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Relative phantom maps

Kouyemon Iriye
Daisuke Kishimoto
Takahiro Matsushita

The de Bruijn–Erdős theorem states that the chromatic number of an infinite graph equals the maximum of the chromatic numbers of its finite subgraphs. Such determination by finite subobjects appears in the definition of a phantom map, which is classical in algebraic topology. The topological method in combinatorics connects these two, which leads us to define the relative version of a phantom map: a map $f\colon X \to Y$ is called a relative phantom map to a map $\varphi\colon B \to Y$ if the restriction of f to any finite subcomplex of X lifts to B through φ, up to homotopy. There are two kinds of maps which are obviously relative phantom maps: (1) the composite of a map $X \to B$ with φ; (2) a usual phantom map $X \to Y$. A relative phantom map of type (1) is called trivial, and a relative phantom map out of a suspension which is a sum of (1) and (2) is called relatively trivial. We study the (relative) triviality of relative phantom maps and, in particular, we give rational homology conditions for the (relative) triviality.

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1 Introduction

1.1 De Bruijn–Erdős theorem

We start with the classical de Bruijn–Erdős theorem on graph colorings. A graph G is called n–colorable if its vertices are colored by n colors in such a way that adjacent vertices have different colors. Then the chromatic number $\chi(G)$ is defined to be the minimum n such that G is n–colorable. De Bruijn and Erdős [3] proved the following:

Theorem 1.1 *The chromatic number of an infinite graph equals the supremum of the chromatic numbers of its finite subgraphs.*

We shall connect this theorem to algebraic topology. Recall that the index of a free $\mathbb{Z}/2$–space X, denoted by $\operatorname{ind}(X)$, is the minimum n such that there is a $\mathbb{Z}/2$–equivariant map $X \to S^n$, where $\mathbb{Z}/2$ acts on S^n by the antipodal map. To a graph G, possibly

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infinite, one associates a free $\mathbb{Z}/2$–complex $\mathtt{B}(G)$ which is called the box complex of G. Here, we do not need to know what the box complex is; all we need to know is that it is a free $\mathbb{Z}/2$–complex having the following property: as in Matoušek and Ziegler [7], the chromatic number of a graph G and the index of the box complex $\mathtt{B}(G)$ are related by the inequality

$$\chi(G) \geq \operatorname{ind}(\mathtt{B}(G)) + 2.$$

Then one may ask whether $\operatorname{ind}(\mathtt{B}(G))$ is determined by the indices of the box complexes of finite subgraphs of G, like the chromatic number as in Theorem 1.1. Let us polish this question to pose a general problem in algebraic topology. First, for any free $\mathbb{Z}/2$–space X, there is a $\mathbb{Z}/2$–equivariant map $X \to S^n$ if and only if the classifying map $X/(\mathbb{Z}/2) \to \mathbb{R}P^\infty$ factors through $\mathbb{R}P^n$, up to homotopy. Next, the box complex of a finite graph is a finite complex, and any finite subcomplex of $\mathtt{B}(G)$ is included in $\mathtt{B}(H)$ for some finite subgraph H of G, where $\mathtt{B}(H)$ is a subcomplex of $\mathtt{B}(G)$. Then the above question is generalized and formalized as the following problem, which we call the topological de Bruijn–Erdős problem:

Problem 1.2 Suppose that a map $f\colon X \to \mathbb{R}P^\infty$ from a CW–complex X factors through $\mathbb{R}P^n$, up to homotopy, whenever it is restricted to any finite subcomplex of X. Then does f itself factor through $\mathbb{R}P^n$?

1.2 Relative phantom maps

Recall that a map $f\colon X \to Y$ from a CW–complex X is called a phantom map if the restriction of f to any finite subcomplex of X is null-homotopic. Phantom maps are classical in algebraic topology and their theory has been developed to a quite high level, as one can see in McGibbon [9]. The determination by finite subobjects of the topological de Bruijn–Erdős problem and phantom maps are quite similar and, actually, the topological de Bruijn–Erdős problem can be rephrased by the following relative version of phantom maps:

Definition 1.3 A map $f\colon X \to Y$ from a CW–complex X is called a relative phantom map from X to $\varphi\colon B \to Y$ if the restriction of f to any finite subcomplex of X lifts to B through φ, up to homotopy.

When B is a point, a relative phantom map to $\varphi\colon B \to Y$ is just a phantom map, so the name "relative" phantom map makes sense. We will call a phantom map an absolute phantom map to distinguish it from a relative phantom map.

It is obvious that any map $X \to B$ becomes a relative phantom map from X to $\varphi\colon B \to Y$ after composition with φ. We call a map $X \to Y$ homotopic to such a composite a trivial relative phantom map from X to $\varphi\colon B \to Y$, which is consistent with the triviality of an absolute phantom map. Then the topological de Bruijn–Erdős problem can be rephrased as follows: is there a nontrivial relative phantom map to the inclusion $\mathbb{R}P^n \to \mathbb{R}P^\infty$? Thus, we aim in this paper to study the "triviality" of relative phantom maps.

1.3 Triviality

Let $\mathrm{Ph}(X,\varphi)$ denote the set of homotopy classes of relative phantom maps from X to $\varphi\colon B \to Y$. We say that $\mathrm{Ph}(X,\varphi)$ is trivial if any relative phantom map from X to $\varphi\colon B \to Y$ is trivial. Note that the triviality of $\mathrm{Ph}(X,\varphi)$ does not imply $\mathrm{Ph}(X,\varphi) = *$. For example, if $\varphi = \mathrm{id}_B$, then $\mathrm{Ph}(X,\varphi)$ is trivial but $\mathrm{Ph}(X,\varphi) = [X, B]$.

We will first consider a condition equivalent to the (non)triviality of $\mathrm{Ph}(X,\varphi)$ when φ extends to a homotopy fibration $B \xrightarrow{\varphi} Y \to Z$. By using this, we will show the following example, which guarantees that there is certainly a nontrivial relative phantom map.

Example 4.9 *Let $u\colon BS^3 \to K(\mathbb{Z},4)$ be a generator of $H^4(BS^3;\mathbb{Z}) \cong \mathbb{Z}$, and extend it to a homotopy fibration sequence*

$$B \xrightarrow{\varphi} Y \to BS^3 \xrightarrow{u} K(\mathbb{Z},4).$$

Then $\mathrm{Ph}(\Sigma\mathbb{C}P^\infty,\varphi)$ is not trivial.

It is well known that any absolute phantom map into a torsion space — that is, a space with π_n finite for any n — is trivial. Next we will generalize this fact to relative phantom maps. As well as absolute phantom maps [9], we consider the class $\mathcal{F}$ of connected CW–complexes having finitely generated π_n for $n \geq 2$.

