$$\int_{\Omega} |e^{2\mu_{s,p}} v^+(x, 2)|^p dx \geq \frac{1}{2} \int_{\Omega} |w|^p dx.$$

Now we can apply Lemma 6.1 repeatedly to $e^{\mu_{s,p}k} v(x, t + k)$ for $k = 2, 3, \dots$ in order to obtain the desired result. □

We are now ready to treat the large time behavior of solutions of equation (1-4).

Proof of Theorem 1.2. Let v be a weak solution of (1-4). By (3-9) in Corollary 3.5,

$$\frac{d}{dt} [e^{\mu_{s,p}t} v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \leq 0 \tag{6-3}$$

for almost every $t \geq 0$. Consequently, the limit

$$S := \lim_{t \rightarrow \infty} [e^{\mu_{s,p}t} v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p$$

exists. If $S = 0$, there is nothing else to prove. Let us assume otherwise.

Let τ_k be an increasing sequence of positive numbers such that $\tau_k \rightarrow \infty$ as $k \rightarrow \infty$, and define, for $k = 1, 2, 3, \dots$,

$$v^k(x, t) = e^{\mu_{s,p}\tau_k} v(x, t + \tau_k). \tag{6-4}$$

Then v^k is a weak solution of (1-4) with initial data

$$g^k(x) := e^{\mu_{s,p}\tau_k} v(x, \tau_k).$$

By (6-3), $g^k \in W_0^{s,p}(\Omega)$ is uniformly bounded in $W^{s,p}(\mathbb{R}^n)$. Hence, it is clear that v^k satisfies the hypotheses of Theorem 3.8. Therefore, we can extract a subsequence $\{v^{k_j}\}_{j \in \mathbb{N}}$ converging to a weak solution w as detailed in Theorem 3.8. We may also assume that $v^{k_j}(\cdot, t)$ converges to $w(\cdot, t)$ in $W^{s,p}(\mathbb{R}^n)$ for almost every time $t \geq 0$ since this occurs for a subsequence.

We now observe that by (6-3)

$$S = e^{\mu_{s,p}t} \lim_{j \rightarrow \infty} [v^{k_j}(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p = e^{\mu_{s,p}t} [w(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \tag{6-5}$$

for almost every time $t \geq 0$. Since $[0, \infty) \ni t \mapsto [w(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p$ is absolutely continuous (by Lemma 3.3),

$$S = e^{\mu_{s,p}t} [w(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p$$

holds for all $t \geq 0$. As w is a solution of (1-4), (3-9) in Corollary 3.5 implies

$$0 = \frac{1}{p} \frac{d}{dt} (e^{\mu_{s,p}t} [w(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p) = e^{(\mu_{s,p}t)} \left(\mu_{s,p} [w(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p - \int_{\Omega} |w_t(x, t)|^p dx \right)$$

for almost every $t \geq 0$.

A more careful inspection of the proof of (3-8) reveals that if

$$\mu_{s,p} [w(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p = \int_{\Omega} |w_t(x, t)|^p dx$$

then we must have $[w(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p = \lambda_{s,p} \int_{\Omega} |w(x, t)|^p dx$. Therefore $w(\cdot, t)$ is a ground state for almost every $t > 0$. The absolute continuity of $[w(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}$ and $\|w(\cdot, t)\|_{L^p(\Omega)}$ then implies that $w(\cdot, t)$ is a ground state for all $t \geq 0$. By [Corollary 3.7](#),

$$w(x, t) = e^{-\mu_{s,p}t} w_0,$$

where $w_0(x) = w(x, 0)$ is a ground state.

