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In this paper, we consider congruences of Hilbert modular forms. Sturm showed that mod ℓ elliptic modular forms of weight k and level $\Gamma_1(N)$ are determined by the first $(k/12)[\Gamma_1(1) : \Gamma_1(N)] \bmod \ell$ Fourier coefficients. We prove an analogue of Sturm's result for Hilbert modular forms associated to totally real number fields. The proof uses the positivity of ample line bundles on toroidal compactifications of Hilbert modular varieties.

1. Introduction

In this paper, we consider congruences of Hilbert modular forms. Sturm [1987, Theorem 1] showed that mod ℓ modular forms of weight k and level $\Gamma_1(N)$ are determined by the first $(k/12)[\Gamma_1(1) : \Gamma_1(N)] \bmod \ell$ Fourier coefficients. We prove an analogue of Sturm's result for Hilbert modular forms associated to totally real number fields not equal to \mathbb{Q} .

Doi and Ohta [1977, Lemma 2.1] showed a result similar to Sturm's theorem for elliptic cusp forms of weight 2 by a geometric method. Sturm improves the result for general weights and general levels by a technical method. For the case that the coefficient field is \mathbb{C} , the similar result was long known [Miyake 1989, Corollary 2.3.4]. Recently, Baba, Chakraborty and Petridis [Baba et al. 2002, Theorem 3] obtained its generalization for complex Hilbert modular forms by using the Rayleigh quotient for the Laplace operator. It seems difficult to apply their method to the mod ℓ case. For the mod ℓ case, Burgos Gil and Pacetti [Dieulefait et al. 2010, Appendix B] showed a generalization for Hilbert modular forms associated to $\mathbb{Q}(\sqrt{5})$ and level $\Gamma_0(6\sqrt{5})$ by a method similar to ours.

As mentioned above, the aim of the article is to prove an analogue of Sturm's theorem for Hilbert modular forms. In other words, we obtain an upper bound of

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the order at zeros of Hilbert modular forms at the exceptional locus of resolution of cusp singularities.

To explain our main result, we prepare several notions (see Sections 2.1–2.2 for more precise definitions). Let $N \geq 3$ be an integer, F a totally real number field of finite degree $g \geq 2$, \mathbb{O}_F the ring of integers of F , d_F the discriminant of F , and \mathfrak{c} a nonzero integral ideal of F . Let \tilde{F} be the Galois closure of F and $\mathbb{O}_{\tilde{F}}$ the ring of integers of \tilde{F} . For a field K that is an $\mathbb{O}_{\tilde{F}}[1/(Nd_F), \mu_N]$ -algebra, $M_K = M_K(\mathfrak{c}, N)$ denotes the open connected Hilbert modular variety over K defined as a moduli of \mathfrak{c} -polarized Hilbert–Blumenthal abelian varieties with $\Gamma(N)$ -structure. Then $\bar{M}_K = \bar{M}_{K, \Sigma}$ denotes the toroidal compactification associated to a collection Σ of cone decompositions, $\underline{\omega}^k = \otimes_i \omega_i^{k_i}$ denotes the automorphic line bundle on \bar{M}_K of weight $k = (k_1, \dots, k_g) \in \mathbb{Z}_{\geq 0}^g$, and M^* denotes the minimal compactification of M . When $k = (1, 1, \dots, 1)$, we write $\underline{\omega}$ instead of $\underline{\omega}^k$. Our main result is the following:

Theorem 1. *Let $N \geq 3$ be an integer, $k = (k_1, k_2, \dots, k_g) \in \mathbb{Z}_{\geq 0}^g$, \mathfrak{c} a nonzero integral ideal of F , K a field that is an $\mathbb{O}_{\tilde{F}}[1/(Nd_F), \mu_N]$ -algebra (a $\mathbb{Z}[1/(Nd_F), \mu_N]$ -algebra if k is parallel), and S a nonempty finite set of irreducible components of codimension 1 in $\bar{M}_K \setminus M_K$. Let f be a \mathfrak{c} -polarized geometric Hilbert modular form over K of weight k and level $\Gamma(N)$, i.e., $f \in H^0(M_K(\mathfrak{c}, N), \underline{\omega}^k)$. Then*

$$f \neq 0 \Rightarrow \min_{E \in S} \{\text{ord}_E(f)\} < \kappa$$

for

$$\kappa = \kappa_S = \kappa_S(k, N) = C^{g-1} \sum_{i=1}^g \frac{k_i \{(\underline{\omega}^{(g-1)} \cdot \omega_i) + (\mathfrak{J}^{(g-1)} \cdot \omega_i)\}}{(\mathfrak{J}^{(g-1)} \cdot \sum_{E \in S} E)},$$

where C is a positive integer independent of k and N , \mathfrak{J} is the inverse image of the ideal sheaf defining $M_K^* \setminus M_K$ by $\pi : \bar{M}_K \rightarrow M_K^*$, and the dot (\cdot) denotes the intersection number (see Section 2.1).

The reason why we call Theorem 1 an “analogue” of Sturm’s theorem is that the constant κ includes a strange constant C . When $g = 2$ and the canonical divisor $K_{\bar{M}}$ of \bar{M} is nef (numerically effective), we may take an explicit constant as κ (Theorem 16). Moreover, when the field K is of positive characteristic and \bar{M} is a minimal surface of general type, using Ekedahl’s result [1988, Chapter III, Proposition 1.13], we may take a slightly better form of κ than Theorem 16 (Theorem 19). Applying Theorem 1 to classical Hilbert modular forms, we obtain the more useful results for the complex case (Corollary 9) and the mod ℓ case (Corollary 12). Corollary 9 gives another proof of [Baba et al. 2002, Theorem 3]. As another application of Theorem 1, we obtain a rough upper bound of the dimension of vector space of Hilbert modular forms (Corollary 15). Because the dimension for weight 1 is unknown, Corollary 15 is not trivial. We remark that

we can also show the results for the congruence subgroup Γ_1 by changing Γ in the proofs to Γ_1 and using theory of the arithmetic compactifications of Hilbert modular varieties for Γ_1 by Dimitrov [2004].

Theorem 1 is proved by a method extending Doi and Ohta's algebraic geometric one. To obtain an upper bound of the order of zeros of modular forms at cusps, Doi and Ohta used Riemann–Roch's theorem on modular curves over finite fields. However, to obtain the bound for Hilbert modular forms, we use the positivity of ample line bundles. The key point of the proof is the construction of a specific ample line bundle on the toroidal compactification. To do this, we use the semiampleness of the automorphic line bundle on the minimal compactification proved by Moret-Bailly [1985, Chapter V, Theorem 2.1]. Combining the inverse image of the ample line bundle onto a toroidal compactification with a certain relatively ample line bundle, a specific ample line bundle on the toroidal compactification is constructed.

This article is organized as follows. In **Section 2.1**, the notation of intersection numbers and the two facts of ampleness are explained. In **Section 2.2**, the definitions and the properties of integral models of Hilbert modular varieties, these arithmetic toroidal and minimal compactifications, and the geometric Hilbert modular forms are recalled. In **Section 3.1**, for Hilbert modular forms associated to totally real number fields that are not \mathbb{Q} , **Theorem 1** is proved. As consequences of **Theorem 1**, **Corollaries 9–15** are obtained. In **Section 3.2**, for the case that F is a real quadratic field, we obtain more explicit forms for κ in **Theorems 16** and **19**.

2. Preliminaries

Notation and conventions.