Proposition 4.6 *Suppose that $B, Y \in \mathcal{F}$ and $\varphi\colon B \to Y$ is an isomorphism in $\pi_n \otimes \mathbb{Q}$ for $n \geq 2$. Then $\mathrm{Ph}(\Sigma X,\varphi)$ is trivial.*

1.4 Relative triviality

Any absolute phantom map is obviously a relative phantom map. So relative phantom maps which are trivial relative phantom maps or absolute phantom maps need no special handling. When the source space is a suspension, any sum of such maps can be understood using the existing theory. So we are led to the following definition: a relative phantom map from a suspension ΣX to $\varphi\colon B \to Y$ is called *relatively trivial* if it is a finite sum of trivial relative phantom maps and absolute phantom maps, and $\mathrm{Ph}(\Sigma X, \varphi)$ is said to be relatively trivial if every relative phantom map from ΣX to φ is relatively trivial.

Example 1.4 Let $\varphi\colon B \to Y$ be as in Example 4.9 above. Since Y is a torsion space, every phantom map into Y is trivial. Then Example 4.9 shows that there is certainly a relatively nontrivial relative phantom map.

The main theorem of this paper is a further generalization of Proposition 4.6 to the relative triviality of relative phantom maps. For a map $\varphi\colon B \to Y$, we put

$$q(\varphi) = \{n \geq 2 \mid \varphi_* \otimes\colon \pi_n(B) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q} \text{ is not injective}\}.$$

Now we state our main theorem.

Theorem 5.8 *Suppose that $B, Y \in \mathcal{F}$ and $H_{n-1}(X;\mathbb{Q}) = 0$ for $n \in q(\varphi)$. Then $\mathrm{Ph}(\Sigma X, \varphi)$ is relatively trivial.*

1.5 Back to triviality

When the source space is not a suspension, we cannot consider the relative nontriviality of relative phantom maps. However, when X is not a suspension and any absolute phantom map $X \to Y$ is trivial, the (non)triviality of relative phantom maps out of X is still our object to study. Let $s_n\colon B \to B_n$ be the n^{th} Postnikov section of a space B. Then it is well known that any absolute phantom map into B_n is trivial, so we will study the following problem:

Problem 1.5 Find whether or not there is a nontrivial relative phantom map to the Postnikov section $s_n\colon B \to B_n$.

Of course, the methods developed for relative phantom maps out of a suspension do not apply to Problem 1.5 if the source space is not a suspension. However, by a sophisticated consideration on $\varprojlim^1$, one can prove the following. Put

$$q(B) = \{n \mid \pi_n(B) \otimes \mathbb{Q} \neq 0\}.$$

Theorem 1.6 *Suppose that $B \in \mathcal{F}$ is nilpotent or has torsion annihilators* (see Definition 6.3)*. If $H_k(X;\mathbb{Q}) = 0$ for $k \in q(B)$, then $\mathrm{Ph}(X, s_n)$ is trivial.*

We finally return to the topological de Bruijn–Erdős problem. Since the inclusion $\mathbb{R}P^n \to \mathbb{R}P^\infty$ is the first Postnikov section of $\mathbb{R}P^n$, Problem 1.5 is actually a generalization of the topological de Bruijn–Erdős problem. Since $\mathbb{R}P^n$ is nilpotent for n odd, one gets an answer to the topological de Bruijn–Erdős problem by Theorem 1.6.

Corollary 6.6 *If n is odd and $H_n(X;\mathbb{Q}) = 0$, then $\mathrm{Ph}(X, i_n)$ is trivial.*

We will construct a space $X(n)$ such that $H_n(X(n);\mathbb{Q}) \neq 0$ and there is a nontrivial phantom map from $X(n)$ to i_n. Then Corollary 6.6 will turn out to be the optimal answer to the topological de Bruijn–Erdős problem in terms of the rational homology of the source space.

Remark Csorba [5] proved that for any free $\mathbb{Z}/2$–complex X, possibly infinite, there is a graph G such that the box complex $\mathtt{B}(G)$ is $\mathbb{Z}/2$–homotopy equivalent to X. Then Corollary 6.6 gives a condition for the positive answer to the original question on the index of box complexes.

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2 Relative phantom maps and inverse limits

2.1 $\varprojlim$ and $\varprojlim^1$ of groups

In this subsection, we recall the definition of $\varprojlim$ and $\varprojlim^1$ of the inverse system of groups, not necessarily abelian. Let

$$G_0 \xleftarrow{f_0} G_1 \xleftarrow{f_1} \cdots \xleftarrow{f_{n-1}} G_n \xleftarrow{f_n} \cdots$$

be an inverse system of groups, and define the left action of $\prod_{n=0}^{\infty} G_n$ on itself by

$$(g_0, \dots, g_n, \dots) \cdot (x_0, \dots, x_n, \dots) = (g_0 x_0 f_0(g_1)^{-1}, \dots, g_n x_n f_n(g_{n+1})^{-1}, \dots).$$

Then $\varprojlim G_n$ and $\varprojlim^1 G_n$ are defined by the isotropy subgroup of $\prod_{n=0}^{\infty} G_n$ at $(1, 1, \dots) \in \prod_{n=0}^{\infty} G_n$ and the orbit space of this action, respectively. By definition, $\varprojlim G_n$ is a group but $\varprojlim^1 G_n$ is just a pointed set in general whose basepoint is the orbit containing $(1, 1, \dots)$. However, if every G_n is abelian, then $\varprojlim^1 G_n$ has a natural abelian group structure.

Next we recall the 6–term exact sequence (Lemma 2.1) involving $\varprojlim$ and $\varprojlim^1$ which will be useful. For a basepoint-preserving map $h\colon S \to T$ between pointed sets, we write $\operatorname{Im} h$ and $\operatorname{Ker} h$ to mean $h(S)$ and $h^{-1}(*)$, respectively. Recall that a sequence of pointed sets $A \xrightarrow{f} B \xrightarrow{g} C$ is exact if $\operatorname{Im} f = \operatorname{Ker} g$. If A, B and C are groups and f and g are group homomorphisms, the exactness coincides with that of groups.

Lemma 2.1 *Let $1 \to \{G_n\} \to \{H_n\} \to \{K_n\} \to 1$ be an exact sequence of inverse systems of groups. Then there is a natural exact sequence of pointed sets*

$$1 \to \varprojlim G_n \to \varprojlim H_n \to \varprojlim K_n \to \varprojlim^1 G_n \to \varprojlim^1 H_n \to \varprojlim^1 K_n \to *.$$

2.2 Absolute phantom maps

Recall that a map $f\colon X \to Y$ from a CW–complex is a phantom map if the restriction of f to any finite subcomplex of X is null-homotopic. Then $f\colon X \to Y$ is a phantom map if and only if the composite $f \circ g$ is null-homotopic for any map $g\colon K \to X$ from a finite complex K. This implies that the definition of a phantom map does not depend on a particular cell structure of the source space. Hereafter, we will assume that the source space of a phantom map is a CW–complex of finite type. Then, in particular, $f\colon X \to Y$ is a phantom map if and only if the restriction of f to any finite-dimensional skeleton of X is null-homotopic. As mentioned in Section 1, we will call a phantom map an absolute phantom map to distinguish it from relative phantom maps. Let $\operatorname{Ph}(X, Y)$ denote the set of homotopy classes of absolute phantom maps from X to Y.