For any $t_0 \in [0, T]$ such that the limit (6-5) holds, we have, by [Proposition 3.6](#),

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{[v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v(x, t)|^p dx} &= \lim_{j \rightarrow \infty} \frac{[v(\cdot, \tau_{k_j} + t_0)]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v(x, \tau_{k_j} + t_0)|^p dx} = \lim_{j \rightarrow \infty} \frac{[v^{k_j}(\cdot, t_0)]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v^{k_j}(x, t_0)|^p dx} \\ &= \frac{[w(\cdot, t_0)]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |w(x, t_0)|^p dx} = \frac{[w_0]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |w_0|^p dx} = \lambda_{s,p}. \end{aligned}$$

Since weak convergence together with the convergence of the norm implies strong convergence, the limit $w_0 = \lim_{j \rightarrow \infty} e^{\mu_{s,p}\tau_{k_j}} v(\cdot, \tau_{k_j})$ holds in $W^{s,p}(\mathbb{R}^n)$. We conclude that $\{e^{\mu_{s,p}\tau_k} v(\cdot, \tau_k)\}_{k \in \mathbb{N}}$ has a subsequence converging in $W^{s,p}(\mathbb{R}^n)$ to a ground state w_0 .

Recall that, due to the simplicity of the ground states, w_0 is determined completely by its sign and the constant

$$S = [w_0]_{W^{s,p}(\mathbb{R}^n)}^p.$$

We may assume $w_0 \geq 0$; if not we can consider $-v$ instead.

Now we note that for any $t \geq 0$,

$$e^{\mu_{s,p}pt} [v(\cdot, t)]_{W^{s,p}(\mathbb{R}^n)}^p \geq \lim_{j \rightarrow \infty} e^{\mu_{s,p}p\tau_{k_j}} [v(\cdot, \tau_{k_j})]_{W^{s,p}(\mathbb{R}^n)}^p = [w_0]_{W^{s,p}(\mathbb{R}^n)}^p = \lambda_{s,p} \int_{\Omega} |w_0|^p. \tag{6-6}$$

Since $v^{k_j}(x, 0) = g^{k_j}(x)$ converges in $W^{s,p}(\mathbb{R}^n)$ to ground state w_0 , we also have

$$\lim_{j \rightarrow \infty} \frac{[v^{k_j}(\cdot, 0)]_{W^{s,p}(\mathbb{R}^n)}^p}{\int_{\Omega} |v^{k_j}(x, 0)|^p dx} = \lambda_{s,p} \tag{6-7}$$

and

$$\int_{\Omega} |(v^{k_j}(x, 0))^+|^p dx \geq \frac{1}{2} \int_{\Omega} |w_0|^p dx \tag{6-8}$$

whenever j is large enough. In addition, due to the monotonicity of the $W^{s,p}$ norm,

$$\int_{\Omega} |v^{k_j}(x, 0)|^p dx \leq \frac{1}{\lambda_{s,p}} [v^{k_j}(\cdot, 0)]_{W^{s,p}(\mathbb{R}^n)}^p \leq \frac{1}{\lambda_{s,p}} [g]_{W^{s,p}(\mathbb{R}^n)}^p, \tag{6-9}$$

where $g = v(\cdot, 0)$. From (6-6)–(6-9), it is clear that for j large enough, v^{k_j} satisfies assumptions (1)–(4) in [Corollary 6.2](#), with $w = w_0$ and $C = [g]_{W^{s,p}(\mathbb{R}^n)}^p / \lambda_{s,p}$. As a result,

$$\int_{\Omega} |e^{\mu_{s,p}t} (v^{k_j})^+(x, t)|^p dx \geq \frac{1}{2} \int_{\Omega} |w_0|^p dx$$

for all $t \geq 0$, which implies

$$\int_{\Omega} |e^{\mu_{s,p}t} (v)^+(x, t)|^p dx \geq \frac{1}{2} \int_{\Omega} |w_0|^p dx \tag{6-10}$$

for t large enough.