- Let F denote a totally real number field and $g = [F : \mathbb{Q}] < \infty$. Let \mathcal{O}_F , $\mathfrak{d} = \mathfrak{d}_F$, and d_F be the ring of integers of F , the different ideal of F/\mathbb{Q} , and the discriminant of F/\mathbb{Q} , respectively. Furthermore, $I = I_F$ denotes the set of the embeddings of F into \mathbb{R} , and $F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$.
- For a nonzero fractional ideal \mathfrak{a} of F , $\mathfrak{a}^* = \mathfrak{a}^{-1}\mathfrak{d}^{-1}$, and $(\mathfrak{a})_+$ denotes the subset of \mathfrak{a} consisting of the totally positive elements.
- Let \mathbf{Sch}/R denote the category of the schemes over a ring R , $R\text{-Alg}$ the category of the R -algebras, and \mathbf{Sets} the category of the sets.
- Let X be a normal variety. For a Cartier divisor D on X , $[D]$ denotes the Weil divisor associated to D . For a rational function f on X , $\text{div}(f)$ denotes the divisor associated to f . A \mathbb{Q} -Cartier divisor D on X is a divisor such that mD is a Cartier divisor for a nonzero integer m .
- For two functions $f, g : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$, $f \ll g$ denotes that there is a positive constant A such that $f(x) \leq Ag(x)$ for all $x \in \mathbb{R}^n$.

2.1. Intersection numbers and ampleness. In this section, we recall some facts in intersection theory [Fulton 1998, Chapter 2; Lazarsfeld 2004, Chapter 1].

Let d and n be positive integers such that $d < n$, K a field, and X a normal proper variety of dimension n over K . For Cartier divisors D_1, D_2, \dots, D_d on X and a d -dimensional irreducible closed subvariety V of X , the intersection number of D_1, D_2, \dots, D_d and V , denoted $(D_1 \cdots D_d \cdot V)$, is defined by several methods. But it is unique [Hartshorne 1977, Appendix A]. If $D = D_1 = \cdots = D_d$, then we write $(D^{(d)} \cdot V)$ instead of $(D_1 \cdots D_d \cdot V)$. For a Cartier divisor E , the intersection number $(D_1 \cdots D_{n-1} \cdot E)$ of D_1, \dots, D_{n-1} and E is defined by $(D_1 \cdots D_{n-1} \cdot [E])$ and linearity. If $D = E = D_1 = \cdots = D_{n-1}$, we write $(D^{(n)})$ instead of $(D^{(n-1)} \cdot D)$. For line bundles $\mathcal{L}_i \simeq \mathcal{O}_X(D_i)$ for $1 \leq i \leq n-1$ and $\mathcal{L}' \simeq \mathcal{O}_X(E)$, the intersection number $(\mathcal{L}_1 \cdots \mathcal{L}_{n-1} \cdot \mathcal{L}')$ is defined as $(D_1 \cdots D_{n-1} \cdot E)$. For \mathbb{Q} -Cartier divisors D_1, \dots, D_{n-1} and a \mathbb{Q} -Cartier divisor E , their intersection number $(D_1 \cdots D_{n-1} \cdot E)$ is defined by $(m_1 D_1 \cdots m_{n-1} D_{n-1} \cdot m_n E) / m_1 m_2 \cdots m_n$, where m_i are integers such that $m_i D_i$ and $m_n E$ are Cartier.

We recall a fact on the ampleness of line bundles:

Lemma 2 [Lazarsfeld 2004, Proposition 1.7.10]. *Let X and Y be proper varieties, $f : X \rightarrow Y$ a proper morphism, \mathcal{L} an f -ample line bundle on X , and \mathcal{M} an ample line bundle on Y . Then the line bundle $(f^* \mathcal{M})^{\otimes m} \otimes \mathcal{L}$ is ample on X for a sufficiently large positive integer m .*

We also recall that ampleness of line bundles is preserved by the pullback of finite morphisms.

Lemma 3 [Lazarsfeld 2004, Proposition 1.2.13, Corollary 1.2.28]. *Let X and Y be two projective varieties, and let $f : X \rightarrow Y$ be a finite morphism. If a line bundle \mathcal{L} on Y is ample, then $f^* \mathcal{L}$ is ample. Moreover, when f is finite and surjective, a line bundle \mathcal{L} on Y is ample if and only if $f^* \mathcal{L}$ is ample.*

2.2. Hilbert modular varieties and geometric modular forms. Let $\mathfrak{c} \subset F$ be a fixed nonzero integral ideal, N a positive integer, and R a $\mathbb{Z}[1/(Nd_F)]$ -algebra. For an R -algebra B , let $\underline{A} = (A, \iota, \lambda, \phi_N)$ be a \mathfrak{c} -polarized Hilbert–Blumenthal abelian variety with $\Gamma(N)$ -structure over B ; i.e.,

- (1) $\rho : A \rightarrow \text{Spec}(B)$ is an abelian scheme,
- (2) $\iota : \mathcal{O}_F \hookrightarrow \text{End}_B(A)$ is an injective ring homomorphism taking 1 to the identity,
- (3) λ is a \mathfrak{c} -polarization (see [Hida 2004, Section 4.1.1] for the definition),
- (4) $\phi_N : A[N] \simeq (\mathcal{O}_F/N\mathcal{O}_F)^2$ is a $\Gamma(N)$ -structure (i.e., an isomorphism as \mathcal{O}_F -modules), and
- (5) $\rho_* \Omega_{A/\text{Spec } B}^1$ is a locally free $\mathcal{O}_F \otimes_{\mathbb{Z}} B$ -module of rank 1.

A suitable rule defines the isomorphism on such schemes [Hida 2004, Section 4.1.1].

We consider the contravariant functor $\mathcal{C} = \mathcal{C}(\mathfrak{c}, \Gamma(N)) : \mathbf{Sch}_{/R}^{\text{op}} \rightarrow \mathbf{Sets}$ defined by

$$\mathcal{C}(\mathfrak{c}, \Gamma(N))(B) = \{(A, \iota, \lambda, \phi_N)_{/B}/\text{isom.}\}.$$

The functor $\mathcal{C}(\mathfrak{c}, \Gamma(N))$ admits a coarse moduli scheme over R . When $N \geq 3$, $\mathcal{C}(\mathfrak{c}, \Gamma(N))$ is representable by a scheme over R . Its fine moduli scheme is denoted $M = M_R = M_N = M_{R,N} = M(\mathfrak{c}, \Gamma(N)) = M_R(\mathfrak{c}, \Gamma(N))$, and the universal object is denoted $A^U \rightarrow M_N$. Remark that M is naturally defined over $R[\mu_N] = R[x]/(\Phi_N(x))$ [Deligne and Rapoport 1973, Section 3.20], where Φ_N is the N th cyclotomic polynomial. Furthermore, M is a smooth, geometrically irreducible scheme of finite type and relative dimension g over $R[\mu_N]$.

Next we define the cusps of M_N . Let \mathfrak{a} and \mathfrak{b} be nonzero fractional ideals of F such that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$, and let

$$\phi_N : \mathfrak{b} \oplus \mathfrak{a}^*/N(\mathfrak{b} \oplus \mathfrak{a}^*) \simeq (\mathbb{O}_F/N\mathbb{O}_F)^2$$

be an isomorphism as \mathbb{O}_F -modules. We set

$$\Gamma(N; \mathfrak{a}, \mathfrak{b}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(F) \mid \begin{array}{l} a, d \in 1 + N\mathbb{O}_F, \\ b \in N(\mathfrak{a}\mathfrak{b})^*, c \in N(\mathfrak{a}\mathfrak{d}_F) \end{array} \right\}.$$

We define the action of $\gamma \in \Gamma(N; \mathfrak{a}, \mathfrak{b})$ on $\mathfrak{b} \oplus \mathfrak{a}^*$ by

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \mathfrak{b} \oplus \mathfrak{a}^* \rightarrow \mathfrak{b} \oplus \mathfrak{a}^* : (\alpha, \beta) \mapsto (\alpha, \beta)\gamma^{-1} = (d\alpha - c\beta, -b\alpha + a\beta).$$

A cusp of $M(\mathfrak{c}, \Gamma(N))$ is defined to be $(\mathfrak{a}, \mathfrak{b}, \phi_N) \bmod \mathcal{B}_{\mathbb{Q}} \cap \Gamma(N; \mathfrak{a}, \mathfrak{b})$, where $\mathcal{B}_{\mathbb{Q}}$ is the standard Borel subgroup of $\text{Res}_{\mathbb{Q}}^F GL_2$.