Let X^n denote the n–skeleton of a CW–complex X. By the Milnor exact sequence (see [2])

$$* \to \varprojlim^1[\Sigma X^n, Y] \to [X, Y] \xrightarrow{\pi_Y} \varprojlim[X^n, Y] \to *, \tag{1}$$

we have the following description of $\operatorname{Ph}(X, Y)$ by $\varprojlim^1$:

Proposition 2.2 *There is an isomorphism of pointed sets*

$$\operatorname{Ph}(X, Y) \cong \varprojlim^1[\Sigma X^n, Y],$$

which is a group isomorphism whenever X is a suspension.

We can dualize this proposition by considering the Postnikov tower of the target space, where the proof is omitted. Let Y_n denote the n^{th} Postnikov section of Y, and let $Y_1 \leftarrow Y_2 \leftarrow \cdots \leftarrow Y_n \leftarrow \cdots$ be the Postnikov tower of Y.

Proposition 2.3 *There is an isomorphism of pointed sets*

$$\mathrm{Ph}(X,Y)\cong \varprojlim{}^1[X,\Omega Y_n],$$

which is a group isomorphism whenever X is a suspension.

We record consequences of the two propositions above on the triviality of $\mathrm{Ph}(X,Y)$ that we are going to use. As in Section 1, let $\mathcal{F}$ denote the class of connected CW–complexes having finitely generated π_n for $n\geq 2$.

Corollary 2.4 (1) *If Y is a finite Postnikov section, then* $\mathrm{Ph}(X,Y)=*$.

(2) *If $Y\in\mathcal{F}$ satisfies that $\pi_*(Y)\otimes\mathbb{Q}=0$ for $*\geq 2$, then* $\mathrm{Ph}(X,Y)=*$.

Proof (1) is immediate from Proposition 2.3. For (2), for any finite connected complex A, the homotopy set $[\Sigma A,Y]$ is a finite set by the assumption on Y, and the inverse system of finite groups satisfies the Mittag-Leffler condition (see [9]). Then $\mathrm{Ph}(X,Y)\cong\varprojlim{}^1[\Sigma X^n,Y]=*$. □

3 Relative phantom maps

In Section 1, we have defined that a map $X\to Y$ from a CW–complex X is a relative phantom to $\varphi\colon B\to Y$ if the restriction of f to any finite subcomplex of X lifts to B through φ, up to homotopy. As well as absolute phantom maps, one can see that the definition of a relative phantom map does not depend on a particular cell structure of the source space. Just as for absolute phantom maps, we will always assume that the source space of a relative phantom map is a connected CW–complex of finite type. In particular, $f\colon X\to Y$ is a relative phantom map to $\varphi\colon B\to Y$ if and only if its restriction to any finite-dimensional skeleton lifts to B through φ, up to homotopy.

Analogous to the absolute case in Proposition 2.3, let us dualize the definition of relative phantom maps. Let $Y_1\leftarrow Y_2\leftarrow\cdots\leftarrow Y_n\leftarrow\cdots$ be the Postnikov tower of Y as in the previous section, and let $s_n\colon Y\to Y_n$ be the n^{th} Postnikov section of Y. By the naturality of Postnikov towers, a map $\varphi\colon B\to Y$ induces a map $\varphi_n\colon B_n\to Y_n$ between the Postnikov sections, satisfying $\varphi_n\circ s_n^B\simeq s_n^Y\circ\varphi$, where s_n^B and s_n^Y are the Postnikov sections of B and Y, respectively.

Proposition 3.1 *The following conditions on a map $f\colon X\to Y$ are equivalent:*

(1) *f is a relative phantom map to φ.*

(2) *For any $n \geq 0$, $s_n \circ f\colon X \to Y_n$ has a lift with respect to $\varphi_n\colon B_n \to Y_n$, up to homotopy.*

Proof Suppose that f is a relative phantom map to φ. We want to show that $s_n \circ f\colon X \to Y_n$ has a lift with respect to φ_n, up to homotopy, for any n. Since f is a relative phantom map to φ, the map $f|_{X^{n+1}}\colon X^{n+1} \to Y$ has a lift $\tilde{f}\colon X^{n+1} \to B$ through φ, up to homotopy. Since the inclusion $X^{n+1} \to X$ induces an isomorphism $[X, B_n] \xrightarrow{\cong} [X^{n+1}, B_n]$ of pointed sets, there is a map $\bar{f}\colon X \to B_n$ satisfying $\bar{f}|_{X^{n+1}} \simeq s_n \circ \tilde{f}$. Now we have

$$\varphi_n \circ \bar{f}|_{X^{n+1}} \simeq \varphi_n \circ s_n \circ \tilde{f} \simeq s_n \circ \varphi \circ \tilde{f} \simeq s_n \circ f|_{X^{n+1}}.$$

Since the inclusion $X^{n+1} \to X$ induces an isomorphism $[X, Y_n] \xrightarrow{\cong} [X^{n+1}, Y_n]$ as pointed sets, we obtain that $\varphi_n \circ \bar{f} \simeq s_n \circ f$. Thus, $\bar{f}$ is a desired lift.

Suppose next that, for any n, $s_{n+1} \circ f\colon X \to Y_{n+1}$ has a lift $g\colon X \to B_{n+1}$ with respect to φ_{n+1}, up to homotopy. We want to show that $f|_{X^n}\colon X^n \to Y$ has a lift with respect to φ, up to homotopy. Since there is an isomorphism $(s_{n+1})_*\colon [X^n, B] \xrightarrow{\cong} [X^n, B_{n+1}]$ of pointed sets, we have a map $\bar{g}\colon X^n \to B$ satisfying $s_{n+1} \circ \bar{g} \simeq g|_{X^n}$. Then we get

$$s_{n+1} \circ \varphi \circ \bar{g} \simeq \varphi_{n+1} \circ s_{n+1} \circ \bar{g} \simeq \varphi_{n+1} \circ g|_{X^n} \simeq s_{n+1} \circ f|_{X^n}.$$

Since the map $(s_{n+1})_*\colon [X^n, Y] \to [X^n, Y_{n+1}]$ is also isomorphic, we get $\varphi \circ \bar{g} \simeq f|_{X^n}$, as required. □

Next we give a description of $\mathrm{Ph}(X, \varphi)$ by using $\mathrm{Ph}(X, Y)$ which will be useful for dealing with $\mathrm{Ph}(X, \varphi)$ algebraically.

Proposition 3.2 *There is an exact sequence of pointed sets*

$$1 \to \mathrm{Ph}(X, Y) \to \mathrm{Ph}(X, \varphi) \xrightarrow{\pi_Y} \varprojlim \varphi_*[X^n, B] \to 1$$

which is an exact sequence of groups whenever X is a suspension.