Suppose now that we pick another convergent subsequence of $\{e^{\mu_{s,p}\tau_k} v(\cdot, \tau_k)\}_{k \in \mathbb{N}}$. Then arguing as above, the sequence converges in $W^{s,p}(\mathbb{R}^n)$ to a ground state w_1 and by (6-5),

$$[w_1]_{W^{s,p}(\mathbb{R}^n)}^p = S.$$

By the simplicity of ground states, $w_1 = w_0$ or $w_1 = -w_0$. Passing $t \rightarrow \infty$ in (6-10), we obtain

$$\int_{\Omega} (w_1^+)^p dx \geq \frac{1}{2} \int_{\Omega} |w_0|^p dx,$$

forcing w_1 to be positive and hence $w_1 = w_0$. As the sequence $\{\tau_k\}_{k \in \mathbb{N}}$ was chosen arbitrarily, it follows that $e^{\mu_{s,p}t} v(\cdot, t) \rightarrow w_0$ as $t \rightarrow \infty$ in $W^{s,p}(\mathbb{R}^n)$. □

We are finally in a position to prove [Theorem 1.3](#).

Proof of Theorem 1.3. Let τ_k be an increasing sequence of positive numbers such that $\tau_k \rightarrow \infty$ as $k \rightarrow \infty$. In [Theorem 1.2](#), we established that $\lim_{k \rightarrow \infty} e^{\mu_{s,p}\tau_k} v(\cdot, \tau_k) = w$ in $W^{s,p}(\mathbb{R}^n)$. In view of [Proposition 4.1](#), it suffices to show that this convergence occurs uniformly on $\bar{\Omega}$.

To this end, define v^k as in (6-4). We also remark that $e^{-\mu_{s,p}t} \Psi$ is a solution of equation (1-1). By the comparison principle,

$$|v^k(x, t)| \leq \Psi(x) \tag{6-11}$$

for $(x, t) \in \mathbb{R}^n \times [-1, 1]$ for all $k \in \mathbb{N}$ large enough. These bounds with [Theorem 1.1](#) give that v^k is uniformly bounded in $C^\alpha(B \times [0, 1])$ for any ball $B \Subset \Omega$. By a routine covering argument, v^k is then uniformly bounded in $C^\alpha(K \times [0, 1])$ for any compact $K \Subset \Omega$. We now claim that the sequence v^k is equicontinuous in $\bar{\Omega} \times [0, 1]$.

Fix $\varepsilon > 0$. Recall that Ψ is continuous up to the boundary of Ω . By (6-11), it follows that there is δ so that $|v^k(x, t)| \leq \frac{1}{2}\varepsilon$ whenever $d(x) := d(x, \partial\Omega) < \delta$ and $t \in [0, 1]$. Now we will show that if $|x - y| + |t - \tau|$ is small enough, then $|v^k(x, t) - v^k(y, \tau)| \leq \varepsilon$. We treat three cases differently as follows.

(1) $d(x) < \frac{1}{2}\delta$ and $d(y) < \frac{1}{2}\delta$: Then

$$|v^k(x, t) - v^k(y, \tau)| \leq |v^k(x, t)| + |v^k(y, \tau)| \leq \varepsilon.$$

(2) $d(x) < \frac{1}{2}\delta$ and $d(y) > \frac{1}{2}\delta$: Then $|x - y| < \frac{1}{2}\delta$ implies $d(y) < \delta$ so that again

$$|v^k(x, t) - v^k(y, \tau)| \leq |v^k(x, t)| + |v^k(y, \tau)| \leq \varepsilon.$$

(3) $d(x) > \frac{1}{2}\delta$ and $d(y) > \frac{1}{2}\delta$: Then by the local Hölder estimates,

$$|v^k(x, t) - v^k(y, \tau)| \leq C_\delta (|x - y| + |t - \tau|)^\alpha \leq \varepsilon$$

if we choose $|x - y| + |t - \tau| \leq (\varepsilon/C_\delta)^{1/\alpha}$.

