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Let Y be a smooth rational surface, and let D be a cycle of rational curves on Y that is an anticanonical divisor, i.e., an element of $|-K_Y|$. Looijenga studied the geometry of such surfaces Y in case D has at most five components and identified a geometrically significant subset R of the divisor classes of square -2 orthogonal to the components of D. Motivated by recent work of Gross, Hacking, and Keel on the global Torelli theorem for pairs (Y, D), we attempt to generalize some of Looijenga's results in case D has more than five components. In particular, given an integral isometry f of $H^2(Y)$ that preserves the classes of the components of D, we investigate the relationship between the condition that f preserves the "generic" ample cone of Y and the condition that f preserves the set R.

Introduction

The ample cone of a del Pezzo surface Y (or rather the associated dual polyhedron) was studied classically by, among others, Gosset, Schoute, Kantor, Coble, Todd, Coxeter, and Du Val. For a brief historical discussion, one can consult the remarks in [Coxeter 1973, §11.x]. From this point of view, the lines on Y are the main object of geometric interest as they are the walls of the ample cone or the vertices of the dual polyhedron. The corresponding root system (in case $K_Y^2 \leq 6$) only manifests itself geometrically by allowing del Pezzo surfaces with rational double points or, equivalently, smooth surfaces Y with $-K_Y$ nef and big but not ample. This is explicitly worked out in [Du Val 1934]. On the other hand, the root system, or rather its Weyl group, appears for a smooth del Pezzo surface as a group of symmetries of the ample cone, a fact which (in a somewhat different guise) was already known to Cartan. Perhaps the culmination of the classical side of the story is [Du Val 1937], where the blowup of $\mathbb{P}^2$ at $n \geq 9$ points is also systematically considered. In modern times, Manin explained the appearance of the Weyl group by noting that the orthogonal complement to K_Y in $H^2(Y; \mathbb{Z})$ is a root lattice Λ. Moreover, given any root of Λ, in other words an element β of square -2, there

exists a deformation of Y for which $\beta = \pm[C]$, where C is a smooth rational curve of self-intersection -2. For modern expositions of the theory, see for example the book of Manin [1986] or the account of Demazure [1980a; 1980b; 1980c; 1980d].

In general, it seems hard to study an arbitrary rational surface Y without imposing some extra conditions. One very natural condition is that $-K_Y$ is effective, i.e., that $-K_Y = D$ for an effective divisor D. In case the intersection matrix of D is negative definite, such pairs (Y, D) arise naturally in the study of minimally elliptic singularities: the case where D is a smooth elliptic curve corresponds to the case of simple elliptic singularities, the case where D is a nodal curve or a cycle of smooth rational curves meeting transversally corresponds to the case of cusp singularities, and the case where D is reduced but has one component with a cusp, two components with a tacnode, or three components meeting at a point corresponds to triangle singularities. From this point of view, the case where D is a cycle of rational curves is the most plentiful. The systematic study of such surfaces in case the intersection matrix of D is negative definite dates back to [Looijenga 1981]. However, for various technical reasons, most of the results of that paper are proved under the assumption that the number of components in the cycle is at most 5. Some of the main points of Looijenga's seminal paper are as follows. Let R denote the set of elements in $H^2(Y; \mathbb{Z})$ of square -2 that are orthogonal to the components of D and that are of the form $\pm[C]$, where C is a smooth rational curve disjoint from D, for some deformation of the pair (Y, D). In terms of deformations of singularities, the set R is related to the possible rational double point singularities that can arise as deformations of the dual cusp to the cusp singularity corresponding to D. Looijenga noted that, in general, there exist elements in $H^2(Y; \mathbb{Z})$ of square -2 that are orthogonal to the components of D but that do not lie in R. Moreover, reflections in elements of the set R give symmetries of the "generic" ample cone (which is the same as the ample cone in case there are no smooth rational curves on Y disjoint from D). Finally, still under the assumption of at most five components, any isometry of $H^2(Y; \mathbb{Z})$ that preserves the positive cone, the classes $[D_i]$, and the set R preserves the generic ample cone.

This paper, which is an attempt to see how much of [Looijenga 1981] can be generalized to the case of arbitrarily many components, is motivated by a question raised by the recent work of Gross, Hacking, and Keel [Gross et al. 2013] on, among other matters, the global Torelli theorem for pairs (Y, D) where D is an anticanonical cycle on the rational surface Y. In order to formulate this theorem in a fairly general way, one would like to characterize the isometries f of $H^2(Y, \mathbb{Z})$, preserving the positive cone and fixing the classes $[D_i]$, which preserve the ample cone of Y. It is natural to ask if, at least in the generic case, the condition that $f(R) = R$ is sufficient. In this paper, we give various criteria on R that insure that, if an isometry f of $H^2(Y; \mathbb{Z})$ preserves the positive cone, the classes $[D_i]$,

and the set R, then f preserves the generic ample cone. Typically, one needs a hypothesis that says that R is large. For example, one such hypothesis is that there is a subset of R that spans a negative definite codimension-1 subspace of the orthogonal complement to the components of D. In theory, at least under various extra hypotheses, such a result gives a necessary and sufficient condition for an isometry to preserve the generic ample cone. In practice, however, the determination of the set R in general is a difficult problem, which seems close in its complexity to the problem of describing the generic ample cone of Y. Finally, we show that some assumptions on (Y, D) are necessary by giving examples where $R = \varnothing$, so that the condition that an isometry f preserves R is automatic, and of isometries f such that f preserves the positive cone, the classes $[D_i]$, and (vacuously) the set R but f does not preserve the generic ample cone. We do not yet have a good understanding of the relationship between preserving the ample cone and preserving the set R.

An outline of this paper is as follows. The preliminary Section 1 reviews standard methods for constructing nef classes on algebraic surfaces and applies this to the study of when the normal surface obtained by contracting a negative definite anticanonical cycle on a rational surface is projective. In Section 2, we analyze the ample cone and generic ample cone of a pair (Y, D) and show that the set R defined by Looijenga is exactly the set of elements β in $H^2(Y; \mathbb{Z})$ of square -2 that are orthogonal to the components of D such that reflection about β preserves the generic ample cone. Much of the material of Section 2 overlaps with results in [Gross et al. 2013], proved there by somewhat different methods. Section 3 is devoted to giving various sufficient conditions for an isometry f of $H^2(Y; \mathbb{Z})$ to preserve the generic ample cone, including the one described above. Section 4 gives examples of pairs (Y, D) satisfying the sufficient conditions of Section 3 where the number of components of D and the multiplicity $-D^2$ are arbitrarily large as well as examples showing that some hypotheses on (Y, D) are necessary.

Notation and conventions. We work over $\mathbb{C}$. If X is a smooth projective surface with $h^1(\mathcal{O}_X) = h^2(\mathcal{O}_X) = 0$ and $\alpha \in H^2(X; \mathbb{Z})$, we let L_α denote the corresponding holomorphic line bundle, i.e., $c_1(L_\alpha) = \alpha$. Given a curve C or divisor class G on X, we let $[C]$ or $[G]$ denote the corresponding element of $H^2(X; \mathbb{Z})$. Intersection pairing on curves or divisors, or on elements in the second cohomology of a smooth surface (viewed as a canonically oriented 4-manifold), is denoted by multiplication.

1. Preliminaries

In this paper, Y denotes a smooth rational surface with $-K_Y = D = \sum_{i=1}^{r} D_i$ a (reduced) cycle of rational curves; i.e., each D_i is a smooth rational curve and D_i meets $D_{i\pm1}$ transversally, where i is taken mod r except for $r = 1$, in which case $D_1 = D$ is an irreducible nodal curve. We note, however, that many of the results

in this paper can be generalized to the case where $D \in |-K_Y|$ is not assumed to be a cycle. The integer $r = r(D)$ is called the *length* of D. An *orientation* of D is an orientation of the dual graph (with appropriate modifications in case $r = 1$). We shall abbreviate the data of the surface Y and the oriented cycle D by (Y, D) and refer to it as an *anticanonical pair*. If the intersection matrix $(D_i \cdot D_j)$ is negative definite, we say that (Y, D) is a *negative definite anticanonical pair*.

Definition 1.1. An irreducible curve E on Y is an *exceptional curve* if $E \cong \mathbb{P}^1$, $E^2 = -1$, and $E \neq D_i$ for any i. An irreducible curve C on Y is a -2-*curve* if $C \cong \mathbb{P}^1$, $C^2 = -2$, and $C \neq D_i$ for any i. Let Δ_Y be the set of all -2-curves on Y, and let $\mathsf{W}(\Delta_Y)$ be the group of integral isometries of $H^2(Y; \mathbb{R})$ generated by the reflections in the classes in the set Δ_Y.

Definition 1.2. Let $\Lambda = \Lambda(Y, D) \subseteq H^2(Y; \mathbb{Z})$ be the orthogonal complement of the lattice spanned by the classes $[D_i]$. Fixing the identification $\mathrm{Pic}^0 D \cong \mathbb{G}_m$ defined by the orientation of the cycle D, we define the *period homomorphism* $\varphi_Y : \Lambda \to \mathbb{G}_m$ as follows: if $\alpha \in \Lambda$ and L_α is the corresponding line bundle, then $\varphi_Y(\alpha) \in \mathbb{G}_m$ is the image of the line bundle of multidegree 0 on D defined by $L_\alpha | D$. Clearly φ_Y is a homomorphism. The *period map* is the function that associates to the pair (Y, D) the homomorphism $\varphi_Y : \Lambda \to \mathbb{G}_m$.

By [Looijenga 1981; Friedman and Scattone 1986; Friedman 1984], we have:

Theorem 1.3. *The period map is surjective. More precisely, given Y as above and given an arbitrary homomorphism $\varphi : \Lambda \to \mathbb{G}_m$, there exists a deformation of the pair (Y, D) over a smooth connected base, which we can take to be $(\mathbb{G}_m)^n$ for some n, such that the monodromy of the family is trivial and there exists a fiber of the deformation, say (Y', D'), such that $\varphi_{Y'} = \varphi$ under the induced identification of $\Lambda(Y', D')$ with Λ.* $\qquad\qquad\square$

For future reference, we recall some standard facts about negative definite curves on a surface.