Let $\Sigma = \{\Sigma_s\}$ be a collection of $\Gamma(N)$ -admissible polyhedral cone decompositions [Hida 2004, Section 4.1.4], and $\bar{M} = \bar{M}_R = \bar{M}_{R,N} = \bar{M}_{R,N,\Sigma}$ denotes the toroidal compactification of $M_R(\mathfrak{c}, \Gamma(N))$ associated to Σ . Then \bar{M} is normal over $R[\mu_N]$. We can take Σ such that \bar{M} is smooth or projective over $R[\mu_N]$ [Hida 2004, Section 4.1.4].

Let \tilde{F} be the Galois closure of F and $\mathbb{O}_{\tilde{F}}$ the ring of integers. Then, for an $\mathbb{O}_{\tilde{F}}[1/(Nd_F), \mu_N]$ -algebra R , we have the isomorphism

$$\mathbb{O}_F \otimes_{\mathbb{Z}} R \simeq \bigoplus_{i=1}^g R : a \otimes b \mapsto (\sigma_1(a)b, \sigma_2(a)b, \dots, \sigma_g(a)b).$$

For the structure morphism $\rho : A^U \rightarrow M_R(\mathfrak{c}, \Gamma(N))$, $\bar{\omega}$ denotes the locally free $\mathbb{O}_F \otimes_{\mathbb{Z}} \mathbb{O}_M$ -module $\rho_* \Omega_{A^U/R}^1$ of rank 1. By the above isomorphism, $\bar{\omega}$ is decomposed $\bar{\omega} \simeq \bigoplus_i \omega_i$, where ω_i is the locally free \mathbb{O}_M -module of rank 1 corresponding to $\sigma_i \in I$. For $k = (k_1, k_2, \dots, k_g) \in \mathbb{Z}^g$, we define the line bundle $\underline{\omega}_R^k = \underline{\omega}_{N,R}^k = \otimes_i \omega_i^{\otimes k_i}$ called

by the automorphic line bundle of weight k . When the weight $k = (k_0, k_0, \dots, k_0)$ is parallel, we can construct the automorphic line bundle $\underline{\omega}_{R'}^k = (\wedge^g \bar{\omega})^{\otimes k_0} = \det(\bar{\omega})^{\otimes k_0}$ over every $\mathbb{Z}[1/(Nd_F), \mu_N]$ -algebra R' .

A \mathfrak{c} -polarized holomorphic *geometric Hilbert modular form* (abbreviated as GHMF) associated to F of weight $k = (k_1, k_2, \dots, k_g)$ and level $\Gamma(N)$ defined over R is an element of $H^0(M_R, \underline{\omega}_R^k)$.

There is the semiabelian scheme \mathcal{G} over \bar{M}_R that is an extension of the universal abelian scheme $\underline{A}^U \rightarrow M_R$. Thus, $\underline{\omega}^k$ is extended on \bar{M} . When $[F : \mathbb{Q}] \geq 2$, by the Koecher principle [Chai 1990, Section 4.3], we have

$$H^0(M_R, \underline{\omega}_R^k) = H^0(\bar{M}_R, \underline{\omega}^k).$$

Let $\rho : \mathcal{G} \rightarrow \bar{M}_R$ be the structure morphism and $\underline{\omega} = \det(\rho_* \Omega_{\mathcal{G}/\bar{M}_R}^1)$.

By Moret-Bailly [1985, Chapter V, Theorem 2.1], $\underline{\omega}$ is semiample; i.e., there is a positive integer n_0 such that $\underline{\omega}^{\otimes n_0}$ is generated by global sections. Thus, the canonical rational map

$$\phi_{\underline{\omega}^{\otimes n_0}} : \bar{M} \rightarrow \mathbb{P}_R^{r'}, \quad P \mapsto (s_0(P) : s_1(P) : \dots : s_r(P))$$

is a morphism, where $s_0, s_1, \dots, s_r \in H^0(\bar{M}, \underline{\omega}^{\otimes n_0})$ are global sections generating $\underline{\omega}^{\otimes n_0}$. We set $\pi = \pi_N = \phi_{\underline{\omega}^{\otimes n_0}}$. The minimal compactification, denoted $M^* = M_R^* = M_{R,N}^* = M_R^*(\mathfrak{c}, \Gamma(N))$, of M is defined to be the image of π , and

$$M^* \simeq \text{Proj} \left(\bigoplus_{n \geq 0} H^0(\bar{M}, \underline{\omega}^{\otimes n_0 n}) \right).$$

By the semiampleness of $\underline{\omega}$, the graded ring $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} H^0(\bar{M}, \underline{\omega}^{\otimes n_0 n})$ is finitely generated over R [Chai 1990, Section 4.4]; in particular, M^* is of finite type over R . And M is isomorphic to an open dense subscheme of M^* , also denoted M . When $g = [F : \mathbb{Q}] \geq 2$, by the Koecher principle M^* does not depend on the choice of cone decompositions. And M^* is also normal. The connected components $M^* \setminus M$ are in one-to-one correspondence with the cusps of M . The direct image $\pi_* \underline{\omega}$ is \mathbb{Q} -Cartier; i.e., there is a positive integer n_0 such that $\pi_* \underline{\omega}^{\otimes n_0}$ is a line bundle. Then $\pi_* \underline{\omega}^{\otimes n_0}$ is ample [Chai 1990, Section 4.3].

There is another useful definition equivalent to the above definition. We consider the covariant functor $\mathcal{P} = \mathcal{P}_R(\mathfrak{c}, \Gamma(N)) : R\text{-Alg} \rightarrow \mathbf{Sets}$ with

$$\mathcal{P}(B) = \left\{ (\underline{A}, \omega)_{/B} / \text{isom.} \mid \underline{A} \in \mathcal{E}(B), \rho : A \rightarrow \text{Spec}(B), \right. \\ \left. H^0(A, \rho_* \Omega_{A/B}^1) \simeq (\mathbb{C}_F \otimes B) \omega \right\}.$$

Then for every R -algebra B , $\mathcal{T}_{\mathbb{Z}}(B) = (\mathbb{C}_F \otimes_{\mathbb{Z}} B)^\times$ acts on $\mathcal{P}(B)$, and $\mathcal{P}/\mathcal{T}_{\mathbb{Z}} \simeq \mathcal{E}$. If \mathcal{E} is representable, so is \mathcal{P} . And \mathcal{M} denotes the moduli scheme over R representing \mathcal{P} . Then \mathcal{M} is a $\mathcal{T}_{\mathbb{Z}}$ -torsor over M . It is known that $H^0(M, \underline{\omega}^k) \subset H^0(\mathcal{M}, \mathbb{C}_{\mathcal{M}})$ [Dimitrov and Tilouine 2004, Remarque 4.5]. Thus, $f \in H^0(M, \underline{\omega}^k)$ is regarded as

a function such that a pair $(\underline{A}, \omega)_{/B}$, for every R -algebra B , associates an element $f((\underline{A}, \omega)_{/B}) \in B$ that satisfies the following:

- (1) The value of f depends only on the isomorphism class of (\underline{A}, ω) .
- (2) Given a base change map $\rho : B \rightarrow B'$, f satisfies

$$\rho(f((\underline{A}, \omega)_{/B})) = f((\underline{A}, \omega)_{/B'}).$$

- (3) For every $\alpha \in (\mathbb{O}_F \otimes_{\mathbb{Z}} B)^\times = \mathcal{T}_{\mathbb{Z}}(B)$,

$$f(\underline{A}, \alpha\omega) = \prod_{i=1}^g \sigma_i(\alpha)^{-k_i} f(\underline{A}, \omega).$$