Hence, if

$$|x - y| + |t - \tau| \leq \min\left(\frac{1}{2}\delta, (\varepsilon/C_\delta)^{1/\alpha}\right)$$

then $|v^k(x, t) - v^k(y, \tau)| \leq \varepsilon$. Therefore, the sequence v^k is equicontinuous on $\bar{\Omega} \times [0, 1]$. By the Arzelà–Ascoli theorem, we can extract a subsequence v^{k_j} converging to $e^{-\mu s \cdot p^t} w$, the limit in (1-5), uniformly in $\bar{\Omega} \times [0, 1]$. \square

Acknowledgements

We thank the referees for carefully reviewing this work and providing many useful suggestions. Hynd was partially supported by NSF grant DMS-1301628. Lindgren was supported by the Swedish Research Council, grant no. 2012-3124, and partially supported by the Royal Swedish Academy of Sciences. Part of this work was carried out when Lindgren was visiting University of Pennsylvania. The math department and its facilities are kindly acknowledged.

References

- [Agueh et al. 2010] M. Agueh, A. Blanchet, and J. A. Carrillo, “Large time asymptotics of the doubly nonlinear equation in the non-displacement convexity regime”, *J. Evol. Equ.* **10**:1 (2010), 59–84. MR 2602927 Zbl 1239.35063
- [Ambrosio et al. 2008] L. Ambrosio, N. Gigli, and G. Savaré, *Gradient flows in metric spaces and in the space of probability measures*, 2nd ed., Birkhäuser, Basel, 2008. MR 2401600 Zbl 1145.35001
- [Armstrong and Trokhimtchouk 2010] S. N. Armstrong and M. Trokhimtchouk, “Long-time asymptotics for fully nonlinear homogeneous parabolic equations”, *Calc. Var. Partial Differential Equations* **38**:3–4 (2010), 521–540. MR 2647131 Zbl 1217.35023
- [Aronson and Peletier 1981] D. G. Aronson and L. A. Peletier, “Large time behaviour of solutions of the porous medium equation in bounded domains”, *J. Differential Equations* **39**:3 (1981), 378–412. MR 612594 Zbl 0475.35059
- [Bjorland et al. 2012] C. Bjorland, L. Caffarelli, and A. Figalli, “Non-local gradient dependent operators”, *Adv. Math.* **230**:4–6 (2012), 1859–1894. MR 2927356 Zbl 1252.35099
- [Brasco and Franzina 2014] L. Brasco and G. Franzina, “Convexity properties of Dirichlet integrals and Picone-type inequalities”, *Kodai Math. J.* **37**:3 (2014), 769–799. MR 3273896 Zbl 1315.47054
- [Brasco and Lindgren 2017] L. Brasco and E. Lindgren, “Higher sobolev regularity for the fractional p -Laplace equation in the superquadratic case”, *Adv. Math.* **304**:2 (2017), 300–354.
- [Brasco and Parini 2015] L. Brasco and E. Parini, “The second eigenvalue of the fractional p -Laplacian”, *Adv. Calc. Var.* (online publication August 2015).
- [Brasco et al. 2014] L. Brasco, E. Lindgren, and E. Parini, “The fractional Cheeger problem”, *Interfaces Free Bound.* **16**:3 (2014), 419–458. MR 3264796 Zbl 1301.49115
- [Brasco et al. 2016] L. Brasco, E. Parini, and M. Squassina, “Stability of variational eigenvalues for the fractional p -Laplacian”, *Discrete Contin. Dyn. Syst.* **36**:4 (2016), 1813–1845. MR 3411543 Zbl 1336.35270
- [Caffarelli and Silvestre 2009] L. Caffarelli and L. Silvestre, “Regularity theory for fully nonlinear integro-differential equations”, *Comm. Pure Appl. Math.* **62**:5 (2009), 597–638. MR 2494809 Zbl 1170.45006
- [Caffarelli and Vasseur 2010] L. A. Caffarelli and A. Vasseur, “Drift diffusion equations with fractional diffusion and the quasi-geostrophic equation”, *Ann. of Math. (2)* **171**:3 (2010), 1903–1930. MR 2680400 Zbl 1204.35063
- [Chambolle et al. 2012] A. Chambolle, E. Lindgren, and R. Monneau, “A Hölder infinity Laplacian”, *ESAIM Control Optim. Calc. Var.* **18**:3 (2012), 799–835. MR 3041665 Zbl 1255.35078
- [Chasseigne and Jakobsen 2015] E. Chasseigne and E. Jakobsen, “On nonlocal quasilinear equations and their local limits”, preprint, 2015, <https://hal.archives-ouvertes.fr/hal-01134618>.