Lemma 1.4. *Let X be a smooth projective surface, and let $G_1, \ldots, G_n$ be irreducible curves on X such that the intersection matrix $(G_i \cdot G_j)$ is negative definite. Let F be an effective divisor on X not necessarily reduced or irreducible and such that, for all i, G_i is not a component of F.*

(i) *Given $r_i \in \mathbb{R}$, if $(F + \sum_i r_i G_i) \cdot G_j = 0$ for all j, then $r_i \geq 0$ for all i, and, for every subset I of $\{1, \ldots, n\}$, if $\bigcup_{i \in I} G_i$ is a connected curve such that $F \cdot G_j \neq 0$ for some $j \in I$, then $r_i > 0$ for $i \in I$.*

(ii) *Given $s_i, t_i \in \mathbb{R}$, if $[F] + \sum_i s_i [G_i] = \sum_i t_i [G_i]$, then $F = 0$ and $s_i = t_i$ for all i.*

The following general result is also well known:

Proposition 1.5. *Let X be a smooth projective surface, and let $G_1, \ldots, G_n$ be irreducible curves on X such that the intersection matrix $(G_i \cdot G_j)$ is negative definite. (We do not, however, assume that $\bigcup_i G_i$ is connected.) Then there exists a nef and big divisor H on X such that $H \cdot G_j = 0$ for all j and, if C is an irreducible curve such that $C \neq G_j$ for any j, then $H \cdot C > 0$. In fact, the set of nef and big $\mathbb{R}$-divisors that are orthogonal to $\{G_1, \ldots, G_n\}$ is a nonempty open subset of $\{G_1, \ldots, G_n\}^{\perp} \otimes \mathbb{R}$.*

Proof. Fix an ample divisor H_0 on X. Since $(G_i \cdot G_j)$ is negative definite, there exist $r_i \in \mathbb{Q}$ such that $\left(\sum_i r_i G_i\right) \cdot G_j = -(H_0 \cdot G_j)$ for every j. Hence, $(H_0 + \sum_i r_i G_i) \cdot G_j = 0$. By Lemma 1.4, $r_i > 0$ for every i. There exists an $N > 0$ such that $N r_i \in \mathbb{Z}$ for all i. Then $H = N(H_0 + \sum_i r_i G_i)$ is an effective divisor satisfying $H \cdot G_j = 0$ for all j. If C is an irreducible curve such that $C \neq G_j$ for any j, then $H_0 \cdot C > 0$ and $G_i \cdot C \geq 0$ for all i. Hence, $H \cdot C > 0$. In particular, H is nef. Finally, H is big since $H^2 = N H \cdot (H_0 + \sum_i r_i G_i) = N(H \cdot H_0) > 0$ as H_0 is ample.

To see the final statement, we apply the above argument to an ample $\mathbb{R}$-divisor x (i.e., an element in the interior of the ample cone) to see that $x + \sum_i r_i G_i$ is a nef and big $\mathbb{R}$-divisor orthogonal to $\{G_1, \ldots, G_n\}$. As $x + \sum_i r_i G_i$ is simply the orthogonal projection p of x onto $\{G_1, \ldots, G_n\}^{\perp} \otimes \mathbb{R}$ and $p : H^2(X; \mathbb{R}) \to \{G_1, \ldots, G_n\}^{\perp} \otimes \mathbb{R}$ is an open map, the image of the interior of the ample cone of X is then a nonempty open subset of $\{G_1, \ldots, G_n\}^{\perp} \otimes \mathbb{R}$ consisting of nef and big $\mathbb{R}$-divisors orthogonal to $\{G_1, \ldots, G_n\}$. □

Applying the above construction to $X = Y$ and $D_1, \ldots, D_r$, we can find a nef and big divisor H such that $H \cdot D_j = 0$ for all j and such that, if C is an irreducible curve such that $C \neq D_j$ for any j, then $H \cdot C > 0$.

Proposition 1.6. *Let (Y, D) be a negative definite anticanonical pair, and let H be a nef and big divisor such that $H \cdot D_j = 0$ for all j and such that, if C is an irreducible curve such that $C \neq D_j$ for any j, then $H \cdot C > 0$. Suppose in addition that $\mathcal{O}_Y(H)|D = \mathcal{O}_D$, i.e., that $\varphi_Y([H]) = 1$. Then the D_i are not fixed components of $|H|$. Hence, if $\overline{Y}$ denotes the normal complex surface obtained by contracting the D_i, then H induces an ample divisor $\overline{H}$ on $\overline{Y}$ and $|3\overline{H}|$ defines an embedding of $\overline{Y}$ in $\mathbb{P}^N$ for some N.*

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_Y(H - D) \to \mathcal{O}_Y(H) \to \mathcal{O}_D \to 0.$$

Looking at the long exact cohomology sequence, as

$$H^1(Y; \mathcal{O}_Y(H - D)) = H^1(Y; \mathcal{O}_Y(H) \otimes K_Y)$$

is Serre dual to $H^1(Y; \mathcal{O}_Y(-H)) = 0$, by Ramanujam's vanishing theorem, there exists a section of $\mathcal{O}_Y(H)$ that is nowhere vanishing on D, proving the first statement.

The second follows from the Nakai–Moishezon criterion and the third from general results on linear series on anticanonical pairs [Friedman 1983].   □

Remark 1.7. By the surjectivity of the period map (Theorem 1.3), for any (Y, D) a negative definite anticanonical pair and H a nef and big divisor on Y such that $H \cdot D_j = 0$ for all j and $H \cdot C > 0$ for all curves $C \neq D_i$, there exists a deformation of the pair (Y, D) such that the divisor corresponding to H has trivial restriction to D. More generally, one can consider deformations such that $\varphi_Y([H])$ is a torsion point of $\mathbb{G}_m$. In this case, if $\overline{Y}$ is the normal surface obtained by contracting D, then $\overline{Y}$ is projective. Note that this implies that the set of pairs (Y, D) such that $\overline{Y}$ is projective is Zariski dense in the moduli space. However, as the set of torsion points is not dense in $\mathbb{G}_m$ in the classical topology, the set of projective surfaces $\overline{Y}$ will not be dense in the classical topology.

2. Roots and nodal classes

Definition 2.1. Let $\mathscr{C} = \mathscr{C}(Y)$ be the positive cone of Y, i.e.,

$$\mathscr{C} = \{x \in H^2(Y; \mathbb{R}) : x^2 > 0\}.$$

Then $\mathscr{C}$ has two components, and exactly one of them, say $\mathscr{C}^+ = \mathscr{C}^+(Y)$, contains the classes of ample divisors. We also define

$$\mathscr{C}_D^+ = \mathscr{C}_D^+(Y) = \{x \in \mathscr{C}^+ : x \cdot [D_i] \geq 0 \text{ for all } i\}.$$

Let $\overline{\mathscr{A}}(Y) \subseteq \mathscr{C}^+ \subseteq H^2(Y; \mathbb{R})$ be (the closure of) the ample (nef, Kähler) cone of Y in $\mathscr{C}^+$. By definition, $\overline{\mathscr{A}}(Y)$ is closed in $\mathscr{C}^+$ but not in general in $H^2(Y; \mathbb{R})$.

Definition 2.2. Let $\alpha \in H^2(Y; \mathbb{Z})$, $\alpha \neq 0$. The *oriented wall* W^α associated to α is the set $\{x \in \mathscr{C}^+ : x \cdot \alpha = 0\}$, i.e., the intersection of $\mathscr{C}^+$ with the orthogonal space to α together with the preferred half space defined by $x \cdot \alpha \geq 0$. If C is a curve on Y, we write W^C for $W^{[C]}$. A standard result (see, for example, [Friedman and Morgan 1988, II (1.8)]) shows that, if I is a subset of $H^2(Y; \mathbb{Z})$ and there exists an $N \in \mathbb{Z}^+$ such that $-N \leq \alpha^2 < 0$ for all $\alpha \in I$, then the collection of walls $\{W^\alpha : \alpha \in I\}$ is locally finite on $\mathscr{C}^+$. Finally, we say that W^α is a *face* of $\overline{\mathscr{A}}(Y)$ if $\partial\overline{\mathscr{A}}(Y) \cap W^\alpha$ contains a nonempty open subset of W^α and $x \cdot \alpha \geq 0$ for all $x \in \overline{\mathscr{A}}(Y)$.

Lemma 2.3. $\overline{\mathscr{A}}(Y)$ *is the set of all* $x \in \mathscr{C}^+$ *such that* $x \cdot [D_i] \geq 0$, $x \cdot [E] \geq 0$ *for all exceptional curves* E, *and* $x \cdot [C] \geq 0$ *for all* -2-*curves* C. *Moreover, if* α *is the class associated to an exceptional or* -2-*curve, or* $\alpha = [D_i]$ *for some* i *such that* $D_i^2 < 0$, *then* W^α *is a face of* $\overline{\mathscr{A}}(Y)$. *If* α *and* β *are two such classes*, $W^\alpha = W^\beta \iff \alpha = \beta$.

Proof. For the first claim, it is enough to show that, if G is an irreducible curve on Y with $G^2 < 0$, then G is either D_i for some i, an exceptional curve, or a -2-curve. This follows immediately from adjunction since, if $G \neq D_i$ for any i, then $G \cdot D \geq 0$

and $-2 \leq 2p_a(G) - 2 = G^2 - G \cdot D < 0$; hence, $p_a(G) = 0$ and either $G^2 = -2$, $G \cdot D = 0$, or $G^2 = G \cdot D = -1$. The last two statements follow from the openness statement in Proposition 1.5 and the fact that no two distinct classes of the types listed above are multiples of each other. $\qquad \square$

As an alternate characterization of the classes in the previous lemma, we have the following:

Lemma 2.4. *Let H be a nef divisor such that $H \cdot D > 0$.*

(i) *If $\alpha \in H^2(Y; \mathbb{Z})$ with $\alpha^2 = \alpha \cdot [K_Y] = -1$, then $\alpha \cdot [H] \geq 0$ if and only if α is the class of an effective curve. In particular, the wall W^α does not pass through the interior of $\bar{\mathcal{A}}(Y)$. (See [Friedman and Morgan 1988, p. 332] for a more general statement.)*

(ii) *If $\beta \in H^2(Y; \mathbb{Z})$ with $\beta^2 = -2$, $\beta \cdot [D_i] = 0$ for all i, $\beta \cdot [H] \geq 0$, and $\varphi_Y(\beta) = 1$, then $\pm\beta$ is the class of an effective curve, and β is effective if $\beta \cdot [H] > 0$.*

Hence, the ample cone $\bar{\mathcal{A}}(Y)$ is the set of all $x \in \mathscr{C}^+$ such that $x \cdot [D_i] \geq 0$ and $x \cdot \alpha \geq 0$ for all classes α and β as described in (i) *and* (ii) *above, where in case* (ii) *we assume in addition that β is effective or equivalently that $\beta \cdot [H] > 0$ for some nef divisor H.*

Proof. (i) Clearly, if α is the class of an effective curve, then $\alpha \cdot [H] \geq 0$ since H is nef. Conversely, assume that $\alpha^2 = \alpha \cdot [K_Y] = -1$ and that $\alpha \cdot [H] \geq 0$. By the Riemann–Roch theorem, $\chi(L_\alpha) = 1$. Hence, either $h^0(L_\alpha) > 0$ or $h^2(L_\alpha) > 0$. But $h^2(L_\alpha) = h^0(L_\alpha^{-1} \otimes K_Y)$ and $[H] \cdot (-\alpha - [D]) < 0$ by assumption. Thus, $h^0(L_\alpha) > 0$ and hence α is the class of an effective curve.