Proof Note that an element f of $[X, Y]$ is a relative phantom map to φ if and only if $\pi_Y(f) \in \varprojlim [X^n, Y]$ is contained in $\varprojlim \varphi_*[X^n, B]$. This means that the diagram

$$\begin{tikzcd}
\mathrm{Ph}(X, \varphi) \arrow[r, "\pi_Y"] \arrow[d] & \varprojlim \varphi_*[X^n, B] \arrow[d] \\
{[X, Y]} \arrow[r, "\pi_Y"] & \varprojlim [X^n, Y]
\end{tikzcd}$$

is a pullback. By the Milnor exact sequence (1), the lower π_Y is surjective, implying that the upper π_Y is surjective too. By (1), we also have that the kernel of the lower π_Y is $\varprojlim^1[\Sigma X^n, Y]$. Thus, the kernel of the upper π_Y is isomorphic to $\varprojlim^1[\Sigma X^n, Y]$, which is isomorphic to $\mathrm{Ph}(X, Y)$ by Proposition 2.2, completing the proof. □

4 Triviality of relative phantom maps out of a suspension

A relative phantom map $f\colon X \to Y$ to $\varphi\colon B \to Y$ is called trivial if the entire map f has a lift with respect to φ, up to homotopy, and $\mathrm{Ph}(X, \varphi)$ is called trivial if every element of $\mathrm{Ph}(X, \varphi)$ is trivial. We consider the triviality of relative phantom maps to $\varphi\colon B \to Y$ when φ is a fiber inclusion, that is, there is a homotopy fibration $B \xrightarrow{\varphi} Y \to W$. This case descends to relative phantom maps out of a suspension as follows. Given a map $\varphi\colon B \to Y$, there is a homotopy fibration $\Omega B \xrightarrow{\Omega\varphi} \Omega Y \to F$, where F is the homotopy fiber of φ. Then $\Omega\varphi$ is a fiber inclusion and by the adjointness, we have

$$\mathrm{Ph}(\Sigma X, \varphi) \cong \mathrm{Ph}(X, \Omega\varphi). \tag{2}$$

The following proposition enables us to detect the (non)triviality of relative phantom maps by that of related absolute phantom maps.

Proposition 4.1 *Let $B \xrightarrow{\varphi} Y \xrightarrow{p} Z$ be a homotopy fibration. Then a map $f\colon X \to Y$ is a relative phantom map to φ if and only if the composite $p \circ f\colon X \to Z$ is an absolute phantom map. Moreover, f is a trivial relative phantom map if and only if $p \circ f$ is null-homotopic.*

Proof For every n, $f|_{X^n}$ has a lift with respect to φ, up to homotopy, if and only if $p \circ f|_{X^n}$ is null-homotopic. This implies that f is a relative phantom map to φ if and only if $p \circ f$ is an absolute phantom map. Similarly, $p \circ f$ is null-homotopic if and only if f has a lift with respect to φ, up to homotopy. Thus, the proof is done. □

We show two applications of Proposition 4.1. The first one is as follows. We denote the adjoint of a map $f\colon \Sigma X \to Y$ by $\mathrm{ad}(f)\colon X \to \Omega Y$.

Corollary 4.2 *Let $\Omega Y \xrightarrow{\delta} F \to B \xrightarrow{\varphi} Y$ be a homotopy fibration sequence. A map $f\colon \Sigma X \to Y$ is a relative phantom map to φ if and only if $\delta \circ \mathrm{ad}(f)$ is an absolute phantom map. Moreover, f is trivial if and only if $\delta \circ \mathrm{ad}(f)$ is null-homotopic.*

Proof Note that $f\colon \Sigma X \to Y$ is a (trivial) relative phantom map to φ if and only if $\mathrm{ad}(f)\colon X \to \Omega Y$ is a (trivial) relative phantom map to $\Omega\varphi$. Then, by applying Proposition 4.1 to the fibration sequence $\Omega B \xrightarrow{\Omega\varphi} \Omega Y \xrightarrow{\delta} F$, the proof is done. □

Corollary 4.3 *Suppose that we have a homotopy fibration $F \to B \xrightarrow{\varphi} Y$ such that the connecting map $\delta\colon \Omega Y \to F$ is null-homotopic. Then $\mathrm{Ph}(\Sigma X, \varphi)$ is trivial.*

Proof Since δ is null-homotopic, so is $\delta \circ \mathrm{ad}(f)$ for any $f \in \mathrm{Ph}(\Sigma X, \varphi)$. Then, by Corollary 4.2, f is trivial, completing the proof. □

Example 4.4 Let

$$F_n(Y) \to G_n(Y) \xrightarrow{p_n} Y$$

be the n^{th} Ganea fibration. Since $\Omega G_n(Y) \to \Omega Y$ has a section (see Chapter 1 of [4]), the connecting map $\delta\colon \Omega Y \to F_n(Y)$ is null-homotopic. Thus, Corollary 4.3 implies that $\mathrm{Ph}(\Sigma X, p_n)$ is trivial.

Although we have seen that $\mathrm{Ph}(\Sigma X, p_n)$ is trivial, we will see in Proposition 6.8 below that there is a nonsuspension space $X(n)$ such that $\mathrm{Ph}(X(n), p_n)$ is not trivial for $Y = \mathbb{R}P^\infty$ with $n > 2$.

The next lemma is a variant of Corollary 4.2 and will be used to prove Proposition 4.6 below, which is a generalization of Corollary 2.4 to the relative case.

Lemma 4.5 *Let F be the homotopy fiber of a map $\varphi\colon B \to Y$. Then $\mathrm{Ph}(\Sigma X, \varphi)$ is trivial whenever $\mathrm{Ph}(X, F) = *$.*

Proof Let $\delta\colon \Omega Y \to F$ be the connecting map of a homotopy fibration $F \to B \xrightarrow{\varphi} Y$. Since $\mathrm{Ph}(X, F) = *$, $\delta \circ \mathrm{ad}(f)$ is a trivial absolute phantom map for any $f \in \mathrm{Ph}(\Sigma X, \varphi)$. Then f is trivial by Corollary 4.2, completing the proof. □

As in Section 1, we will write $\mathcal{F}$ to denote the class of connected CW–complexes each of which has finitely generated π_n for $n \geq 2$.

Proposition 4.6 *Let $B, Y \in \mathcal{F}$. Suppose that $\varphi\colon B \to Y$ is an isomorphism in $\pi_n \otimes \mathbb{Q}$ for $n \geq 2$. Then $\mathrm{Ph}(\Sigma X, \varphi)$ is trivial.*

Proof By assumption, the homotopy fiber F of φ satisfies the condition of Corollary 2.4, implying $\mathrm{Ph}(X, F) = *$. Then we get the desired result by Lemma 4.5. □

Next we show the second application of Proposition 4.1.

Proposition 4.7 *Suppose that there is a homotopy fibration sequence*

$$B \xrightarrow{\varphi} Y \xrightarrow{\alpha} V \xrightarrow{\beta} W$$

such that either β is null-homotopic or $\mathrm{Ph}(X, W) = *$. *Then* $\mathrm{Ph}(X, \varphi)$ *is trivial if and only if* $\mathrm{Ph}(X, V) = *$.