- [Crandall et al. 1992] M. G. Crandall, H. Ishii, and P.-L. Lions, “User’s guide to viscosity solutions of second order partial differential equations”, *Bull. Amer. Math. Soc. (N.S.)* **27**:1 (1992), 1–67. MR 1118699 Zbl 0755.35015
- [Del Pezzo and Salort 2015] L. M. Del Pezzo and A. M. Salort, “The first non-zero Neumann p -fractional eigenvalue”, *Nonlinear Anal.* **118** (2015), 130–143. MR 3325609 Zbl 1328.35274
- [Di Castro et al. 2014] A. Di Castro, T. Kuusi, and G. Palatucci, “Nonlocal Harnack inequalities”, *J. Funct. Anal.* **267**:6 (2014), 1807–1836. MR 3237774 Zbl 1302.35082
- [Di Castro et al. 2016] A. Di Castro, T. Kuusi, and G. Palatucci, “Local behavior of fractional p -minimizers”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **33**:5 (2016), 1279–1299. MR 3542614
- [Di Nezza et al. 2012] E. Di Nezza, G. Palatucci, and E. Valdinoci, “Hitchhiker’s guide to the fractional Sobolev spaces”, *Bull. Sci. Math.* **136**:5 (2012), 521–573. MR 2944369 Zbl 1252.46023
- [Evans 2010] L. C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, RI, 2010. MR 2597943 Zbl 1194.35001
- [Franzina and Palatucci 2014] G. Franzina and G. Palatucci, “Fractional p -eigenvalues”, *Riv. Math. Univ. Parma (N.S.)* **5**:2 (2014), 373–386. MR 3307955 Zbl 1327.35286
- [Gal and Warma 2016] C. G. Gal and M. Warma, “Bounded solutions for nonlocal boundary value problems on Lipschitz manifolds with boundary”, *Adv. Nonlinear Stud.* **16**:3 (2016), 529–550. MR 3518345 Zbl 1342.35435
- [Hynd 2016] R. Hynd, “Compactness methods for doubly nonlinear parabolic systems”, preprint, 2016. To appear in *Trans. of the AMS*. arXiv 1410.7844
- [Hynd and Lindgren 2016] R. Hynd and E. Lindgren, “A doubly nonlinear evolution for the optimal Poincaré inequality”, *Calc. Var. Partial Differential Equations* **55**:4 (2016), 55:100. MR 3528441
- [Iannizzotto and Squassina 2014] A. Iannizzotto and M. Squassina, “Weyl-type laws for fractional p -eigenvalue problems”, *Asymptot. Anal.* **88**:4 (2014), 233–245. MR 3245079 Zbl 1296.35103
- [Iannizzotto et al. 2016] A. Iannizzotto, S. Mosconi, and M. Squassina, “Global Hölder regularity for the fractional p -Laplacian”, preprint, 2016. To appear in *Rev. Mat. Iberoam.*
- [Ishii and Nakamura 2010] H. Ishii and G. Nakamura, “A class of integral equations and approximation of p -Laplace equations”, *Calc. Var. Partial Differential Equations* **37**:3–4 (2010), 485–522. MR 2592984 Zbl 1198.45005
- [Juutinen 2001] P. Juutinen, “On the definition of viscosity solutions for parabolic equations”, *Proc. Amer. Math. Soc.* **129**:10 (2001), 2907–2911. MR 1840092 Zbl 0979.35071
- [Kamin and Vázquez 1988] S. Kamin and J. L. Vázquez, “Fundamental solutions and asymptotic behaviour for the p -Laplacian equation”, *Rev. Mat. Iberoamericana* **4**:2 (1988), 339–354. MR 1028745 Zbl 0699.35158
- [Kilpeläinen and Lindqvist 1996] T. Kilpeläinen and P. Lindqvist, “On the Dirichlet boundary value problem for a degenerate parabolic equation”, *SIAM J. Math. Anal.* **27**:3 (1996), 661–683. MR 1382827 Zbl 0857.35071
- [Kim and Lee 2013] S. Kim and K.-A. Lee, “Asymptotic behavior in degenerate parabolic fully nonlinear equations and its application to elliptic eigenvalue problems”, *J. Differential Equations* **254**:8 (2013), 3259–3306. MR 3020876 Zbl 1264.35044
- [Korvenpää et al. 2016] J. Korvenpää, T. Kuusi, and G. Palatucci, “Fractional superharmonic functions and the perron method for nonlinear integro-differential equations”, preprint, 2016, <http://mipa.unimes.fr/preprints/MIPA-Preprint02-2016.pdf>.
- [Kuusi et al. 2015a] T. Kuusi, G. Mingione, and Y. Sire, “Nonlocal equations with measure data”, *Comm. Math. Phys.* **337**:3 (2015), 1317–1368. MR 3339179 Zbl 1323.45007
- [Kuusi et al. 2015b] T. Kuusi, G. Mingione, and Y. Sire, “Nonlocal self-improving properties”, *Anal. PDE* **8**:1 (2015), 57–114. MR 3336922 Zbl 1317.35284
- [Lara and Dávila 2014] H. C. Lara and G. Dávila, “Regularity for solutions of non local parabolic equations”, *Calc. Var. Partial Differential Equations* **49**:1–2 (2014), 139–172. MR 3148110 Zbl 1292.35068
- [Lieb 1983] E. H. Lieb, “On the lowest eigenvalue of the Laplacian for the intersection of two domains”, *Invent. Math.* **74**:3 (1983), 441–448. MR 724014 Zbl 0538.35058
- [Lindgren 2016] E. Lindgren, “Hölder estimates for viscosity solutions of equations of fractional p -Laplace type”, *NoDEA Nonlinear Differential Equations Appl.* **23**:5 (2016), 23–55.