(ii) As in (i), $H \cdot (-\beta - [D]) < 0$, and hence, $h^0(L_\beta^{-1} \otimes K_Y) = 0$. Thus, $h^2(L_\beta) = 0$. Suppose that $h^0(L_\beta) = 0$. Then, by the Riemann–Roch theorem, $\chi(L_\beta) = 0$ and hence $h^1(L_\beta) = 0$. Hence, $h^1(L_\beta^{-1} \otimes K_Y) = 0$. Since $\varphi_Y(\beta) = 1$, $L_\beta^{\pm 1} | D = \mathcal{O}_D$. Thus, there is an exact sequence

$$0 \to L_\beta^{-1} \otimes \mathcal{O}_Y(-D) \to L_\beta^{-1} \to \mathcal{O}_D \to 0.$$

Since $H^1(L_\beta^{-1} \otimes K_Y) = H^1(L_\beta^{-1} \otimes \mathcal{O}_Y(-D)) = 0$, the map $H^0(L_\beta^{-1}) \to H^0(\mathcal{O}_D)$ is surjective and hence $-\beta$ is the class of an effective curve. $\qquad \square$

Definition 2.5. Let $\alpha \in H^2(Y; \mathbb{Z})$. Then α is a *numerical exceptional curve* if $\alpha^2 = \alpha \cdot [K_Y] = -1$. The numerical exceptional curve α is *effective* if $h^0(L_\alpha) > 0$, i.e., if $\alpha = [G]$, where G is an effective curve.

A minor variation of the proof of Lemma 2.4 shows the following:

Lemma 2.6. *Let H be a nef and big divisor such that $H \cdot G > 0$ for all irreducible curves G not equal to D_i for some i, and let α be a numerical exceptional curve.*

(i) *Suppose that $[H] \cdot \alpha \geq 0$. Then either $[H] \cdot \alpha > 0$ and α is effective or $H \cdot D = [H] \cdot \alpha = 0$ and α is an integral linear combination of the $[D_i]$.*

(ii) *If (Y, D) is negative definite and α is an integral linear combination of the $[D_i]$, then either some component D_i is a smooth rational curve of self-intersection -1 or $K_Y^2 = -1$, $\alpha = K_Y$, and hence α is not effective.*

(iii) *If no component D_i is a smooth rational curve of self-intersection -1, then α is effective if and only if $[H] \cdot \alpha > 0$.*

Proof. (i) As in the proof of Lemma 2.4, either α or $-\alpha - [D]$ is the class of an effective divisor. If $-\alpha - [D]$ is the class of an effective divisor, then $0 \leq [H] \cdot (-\alpha - [D]) \leq 0$, so $[H] \cdot \alpha = H \cdot D = 0$. In particular, (Y, D) is negative definite. Moreover, if G is an effective divisor with $[G] = -\alpha - [D]$, then every component of G is equal to some D_i. Hence, $[G]$ and therefore $\alpha = -[G] - [D]$ are integral linear combinations of the $[D_i]$.

(ii) Suppose that α is an integral linear combination of the $[D_i]$ but that no D_i is a smooth rational curve of self-intersection -1. We shall show that $K_Y^2 = -1$ and $\alpha = K_Y$. First suppose that $K_Y^2 = -1$. Then $\bigoplus_i \mathbb{Z} \cdot [D_i] = \mathbb{Z} \cdot [K_Y] \oplus L$, where L, the orthogonal complement of $[K_Y]$ in $\bigoplus_i \mathbb{Z} \cdot [D_i]$, is even and negative definite. Thus, $\alpha = a[K_Y] + \beta$, with either $\beta = 0$ or $\beta^2 \leq -2$, and $\alpha^2 = -a^2 + \beta^2$. Hence, if $\alpha^2 = \alpha \cdot [K_Y] = -1$, the only possibility is $\beta = 0$ and $a = 1$. In case $K_Y^2 < -1$, D is reducible, and no D_i is a smooth rational curve of self-intersection -1, then $D_i^2 \leq -2$ for all i and either $D_i^2 \leq -4$ for some i or there exist $i \neq j$ such that $D_i^2 = D_j^2 = -3$. In this case, it is easy to check that, for all integers a_i such that $a_i \neq 0$ for some i, $\left(\sum_i a_i D_i \right)^2 < -1$. This contradicts $\alpha^2 = -1$.

(iii) If $[H] \cdot \alpha > 0$, then α is effective by (i). If $[H] \cdot \alpha < 0$, then clearly α is not effective. Suppose that $[H] \cdot \alpha = 0$; we must show that, again, α is not effective. Suppose that $\alpha = [G]$ is effective. By the hypothesis on H, every component of G is a D_i for some i so that $\alpha = \sum_i a_i [D_i]$ for some $a_i \in \mathbb{Z}$, $a_i \geq 0$. Let $I \subseteq \{1, \ldots, r\}$ be the set of i such that $a_i > 0$. Then $H \cdot D_i = 0$ for all $i \in I$. If $I = \{1, \ldots, r\}$, then (Y, D) is negative definite and we are done by (ii). Otherwise, $\bigcup_{i \in I} D_i$ is a union of chains of curves whose components D_i satisfy $D_i^2 \leq -2$. It is then easy to check that $\alpha^2 < -1$ in this case, a contradiction. Hence, α is not effective. $\square$

Definition 2.7. Let Y_t be a generic small deformation of Y, and identify $H^2(Y_t; \mathbb{R})$ with $H^2(Y; \mathbb{R})$. Define $\overline{\mathscr{A}}_{\text{gen}} = \overline{\mathscr{A}}_{\text{gen}}(Y)$ to be the ample cone $\overline{\mathscr{A}}(Y_t)$ of Y_t, viewed as a subset of $H^2(Y; \mathbb{R})$.

Lemma 2.8. *With notation as above, the following are true*:

(i) *If there do not exist any -2-curves on Y, then $\overline{\mathscr{A}}(Y) = \overline{\mathscr{A}}_{\text{gen}}$. More generally, $\overline{\mathscr{A}}_{\text{gen}}$ is the set of all $x \in \mathscr{C}^+$ such that $x \cdot [D_i] \geq 0$ and $x \cdot \alpha \geq 0$ for all effective numerical exceptional curves. In particular, $\overline{\mathscr{A}}(Y) \subseteq \overline{\mathscr{A}}_{\text{gen}}$.*

(ii) *We have* $\overline{\mathscr{A}}(Y) = \{x \in \overline{\mathscr{A}}_{\text{gen}} : x \cdot [C] \geq 0 \text{ for all } -2\text{-curves } C\}.$

Proof. Let Y be a surface with no -2-curves (such surfaces exist and are generic by the surjectivity of the period map (Theorem 1.3)). Fix a nef divisor H on Y with $H \cdot D > 0$. Then $\overline{\mathscr{A}}(Y)$ is the set of all $x \in \mathscr{C}^+$ such that $x \cdot [D_i] \geq 0$ and $x \cdot [E] \geq 0$ for all exceptional curves E, and this last condition is equivalent to $x \cdot \alpha \geq 0$ for all $\alpha \in H^2(Y; \mathbb{Z})$ such that $\alpha^2 = \alpha \cdot [K_Y] = -1$ and $\alpha \cdot [H] \geq 0$ by Lemma 2.4. Since this condition is independent of the choice of Y, because we can choose the divisor H to be ample and to vary in a small deformation, the first part of (i) follows, and the remaining statements are clear. $\qquad \square$

In fact, the argument above shows the following:

Lemma 2.9. *The set of effective numerical exceptional curves and the set $\overline{\mathscr{A}}_{\text{gen}}$ are locally constant and hence are invariant in a global deformation with trivial monodromy under the induced identifications.* $\qquad \square$

Lemma 2.10. *If C is a -2-curve on Y, then the wall W^C meets the interior of $\overline{\mathscr{A}}_{\text{gen}}$, and in fact, $r_C(\overline{\mathscr{A}}_{\text{gen}}) = \overline{\mathscr{A}}_{\text{gen}}$, where $r_C : H^2(Y; \mathbb{R}) \to H^2(Y; \mathbb{R})$ is reflection in the class $[C]$. Hence, $\overline{\mathscr{A}}(Y)$ is a fundamental domain for the action of the group $W(\Delta_Y)$ on $\overline{\mathscr{A}}_{\text{gen}}$, where $W(\Delta_Y)$ is the group generated by the reflections in the classes in the set Δ_Y of -2-curves on Y.*

Proof. Clearly, if $r_C(\overline{\mathscr{A}}_{\text{gen}}) = \overline{\mathscr{A}}_{\text{gen}}$, then W^C meets the interior of $\overline{\mathscr{A}}_{\text{gen}}$. To see that $r_C(\overline{\mathscr{A}}_{\text{gen}}) = \overline{\mathscr{A}}_{\text{gen}}$, assume first more generally that $\beta \in \Lambda$ is any class with $\beta^2 = -2$, and let r_β be the corresponding reflection. Then r_β permutes the set of $\alpha \in H^2(Y; \mathbb{Z})$ such that $\alpha^2 = \alpha \cdot [K_Y] = -1$ but does not necessarily preserve the condition that α is effective, i.e., that $\alpha \cdot [H] \geq 0$ for some nef divisor H on Y with $H \cdot D > 0$. However, for $\beta = [C]$, there exists by Proposition 1.5 a nef and big divisor H_0 such that $H_0 \cdot C = 0$ and $H \cdot D > 0$. Hence, $[H_0]$ is invariant under r_C, and so r_C permutes the set of $\alpha \in H^2(Y; \mathbb{Z})$ such that $\alpha^2 = \alpha \cdot [K_Y] = -1$ and $\alpha \cdot [H_0] \geq 0$. Thus, r_C permutes the set of effective numerical exceptional curves and hence the faces of $\overline{\mathscr{A}}_{\text{gen}}$ so that $r_C(\overline{\mathscr{A}}_{\text{gen}}) = \overline{\mathscr{A}}_{\text{gen}}$. Since $\overline{\mathscr{A}}(Y) \subseteq \overline{\mathscr{A}}_{\text{gen}}$ is given by Lemma 2.8(ii), the final statement is then a general result in the theory of reflection groups [Bourbaki 1981, V §3]. $\square$

Remark 2.11. (i) The argument for the first part of Lemma 2.10 essentially boils down to the following. Let $\overline{Y}$ be the normal surface obtained by contracting C. Then the reflection r_C is the monodromy associated to a generic smoothing of the singular surface $\overline{Y}$, and the cone $\overline{\mathscr{A}}_{\text{gen}}$ is invariant under monodromy.

(ii) If E is an exceptional curve, then W^E is a face of $\overline{\mathscr{A}}(Y)$. For a generic Y (i.e., no -2-curves), Lemma 2.10 then says that the set of exceptional curves on Y is invariant under the reflection group generated by all classes of square -2 that become the classes of a -2-curve under some specialization. A somewhat more involved statement holds in the nongeneric case.

Lemma 2.12. *With* $W(\Delta_Y)$ *as in Definition 1.1, for all* $w \in W(\Delta_Y)$ *and all* $\beta \in \Lambda$, $\varphi_Y(w(\alpha)) = \varphi_Y(\alpha)$.