Proof It clearly follows from Proposition 4.1 that $\mathrm{Ph}(X, V) = *$ implies that $\mathrm{Ph}(X, \varphi)$ is trivial. On the other hand, let $f\colon X \to V$ be an absolute phantom map. Then $\beta \circ f\colon X \to W$ is an absolute phantom map, so by the assumption, $\beta \circ f$ is null-homotopic. Thus, f has a lift $\tilde{f}$ with respect to α, up to homotopy. By Proposition 4.1, $\tilde{f}$ is a relative phantom map from X to φ which is trivial if and only if $f\colon X \to V$ is null-homotopic. Therefore the proof is completed. □

Corollary 4.8 *Let $F \xrightarrow{j} B \xrightarrow{\varphi} Y$ be a homotopy fibration such that either j is null-homotopic or* $\mathrm{Ph}(X, B) = *$. *Then* $\mathrm{Ph}(\Sigma X, \varphi)$ *is trivial if and only if* $\mathrm{Ph}(X, F)$ *is trivial.*

Proof Apply Proposition 4.7 to the homotopy fibration sequence $\Omega B \xrightarrow{\Omega\varphi} \Omega Y \xrightarrow{s} F \xrightarrow{j} B$ together with the adjoint congruence (2). □

Example 4.9 We give an example of a space X and a map φ such that $\mathrm{Ph}(\Sigma X, \varphi)$ is nontrivial although $\mathrm{Ph}(\Sigma X, Y)$ is trivial. Let $u\colon BS^3 \to K(\mathbb{Z}, 4)$ be a generator of $H^4(BS^3; \mathbb{Z}) \cong \mathbb{Z}$, and extend it to a homotopy fibration sequence

$$S^3 \xrightarrow{\Omega u} K(\mathbb{Z}, 3) = B \xrightarrow{\varphi} Y \to BS^3 \xrightarrow{u} K(\mathbb{Z}, 4).$$

By Corollary 2.4, we have $\mathrm{Ph}(X, B) = *$ for any space X. So we can apply Corollary 4.8 to the homotopy fibration sequence $S^3 \xrightarrow{\Omega u} K(\mathbb{Z}, 3) = B \xrightarrow{\varphi} Y$. By [6], we have $\mathrm{Ph}(\mathbb{C}P^\infty, S^3) \neq *$, and thus we obtain that $\mathrm{Ph}(\Sigma\mathbb{C}P^\infty, \varphi)$ is not trivial. On the other hand, it follows from Corollary 2.4 that $\mathrm{Ph}(\Sigma\mathbb{C}P^\infty, Y)$ is trivial.

5 Relative triviality of relative phantom maps out of a suspension

Any absolute phantom map is a relative phantom map and it is not an object that we would like to study in this paper. So, as in Section 1, a relative phantom map out of a suspension is called relatively trivial if it is a finite sum of trivial relative phantom maps and absolute phantom maps, and we will investigate conditions for the existence of a relatively nontrivial phantom map. By Example 1.4, there is certainly a relatively nontrivial phantom map. We say that $\mathrm{Ph}(\Sigma X,\varphi)$ is relatively trivial if it consists only of relatively trivial relative phantom maps. We first observe basic properties of relatively trivial relative phantom maps. Note that the set $\mathrm{Ph}(\Sigma X,\varphi)$ is a group.

Proposition 5.1 (1) *Any relatively trivial relative phantom map is homotopic to the sum $f+g$, where f is a trivial phantom map and g is an absolute phantom map.*

(2) *The set of relatively trivial relative phantom maps from ΣX to $\varphi\colon B\to Y$ is a subgroup of $\mathrm{Ph}(\Sigma X,\varphi)$.*

Proof The map $\pi_Y\colon \mathrm{Ph}(\Sigma X,\varphi)\to \varprojlim \varphi_*[\Sigma X^n,B]$ in Proposition 3.2 is a group homomorphism whose kernel is $\mathrm{Ph}(\Sigma X,Y)$. In particular, $\mathrm{Ph}(\Sigma X,Y)$ is a normal subgroup of $\mathrm{Ph}(\Sigma X,\varphi)$, implying (1). We also get that the set of relatively trivial relative phantom maps from X to φ is the subgroup $\varphi_*[\Sigma X,B]+\mathrm{Ph}(\Sigma X,Y)$ of $\mathrm{Ph}(\Sigma X,\varphi)$. Thus, the proof is done. □

We investigate conditions which guarantee that $\mathrm{Ph}(\Sigma X,\varphi)$ is relatively trivial.

Lemma 5.2 *$\mathrm{Ph}(\Sigma X,\varphi)$ is relatively trivial if and only if the composite*

$$[\Sigma X,B]\xrightarrow{\varphi_*}\mathrm{Ph}(\Sigma X,\varphi)\xrightarrow{\pi_Y}\varprojlim \varphi_*[\Sigma X^n,B]$$

is surjective, where the map π_Y is as Proposition 3.2.

Proof Suppose first that $\mathrm{Ph}(\Sigma X,\varphi)$ is relatively trivial. There is a commutative diagram of groups

$$\begin{array}{ccc}
[\Sigma X,B] & \xrightarrow{\pi_B} & \varprojlim[\Sigma X^n,B] \\
\downarrow{\scriptstyle\varphi_*} & & \downarrow{\scriptstyle\varphi_*} \\
\mathrm{Ph}(\Sigma X,\varphi) & \xrightarrow{\pi_Y} & \varprojlim \varphi_*[\Sigma X^n,B]
\end{array} \tag{3}$$

where π_B and π_Y denote the natural projections as in (1) and Proposition 3.2. Then, by Proposition 3.2, the bottom arrow π_Y of (3) is surjective, so for any $f\in\varprojlim\varphi_*[\Sigma X^n,B]$, there is $\tilde f\in\mathrm{Ph}(\Sigma X,\varphi)$ satisfying $\pi_Y(\tilde f)=f$. By the assumption, $\tilde f$ is relatively trivial, so there are $g\in[\Sigma X,B]$ and $h\in\mathrm{Ph}(\Sigma X,Y)$ such that $\tilde f=\varphi_*(g)+h$. Now we have

$$f=\pi_Y(\tilde f)=\pi_Y(\varphi_*(g)+h)=\pi_Y\circ\varphi_*(g)+\pi_Y(h),$$

where π_Y is a group homomorphism. By definition, we have $\pi_Y(h)=0$, and then we have proved that $\pi_Y\circ\varphi_*$ is surjective.