- [Lindgren and Lindqvist 2014] E. Lindgren and P. Lindqvist, “Fractional eigenvalues”, *Calc. Var. Partial Differential Equations* **49**:1–2 (2014), 795–826. [MR 3148135](#) [Zbl 1292.35193](#)
- [Lindqvist 1990] P. Lindqvist, “On the equation $\operatorname{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$ ”, *Proc. Amer. Math. Soc.* **109**:1 (1990), 157–164. [MR 1007505](#) [Zbl 0714.35029](#)
- [Lindqvist 2008] P. Lindqvist, “A nonlinear eigenvalue problem”, pp. 175–203 in *Topics in mathematical analysis*, Ser. Anal. Appl. Comput. **3**, World Sci. Publ., Hackensack, NJ, 2008. [MR 2462954](#) [Zbl 1160.35058](#)
- [Mazón et al. 2016] J. M. Mazón, J. D. Rossi, and J. Toledo, “Fractional p -Laplacian evolution equations”, *J. Math. Pures Appl.* (9) **105**:6 (2016), 810–844. [MR 3491533](#) [Zbl 1338.45009](#)
- [Puhst 2015] D. Puhst, “On the evolutionary fractional p -Laplacian”, *Appl. Math. Res. Express. AMRX* **2** (2015), 253–273. [MR 3394266](#) [Zbl 1332.35383](#)
- [Silvestre 2010] L. Silvestre, “Hölder continuity for integro-differential parabolic equations with polynomial growth respect to the gradient”, *Discrete Contin. Dyn. Syst.* **28**:3 (2010), 1069–1081. [MR 2644779](#) [Zbl 1193.35240](#)
- [Silvestre 2012] L. Silvestre, “Hölder estimates for advection fractional-diffusion equations”, *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **11**:4 (2012), 843–855. [MR 3060702](#) [Zbl 1263.35056](#)
- [Simon 1987] J. Simon, “Compact sets in the space $L^p(0, T; B)$ ”, *Ann. Mat. Pura Appl.* (4) **146** (1987), 65–96. [MR 916688](#) [Zbl 0629.46031](#)
- [Stan and Vázquez 2013] D. Stan and J. L. Vázquez, “Asymptotic behaviour of the doubly nonlinear diffusion equation $u_t = \Delta_p u^m$ on bounded domains”, *Nonlinear Anal.* **77** (2013), 1–32. [MR 2988757](#) [Zbl 1258.35130](#)
- [Vázquez 2016] J. L. Vázquez, “The Dirichlet problem for the fractional p -Laplacian evolution equation”, *J. Differential Equations* **260**:7 (2016), 6038–6056. [MR 3456825](#) [Zbl 06539959](#)
- [Warma 2016] M. Warma, “Local Lipschitz continuity of the inverse of the fractional p -Laplacian, Hölder type continuity and continuous dependence of solutions to associated parabolic equations on bounded domains”, *Nonlinear Anal.* **135** (2016), 129–157. [MR 3473114](#) [Zbl 1333.35332](#)