Proof. This is clear since $\varphi_Y([C]) = 1$ implies $\varphi_Y(r_C(\alpha)) = \varphi_Y(\alpha)$ for all $\alpha \in \Lambda$. □

Lemma 2.13. *Suppose that* $C = \sum_i a_i C_i$, *where the* C_i *are* -2*-curves*, $a_i \in \mathbb{Z}$, $C^2 = -2$, *the support of* C *is connected, and* $(C_i \cdot C_j)$ *is negative definite. Then there exists an element* w *in the group generated by reflections in the* $[C_i]$ *such that* $w([C]) = [C_i]$ *for some* i.

Proof. This follows from the well known fact that, if R is an irreducible root system such that all roots have the same length, then the Weyl group $W(R)$ acts transitively on the set of roots. □

Theorem 2.14. *Let* $\beta \in \Lambda$ *with* $\beta^2 = -2$. *Then the following are equivalent*:

(i) *Let* Y_1 *be a deformation of* Y *with trivial monodromy such that* $\varphi_{Y_1}(\beta) = 1$. *Then, with* $W(\Delta_{Y_1})$ *as in Definition 1.1, there exists* $w \in W(\Delta_{Y_1})$ *such that* $w(\beta) = [C]$, *where* C *is a* -2*-curve on* Y_1. *In particular, if* Y_1 *is generic subject to the condition that* $\varphi_{Y_1}(\beta) = 1$ (*i.e., if* $\mathrm{Ker}\,\varphi_{Y_1} = \mathbb{Z} \cdot \beta$), *then* $\pm\beta = [C]$ *for a* -2*-curve* C.

(ii) *The wall* W^β *meets the interior of* $\overline{\mathscr{A}}_{\mathrm{gen}}$.

(iii) *If* r_β *is reflection in the class* β, *then* $r_\beta(\overline{\mathscr{A}}_{\mathrm{gen}}) = \overline{\mathscr{A}}_{\mathrm{gen}}$.

Proof. Lemma 2.10 implies that (i) $\Longrightarrow$ (iii) in case $Y = Y_1$ and $\beta = [C]$ where C is a -2-curve. The case where $w(\beta) = [C]$ follows easily from this since, for all $w \in W(\Delta_{Y_1})$, $w \circ r_\beta \circ w^{-1} = r_{w(\beta)}$. Lemma 2.9 then handles the case where Y_1 is replaced by a general deformation Y. Also, clearly (iii) $\Longrightarrow$ (ii). So it is enough to show that (ii) $\Longrightarrow$ (i). In fact, by Lemma 2.13, it is enough to show that, if Y is any surface such that $\varphi_Y(\beta) = 1$ and W^β meets the interior of $\overline{\mathscr{A}}_{\mathrm{gen}}$, then there exists a $w \in W(\Delta_Y)$ such that $w(\beta) = [\sum_i a_i C_i]$ where $a_i \in \mathbb{Z}^+$, the C_i are curves disjoint from D, and $\bigcup_i C_i$ is connected.

By hypothesis, there exists an x in the interior of $\overline{\mathscr{A}}_{\mathrm{gen}}$ such that $x \cdot \beta = 0$. In particular, $x \cdot [D_i] > 0$ for all i. We can assume that $x = [H]$ is the class of a divisor H. After replacing x by $w(x)$ and β by $w(\beta)$ for some $w \in W(\Delta_Y)$, we can assume that x (and hence H) lies in $\overline{\mathscr{A}}(Y)$ so that H is a nef and big divisor with $H \cdot D_i > 0$ for all i, and we still have $\varphi_Y(\beta) = 1$ by Lemma 2.12. By Lemma 2.4, possibly after replacing β by $-\beta$, $\beta = [\sum_i a_i C_i]$ where the C_i are irreducible curves and $a_i \in \mathbb{Z}^+$. Since $\beta \cdot [H] = \sum_i a_i (C_i \cdot H) = 0$, $C_i \cdot H \geq 0$, and $D_j \cdot H > 0$, $C_i \cdot H = 0$ for all i, and no C_i is equal to D_j for any j. Hence, the C_i are curves meeting each D_j in at most finitely many points and $\sum_i a_i (C_i \cdot D_j) = 0$ so that $C_i \cap D_j = \varnothing$. Finally, each $(C_i)^2 < 0$ by Hodge index, and so each C_i is a -2-curve. Moreover, the C_i span a negative definite lattice, and in particular, their classes are independent. From this, the statement about the connectedness of $\bigcup_i C_i$ is clear. □

Definition 2.15. Let $R = R_Y$ be the set of all $\beta \in \Lambda$ such that $\beta^2 = -2$ and such that there exists some deformation of Y for which β becomes the class of a -2-curve. Following [Gross et al. 2013], we call R the set of *Looijenga roots* (or briefly *roots*) of Y. Note that R only depends on the deformation type of Y.

The definition of R is slightly ill-posed since we have not specified an identification of the cohomologies of the fibers along the deformation. In particular, if $\beta = [C]$ is a -2-curve on Y, then by Remark 2.11(i) if Y' is a nearby deformation of Y, then a general smoothing of the ordinary double point on the contraction of C on Y has monodromy that sends $[C]$ to $-[C]$, and hence, $-\beta \in R$ as well. To avoid this issue, it is simpler to define R to be the set of $\beta \in \Lambda$, $\beta^2 = -2$, which satisfy either of the equivalent conditions Theorem 2.14(ii)–(iii).

Given Y, let Δ_Y be the set of classes of -2-curves on Y and $W(\Delta_Y)$ the reflection group generated by Δ_Y. Finally set R^{nod}, the set of *nodal classes*, to be $W(\Delta_Y) \cdot \Delta_Y$. Then $R^{\mathrm{nod}} \subseteq R$.

Corollary 2.16.   (i) *If $f : H^2(Y; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$ is an integral isometry preserving the classes $[D_i]$ such that $f(\bar{\mathscr{A}}_{\mathrm{gen}}) = \bar{\mathscr{A}}_{\mathrm{gen}}$, then $f(R) = R$.*

(ii) *If $W(R)$ is the reflection group generated by reflections in the elements of R, then $W(R) \cdot R = R$ and $w(\bar{\mathscr{A}}_{\mathrm{gen}}) = \bar{\mathscr{A}}_{\mathrm{gen}}$ for all $w \in W(R)$.* □

Remark 2.17. A result similar to Theorem 2.14 classifies the elements of $H^2(Y; \mathbb{Z})$ that are represented by the class of a smoothly embedded 2-sphere of self-intersection -2 in terms of the "super P-cell" of [Friedman and Morgan 1988].

For the case where the length $r(D) \leq 5$, Looijenga [1981] defines a subset R_L of Λ by starting with a particular configuration B of elements of square -2 (a *root basis* in his terminology) and setting $R_L = W(B) \cdot B$, where $W(B)$ is the reflection group generated by B. In fact, the set R_L is just the set R of Looijenga roots.

Proposition 2.18. *In the above notation, $R_L = R$.*

Proof. It is easy to see from the construction of [Looijenga 1981, I §2] that $B \subseteq R$. Hence, $R_L \subseteq R$. Conversely, if $\alpha \in R$, then, by Corollary 2.16(ii), $r_\alpha(\bar{\mathscr{A}}_{\mathrm{gen}}) = \bar{\mathscr{A}}_{\mathrm{gen}}$. It then follows from [Looijenga 1981, Proposition I (4.7)] that $r_\alpha \in W(B)$. By a general result in the theory of reflection groups [Bourbaki 1981, V §3.2, Theorem 1(iv)], $r_\alpha = r_\beta$ for some $\beta \in R_L$. Thus, $\alpha = \pm\beta$ so that $\alpha \in R_L$. Hence, $R \subseteq R_L$, and therefore, $R_L = R$. □

Example 2.19. Let (Y, D) be the blowup of $\mathbb{P}^2$ at $N \geq 10$ general points on an irreducible nodal cubic curve. We let h be the pullback of the class of a line on $\mathbb{P}^2$ and $e_1, \ldots, e_N$ be the classes of the exceptional curves.

(i) Let $\alpha = -3h + \sum_{i=1}^{10} e_i$. Then $\alpha^2 = \alpha \cdot [K_Y] = -1$ so that α is a numerical exceptional curve. But there exists a nef and big divisor H (for example h) such that $\alpha \cdot [H] < 0$ so that α is not effective. Hence, $\alpha \cdot x \leq 0$ for all $x \in \bar{\mathscr{A}}(Y) = \bar{\mathscr{A}}_{\mathrm{gen}}$ since W^α

does not pass through the interior of $\overline{\mathscr{A}}_{\text{gen}}$. Note that W^{α} is never a face of $\overline{\mathscr{A}}_{\text{gen}}$. For $N = 10$, $W^{-\alpha}$ is a face of $\overline{\mathscr{A}}_{\text{gen}}$, but this is no longer the case for $N \geq 11$. Thus, the condition $\alpha \cdot [H] \geq 0$ for some H such that $H \cdot D > 0$ is necessary for α to be effective.

More generally, let $f = 3h - \sum_{i=1}^{9} e_i$ and set $\alpha = kf + e_{10}$ (the case above corresponds to $k = -1$). As above, α is a numerical exceptional curve. For $k \leq -1$, $h \cdot \alpha < 0$. Hence, α is not effective. For $k \geq 1$, α is effective but it is not the class of an exceptional curve: for all $x \in \overline{\mathscr{A}}_{\text{gen}}$, $x \cdot f > 0$, and $x \cdot e_{10} \geq 0$. Hence, $x \cdot \alpha > 0$ for all $x \in \overline{\mathscr{A}}_{\text{gen}}$. Thus, W^{α} is not a face of $\overline{\mathscr{A}}_{\text{gen}}$ and so α is not the class of an exceptional curve.

(ii) With α any of the classes as above, suppose that $N \geq 11$ and $k \neq 0$ and set $\beta = \alpha - e_{11}$. Then $\beta^2 = -2$ and $\beta \cdot [K_Y] = 0$. However,

$$r_{\beta}(e_{11}) = e_{11} + (e_{11} \cdot \beta)\beta = \alpha.$$

Since $W^{e_{11}}$ is a face of $\overline{\mathscr{A}}_{\text{gen}}$ and W^{α} is not a face of $\overline{\mathscr{A}}_{\text{gen}}$, $r_{\beta}(\overline{\mathscr{A}}_{\text{gen}}) \neq \overline{\mathscr{A}}_{\text{gen}}$. Hence, β does not satisfy any of the equivalent conditions of Theorem 2.14 so that $\beta \notin R$.

Remark 2.20. In the situation of the example above, it is well known that if D is irreducible, $N \leq 9$ (i.e., $D^2 \geq 0$), and there are no -2-curves on Y, then every numerical exceptional curve is the class of an exceptional curve, so (i) above is best possible. A generalization is given in Proposition 3.3 below. We shall show in Proposition 3.5 that the example in (ii) is best possible as well.