Next suppose that $\pi_Y \circ \varphi_*$ is surjective, and take any $f \in \mathrm{Ph}(\Sigma X, \varphi)$. Then there is $g \in [\Sigma X, B]$ such that $\pi_Y \circ \varphi_*(g) = \pi_Y(f)$, implying $f - \varphi_*(g) \in \operatorname{Ker} \pi_Y$. Since $\operatorname{Ker} \pi_Y = \mathrm{Ph}(\Sigma X, Y)$ by Proposition 3.2, there is $h \in \mathrm{Ph}(\Sigma X, Y)$ satisfying $f - \varphi_*(g) = h$ or, equivalently, $f = h + \varphi_*(g)$. Thus, $\mathrm{Ph}(\Sigma X, \varphi)$ is relatively trivial. Therefore the proof is completed. $\square$

Let K_n be the kernel of the group homomorphism $\varphi_*\colon [\Sigma X^n, B] \to [\Sigma X^n, Y]$. Then we have the exact sequence of inverse systems of groups

$$1 \to \{K_n\} \to \{[\Sigma X^n, B]\} \to \{\varphi_*[\Sigma X^n, B]\} \to 1.$$

Proposition 5.3 $\mathrm{Ph}(\Sigma X, \varphi)$ *is relatively trivial if and only if the kernel of the map*

$$\varprojlim{}^1 K_n \to \varprojlim{}^1 [\Sigma X^n, B]$$

is trivial.

Proof Consider the commutative diagram (3). Since the top map π_B is surjective by the Milnor exact sequence (1), the map $\varphi_* \circ \pi_B\colon [\Sigma X, B] \to \varprojlim \varphi_*[\Sigma X^n, B]$ is surjective if and only if $\varphi_*\colon \varprojlim [\Sigma X^n, B] \to \varprojlim \varphi_*[\Sigma X^n, B]$ is surjective. Applying Lemma 2.1 to the short exact sequence

$$1 \to \{K_n\} \to \{[\Sigma X^n, B]\} \xrightarrow{\varphi_*} \{\varphi_*[\Sigma X^n, B]\} \to 1$$

of inverse systems of groups, we get an exact sequence

$$\varprojlim [\Sigma X^n, B] \xrightarrow{\varphi_*} \varprojlim \varphi_*[\Sigma X^n, B] \to \varprojlim{}^1 K_n \to \varprojlim{}^1 [\Sigma X^n, B]$$

of pointed sets. Thus, the map $\varphi_*\colon \varprojlim [\Sigma X^n, B] \to \varprojlim \varphi_*[\Sigma X^n, B]$ is surjective if and only if the kernel of the map $\varprojlim^1 K_n \to \varprojlim^1 [\Sigma X^n, B]$ is trivial. This completes the proof. $\square$

The assumption of the following corollary trivially implies that of Proposition 5.3.

Corollary 5.4 $\mathrm{Ph}(\Sigma X, \varphi)$ *is relatively trivial whenever* $\varprojlim^1 K_n = *$.

We then consider practical conditions which guarantee $\varprojlim^1 K_n = *$. We first translate the condition $\varprojlim^1 K_n = *$ to that of absolute phantom maps.

Lemma 5.5 *Let* $F \xrightarrow{j} B \xrightarrow{\varphi} Y$ *be a homotopy fibration with the connecting map* $\delta\colon \Omega Y \to F$. *For any space* X, $\varprojlim^1 K_n = *$ *if and only if the map* $\delta_*\colon \mathrm{Ph}(X, \Omega Y) \to \mathrm{Ph}(X, F)$ *is surjective.*

Proof Put $L_n = \mathrm{Ker}\{j_*\colon [\Sigma X^n, F] \to [\Sigma X^n, B]\}$. By the exactness of the sequence

$$[\Sigma X^n, F] \xrightarrow{j_*} [\Sigma X^n, B] \xrightarrow{\varphi_*} [\Sigma X^n, Y],$$

we have an exact sequence of inverse systems of groups

$$1 \to \{L_n\} \to \{[\Sigma X^n, F]\} \to \{K_n\} \to 1.$$

Then, by Lemma 2.1, we get an exact sequence of pointed sets

$$\varprojlim{}^1 L_n \to \varprojlim{}^1 [\Sigma X^n, F] \to \varprojlim{}^1 K_n \to *.$$

Thus, $\varprojlim{}^1 K_n = *$ if and only if the map $\varprojlim{}^1 L_n \to \varprojlim{}^1[\Sigma X^n, F]$ is surjective.

Next we put $M_n = \mathrm{Ker}\{\delta_*\colon [\Sigma X^n, \Omega Y] \to [\Sigma X^n, F]\}$. Similarly to the above, from the exact sequence of groups

$$[\Sigma X^n, \Omega Y] \xrightarrow{\delta_*} [\Sigma X^n, F] \xrightarrow{j_*} [\Sigma X^n, Y],$$

we get an exact sequence of inverse systems of groups

$$1 \to \{M_n\} \to \{[\Sigma X^n, \Omega Y]\} \to \{L_n\} \to 1.$$

Thus, by Lemma 2.1, we have that $\varprojlim{}^1[\Sigma X^n, \Omega Y] \to \varprojlim{}^1 L_n$ is surjective. Then $\varprojlim{}^1 K_n = *$ if and only if the composite $\varprojlim{}^1[\Sigma X^n, \Omega Y] \to \varprojlim{}^1 L_n \to \varprojlim{}^1[\Sigma X^n, F]$ is surjective. By Proposition 2.2, this composite is identified with $\delta_*\colon \mathrm{Ph}(X, \Omega Y) \to \mathrm{Ph}(X, F)$. Thus, the proof is completed. □

As we have given a rational homotopy condition for the triviality of $\mathrm{Ph}(\Sigma X, \varphi)$ in Proposition 4.6, we expect to find a rational homotopy condition for the relative triviality of $\mathrm{Ph}(\Sigma X, \varphi)$. McGibbon and Roitberg [10] gave a necessary and sufficient rational homotopy condition which guarantees that every phantom map $X \to Y$ is null-homotopic, and we are motivated by their result to consider a rational homotopy condition for the relative triviality of $\mathrm{Ph}(\Sigma X, \varphi)$. We first recall the result of Roitberg and Touhey [12].

Theorem 5.6 [12] *For $Y \in \mathcal{F}$, there is an isomorphism of pointed sets*

$$\mathrm{Ph}(X, Y) \cong \prod_{n \ge 1} H^n(X; \pi_{n+1}(Y) \otimes \widehat{\mathbb{Z}}/\mathbb{Z})/[X, \Omega \widehat{Y}] \tag{4}$$

which is natural with respect to X and Y, where $\widehat{\mathbb{Z}}$ is the profinite completion of the integer ring $\mathbb{Z}$ and $\widehat{Y}$ is the profinite completion of a space Y in the sense of Sullivan.

Remark Although more conditions on Y are assumed in [12], we may replace Y with its universal cover by Proposition 2.2, so that the conditions reduce to that $Y \in \mathcal{F}$.