Let $R = \mathbb{O}_{\bar{F}}[1/(Nd_F), \mu_N]$, and let B be an R -algebra and f a c -polarized GHMF of weight $k = (k_1, k_2, \dots, k_g)$ and level $\Gamma(N)$ defined over B . Let \mathfrak{a} and $\mathfrak{b} \subset F$ be nonzero fractional ideals such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$, $\phi_N : (\mathfrak{b} \oplus \mathfrak{a}^*)/N(\mathfrak{b} \oplus \mathfrak{a}^*) \simeq (\mathbb{O}_F/N\mathbb{O}_F)^2$ an isomorphism, s a cusp of $\Gamma(N)$ parametrized by $(\mathfrak{a}, \mathfrak{b}, \phi_N)$, and σ a cone in the cone decomposition Σ_s . We set

$$R_\sigma(N) = R[q^\xi]_{\xi \in N^{-1}\mathfrak{ab} \cap \sigma^\vee} \quad \text{and} \quad R_0(N) = R[q^\xi]_{\xi \in N^{-1}(\mathfrak{ab})_+ \cup \{0\}},$$

where σ^\vee is the dual of σ . Let $S_\sigma(N) = \text{Spec } R_\sigma(N)$ and $S_0(N) = \text{Spec } R_0(N)$. Let $\hat{S}_\sigma(N)$ be the formal completion of $S_\sigma(N)$ along $S_\sigma^\infty(N) = S_\sigma(N) \setminus S_0(N)$. The formal scheme $\hat{S}_\sigma(N)$ is affine, and we define $\hat{R}_\sigma(N)$ as its coordinate ring. Then we can show that

$$R[[q^\xi]]_{\xi \in N^{-1}(\mathfrak{ab})_+ \cup \{0\}} \xrightarrow{\sim} \hat{R}_\sigma(N).$$

The Tate object $\text{Tate}_{(\mathfrak{a}, \mathfrak{b})}(q)$ corresponding to the cusp s is a Hilbert–Blumenthal abelian variety defined over the quotient field $\mathcal{Q}(\hat{R}_\sigma(N) \otimes_R B)$. The q -expansion of f at the cusp s is defined to be the value of f at the Tate object corresponding to s :

$$f(\text{Tate}_{(\mathfrak{a}, \mathfrak{b})}(q), \phi_N, \omega) = a_0 + \sum_{\xi \in N^{-1}\mathfrak{ab} \cap \sigma^\vee} a_\xi q^\xi \in \mathcal{Q}(\hat{R}_\sigma(N) \otimes_R B),$$

where ω is the canonical nonvanishing differential form on $\text{Tate}_{(\mathfrak{a}, \mathfrak{b})}(q)$.

The following fact is known as the q -expansion principle:

Lemma 4. *Let B' be a subring of B such that B' is an R -algebra, and let f be as above. Then f is defined over B' if and only if $a_\xi \in B'$ for all $\xi \in (\mathfrak{ab})_+ \cup \{0\}$.*

Proof. See [Rapoport 1978, Theorem 6.7]. □

There is a correspondence between the classical Hilbert modular forms and the geometric Hilbert modular forms defined over \mathbb{C} . Let $\mathfrak{H} = \mathcal{H}^g$ be the g -tuple product of the complex upper half-plane. Then the holomorphic function $f : \mathfrak{H} \rightarrow \mathbb{C}$ is a

classical Hilbert modular form of weight $k = (k_1, \dots, k_g)$ of level $\Gamma(N; \mathfrak{a}, \mathfrak{b})$ if f satisfies the following equation:

$$f(\gamma z) = \prod_{\sigma_i \in I} (\sigma_i(c)z_i + \sigma_i(d))^{k_i} f(z) \quad \text{for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(N; \mathfrak{a}, \mathfrak{b}).$$

Here $M_k(\Gamma(N; \mathfrak{a}, \mathfrak{b}))$ denotes the complex vector space of the classical Hilbert modular forms of weight k and level $\Gamma(N; \mathfrak{a}, \mathfrak{b})$.

To show corollaries of our main result, we need the following lemma:

Lemma 5. *As \mathbb{C} -vector spaces, $M_k(\Gamma(N; \mathfrak{a}, \mathfrak{b}))$ and $H^0(\overline{M}_{\mathbb{C}}(\mathfrak{a}\mathfrak{b}^{-1}, \Gamma(N)), \underline{\omega}_{\mathbb{C}}^k)$ are canonically isomorphic.*

Proof. See [Rapoport 1978, Lemme 6.12]. □

For $f \in M_k(\Gamma(N; \mathfrak{a}, \mathfrak{b}))$, let $f' \in H^0(\overline{M}_{\mathbb{C}}, \underline{\omega}_{\mathbb{C}}^k)$ be the form corresponding to f . We remark that replacing q^ξ by $e^{2\pi i \text{Tr}(\xi z)}$, the q -expansion of f' at a cusp s corresponds to the Fourier expansion of f at the cusp s .

Remark 6. Let K be a field that is a $\mathbb{Z}[1/(Nd_F), \mu_N]$ -algebra. For the open Hilbert modular variety $M_K \subset \overline{M}_K$, the Kodaira–Spencer isomorphism [Katz 1978, Section 1.0] gives $\overline{\omega} \otimes_{\mathbb{C}_F \otimes_{\mathbb{C}_M}} \overline{\omega} \simeq \Omega_{M/K}^1$. Therefore, $\underline{\omega}_K^{\otimes 2} = \underline{\omega}_K \otimes_{\mathbb{C}_M} \underline{\omega}_K \simeq \det(\overline{\omega} \otimes_{\mathbb{C}_F \otimes_{\mathbb{C}_M}} \overline{\omega}) \simeq \Omega_{M/K}^g$. Observing the behaviors of sections at cusps, we have

$$\underline{\omega}^{\otimes 2} \simeq \Omega_{\overline{M}/K}^g(\log D_\infty) \simeq \mathbb{C}_{\overline{M}}(K_{\overline{M}} + D_\infty),$$

where $K_{\overline{M}}$ is the canonical divisor of \overline{M} , and the Cartier divisor $D_\infty = \sum E$, where E runs over the irreducible components of codimension 1 in $\overline{M}_K \setminus M_K$.

3. Proofs of the main theorems

3.1. Main result. We assume $g = [F : \mathbb{Q}] \geq 2$ so that we use intersection theory. Let N be a positive integer such that $N \geq 3$, $k = (k_1, \dots, k_g) \in \mathbb{Z}_{\geq 0}^g$, \mathfrak{c} a nonzero integral ideal of F , and K a field that is a $\mathbb{Z}[1/(Nd_F), \mu_N]$ -algebra if k is parallel or an $\mathbb{C}_{\overline{F}}[1/(Nd_F), \mu_N]$ -algebra otherwise. Let $M_K = M_{K,N} = M_K(\mathfrak{c}, \Gamma(N))$ be the moduli scheme over K defined in Section 2.2. We choose a collection of projective $\Gamma(N)$ -admissible polyhedral cone decompositions $\Sigma = \{\Sigma_s\}$. Let $\overline{M}_K = \overline{M}_{K,N} = \overline{M}_{K,N,\Sigma}$ be the toroidal compactification of M_K associated to Σ , $M_K^* = M_{K,N}^*$ the minimal compactification of M_K , and $\pi = \pi_N : \overline{M}_K \rightarrow M_K^*$ the canonical morphism defined in Section 2.2. We set $D_\infty = \sum E$, where E runs over the irreducible components of codimension 1 in $\overline{M}_K \setminus M_K$. Let S be a nonempty finite set of irreducible components of codimension 1 in $\overline{M}_K \setminus M_K$. For a closed point $P \in M_K^*$, \mathcal{I}_P denotes the ideal sheaf on M_K^* defining P . And $\mathfrak{I} = \mathfrak{I}_N$ denotes the inverse image ideal sheaf $\pi_N^{-1}(\otimes_{P \in M_K^* \setminus M_K} \mathcal{I}_P) \cdot \mathbb{C}_{\overline{M}_K}$.

For convenience, we restate Theorem 1.

Theorem 1. *Retain the notation above and let f be a c -polarized geometric Hilbert modular form over K of weight k and level $\Gamma(N)$, i.e., $f \in H^0(\overline{M}_K(\Gamma(N), c), \underline{\omega}^k)$. Then, if $f \neq 0$, we have $\min_{E \in S} \{\text{ord}_E(f)\} < \kappa$ for*

$$\kappa = \kappa_S = \kappa_S(k, N) = C^{g-1} \sum_{i=1}^g \frac{k_i \{(\underline{\omega}^{(g-1)} \cdot \underline{\omega}_i) + (\mathfrak{J}^{(g-1)} \cdot \underline{\omega}_i)\}}{(\mathfrak{J}^{(g-1)} \cdot \sum_{E \in S} E)},$$

where C is a positive integer which is independent of k or N .