The numerical exceptional curves given in Example 2.19(i) were known to Du Val. In fact, he showed that they are essentially the only numerical exceptional curves in case Y is the blowup of $\mathbb{P}^2$ at ten points [Du Val 1937, pp. 46–47].

Proposition 2.21. *Suppose that (Y, D) is the blowup of $\mathbb{P}^2$ at ten points lying on an irreducible cubic, that Y is generic in the sense that there are no -2-curves on Y, and that α is a numerical exceptional curve. Then there exists an exceptional curve E on Y and an integer k such that α is the class of $k(D + E) + E$.*

Proof. Suppose that α is a numerical exceptional curve on Y. Then, since $K_Y^2 = -1$, $\lambda = \alpha + [D] = \alpha - [K_Y]$ satisfies $\lambda^2 = \lambda \cdot \alpha = \lambda \cdot [K_Y] = 0$. In particular, $\lambda \in \Lambda$. Conversely, given an isotropic vector $\lambda \in \Lambda$, if we set $\alpha = \lambda + [K_Y]$, then α is a numerical exceptional curve.

Any isotropic vector $\lambda \in \Lambda$ can be uniquely written as $n\lambda_0$, where $n \in \mathbb{Z}$ and λ_0 is primitive and lies in $\overline{\mathscr{C}^+}$. Note that $H^2(Y; \mathbb{Z}) = \mathbb{Z}[K_Y] \oplus \Lambda$ and that $\Lambda = U \oplus (-E_8)$ (both sums orthogonal). An easy exercise shows that, if $\text{Aut}^+(\Lambda)$ is the group of integral isometries A of Λ such that $A(\mathscr{C}^+ \cap \Lambda) = \mathscr{C}^+ \cap \Lambda$, i.e., A has real spinor norm equal to 1, then every $A \in \text{Aut}^+(\Lambda)$ extends uniquely to an integral isometry of $H^2(Y; \mathbb{Z})$ fixing $[K_Y]$ and hence $[D]$ and moreover that $\text{Aut}^+(\Lambda)$ acts transitively on the set of (nonzero) primitive isotropic vectors in $\overline{\mathscr{C}^+} \cap \Lambda$. Hence, there exists

an $A \in \mathrm{Aut}^+(\Lambda)$ such that $A(\lambda_0) = f$ in the notation of Example 2.19. If we continue to let A denote the extension of A to an isometry of $H^2(Y; \mathbb{Z})$, then $A(\alpha) = nf + [K_Y] = (n-1)f + e_{10}$ since $f = -[K_Y] + e_{10}$. It follows that $\alpha = (n-1)\lambda_0 + A^{-1}(e_{10})$. Using Proposition 3.5 below, A^{-1} preserves the walls of the ample cone of Y, and thus, $A^{-1}(e_{10}) = e$ is the class of an exceptional curve E, and $\lambda_0 = A^{-1}(f) = A^{-1}([D] + e_{10}) = [D] + E$. Hence, setting $k = n-1$, α is the class of $k(D + E) + E$ as claimed. $\qquad\square$

The proof above shows the following:

Corollary 2.22. *Let (Y, D) be the blowup of $\mathbb{P}^2$ at ten points lying on an irreducible cubic and such that there are no -2-curves on Y, let α be a numerical exceptional curve on Y, and let $\lambda = \alpha - [K_Y]$. Then*

(i) *α is effective if and only if $\lambda \in (\overline{\mathscr{C}^+} - \{0\}) \cap \Lambda$,*

(ii) *α is not effective if and only if $\lambda \in (-\overline{\mathscr{C}^+}) \cap \Lambda$, and*

(iii) *α is the class of an exceptional curve if and only if λ is a primitive isotropic vector in $\overline{\mathscr{C}^+} \cap \Lambda$. Thus, there is a bijection from the set of exceptional curves on Y to the set of primitive isotropic vectors in $\overline{\mathscr{C}^+} \cap \Lambda$.* $\qquad\square$

Remark 2.23. In the above situation, let W be the group generated by the reflections in the classes $e_1 - e_2, \ldots, e_9 - e_{10}, h - e_2 - e_2 - e_3$, which are easily seen to be Looijenga roots. A classical argument (usually called Noether's inequality) shows that, if λ_0 is a primitive integral isotropic vector in Λ lying in $\overline{\mathscr{C}^+}$, then there exists $w \in W$ such that $w(\lambda_0) = f = 3h - \sum_{i=1}^9 e_i$ in the notation of Example 2.19. Thus, W acts transitively on the set of such vectors. Using standard results about the affine Weyl group of E_8, it is then easy to see that $W = \mathrm{Aut}^+(\Lambda)$. This was already noted in [Du Val 1937].

3. Roots and the ample cone

By Corollary 2.16, if $f : H^2(Y; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$ is an integral isometry preserving the classes $[D_i]$ such that $f(\overline{\mathscr{A}}_{\mathrm{gen}}) = \overline{\mathscr{A}}_{\mathrm{gen}}$, then $f(R) = R$. In this section, we find criteria for when the converse holds.

Lemma 3.1. *Let $f : H^2(Y; \mathbb{Z}) \to H^2(Y; \mathbb{Z})$ be an integral isometry preserving $\mathscr{C}^+$ and the classes $[D_i]$. If $f(\overline{\mathscr{A}}_{\mathrm{gen}}) \cap \overline{\mathscr{A}}_{\mathrm{gen}}$ contains an open set, then $f(\overline{\mathscr{A}}_{\mathrm{gen}}) = \overline{\mathscr{A}}_{\mathrm{gen}}$.*

Proof. Choosing $x \in f(\overline{\mathscr{A}}_{\mathrm{gen}}) \cap \overline{\mathscr{A}}_{\mathrm{gen}}$ corresponding to an ample divisor, it is easy to see that $f(\overline{\mathscr{A}}_{\mathrm{gen}})$ and $\overline{\mathscr{A}}_{\mathrm{gen}}$ have the same set of walls and hence are equal. $\qquad\square$

Next we deal with the case where one component of D is a smooth rational curve of self-intersection -1.

Lemma 3.2. *Suppose that D is reducible and that $D_r^2 = -1$. Let $(\overline{Y}, \overline{D})$ be the anticanonical pair obtained by contracting D_r. Then any isometry f of $H^2(Y; \mathbb{Z})$ preserving the classes $[D_i]$, $1 \le i \le r$, defines an isometry $\overline{f}$ of $H^2(\overline{Y}; \mathbb{Z})$ preserving the classes $[\overline{D}_i]$, $1 \le i \le r-1$, and conversely. Moreover, f preserves $\overline{\mathscr{A}}_{\mathrm{gen}}(Y)$ if and only if $\overline{f}$ preserves $\overline{\mathscr{A}}_{\mathrm{gen}}(\overline{Y})$, and R_Y is naturally identified with the roots $R_{\overline{Y}}$ of $\overline{Y}$.*

Proof. The first statement is clear. Identifying $H^2(\overline{Y}, \mathbb{Z})$ with $[D_r]^\perp \subseteq H^2(Y; \mathbb{Z})$, it is clear that $\overline{\mathscr{A}}_{\mathrm{gen}}(Y) \cap [D_r]^\perp = \overline{\mathscr{A}}_{\mathrm{gen}}(\overline{Y})$. Hence, if f preserves $\overline{\mathscr{A}}_{\mathrm{gen}}(Y)$, then $\overline{f}$ preserves $\overline{\mathscr{A}}_{\mathrm{gen}}(\overline{Y})$. Since a divisor $\overline{H}$ on $\overline{Y}$ is ample if and only if $N\overline{H} - D_r$ is ample for all $N \gg 0$, it follows that, if $\overline{f}$ preserves $\overline{\mathscr{A}}_{\mathrm{gen}}(\overline{Y})$, then $f(\overline{\mathscr{A}}_{\mathrm{gen}}(Y)) \cap \overline{\mathscr{A}}_{\mathrm{gen}}(Y)$ contains an open set, and hence, $f(\overline{\mathscr{A}}_{\mathrm{gen}}(Y)) = \overline{\mathscr{A}}_{\mathrm{gen}}(Y)$ by Lemma 3.1. It follows from this and from Theorem 2.14 that R_Y is naturally identified with $R_{\overline{Y}}$ (or directly from the definition by noting that there is a bijection from the set of deformations of (Y, D) to those of $(\overline{Y}, \overline{D})$). $\qquad\square$

Henceforth, then, we shall always assume if need be that no component of D is a smooth rational curve of self-intersection -1.

We turn to the straightforward case where (Y, D) is not negative definite.

Proposition 3.3. *Suppose that (Y, D) and (Y', D') are two anticanonical pairs with $r(D) = r(D')$ and neither pair is negative definite. If $f : H^2(Y; \mathbb{Z}) \to H^2(Y'; \mathbb{Z})$ is an integral isometry with $f([D_i]) = [D_i']$ for all i, then $f(\overline{\mathscr{A}}_{\mathrm{gen}}(Y)) = \overline{\mathscr{A}}_{\mathrm{gen}}(Y')$ and hence $f(R_Y) = R_{Y'}$. Moreover,*

$$R_Y = \{\beta \in \Lambda(Y, D) : \beta^2 = -2\}.$$

Proof. By Lemma 3.2, we may assume that no D_i has self-intersection -1. The statement that the cycle is not negative definite is then equivalent to the statement that either $D_j^2 \ge 0$ for some j or $D_i^2 = -2$ for all i and $r \ge 2$. In the first case, D_j is nef and $D_j \cdot D > 0$. Hence, if α is a numerical exceptional curve such that $\alpha \cdot [D_j] \ge 0$, then α is effective by Lemma 2.4. Thus, $\overline{\mathscr{A}}_{\mathrm{gen}}(Y)$ is the set of all $x \in \mathscr{C}_D^+(Y)$ such that $x \cdot \alpha \ge 0$ for all numerical exceptional curves α such that $\alpha \cdot [D_j] \ge 0$. Since $f(\alpha)^2 = \alpha^2$, $f([D_j]) = [D_j']$, and $f(\alpha) \cdot [K_{Y'}] = \alpha \cdot [K_Y]$, it follows that $f(\overline{\mathscr{A}}_{\mathrm{gen}}(Y)) = \overline{\mathscr{A}}_{\mathrm{gen}}(Y')$. Applying this to reflection in a class β of square -2 in $\Lambda(Y, D)$ then implies that $\beta \in R_Y$.

The case where $D_i^2 = -2$ for every i is similar, using the nef divisor $D = \sum_i D_i$ with $D^2 = 0$. If α is a numerical exceptional curve, then α is effective since $(-\alpha + [K_Y]) \cdot [D] = \alpha \cdot [K_Y] = -1$. The rest of the argument proceeds as before. $\qquad\square$

Remark 3.4. If D is irreducible and not negative definite (i.e., $D^2 \ge 0$) and there are no -2-curves on Y, then, as is well known and noted in Remark 2.20, every numerical exceptional curve is the class of an exceptional curve. However, if D is reducible but not negative definite, then, even if there are no -2-curves on Y, there

may well exist numerical exceptional curves that are not effective and effective numerical exceptional curves that are not the class of an exceptional curve.