Next we apply Theorem 5.6 to the induced map between absolute phantom maps. For a map $g\colon V \to W$, we put

$$\widehat{q}(g) = \{n \geq 2 \mid g_*\colon \pi_n(V) \otimes \mathbb{Q} \to \pi_n(W) \otimes \mathbb{Q} \text{ is not surjective}\}.$$

Lemma 5.7 *Given a map $g\colon V \to W$ for $V, W \in \mathcal{F}$, suppose that $H_{n-1}(X;\mathbb{Q}) = 0$ for $n \in \widehat{q}(g)$. Then $g_*\colon \mathrm{Ph}(X, V) \to \mathrm{Ph}(X, W)$ is surjective.*

Proof Since the isomorphism of Theorem 5.6 is natural with respect to Y, the lemma immediately follows from the fact that $\widehat{\mathbb{Z}}/\mathbb{Z}$ is a $\mathbb{Q}$–vector space. □

Put

$$q(\varphi) = \{n \geq 2 \mid \varphi_*\colon \pi_n(B) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q} \text{ is not injective}\}.$$

Now we give a rational homotopy condition for the relative triviality of $\mathrm{Ph}(\Sigma X, \varphi)$.

Theorem 5.8 *Let $B, Y \in \mathcal{F}$. If $H_{n-1}(X;\mathbb{Q}) = 0$ for $n \in q(\varphi)$, then $\mathrm{Ph}(\Sigma X, \varphi)$ is relatively trivial.*

Proof Let F be the homotopy fiber of $\varphi\colon B \to Y$ and $\delta\colon \Omega Y \to F$ be the corresponding connecting map. By the homotopy exact sequence, $\pi_n(\Omega Y) \otimes \mathbb{Q} \to \pi_n(F) \otimes \mathbb{Q}$ is surjective if and only if $\varphi_*\colon \pi_n(B) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}$ is injective for $n \geq 2$. Then we have $q(\varphi) = \widehat{q}(\delta)$. Thus, the proof is completed by Corollary 5.4 and Lemmas 5.5 and 5.7. □

We give three corollaries of this theorem.

Corollary 5.9 *Let $B, Y \in \mathcal{F}$. If $\varphi_*\colon \pi_n(B) \otimes \mathbb{Q} \to \pi_n(Y) \otimes \mathbb{Q}$ is injective for $n \geq 2$, then $\mathrm{Ph}(\Sigma X, \varphi)$ is relatively trivial.*

For a space A, we put

$$q(A) = \{n \geq 2 \mid \pi_n(A) \otimes \mathbb{Q} \neq 0\}.$$

Corollary 5.10 *Let $B, Y \in \mathcal{F}$. If $H_{n-1}(X;\mathbb{Q}) = 0$ for $n \in q(F)$, then $\mathrm{Ph}(\Sigma X, \varphi)$ is relatively trivial, where F is the homotopy fiber of $\varphi\colon B \to Y$.*

Proof By the homotopy exact sequence of the homotopy fibration $F \to Y \xrightarrow{\varphi} B$, we see that $q(\varphi) \subset q(F)$. Thus, the proof is done by Theorem 5.8. □

Corollary 5.11 *Let $B, Y \in \mathcal{F}$ and $F \xrightarrow{j} B \xrightarrow{\varphi} Y$ be a homotopy fibration such that j is null-homotopic. Then* $\mathrm{Ph}(\Sigma X, \varphi)$ *is relatively trivial.*

We close this section with the following example:

Example 5.12 By definition, if $\mathrm{Ph}(\Sigma X, \varphi)$ is trivial, then it is relatively trivial. Here we give a space X and a map φ such that the converse of this implication does not hold, that is, $\mathrm{Ph}(\Sigma X, \varphi)$ is relatively trivial and is not trivial.

Let $S^3 \to S^{4n+3} \xrightarrow{p_n} \mathbb{H}P^n$ be the Hopf fibration. Since the fiber inclusion $S^3 \to S^{4n+3}$ is null-homotopic, $\mathrm{Ph}(\Sigma X, p_n)$ is relatively trivial by Corollary 5.11. By Corollary 4.8, we also have that $\mathrm{Ph}(\Sigma X, p_n)$ is trivial if and only if $\mathrm{Ph}(X, S^3) = *$. Then, since $\mathrm{Ph}(\mathbb{C}P^\infty, S^3) \neq *$ by [6], we get that $\mathrm{Ph}(\Sigma\mathbb{C}P^\infty, p_n)$ is not trivial. Thus, we have obtained that $\mathrm{Ph}(\Sigma\mathbb{C}P^\infty, p_n)$ is relatively trivial and is not trivial.

6 Triviality of relative phantom maps out of a nonsuspension

In this section, we consider Problem 1.5. By Corollary 2.4, we have $\mathrm{Ph}(X, B_n) = *$ for all X, so the triviality and the relative triviality of phantom maps out of a suspension to $s_n\colon B \to B_n$ are the same. The case of relative phantom maps out of a suspension in Problem 1.5 has been studied in the previous sections. In particular, by Example 4.4, $\mathrm{Ph}(\Sigma X, i_n)$ is trivial for the inclusion $i_n\colon \mathbb{R}P^n \to \mathbb{R}P^\infty$. Thus, we consider relative phantom maps out of a nonsuspension for Problem 1.5. When X is not a suspension, the Puppe exact sequence associated with skeleta of X is not an exact sequence of groups, so we cannot use Lemma 2.1, which has been fundamental in many places above. Instead, we will use the following lemma:

Lemma 6.1 (cf [11, Lemma 1.1.5]) *Let $\{f_n\}\colon \{G_n\} \to \{H_n\}$ be a continuous map between inverse systems of compact Hausdorff topological spaces. Then the map $\varprojlim f_n\colon \varprojlim G_n \to \varprojlim H_n$ is surjective whenever each $f_n\colon G_n \to H_n$ is so.*

Let V be a finite complex and W be a torsion space, that is, $\widetilde{H}_n(W; \mathbb{Q}) = 0$ for any n. Then it is well known that the homotopy set $[V, W]$ is finite. We generalize this fact in two cases. The first case is the following:

Lemma 6.2 *If $B \in \mathcal{F}$ is nilpotent with finite π_1 and a finite complex Z satisfies $H_k(Z; \mathbb{Q}) = 0$ for $k \in q(B)$, then $[Z, B]$ is finite.*

Proof Let

$$\cdots \xrightarrow{q_{k+1}} B(k+1) \xrightarrow{q_k} B(k) \xrightarrow{q_{k-1}} \cdots \xrightarrow{q_0} B(0) = *$$

be a principal replacement of the Postnikov tower of B. Since Z is a finite complex, we have $[Z, B] \cong [Z, B(k)]$ for large k. Then it suffices to show that $[Z, B(k)]$ is finite for any k. We prove this by induction on k.

Each arrow $q_k\colon B(k+1) \to B(k)$ is a principal fibration with fiber $K(A_k, m_k)$ such that A_k is an abelian group. Then we have an exact sequence of pointed sets

$$H^{m_k}(Z; A_k) \to [Z, B(k)] \xrightarrow{(q_{k-1})_*} [Z, B(k-1)].$$

Since $q_{k-1}\colon B(k) \to B(k-1)$ is principal, we have $|(q_{k-1})_*^{-1}(a)| \le |H^{m_k}(Z; A_k)|$ for any $a \in [Z, B(k-1)]$. Moreover, by the assumption on X, $\widetilde{H}^{m_k}(Z; A_k)$ is finite for any k. Then the proof is done by induction on k starting with $[Z, B(0)] = *$ for $B(0) = *$. $\square$

To consider the second case, we introduce:

Definition 6.3 We say that a space Z has torsion annihilators if it has the following properties:

(1) $\pi_1(Z)$ is an abelian group.