Remark 6. If f is a cusp form, κ can be smaller. More generally, for a nonnegative integer a , if f has zeros of order a at all irreducible components of codimension 1 in $\overline{M} \setminus M$, we may take κ as

$$\kappa = C^{g-1} \sum_{i=1}^g \frac{k_i \{(\underline{\omega}^{(g-1)} \cdot \underline{\omega}_i) + (\mathfrak{J}^{(g-1)} \cdot \underline{\omega}_i)\}}{(\mathfrak{J}^{(g-1)} \cdot \sum_{E \in S} E)} - a \left(\frac{(\mathfrak{J}^{(g-1)} \cdot D_\infty)}{(\mathfrak{J}^{(g-1)} \cdot \sum_{E \in S} E)} - 1 \right).$$

Remark 7. When the weight is parallel, $k = (k_0, \dots, k_0)$, we have

$$\sum_i k_0 (\mathfrak{J}^{(g-1)} \cdot \underline{\omega}_i) = k_0 (\mathfrak{J}^{(g-1)} \cdot \underline{\omega}) = 0$$

by the projection formula. This implies

$$\kappa = \frac{k_0 C^{g-1} (\underline{\omega}^{(g)})}{(\mathfrak{J}^{(g-1)} \cdot \sum_{E \in S} E)}.$$

By Hirzebruch’s proportionality theorem [Mumford 1977, Theorem 3.2], we have

$$(\underline{\omega}^{(g)}) = A \text{vol}(\overline{M}) c_1(\Omega_{\mathbb{P}^g}^1)^g = A' (-1)^g \zeta_F(-1) [\Gamma(1; \mathbf{a}, \mathbf{b}) : \Gamma(N; \mathbf{a}, \mathbf{b})],$$

where A and A' are positive constants not depending on the level N . In this case, we obtain

$$\kappa = k_0 C^{g-1} A' (-1)^g \zeta_F(-1) \frac{[\Gamma(1; \mathbf{a}, \mathbf{b}) : \Gamma(N; \mathbf{a}, \mathbf{b})]}{(\mathfrak{J}^{(g-1)} \cdot \sum_{E \in S} E)}.$$

To prove Theorem 1, we introduce the following lemma:

Lemma 8. *For a sufficiently large integer C , $\underline{\omega}^{\otimes C} \otimes \mathfrak{J}$ is ample on \overline{M}_K .*

Proof. We know that $\pi_* \underline{\omega}$ is \mathbb{Q} -Cartier; i.e., $\pi_* \underline{\omega}^{\otimes k_0}$ is a line bundle on M_K^* for a sufficiently large k_0 . Then this is ample [Chai 1990, Section 4.3]. Since \overline{M}_K is the normalization of the blowing-up of M_K^* along the ideal sheaf $\otimes_{P \in M_K^* \setminus M_K} \mathcal{I}_P$ [Ash et al. 1975, Chapter IV; Faltings and Chai 1990, Chapter V, Theorem 5.8], \mathfrak{J} is π -ample.

Lemma 2 implies the line bundle $(\pi^* \pi_* \underline{\omega})^{\otimes k_0 n} \otimes \mathfrak{J}$ is ample for an integer $n \gg 0$. Because $\underline{\omega}$ is semiample, $\pi^* \pi_* \underline{\omega}^{\otimes k_0} \simeq \underline{\omega}^{\otimes k_0}$. Thus, $\underline{\omega}^{\otimes k_0 n} \otimes \mathfrak{J}$ is also ample for n . Replacing $k_0 n$ with C , we obtain the lemma. \square

Proof of Theorem 1. We assume that $f \neq 0$ and set

$$v = \min_{E \in S} \{\text{ord}_E(f)\}.$$

Then $f \in H^0(\overline{M}_K, \underline{\omega}^k(-v \sum_{E \in S} E))$. Thus,

$$\mathbb{O}_{\overline{M}_K}(\text{div}(f)) \subset \underline{\omega}^k\left(-v \sum_{E \in S} E\right),$$

and they are effective. By positivity of ample line bundles,

$$\left((\underline{\omega}^{\otimes C} \otimes \mathcal{J})^{(g-1)} \cdot \underline{\omega}^k\left(-v \sum_{E \in S} E\right) \right) > 0$$

for the integer C in Lemma 8. Thus, we have

$$v < C^{g-1} \sum_{i=1}^g \frac{k_i \{(\underline{\omega}^{(g-1)} \cdot \underline{\omega}_i) + (\mathcal{J}^{(g-1)} \cdot \underline{\omega}_i)\}}{(\mathcal{J}^{(g-1)} \cdot \sum_{E \in S} E)}.$$

The independence of C and k and N refers to Lemma 14. □

Next we apply Theorem 1 for classical Hilbert modular forms. For this purpose, we start with a review of the relation between smooth cone decompositions and local structures of \overline{M} at cusps.

We assume that the cone decompositions in Σ are smooth. Let s be a cusp of $M(\mathfrak{c}, \Gamma(N))$ parametrized by $(\mathfrak{a}, \mathfrak{b}, \phi_N)$, where \mathfrak{a} and \mathfrak{b} are two nonzero fractional ideals such that $\mathfrak{a}\mathfrak{b}^{-1} = \mathfrak{c}$ and $\phi_N : \mathfrak{b} \oplus \mathfrak{a}^* / N(\mathfrak{b} \oplus \mathfrak{a}^*) \simeq (\mathbb{O}_F / N\mathbb{O}_F)^2$ is an isomorphism (see Section 2.2). We can take the quotient

$$\Sigma_s / U_N = \bigcup_{i=1}^g \{\overline{\sigma}_{i,j}\}_{j=1}^{r_i},$$

where $U_N = \{\epsilon \in \mathbb{O}_F^\times \mid \epsilon \equiv 1 \pmod{N}\}$, r_i is a positive integer, and $\overline{\sigma}_{i,j} \equiv \sigma_{i,j} \pmod{U_N}$ with an i -dimensional cone $\sigma_{i,j} \in \Sigma_s$. Let $1 \leq d \leq g$ and $\sigma \in \Sigma_s$ be a smooth d -dimensional cone. Since σ is smooth, we can take part of a \mathbb{Z} -basis $\alpha_1, \dots, \alpha_d$ of $N^{-1}\mathfrak{a}\mathfrak{b} \cap \sigma^\vee$ such that $N^{-1}\mathfrak{a}\mathfrak{b} \cap \sigma^\vee = \mathbb{Z}_{\geq 0}\alpha_1 + \dots + \mathbb{Z}_{\geq 0}\alpha_d + \mathbb{Z}\beta_{d+1} + \dots + \mathbb{Z}\beta_g$ with some part of a \mathbb{Z} -basis $\beta_{d+1}, \dots, \beta_g$ [Hida 2004, Section 4.1.4]. We regard 0 as a cone in Σ_s . Then $0^\vee = F_\infty = F \otimes_{\mathbb{Q}} \mathbb{R}$. We remark that $0^\vee = \bigcup_{\sigma \in \Sigma_s} \sigma^\vee$; in particular, $0^\vee \supset \sigma^\vee$. Thus, we have $N^{-1}\mathfrak{a}\mathfrak{b} \cap 0^\vee = \mathbb{Z}\alpha_1 + \dots + \mathbb{Z}\alpha_d + \mathbb{Z}\beta_{d+1} + \dots + \mathbb{Z}\beta_g$ for the above \mathbb{Z} -basis α_1, \dots, β_g .

For the above cone $\sigma \in \Sigma_s$ and the ring $R = \mathbb{Z}[1/(Nd_F), \mu_N]$, the ring $\hat{R}_\sigma(N)$ defined in Section 2.2 is the completion of $R_\sigma(N)$ by the principal ideal $(q^{\alpha_1} \dots q^{\alpha_d})$. We set

$$\hat{R}_\sigma^0(N) = \hat{R}_\sigma(N) \otimes_{R_\sigma(N)} R_0(N).$$

It is easy to show that, if $\tau \in \Sigma_s$ is a face of σ , $\text{Spec}(\hat{R}_\tau(N)) \subset \text{Spec}(\hat{R}_\sigma(N))$. The exceptional locus of $\pi : \bar{M} \rightarrow M^*$ is defined by gluing $\text{Spec}(\hat{R}_\sigma(N)) \setminus \text{Spec}(\hat{R}_\sigma^0(N))$ along Σ_s/U_N by the above rule.