From now on, we assume that D is negative definite. The case $K_Y^2 = -1$ can also be handled by straightforward methods as noted in [Looijenga 1981]. (See also [Friedman and Morgan 1988, II (2.7)(c)] in case D is irreducible.)

Proposition 3.5. *Let (Y, D) and (Y', D') be two negative definite anticanonical pairs with $r(D) = r(D')$ and $K_Y^2 = K_{Y'}^2 = -1$. Let $f : H^2(Y; \mathbb{Z}) \to H^2(Y'; \mathbb{Z})$ be an isometry such that $f([D_i]) = [D_i']$ for all i and $f(\mathscr{C}^+(Y)) = \mathscr{C}^+(Y')$. Then $f(\overline{\mathscr{A}}_{\mathrm{gen}}(Y)) = \overline{\mathscr{A}}_{\mathrm{gen}}(Y')$. Moreover,*

$$R_Y = \{\beta \in \Lambda(Y, D) : \beta^2 = -2\}.$$

Hence, $f(R_Y) = R_{Y'}$.

Proof. Since (Y, D) is negative definite, no component of D is a smooth rational curve of self-intersection -1. Fix a nef and big divisor H such that $H \cdot D_i = 0$ for all i and $H \cdot G > 0$ for every irreducible curve $G \neq D_i$. If α is a numerical exceptional curve, $(\alpha - [K_Y])^2 = (\alpha + [D])^2 = 0$. By Lemma 2.6, α is effective if and only if $[H] \cdot \alpha > 0$ if and only if $[H] \cdot (\alpha + [D]) > 0$. By the light cone lemma [Friedman and Morgan 1988, p. 320], this last condition is equivalent to $\alpha + [D] \in \overline{\mathscr{C}^+} - \{0\}$. Since this condition is clearly preserved by an isometry f as in the statement of the proposition, we see that $f(\overline{\mathscr{A}}_{\mathrm{gen}}(Y)) = \overline{\mathscr{A}}_{\mathrm{gen}}(Y')$. The final statement then follows as in the proof of Proposition 3.3. $\qquad\square$

Remark 3.6. The hypothesis $K_Y^2 = -1$ implies that $r(D) \leq 10$, so there are only finitely many examples of the above type. For $r(D) = 10$, there is essentially just one combinatorial possibility for (Y, D) neglecting the orientation [Friedman and Miranda 1983, (4.7)], where it is easy to check that this is the only possibility. For $r(D) = 9$, however, there are two different possibilities for the combinatorial type of (Y, D) (again ignoring the orientation). Begin with an anticanonical pair $(\bar{Y}, \bar{D})$, where $\bar{Y}$ is a rational elliptic surface and $\bar{D} = \bar{D}_0 + \cdots + \bar{D}_8$ is a fiber of type $\tilde{A}_8$ (or I_9 in Kodaira's notation). There is a unique such rational elliptic surface $\bar{Y}$, and its Mordell–Weil group has order 3 (see, for example, [Miranda and Persson 1986]). In particular, possibly after relabeling the components, there is an exceptional curve meeting $\bar{D}_i$ if and only if $i = 0, 3, 6$. It is easy to see that blowing up a point on a component $\bar{D}_i$ meeting an exceptional curve leads to a different combinatorial possibility for an anticanonical pair (Y, D) with $K_Y^2 = -1$ and $r(D) = 9$ than blowing up a point on a component $\bar{D}_i$ that does not meet an exceptional curve.

We turn now to the case where (Y, D) is negative definite but with no assumption on K_Y^2.

Definition 3.7. A point $x \in \mathscr{C}^+ \cap \Lambda$ is *R-distinguished* if there exists a codimension-1 negative definite subspace V of $\Lambda \otimes \mathbb{R}$ spanned by elements of R such that $x \in V^\perp$. Note that the definition only depends on the deformation type of the pair (Y, D).

Remark 3.8. Clearly, if V is a codimension-1 negative definite subspace of $\Lambda \otimes \mathbb{R}$ spanned by elements of R, then V is defined over $\mathbb{Q}$ and $V^\perp \cap (\Lambda \otimes \mathbb{R})$ is a one-dimensional subspace of $H^2(Y; \mathbb{R})$ defined over $\mathbb{Q}$ and spanned by an $h \in H^2(Y; \mathbb{Z})$ with $h^2 > 0$, $h \cdot [D_i] = 0$, and $h \cdot \beta = 0$ for all $\beta \in R \cap V$. Hence, if $h \in \mathscr{C}^+ \cap \Lambda$, then h is R-distinguished.

Also, if the rank of Λ is one, then $\{0\}$ is a codimension-1 negative definite subspace of $\Lambda \otimes \mathbb{R}$, and hence, every point of $\mathscr{C}^+ \cap \Lambda$ is R-distinguished.

However, as we shall see, there exist deformation types (Y, D) with no R-distinguished points.

The following is also clear:

Lemma 3.9. *Let (Y, D) and (Y', D') be two anticanonical pairs with $r(D) = r(D')$, and let $f: H^2(Y; \mathbb{Z}) \to H^2(Y'; \mathbb{Z})$ be an isometry such that $f([D_i]) = [D_i']$ for all i, $f(\mathscr{C}^+(Y)) = \mathscr{C}^+(Y')$, and $f(R_Y) = R_{Y'}$. Then, if x is an R_Y-distinguished point of $\mathscr{C}^+(Y) \cap \Lambda(Y, D)$, $f(x)$ is an $R_{Y'}$-distinguished point of $\mathscr{C}^+(Y') \cap \Lambda(Y', D')$.*

Our goal now is to prove this:

Theorem 3.10. *Suppose that (Y, D) and (Y', D') are two anticanonical pairs such that $r(D) = r(D')$. Let $f: H^2(Y; \mathbb{Z}) \to H^2(Y'; \mathbb{Z})$ be an isometry such that $f([D_i]) = [D_i']$ for all i, $f(\mathscr{C}^+(Y)) = \mathscr{C}^+(Y')$, and $f(R_Y) = R_{Y'}$. If there exists an R-distinguished point of $\mathscr{C}^+ \cap \Lambda$, then $f(\bar{\mathscr{A}}_{\mathrm{gen}}(Y)) = \bar{\mathscr{A}}_{\mathrm{gen}}(Y')$.*

We begin by showing:

Proposition 3.11. *Let x be an R-distinguished point of $\mathscr{C}^+ \cap \Lambda$. Then $x \in \bar{\mathscr{A}}_{\mathrm{gen}}$. Moreover, if α is a numerical exceptional curve and α is not in the span of the $[D_j]$, then α is effective if and only if $\alpha \cdot x \geq 0$.*

Proof. It is enough by Lemma 2.9 to check this on some (global) deformation of (Y, D) with trivial monodromy. By Theorem 1.3, we can assume that

$$\mathrm{Ker}\, \varphi_Y = V \cap \Lambda,$$

where V is as in the definition of R-distinguished. In particular, if $C \in \Delta_Y$, i.e., C is a -2-curve on Y, then $[C] \in V$. It follows from Theorem 2.14(i) that every $\beta \in V \cap R$ is a sum of elements of Δ_Y so that Δ_Y spans V over $\mathbb{Q}$. Thus, there exist -2-curves $C_1, \ldots, C_k$ such that V is spanned by the classes $[C_i]$, and the intersection matrix $(C_i \cdot C_j)$ is negative definite. The classes $[C_1], \ldots, [C_k], [D_1], \ldots, [D_r]$ span a negative definite sublattice of $H^2(Y; \mathbb{Z})$. By Proposition 1.5, there exists a nef and big divisor H such that H is perpendicular to the curves $C_1, \ldots, C_k, D_1, \ldots, D_r$.

Clearly, then $[H] \in \overline{\mathcal{A}}(Y) \subseteq \overline{\mathcal{A}}_{\text{gen}}$ and $[H] = tx$ for some $t \in \mathbb{R}^+$. Hence, $x \in \overline{\mathcal{A}}_{\text{gen}}$ as well. Note that $[H]^\perp$ is spanned over $\mathbb{Q}$ by $[C_1], \ldots, [C_k], [D_1], \ldots, [D_r]$.

Since $x \in \overline{\mathcal{A}}(Y)$, if α is effective, $x \cdot \alpha \geq 0$. Conversely, suppose that α is a numerical exceptional curve with $x \cdot \alpha \geq 0$ and that α is not effective. Then $-\alpha + [K_Y] = [G]$, where G is effective, and $H \cdot (-\alpha + [K_Y]) = -\alpha \cdot [H] \leq 0$. Hence, $(-\alpha + [K_Y]) \cdot [H] = 0$.

Claim 3.12. *We have* $-\alpha + [K_Y] = \sum_i a_i[C_i] + \sum_j b_j[D_j]$, *where* $a_i, b_j \in \mathbb{Z}$.

Proof. In any case, since $-\alpha + [K_Y]$ is perpendicular to $[H]$, there must exist $a_i, b_j \in \mathbb{Q}$ such that $-\alpha + [K_Y] = \sum_i a_i[C_i] + \sum_j b_j[D_j]$. There exist $n_i, m_j \in \mathbb{Z}$ such that $-\alpha + [K_Y] = [G] = \sum_i n_i[C_i] + \sum_j m_j[D_j] + [F]$, where F is an effective curve not containing C_i or D_j in its support for any i, j. By Lemma 1.4(ii), $F = 0$, $a_i = n_i$, and $b_j = m_j$ for all i, j. Hence, $a_i, b_j \in \mathbb{Z}$. $\square$

Because $-\alpha + [K_Y]$ is an integral linear combination of the $[C_i]$ and $[D_j]$, the same holds for α. Then $\alpha = \sum_i c_i[C_i] + \sum_j d_j[D_j]$ with $c_i, d_j \in \mathbb{Z}$. However, $\alpha^2 = -1 = \left(\sum_i c_i C_i\right)^2 + \left(\sum_j d_j D_j\right)^2$. Both terms are nonpositive, and so $\left(\sum_i c_i C_i\right)^2 \geq -1$. But if $\sum_i c_i C_i \neq 0$, then $\left(\sum_i c_i C_i\right)^2 \leq -2$. Thus, $\sum_i c_i C_i = 0$ and α lies in the span of the $[D_j]$. Conversely, if α is not in the span of the $[D_j]$ and $\alpha \cdot x \geq 0$, then α is the class of an effective curve. $\square$