Received 18 Nov 2015. Revised 16 May 2016. Accepted 17 Jun 2016.

RYAN HYND: rhynd@math.upenn.edu

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, United States

ERIK LINDGREN: eriklin@kth.se

Department of Mathematics, Royal Institute of Technology, 100 44 Stockholm, Sweden

Analysis & PDE

msp.org/apde

EDITORS

EDITOR-IN-CHIEF

Patrick Gérard

patrick.gerard@math.u-psud.fr

Université Paris Sud XI

Orsay, France

BOARD OF EDITORS

Nicolas Burq	Université Paris-Sud 11, France nicolas.burq@math.u-psud.fr	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Yuval Peres	University of California, Berkeley, USA peres@stat.berkeley.edu
Sun-Yung Alice Chang	Princeton University, USA chang@math.princeton.edu	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Wilhelm Schlag	University of Chicago, USA schlag@math.uchicago.edu
Vaughan Jones	U.C. Berkeley & Vanderbilt University vaughan.f.jones@vanderbilt.edu	Sylvia Serfaty	New York University, USA serfaty@cims.nyu.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Gilles Lebeau	Université de Nice Sophia Antipolis, France lebeau@unice.fr	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbb@math.mit.edu	András Vasy	Stanford University, USA andras@math.stanford.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu	Steven Zelditch	Northwestern University, USA zelditch@math.northwestern.edu
Clément Mouhot	Cambridge University, UK c.mouhot@dpms.cam.ac.uk	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2016 is US \$235/year for the electronic version, and \$430/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 9 No. 6 2016

A complete study of the lack of compactness and existence results of a fractional Nirenberg equation via a flatness hypothesis, I	1285
Wael Abdelhedi, Hichem Chtioui and Hichem Hajaiej	
On positive solutions of the (p, A) -Laplacian with potential in Morrey space	1317
Yehuda Pinchover and Georgios Psaradakis	
Geometric optics expansions for hyperbolic corner problems, I: Self-interaction phenomenon	1359
Antoine Benoit	
The interior C^2 estimate for the Monge–Ampère equation in dimension $n = 2$	1419
Chuanqiang Chen, Fei Han and Qianzhong Ou	
Bounded solutions to the Allen–Cahn equation with level sets of any compact topology	1433
Alberto Enciso and Daniel Peralta-Salas	
Hölder estimates and large time behavior for a nonlocal doubly nonlinear evolution	1447
Ryan Hynd and Erik Lindgren	
Boundary $C^{1,\alpha}$ regularity of potential functions in optimal transportation with quadratic cost	1483
Elina Andriyanova and Shibing Chen	
Commutators with fractional differentiation and new characterizations of BMO-Sobolev spaces	1497
Yanping Chen, Yong Ding and Guixiang Hong	