For one element α_i of the basis, the divisor E_i associated to α_i is defined by the closed irreducible subvariety of codimension 1 in \bar{M} that contains the affine subvariety defined by $\{q^{\alpha_i} = 0\}$ in the exceptional locus.

For a GHMF f over a field K that is an $\mathbb{O}_{\tilde{F}}[1/(Nd_F), \mu_N]$ -algebra and its q -expansion

$$f = \sum_{\xi \in N^{-1}(\mathfrak{ab})_+ \cup \{0\}} a_\xi q^\xi$$

at the cusp s , the order of f at $E_{i,K} = E_i \times \text{Spec}(K)$ is

$$\text{ord}_{E_{i,K}}(f) = \min\{m_i \in \mathbb{Z} \mid a_{m_1\alpha_1 + \dots + m_g\alpha_g} \neq 0 \text{ in } K\}.$$

Corollary 9. *Let $f \in M_k(\Gamma(N; \mathfrak{a}, \mathfrak{b}))$ and*

$$f(z) = \sum_{\xi \in N^{-1}(\mathfrak{ab})_+ \cup \{0\}} a_\xi e^{2\pi i \text{Tr}(\xi z)}$$

be the Fourier expansion of f at cusp s . We fix a g -dimensional cone $\sigma \in \Sigma_s$. Let $\{\alpha_1, \dots, \alpha_g\}$ be a \mathbb{Z} -basis of $N^{-1}(\mathfrak{ab})$ corresponding to σ and E_i the divisor associated to α_i . For a fixed $i \in \{1, 2, \dots, g\}$, we set

$$\kappa_i = C^{g-1} \sum_{j=1}^g \frac{k_j \{(\underline{\omega}^{(g-1)} \cdot \underline{\omega}_j) + (\mathcal{J}^{(g-1)} \cdot \underline{\omega}_j)\}}{(\mathcal{J}^{(g-1)} \cdot E_{i,\mathbb{C}})},$$

$$\mathcal{S}_i = \{ \xi \in m_1\alpha_1 + \dots + m_g\alpha_g \in N^{-1}(\mathfrak{ab})_+ \cup \{0\} \mid 0 \leq m_i \leq \kappa_i \},$$

where C is the integer in [Theorem 1](#). Then, if $a_\xi = 0$ for every $\xi \in \mathcal{S}_i$, we have $a_\xi = 0$ for every $\xi \in N^{-1}(\mathfrak{ab})_+ \cup \{0\}$.

Proof. By [Lemma 5](#), f is regarded as a global section of $H^0(\bar{M}_{\mathbb{C}}, \underline{\omega}^k)$, and the q -expansion of f at cusp $s = (\mathfrak{a}, \mathfrak{b}, \phi_N)$ is

$$f(\text{Tate}_{(\mathfrak{a}, \mathfrak{b})}, \phi_N, \omega) = \sum_{\xi \in N^{-1}(\mathfrak{ab})_+ \cup \{0\}} a_\xi q^\xi,$$

where ω is a nonvanishing differential form on $\text{Tate}_{(\mathfrak{a}, \mathfrak{b})}$. Applying [Theorem 1](#) for f as $K = \mathbb{C}$ and $S = \{E_{i,\mathbb{C}}\}$, the corollary is proved. □

Remark 10. We can easily show that the subset \mathcal{S}_i is a finite set. Indeed, by [Proposition 13](#) and an easy calculation, we have an upper bound

$$\#\mathcal{S}_i \ll \max\{k_i\}^g N^{3g^2}.$$

Remark 11. [Corollary 9](#) gives a different proof of the result of [\[Baba et al. 2002\]](#).

Next we discuss the case of positive characteristic. The following corollary is an analogue of Sturm’s theorem:

Corollary 12. *Let $f \in M_k(\Gamma(N; \mathbf{a}, \mathbf{b}))$ and*

$$f(z) = \sum_{\xi \in N^{-1}(\mathbf{ab})_+ \cup \{0\}} a_\xi e^{2\pi i \operatorname{Tr}(\xi z)}$$

be the Fourier expansion of f at the cusp $s = (\mathbf{a}, \mathbf{b}, \phi_N)$. We assume $g = [F : \mathbb{Q}] \geq 2$ and fix a g -dimensional cone $\sigma \in \Sigma_s$. Let $\{\alpha_1, \dots, \alpha_g\}$ be a \mathbb{Z} -basis of $N^{-1}(\mathbf{ab})$ corresponding to σ and E_i the divisor associated to α_i . Let L be a number field, \mathbb{O}_L the ring of integers of L , and λ a prime ideal in \mathbb{O}_L such that $\lambda \nmid Nd_F \mathbb{O}_L$. Assume $a_\xi \in \mathbb{O}_L$ for every ξ . For a fixed $i \in \{1, 2, \dots, g\}$, we set

$$\kappa_i = C^{g-1} \sum_{j=1}^g \frac{k_j \{(\omega^{(g-1)} \cdot \omega_j) + (\mathfrak{J}^{(g-1)} \cdot \omega_j)\}}{(\mathfrak{J}^{(g-1)} \cdot E_{i, \mathbb{F}})},$$

where C is the integer in [Theorem 1](#) and \mathbb{F} is a finite extension of \mathbb{O}_L/λ (defined in the proof), and set \mathcal{S}_i to be same as in [Corollary 9](#).

Then, if $a_\xi \equiv 0 \pmod{\lambda}$ for every $\xi \in \mathcal{S}_i$, we have $a_\xi \equiv 0 \pmod{\lambda}$ for every ξ in $N^{-1}(\mathbf{ab})_+ \cup \{0\}$.

Proof. [Lemma 5](#) implies that f is regarded as an element of $H^0(\overline{M}_{\mathbb{C}}, \omega^k)$. Let \tilde{L} be the composition field of L and \tilde{F} . By the assumption, all the Fourier coefficients of f are in \mathbb{O}_L and thus in $\mathbb{O}_{\tilde{L}}[1/(Nd_F), \mu_N]$. Let λ' be a maximal ideal of ring $\mathbb{O}_{\tilde{L}}[1/(Nd_F), \mu_N]$ such that $\lambda' \mid \lambda \mathbb{O}_{\tilde{L}}[1/(Nd_F), \mu_N]$. Remark that $\lambda' \cap \mathbb{O}_L = \lambda$. By [Lemma 4](#) and the commutativity with base change maps of GHMF (see [Section 2.2](#)), we regard f as a GHMF defined over the field $\mathbb{O}_{\tilde{L}}[1/(Nd_F), \mu_N]/\lambda'$, and the q -expansion of f at cusp $s = (\mathbf{a}, \mathbf{b}, \phi_N)$ is

$$f(\operatorname{Tate}_{(\mathbf{a}, \mathbf{b})}, \phi_N, \omega_{\text{can}}) = \sum_{\xi \in N^{-1}(\mathbf{ab})_+ \cup \{0\}} (a_\xi \pmod{\lambda'}) q^\xi.$$

Applying [Theorem 1](#) for f as $K = \mathbb{F} = \mathbb{O}_{\tilde{L}}[1/(Nd_F), \mu_N]/\lambda'$ and $S = \{E_{i, \mathbb{F}}\}$, if $f \not\equiv 0 \pmod{\lambda'}$, an integer m_i such that $0 \leq m_i \leq \kappa_i$ and $a_{m_1 \alpha_1 + \dots + m_g \alpha_g} \not\equiv 0 \pmod{\lambda'}$ exists. Thus, $a_{m_1 \alpha_1 + \dots + m_g \alpha_g} \notin \lambda' \cap \mathbb{O}_K = \lambda$, and the contrapositive of the corollary is proved. \square

Next we examine the growth of $\kappa = \kappa(k, N)$ associated to weight $k = (k_1, \dots, k_g)$ and level N .