Proof of Theorem 3.10. It follows from Proposition 3.11 that, if $x \in \mathscr{C}^+(Y) \cap \Lambda(Y, D)$ is R_Y-distinguished, then $\overline{\mathcal{A}}_{\text{gen}}(Y)$ is the set of all $y \in \mathscr{C}_D^+(Y)$ such that $\alpha \cdot y \geq 0$ for all α a numerical exceptional curve on Y, not in the span of the $[D_i]$, such that $\alpha \cdot x \geq 0$. Let f be an isometry satisfying the conditions of the theorem. Then $f(x)$ is $R_{Y'}$-distinguished, and $f(\overline{\mathcal{A}}_{\text{gen}}(Y))$ is clearly the set of all $y \in \mathscr{C}_{D'}^+(Y')$ such that $\alpha \cdot y \geq 0$ for all α a numerical exceptional curve on Y', not in the span of the $[D_i']$, such that $\alpha \cdot f(x) \geq 0$. Again by Proposition 3.11, this set is exactly $\overline{\mathcal{A}}_{\text{gen}}(Y')$. $\square$

Theorem 3.10 covers all of the cases in [Looijenga 1981] except for the case of five components: by inspection of the root diagrams on [Looijenga 1981, pp. 275–277], the complement of any trivalent vertex spans a negative definite codimension-1 subspace except in the case of five components. To give a direct argument along the above lines that also handles this case (and all of the other cases in [Looijenga 1981]), we recall the basic setup there: there exists a subset $B = \{\beta_1, \ldots, \beta_n\} \subseteq R$ such that B is a basis for $\Lambda \otimes \mathbb{R}$, and there exist $n_i \in \mathbb{Z}^+$ such that $\left(\sum_i n_i \beta_i\right) \cdot \beta_j > 0$ for all j (compare also [Looijenga 1980, (1.18)]). In particular, note that the intersection matrix $(\beta_i \cdot \beta_j)$ is nonsingular. Finally, by the classification of [Looijenga 1981, Theorem (1.1)], there exists a deformation of (Y, D) for which $\beta_i = [C_i]$ is the class of a -2-curve for all i. (With some care, this explicit argument could be avoided by appealing to the surjectivity of the period map and Theorem 2.14(i).)

Theorem 3.13. *Let (Y, D) and (Y', D') be two anticanonical pairs satisfying the hypotheses of the preceding paragraph, both negative definite, with $r(D) = r(D')$, and let $f : H^2(Y; \mathbb{Z}) \to H^2(Y'; \mathbb{Z})$ be an isometry such that $f([D_i]) = [D_i']$ for all i, $f(\mathscr{C}^+(Y)) = \mathscr{C}^+(Y')$, and $f(R_Y) = R_{Y'}$. Then $f(\bar{\mathscr{A}}_{\mathrm{gen}}(Y)) = \bar{\mathscr{A}}_{\mathrm{gen}}(Y')$.*

Sketch of the proof. With notation as in the paragraph preceding the statement of the theorem, let $h = \sum_i n_i \beta_i$ have the property that $h \cdot \beta_i > 0$. By the arguments used in the proof of Theorem 3.10, it is enough to show that $h \in \bar{\mathscr{A}}_{\mathrm{gen}}$ and that, if α is a numerical exceptional curve and α is not in the span of the $[D_j]$, then α is effective if and only if $\alpha \cdot h \geq 0$. Also, it is enough to prove this for some deformation of (Y, D), so we can assume $\beta_i = [C_i]$ is the class of a -2-curve for all i and hence that h is the class of $H = \sum_i n_i C_i$. By construction, $H \cdot C_j > 0$ for every j. Hence, H is nef and big. By Lemma 2.6, it is enough to show that, if G is an irreducible curve not equal to D_i for any i, then $H \cdot G > 0$. Since H is nef, it suffices to rule out the case $H \cdot G = 0$, in which case $G^2 < 0$. As $G \neq D_j$ for any j, then G is either a -2-curve or an exceptional curve. The case where G is a -2-curve is impossible since then G is orthogonal to the span of the $[C_i]$, but the $[C_i]$ span Λ over $\mathbb{Q}$ and the intersection form is nondegenerate. So $G = E$ is an exceptional curve disjoint from the C_i. If $(\bar{Y}, \bar{D})$ is the anticanonical pair obtained by contracting E, then the $[C_i]$ define classes in $\bar{\Lambda} = \Lambda(\bar{Y}, \bar{D})$. Since the intersection form $(C_i \cdot C_j)$ is nondegenerate, the rank of $\bar{\Lambda}$ is at least that of Λ. It is easy to check that the classes of $\bar{D}_1, \ldots, \bar{D}_r$ are linearly independent: if say E meets D_1, then the intersection matrix of $\bar{D}_2, \ldots, \bar{D}_r$ is still negative definite and then Lemma 1.4(ii) (with $F = \bar{D}_1$ and $G_1, \ldots, G_n = \bar{D}_2, \ldots, \bar{D}_r$) shows that the classes of $\bar{D}_1, \ldots, \bar{D}_r$ are linearly independent. Hence, the rank of $H^2(\bar{Y}; \mathbb{Z})$ is greater than or equal to the rank of $H^2(Y; \mathbb{Z})$, which contradicts the fact that $\bar{Y}$ is obtained from Y by contracting an exceptional curve. $\qquad\square$

4. Some examples

Example 4.1. We provide a series of examples that satisfy the hypotheses of Theorem 3.10, where the number of components and the multiplicities are arbitrarily large. Let $(\bar{Y}, \bar{D})$ be the anticanonical pair obtained by making $k + 6$ infinitely near blowups starting with the double point of a nodal cubic. Thus, $\bar{D} = \bar{D}_0 + \cdots + \bar{D}_{k+6}$, where $\bar{D}_0^2 = -k$, $\bar{D}_i^2 = -2$, $1 \leq i \leq k+5$, and $\bar{D}_{k+6}^2 = -1$. Now blow up $N \geq 1$ points $p_1, \ldots, p_N$ on $\bar{D}_{k+6}$, and let (Y, D) be the resulting anticanonical pair. Note that (Y, D) is negative definite as long as $k \geq 3$ or $k = 2$ and $N \geq 2$. Clearly $r(D) = k + 7$ and $K_Y^2 = 3 - k - N$. It follows that $\Lambda = \Lambda(Y, D)$ has rank N. If $E_1, \ldots, E_N$ are the exceptional curves corresponding to $p_1, \ldots, p_N$, then the classes $[E_i] - [E_{i+1}]$ span a negative definite root lattice of type A_{N-1} in Λ. By making all of the blowups

infinitely near to the first point, we see that all of the classes $[E_i] - [E_{i+1}]$ lie in R. Hence, (Y, D) satisfies the hypotheses of Theorem 3.10.

Next we turn to examples where the rank of Λ is small. The case where the rank of Λ is 1 is covered by Theorem 3.10 as well as the case where the rank of Λ is 2 and $R \neq \varnothing$. Note that, conjecturally at least, the case where $R \neq \varnothing$ should be related to the question of whether the dual cusp singularity deforms to an ordinary double point. It is easy to construct examples where the rank of Λ is 2 and with $R \neq \varnothing$: begin with an anticanonical pair $(\hat{Y}, \hat{D})$ where the rank of $\Lambda(\hat{Y}, \hat{D})$ is 1, locate a component $\hat{D}_i$ such that there exists an exceptional curve E on $\hat{Y}$ with $E \cdot \hat{D}_i = 1$, and blow up a point of $\hat{D}_i$ to obtain a new anticanonical pair (Y, D) together with exceptional curves E and E' (where we continue to let E denote the pullback to Y and E' the new exceptional curve) such that $[E] - [E'] \in R$. So our interest is in finding examples where $R = \varnothing$.

Remark 4.2. In case the rank of Λ is 2 and $R \neq \varnothing$, it is easy to see that either $(\overline{\mathscr{A}}_{\text{gen}} \cap \Lambda)/\mathbb{R}^+$ is a closed (compact) interval or $\overline{\mathscr{A}}_{\text{gen}} \cap \Lambda = \mathscr{C}^+ \cap \Lambda$ (and in fact both cases arise). In either case, there is at most one wall W^β with $\beta \in R$ passing through the interior of $\overline{\mathscr{A}}_{\text{gen}} \cap \Lambda$, and hence, either $R = \varnothing$ or $R = \{\pm\beta\}$.

Example 4.3. We give an example where the rank of Λ is 2 and there are no $\beta \in \Lambda$ such that $\beta^2 = -2$ (in particular, $R = \varnothing$; hence, the condition $f(R) = R$ is automatic for every isometry f) and of an isometry f that preserves $\mathscr{C}^+$ and the classes $[D_i]$ but not the generic ample cone. Let $(\overline{Y}, \overline{D})$ be the anticanonical pair obtained by making nine infinitely near blowups starting with the double point of a nodal cubic. Thus, $\overline{D} = \overline{D}_0 + \cdots + \overline{D}_9$, where $\overline{D}_0 = 3H - 2E_1 - \sum_{i=2}^{9} E_i$, $\overline{D}_i = E_i - E_{i+1}$, $1 \leq i \leq 8$, and $\overline{D}_9 = E_9$. Make two more blowups, one at a point p_{10} on $\overline{D}_9$ and one at a point p_{11} on $\overline{D}_4$. This yields an anticanonical pair (Y, D) with $D_0 = 3H - 2E_1 - \sum_{i=2}^{9} E_i$, $D_i = E_i - E_{i+1}$, $i > 0$ and $i \neq 4$, and $D_4 = E_4 - E_5 - E_{11}$. Thus,

$$(-d_0, \ldots, -d_9) = (3, 2, 2, 2, 3, 2, 2, 2, 2, 2),$$

i.e., D is of type $\left(\begin{smallmatrix} 3 & 3 \\ 3 & 5 \end{smallmatrix}\right)$, with dual cycle $\left(\begin{smallmatrix} 6 & 8 \\ 0 & 0 \end{smallmatrix}\right)$ in the notation of [Friedman and Miranda 1983]. Set

$$G_1 = 5H - 2\sum_{i=1}^{4} E_i - \sum_{i=5}^{10} E_i - E_{11} \quad \text{and} \quad G_2 = 10H - 5\sum_{i=1}^{4} E_i - \sum_{i=5}^{10} E_i - 4E_{11}.$$

It is straightforward to check that $(G_i \cdot D_j) = 0$ for $i = 1, 2$ and $0 \leq j \leq 9$. Hence, $G_1, G_2 \in \Lambda$. Also,

$$G_1^2 = 2, \quad G_2^2 = -22, \quad G_1 \cdot G_2 = 0.$$

The corresponding quadratic form

$$q(n, m) = (nG_1 + mG_2)^2 = 2n^2 - 22m^2$$

has discriminant $-44 = -2^2 \cdot 11$. Note that this is consistent with the fact that the discriminant of the dual cycle is

$$\det \begin{pmatrix} -6 & 2 \\ 2 & -8 \end{pmatrix} = 44.$$

It is easy to see that G_1 and G_2 are linearly independent mod 2 and hence span a primitive lattice, which must therefore equal Λ.