Proposition 13. *We may take κ in [Theorem 1](#) as*

$$\kappa(k, N) \ll \max_i \{k_i\} N^{3g}.$$

Let ω_N be the automorphic line bundle on \overline{M}_N . To show [Proposition 13](#), we introduce the following lemma:

Lemma 14. *Let ℓ be the characteristic of K , N and N' two positive integers such that $N \geq 3$ and $N \mid N'$ and $\ell \nmid N'$, and C a positive integer. Then $\underline{\omega}_N^{\otimes C} \otimes \mathfrak{I}_N$ is ample if and only if $\underline{\omega}_{N'}^{\otimes C} \otimes \mathfrak{I}_{N'}$ is ample.*

Proof. We can take a finite étale morphism $h : M_{N',K} \rightarrow M_{N,K}$ over $\mathbb{Z}[1/(N')]$ [Chai 1990, Section 2.2]. Let Σ be a collection of cone decompositions of level N and $\bar{M}_{N,K,\Sigma}$ the toroidal compactification associated to Σ . We define $\bar{M}_{N',K}$ to be the normalization of $\bar{M}_{N,K}$ in $M_{N',K}$, and $\tilde{h} : \bar{M}_{N',K} \rightarrow \bar{M}_{N,K}$ denotes the normalization morphism. Then the normalization $\bar{M}_{N',K}$ is the toroidal compactification of $M_{N',K}$ associated to Σ [Faltings and Chai 1990, Chapter IV, Theorem 6.7(1)]. Here Σ is regarded as a collection of cone decompositions of level N' . Remark that $\bar{M}_{N',K}$ may not be smooth. Then \tilde{h} is an extension of h to toroidal compactifications [Faltings and Chai 1990, Chapter IV, Theorem 6.7(2)]. In particular, $\tilde{h}|_{M_{N',K}}$ is étale. We can show that $\tilde{h}^* \underline{\omega}_N^{\otimes 2} \simeq \underline{\omega}_{N'}^{\otimes 2}$. Indeed by Remark 6,

$$\underline{\omega}_N^{\otimes 2} \simeq \mathcal{O}_{\bar{M}_{N,K}}(K_{\bar{M}_{N,K}} + D_{\infty,N}),$$

where $D_{\infty,N} = \sum E$ such that E runs over the irreducible components of codimension 1 in $\bar{M}_{N,K} \setminus M_{N,K}$. Since $\tilde{h}|_{M_{N',K}}$ is étale, the ramification of \tilde{h} occurs only at $\bar{M}_{N,K} \setminus M_{N,K}$. Thus, we can show that

$$\tilde{h}^* \underline{\omega}_N^{\otimes 2} \simeq \tilde{h}^* \mathcal{O}_{\bar{M}_{N,K}}(K_{\bar{M}_{N,K}} + D_{\infty,N}) \simeq \mathcal{O}_{\bar{M}_{N',K}}(K_{\bar{M}_{N',K}} + D_{\infty,N'}) \simeq \underline{\omega}_{N'}^{\otimes 2}.$$

It is known that $\tilde{h}^* \mathfrak{I}_N \simeq \mathfrak{I}_{N'}$ [Faltings and Chai 1990, Chapter V, proof of Corollary 5.14]. Assume that C is even and that $\underline{\omega}_N^{\otimes C} \otimes \mathfrak{I}_N$ is ample. Since \tilde{h} is finite and surjective, $\underline{\omega}_N^{\otimes C} \otimes \mathfrak{I}_N$ is ample if and only if $\underline{\omega}_{N'}^{\otimes C} \otimes \mathfrak{I}_{N'}$ is ample by Lemma 3. \square

Proof of Proposition 13. Let N and N' be positive integers such that $N, N' \geq 3$ and $\text{char}(K) \nmid NN'$ and C a positive integer such that $\underline{\omega}_N^{\otimes C} \otimes \mathfrak{I}_N$ is ample. According to Lemma 14, $\underline{\omega}_{N'}^{\otimes C} \otimes \mathfrak{I}_{N'}$ is ample. Thus, $\underline{\omega}_{N'}^{\otimes C} \otimes \mathfrak{I}_{N'}$ is also ample. This implies that we may take C independent of N . By Remark 7, we have

$$\begin{aligned} \kappa(k, N) &= C^{g-1} \sum_{i=1}^g \frac{k_i \{(\underline{\omega}^{(g-1)} \cdot \underline{\omega}_i) + (\mathfrak{I}_N^{(g-1)} \cdot \underline{\omega}_i)\}}{(\mathfrak{I}_N^{(g-1)}) \cdot \sum_{E \in S} E} \\ &\leq C^{g-1} \max_i \{k_i\} (\underline{\omega}^{(g)}) \\ &= C^{(g-1)} A \max_i \{k_i\} \zeta_F(-1)[\Gamma(1; \mathbf{a}, \mathbf{b}) : \Gamma(N; \mathbf{a}, \mathbf{b})] \\ &\ll \max_i \{k_i\} N^{3g}. \end{aligned} \quad \square$$

As a consequence of Theorem 1, Proposition 13, and Remark 10, we obtain an upper bound for the dimension of the vector space of the Hilbert modular forms.

Corollary 15. *Let N be an integer such that $N \geq 3$, $k = (k_1, \dots, k_g) \in \mathbb{Z}_{\geq 0}^g$ a weight vector, and K a field that is an $\mathbb{C}_{\bar{F}}[1/(Nd_F), \mu_N]$ -algebra (a $\mathbb{Z}[1/(Nd_F), \mu_N]$ -algebra if k is parallel). Then*

$$\dim_K H^0(\bar{M}_K, \underline{\omega}^k) \ll (\max_i \{k_i\})^g N^{3g^2}.$$

3.2. The case that F is a real quadratic field. In this section, for some special situations, we investigate the strange constant C appearing in [Theorem 1](#).

Assume that $g = 2$ and the canonical bundle $K_{\bar{M}_K}$ is nef. For the invertible sheaf \mathfrak{I}_N in [Theorem 1](#), we set

$$\mathbb{C}_{\bar{M}_K} \left(- \sum_i n_i E_i \right) = \mathfrak{I}_N,$$

where $\{E_i\}$ are the exceptional curves of $\pi : \bar{M}_K \rightarrow M_K^*$ and n_i are positive integers. We set $n_{\max} = \max_i \{n_i\}$. Then we may take $2n_{\max}$ as the constant C .

Theorem 16. *In the above setting, we may take $C = 2n_{\max}$. Thus, in this situation, we may take*

$$\kappa = 2n_{\max} \frac{\sum_{j=1}^2 k_j \{(\underline{\omega} \cdot \underline{\omega}_j) - (\sum_i n_i E_i \cdot \underline{\omega}_j)\}}{-\left(\sum_i n_i E_i \cdot \sum_{E \in S} E\right)}.$$

Proof. For our purpose, we need a line bundle \mathcal{L} on \bar{M} that is nef and $(\mathcal{L} \cdot E) > 0$ for all exceptional curves E . Thus, it is sufficient to prove that $(\omega^{\otimes 2n_{\max}} \otimes \mathfrak{I}_N \cdot \mathcal{C}) \geq 0$ for every irreducible curve \mathcal{C} in \bar{M}_K . In particular, it is not necessary to prove that the line bundle is ample. First, assume that the curve \mathcal{C} is exceptional. Then, by the π -ampleness of \mathfrak{I}_N , we have

$$\begin{aligned} (\omega^{\otimes 2n_{\max}} \otimes \mathfrak{I}_N \cdot \mathcal{C}) &= (\pi^* \pi_* \omega^{\otimes 2n_{\max}} \cdot \mathcal{C}) + (\mathfrak{I}_N \cdot \mathcal{C}) \\ &= (\pi_* \omega^{\otimes 2n_{\max}} \cdot \pi_* \mathcal{C}) + (\mathfrak{I}_N \cdot \mathcal{C}) = (\mathfrak{I}_N \cdot \mathcal{C}) > 0. \end{aligned}$$

Next assume \mathcal{C} is an irreducible curve such that $\pi(\mathcal{C})$ is an irreducible curve. Then

$$\begin{aligned} (\omega^{\otimes 2n_{\max}} \otimes \mathfrak{I}_N \cdot \mathcal{C}) &= \left(\mathbb{C}_{\bar{M}_K} \left(n_{\max} K_{\bar{M}_K} + n_{\max} D_{\infty} - \sum_i n_i E_i \right) \cdot \mathcal{C} \right) \\ &= (n_{\max} K_{\bar{M}_K} \cdot \mathcal{C}) + \left(\sum_i (n_{\max} - n_i) E_i \cdot \mathcal{C} \right) \geq 0. \quad \square \end{aligned}$$

We give an example for [Theorem 16](#) in the setting of [\[Dieulefait et al. 2010, Appendix B\]](#).

Example 17. Let $F = \mathbb{Q}(\sqrt{5})$, $N = 3$, and $k_1 = k_2 = 2k \in 2\mathbb{Z}_{>0}$. Then there are the exceptional curves E_i ($i = 1, 2, \dots, 10$) and the curve F_1 defined in [\[van der](#)

Geer 1988, page 88]. These intersection numbers are as follows:

$$\begin{aligned} (E_i \cdot E_j) &= \begin{cases} 0 & \text{if } i \neq j, \\ -4 & \text{if } i = j, \end{cases} \\ (F_1 \cdot F_1) &= -60, \\ (E_i \cdot F_1) &= 12 \quad \text{for } i = 1, 2, \dots, 10. \end{aligned}$$

We set

$$D' = \frac{1}{5} \left(\sum_{i=1}^{10} E_i + 2F_1 \right),$$

then D' is nef and $(D' \cdot D') = 8$, and it is known that D' equals the canonical divisor $K_{\bar{M}}$. In particular, \bar{M} is a minimal surface of general type. Then we may take 1 as n_{\max} because we may use $\mathcal{O}(-D_\infty)$ as the ideal sheaf \mathfrak{I}_3 , where $D_\infty = \sum_{i=1}^{10} E_i$. Thus, by easy calculation, we have $\kappa = 12k$. This estimate is the same as [Dieulefait et al. 2010, Theorem B.3].

Next we consider a more particular case. Assume that the toroidal compactification $\bar{M}_{\mathbb{F}}$ is a minimal surface of general type (i.e., the canonical divisor $K_{\bar{M}_{\mathbb{F}}}$ of $\bar{M}_{\mathbb{F}}$ is nef and $(K_{\bar{M}_{\mathbb{F}}} \cdot K_{\bar{M}_{\mathbb{F}}}) > 0$) over a field \mathbb{F} of positive characteristic ℓ . Under the above assumption, we have:

Lemma 18 [Ekedahl 1988, Chapter III, Proposition 1.13]. *Let \mathbb{F} be a field of positive characteristic, X a minimal surface of general type over \mathbb{F} , and Z the fundamental cycle on X . Then $2K_X - Z$ is numerically positive.*

Here the fundamental cycle of X is the fundamental cycle associated to the canonical morphism $\phi = \phi_{mK_X} : X \rightarrow X_{\text{can}}$ with a sufficiently large integer m , where X_{can} is the canonical model of X . The Weil divisor Z is the fundamental cycle of ϕ if Z is the smallest element in the set

$$\left\{ D = \sum m_i E_i \mid E_i \in \text{Ex}(\phi), m_i > 0, (D \cdot E) \leq 0 \text{ for all } E \in \text{Ex}(\phi) \right\},$$

where $\text{Ex}(\phi)$ is the set of the exceptional curves of ϕ (i.e., the irreducible curves contracted by ϕ) on X . For an irreducible curve $\mathcal{C} \subset X$, \mathcal{C} is contracted by ϕ if and only if $(K_X \cdot \mathcal{C}) = 0$. If $\mathcal{C} \simeq \mathbb{P}^1$ and \mathcal{C} is contracted by ϕ , by the adjunction formula

$$p_a(\mathcal{C}) = 1 + \frac{1}{2}((\mathcal{C} \cdot \mathcal{C}) + (K_X \cdot \mathcal{C})),$$

we have $(\mathcal{C} \cdot \mathcal{C}) = -2$. Thus, X_{can} is obtained by contracting all (-2) -curves on X .

Return to our case. Let π be the morphism $\bar{M}_{\mathbb{F}} \rightarrow M_{\mathbb{F}}^*$, $\phi = \phi_{mK_{\bar{M}_{\mathbb{F}}}} : \bar{M}_{\mathbb{F}} \rightarrow \bar{M}_{\mathbb{F},\text{can}}$ with a sufficiently large integer m , and $\text{Ex}(\pi)$ and $\text{Ex}(\phi)$ the sets of the exceptional curves of π and ϕ , respectively. Then we have

$$\text{Ex}(\phi) = \{ E \in \text{Ex}(\pi) \mid (E \cdot E) = -2 \}.$$

We can take $Z = \sum_{E \in \text{Ex}(\phi)} E$ as the fundamental cycle of ϕ . [Remark 6](#) implies

$$\mathcal{O}_{\bar{M}_{\mathbb{F}}}(2K_{\bar{M}_{\mathbb{F}}} - Z) \simeq \underline{\omega}^{\otimes 4} \otimes \mathcal{O}_{\bar{M}_{\mathbb{F}}}(-2D_{\infty} - Z).$$

Thus, we obtain:

Theorem 19. *Let $N \geq 3$ be an integer, $k = (k_1, k_2) \in \mathbb{Z}_{\geq 0}^2$, ℓ an odd prime number such that $\ell \nmid Nd_F$, \mathfrak{c} a nonzero integral ideal of F , and \mathbb{F} a field of characteristic ℓ . Assume that $K_{\bar{M}_{\mathbb{F}}}$ is nef and $(K_{\bar{M}_{\mathbb{F}}} \cdot K_{\bar{M}_{\mathbb{F}}}) > 0$. Let S be nonempty finite set of exceptional curves of $\pi : \bar{M}_{\mathbb{F}} \rightarrow M_{\mathbb{F}}^*$. Let f be a \mathfrak{c} -polarized geometric Hilbert modular form over \mathbb{F} of weight (k_1, k_2) and level $\Gamma(N)$. Then, if $f \neq 0$, we have $\min_{E \in S} \{\text{ord}_E(f)\} < \kappa$. Here*

$$\kappa = \sum_{i=1}^2 \frac{k_i \{4(\underline{\omega} \cdot \underline{\omega}_i) - ((2D_{\infty} + Z) \cdot \underline{\omega}_i)\}}{((-2D_{\infty} - Z) \cdot \sum_{E \in S} E)},$$

where D_{∞} and Z are the above Weil divisors.

Remark 20. In the settings of [Theorem 19](#), $2n_{\max} = 6$ if $\text{Ex}(\phi)$ is nonempty. Thus, when the weight is parallel (k, k) and $\text{Ex}(\phi)$ is not empty, [Theorem 16](#) implies

$$\kappa = \frac{6k(\underline{\omega} \cdot \underline{\omega})}{((-2D_{\infty} - Z) \cdot \sum_{E \in S} E)}.$$

But on the same assumptions, [Theorem 19](#) implies

$$\kappa = \frac{4k(\underline{\omega} \cdot \underline{\omega})}{((-2D_{\infty} - Z) \cdot \sum_{E \in S} E)}.$$

Therefore, the estimate of κ in [Theorem 19](#) is slightly better than in [Theorem 16](#).

Remark 21. In the situation of [Theorems 16](#) and [19](#), if f is a cusp form, we can take a smaller bound as [Remark 6](#). For example, under the assumption of [Theorem 16](#), for f having zeros of order a at all the irreducible components of codimension 1 in $\bar{M} \setminus M$, the bound can be taken as

$$\kappa - a \left(\frac{(\sum_i n_i E_i \cdot D_{\infty})}{(\sum_i n_i E_i \cdot \sum_{E \in S} E)} - 1 \right).$$

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