First, we claim that there is no element of Λ of square -2. This is equivalent to the statement that there is no solution in integers to the equation $n^2 - 11m^2 = -1$, i.e., that the fundamental unit in $\mathbb{Z}[\sqrt{11}]$ has norm 1. But clearly if there were an integral solution to $n^2 - 11m^2 = -1$, then since $-11 \equiv 1 \bmod 4$, we could write -1 as a sum of two squares mod 4, which is impossible. In fact, the fundamental unit in $\mathbb{Z}[\sqrt{11}]$ is $10 + 3\sqrt{11}$. Thus, if R is the set of roots for (Y, D), then $R = \varnothing$. In particular, any isometry f trivially satisfies $f(R) = R$.

Finally, we claim there is an isometry f of $H^2(Y; \mathbb{Z})$ such that $f([D_i]) = [D_i]$ for all i and $f(\mathscr{C}^+) = \mathscr{C}^+$ but such that f does not preserve the generic ample cone. Note the unit group U of $\mathbb{Z}[\sqrt{11}]$ acts as a group of isometries on Λ and hence acts as a group of isometries (with $\mathbb{Q}$-coefficients) of the lattice

$$H^2(Y; \mathbb{Q}) = (\Lambda \otimes \mathbb{Q}) \oplus \bigoplus_i \mathbb{Q}[D_i],$$

fixing the classes $[D_i]$. Also, any isometry of Λ that is trivial on the discriminant group $\Lambda^\vee/\Lambda$ extends to an integral isometry of $H^2(Y; \mathbb{Z})$ fixing the $[D_i]$. Concretely, the discriminant form $\Lambda^\vee/\Lambda \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/22\mathbb{Z}$. If $\mu = 10 + 3\sqrt{11}$, then it is easy to check that the automorphism of Λ corresponding to $\mu^2 = 199 + 60\sqrt{11}$ acts trivially on $\Lambda^\vee/\Lambda$ and hence defines an isometry f of $H^2(Y; \mathbb{Z})$ fixing the $[D_i]$. Then f acts freely on $(\mathscr{C}^+ \cap \Lambda)/\mathbb{R}^+$, which is just a copy of $\mathbb{R}$ (and f acts on it via translation). But the intersection of the generic ample cone with Λ has the nontrivial wall $W^{E_{11}}$ so that the intersection cannot be all of $\mathscr{C}^+ \cap \Lambda$. It follows that $f^{\pm 1}$ does not preserve the generic ample cone. Explicitly, let $(\hat{Y}, \hat{D})$ be the surface obtained by contracting E_{11} and let $\hat{G}_1 = 4G_1 - G_2 = 10H - 3\sum_{i=1}^{10} E_i$ be the pullback of the positive generator of $\Lambda(\hat{Y}, \hat{D})$. Thus, $\hat{G}_1$ is nef and big so that $\hat{G}_1 \in \bar{\mathscr{A}}_{\text{gen}}$. Clearly $\hat{G}_1 \in W^{E_{11}}$. If $A = \left(\begin{smallmatrix} a & 11b \\ b & a \end{smallmatrix}\right)$ is the isometry of Λ corresponding to multiplication by the unit $a + b\sqrt{11}$, then $A(G_1) = aG_1 + bG_2$, $A(G_2) = 11bG_1 + aG_2$, and $A(\hat{G}_1) = (4a - 11b)G_1 + (4b - a)G_2$. Thus,

$$E_{11} \cdot A(\hat{G}_1) = (4a - 11b) + 4(4b - a) = 5b.$$

Hence, $E_{11} \cdot A(\hat{G}_1) < 0$ if $b < 0$. Taking f^{-1}, which corresponds to $199 - 60\sqrt{11}$, we see that $f^{-1}(\hat{G}_1) \notin \bar{\mathscr{A}}_{\text{gen}}$.

Example 4.4. In this example, the rank of Λ is 2 and $R = \varnothing$, but there exist infinitely many $\beta \in \Lambda$ such that $\beta^2 = -2$. The condition $f(R) = R$ is again automatic for every isometry f, and reflection about every $\beta \in \Lambda$ with $\beta^2 = -2$ is an isometry that preserves $\mathscr{C}^+$ and the classes $[D_i]$ but not the generic ample cone.

As in the previous example, let $(\bar{Y}, \bar{D})$ be the anticanonical pair obtained by making nine infinitely near blowups starting with the double point of a nodal cubic. Thus, $\bar{D} = \bar{D}_0 + \cdots + \bar{D}_9$, where $\bar{D}_0 = 3H - 2E_1 - \sum_{i=2}^{9} E_i$, $\bar{D}_i = E_i - E_{i+1}$, $1 \le i \le 8$, and $\bar{D}_9 = E_9$. Make two more blowups, one at a point p_{10} on $\bar{D}_9$ and one at a point p_{11} on $\bar{D}_0$. This yields an anticanonical pair (Y, D) with $D_0 = 3H - 2E_1 - \sum_{i=2}^{9} E_i - E_{11}$ and $D_i = E_i - E_{i+1}$, $1 \le i \le 9$. Thus,

$$(-d_0, \ldots, -d_9) = (4, 2, 2, 2, 2, 2, 2, 2, 2, 2),$$

i.e., D is of type $\binom{4}{9}$, with dual cycle $\binom{12}{1}$ in the notation of [Friedman and Miranda 1983]. Set

$$G_1 = 10H - 3\sum_{i=1}^{10} E_i \quad \text{and} \quad G_2 = 3H - \sum_{i=1}^{10} E_i + E_{11}.$$

It is straightforward to check that $(G_i \cdot D_j) = 0$ for $i = 1, 2$ and $0 \le j \le 9$. Hence, $G_1, G_2 \in \Lambda$. Also,

$$G_1^2 = 10, \quad G_2^2 = -2, \quad \text{and} \quad G_1 \cdot G_2 = 0.$$

The corresponding quadratic form

$$q(n, m) = (nG_1 + mG_2)^2 = 10n^2 - 2m^2$$

has discriminant $-20 = -2^2 \cdot 5$. Note that this is consistent with the fact that the discriminant of the dual cycle is

$$\det \begin{pmatrix} -12 & 2 \\ 2 & -2 \end{pmatrix} = 20.$$

It is easy to see that G_1 and G_2 are linearly independent mod 2 and hence span a primitive lattice, which must therefore equal Λ.

To give a partial description of $\bar{\mathscr{A}}_{\text{gen}} \cap \Lambda$, note that (as for $\hat{G}_1$ in the previous example) G_1 is the pullback to Y of a positive generator for $\Lambda(\hat{Y}, \hat{D})$, where $\hat{Y}$ denotes the surface obtained by contracting E_{11}. Thus, G_1 is nef and big so that $G_1 \in \bar{\mathscr{A}}_{\text{gen}}$ and also $G_1 \in W^{E_{11}}$. Hence,

$$\mathscr{C}^+ \cap \Lambda = \{nG_1 + mG_2 : 5n^2 - m^2 > 0, \ n > 0\},$$

i.e., $n > 0$ and $-n\sqrt{5} < m < n\sqrt{5}$. The condition $E_{11} \cdot (nG_1 + mG_2) \geq 0$ gives $m \leq 0$. To get a second inequality on n and m, let

$$E' = 5H - 4E_{11} - \sum_{i=1}^{10} E_i.$$

Then $(E')^2 = E' \cdot K_Y = -1$, and $H \cdot E' > 0$. Hence, E' is effective. (In fact, one can show that E' is generically the class of an exceptional curve.) Thus, for all $nG_1 + mG_2 \in \bar{\mathcal{A}}_{\text{gen}}$,

$$E' \cdot (nG_1 + mG_2) = 20n + 9m \geq 0;$$

hence,

$$\bar{\mathcal{A}}_{\text{gen}} \cap \Lambda \subseteq \{nG_1 + mG_2 : n > 0, \ -\tfrac{20}{9}n \leq m \leq 0\}.$$

Next we describe the classes $\beta \in \Lambda$ with $\beta^2 = -2$. The element $\beta = aG_1 + bG_2 \in \Lambda$ satisfies $\beta^2 = -2$ if and only if $5a^2 - b^2 = -1$, i.e., if and only if $b + a\sqrt{5}$ is a unit in the (nonintegrally closed) ring $\mathbb{Z}[\sqrt{5}]$. For example, the class G_2 corresponds to 1; as we have seen, the wall $W^{G_2} = W^{E_{11}}$. The fundamental unit in $\mathbb{Z}[\sqrt{5}]$ is easily checked to be $9 + 4\sqrt{5}$. However, since we are only concerned with walls that are rays in the fourth quadrant $\{(nG_1 + mG_2) : n > 0, \ m < 0\}$, we shall consider instead $\pm(9 - 4\sqrt{5})$ and shall choose the sign corresponding to $\beta = 4G_1 - 9G_2$. Note that

$$\beta \cdot (nG_1 + mG_2) = 40n + 18m = 0 \iff E' \cdot (nG_1 + mG_2) = 0.$$

Hence, $W^\beta = W^{E'}$. Moreover, for every $\gamma \in \Lambda$ such that $\gamma^2 = -2$ and such that the wall W^γ passes through the fourth quadrant, either $W^\gamma = W^\beta$ or the corresponding ray W^γ lies below W^β. Thus, for every $\gamma \in \Lambda$ with $\gamma^2 = -2$, r_γ does not preserve $\bar{\mathcal{A}}_{\text{gen}} \cap \Lambda$. Hence, $R = \varnothing$.

Note that, aside from the isometries r_β, where $\beta^2 = -2$, one can also construct isometries of infinite order preserving $\mathscr{C}^+$ and the classes $[D_i]$ that do not preserve $\bar{\mathcal{A}}_{\text{gen}}$ using multiplication by fundamental units in $\mathbb{Z}[\sqrt{5}]$ as in the previous example.

Remark 4.5. The exceptional curve E' used in the above example is part of a general series of such. For $n \geq 0$, let Y be the blowup of $\mathbb{P}^2$ at $2n + 1$ points $p_0, \ldots, p_{2n}$ with corresponding exceptional curves $E_0, \ldots, E_{2n}$, and consider the divisor

$$A = nH - (n-1)E_0 - \sum_{i=1}^{2n} E_i.$$

Then $A^2 = A \cdot K_Y = -1$, and it is easy to see that there exist $p_0, \ldots, p_{2n}$ such that A is the class of an exceptional curve. In fact, if $\mathbb{F}_1$ is the blowup of $\mathbb{P}^2$ at p_0, then $\Sigma = nH - (n-1)E_0$ is very ample on $\mathbb{F}_1$ and, for an anticanonical divisor $D \in |-K_{\mathbb{F}_1}| = |3H - E_0|$, $\Sigma \cdot D = 2n + 1$. From this, it is easy to see that we

can choose the points $p_1, \ldots, p_{2n}$ to lie on the image of D in $\mathbb{P}^2$, and hence, we can arrange the blowup Y to have (for example) an irreducible anticanonical nodal curve.

Acknowledgements

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rf@math.columbia.edu   *Department of Mathematics, Columbia University,*
                                      2990 Broadway, New York, NY, 10027, United States

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