(2) For any given integers n and N, there is a self-map $g\colon Z \to Z$ such that

  (a) $g_* \otimes \mathbb{Q}\colon \pi_*(Z) \otimes \mathbb{Q} \to \pi_*(Z) \otimes \mathbb{Q}$ is an isomorphism, and

  (b) for each $i \le n$, the map $g_*\colon \pi_i(Z) \to \pi_i(Z)$ is multiplication by an integer m_i with $N \mid m_i$.

For example, $S^n \vee \mathbb{R}P^\infty$ is a space which has torsion annihilators but is not nilpotent.

Lemma 6.4 *If $B \in \mathcal{F}$ has torsion annihilators and a finite complex Z satisfies $H_k(Z; \mathbb{Q}) = 0$ for $k \in q(B)$, then $[Z, B]$ is finite.*

Proof Since Z is a finite complex, we have $[Z, B] \cong [Z, B_n]$ for large n. Then it suffices to show that $[Z, B_n]$ is finite for any n. To see this, we prove by induction that there is a self-map $g\colon B \to B$ such that g is an isomorphism in rational homotopy groups and $(g_n)_*\colon [Z, B_n] \to [Z, B_n]$ is the constant map. When $B_0 = *$, this condition is satisfied.

Suppose that $[Z, B_{n-1}]$ is finite and there is a self-map $h\colon B \to B$ such that h is an isomorphism in rational homotopy groups and $(h_i)_*\colon [Z, B_i] \to [Z, B_i]$ is the constant map for $i < n$. By the naturality of Postnikov towers, we have the homotopy commutative diagram:

$$\begin{tikzcd} K(\pi_n(B), n) \arrow[r] \arrow[d, "h_*"] & B_n \arrow[r, "p_n"] \arrow[d, "h_n"] & B_{n-1} \arrow[d, "h_{n-1}"] \\ K(\pi_n(B), n) \arrow[r] & B_n \arrow[r, "p_n"] & B_{n-1} \end{tikzcd}$$

Then any map $f\colon Z \to B_n$ satisfies $p_n \circ h_n \circ f \simeq h_{n-1} \circ p_n \circ f \simeq *$, so $h_n \circ f$ has a lift $e\colon Z \to K(\pi_n(B), n)$, up to homotopy. By the assumption on Z, there is an integer N such that $N \cdot H^n(Z; \pi_n(B)) = 0$, so $Ne = 0$. Since B has torsion annihilators, there is a self-map $\ell\colon B \to B$ such that ℓ is an isomorphism in rational homotopy groups and the map $\ell_*\colon \pi_n(B) \to \pi_n(B)$ is the multiplication by an integer M with $N \mid M$. Then we see that $\ell_n \circ h_n \circ f \simeq *$ for any $f \in [Z, B_n]$. Let F be the homotopy fiber of $\ell_n \circ h_n$. Then F is a torsion space and $[Z, F] \to [Z, B_n]$ is surjective. Since Z is a finite complex, $[Z, F]$ is a finite set, so $[Z, B_n]$ too is a finite set. This completes the proof. □

Now we give our answer to Problem 1.5.

Theorem 6.5 *Let $s_n\colon B \to B_n$ be the n^{th} Postnikov section, and suppose that $B \in \mathcal{F}$ is nilpotent or has torsion annihilators. If $H_k(X; \mathbb{Q}) = 0$ for $k \in q(B)$, then $\mathrm{Ph}(X, s_n)$ is trivial for any n.*

Proof Consider a map between the inverse systems of pointed sets $\{[X^k, B]\} \to \{[X^k, B_n]\}$ induced by the Postnikov section $s_n\colon B \to B_n$. There is a commutative diagram

$$\begin{tikzcd} {[X, B]} \arrow[r, "\pi_B"] \arrow[d, "(s_n)_*"'] & \varprojlim [X^k, B] \arrow[d, "(s_n)_*"] \\ \mathrm{Ph}(X, s_n) \arrow[r, "\pi_{B_n}"] & \varprojlim (s_n)_*[X^k, B] \end{tikzcd}$$

where the horizontal arrows are surjective by (1) and Proposition 3.2. Since $[X^k, B_n] \cong [X, B_n]$ for $k > n$, the map $\pi_{B_n}\colon [X, B_n] \to \varprojlim [X^k, B_n]$ is injective. Then, since $\mathrm{Ph}(X, s_n)$ is a subset of $[X, B_n]$ and the lower π_{B_n} is the restriction of $\pi_{B_n}\colon [X, B_n] \to \varprojlim [X^k, B_n]$, the lower π_{B_n} is injective, so it is bijective. Then it follows that $\mathrm{Ph}(X, s_n)$

is trivial if and only if the right $(s_n)_*$ is surjective. Thus, we shall show that the right $(s_n)_*$ is surjective.

Note that the map $(s_n)_*\colon [X^k, B] \to (s_n)_*[X^k, B]$ is surjective for any k and that, by Lemmas 6.2 and 6.4, $[X^k, B]$ is a finite set for any k. It follows from Lemma 6.1 that the right $(s_n)_*$ is surjective, as desired. This completes the proof. $\square$

Finally, we deal with the case that φ is the inclusion $i_n\colon \mathbb{R}P^n \hookrightarrow \mathbb{R}P^\infty$. Since $\mathbb{R}P^n$ is nilpotent for an odd n, Theorem 6.5 implies the following corollary:

Corollary 6.6 *If $H_{2n+1}(X;\mathbb{Q}) = 0$ then $\mathrm{Ph}(X, i_{2n+1})$ is trivial.*

We finally show that Corollary 6.6 is optimal by giving an example of a space X such that $H_n(X;\mathbb{Q}) \neq 0$ and there is a nontrivial relative phantom map from X to $i_n\colon \mathbb{R}P^n \to \mathbb{R}P^\infty$. We will use the following simple lemma:

Lemma 6.7 *Let $\mathbb{Z}/2$ act on S^n by the antipodal map. For every odd integer k, there is a $\mathbb{Z}/2$–map $f\colon S^n \to S^n$ of degree k.*

Proof The case $n = 1$ is trivial, and for $n > 1$, take the $(n-1)$–fold suspension of the $\mathbb{Z}/2$–map on S^1. $\square$

Remark Lemma 6.7 implies that there is a mistake in the calculation of the homotopy set $[\mathbb{R}P^n, \mathbb{R}P^n]$ for n even due to McGibbon [8]. It is calculated as follows. Consider the homotopy cofibration sequence

$$S^{n-1} \xrightarrow{p_{n-1}} \mathbb{R}P^{n-1} \xrightarrow{i_{n-1}} \mathbb{R}P^n \xrightarrow{q_n} S^n,$$

where p_{n-1} is the universal covering, i_{n-1} is the inclusion and q_n is the pinch map to the top cell. Then, for $n-k > 0$ and $k > 0$, there is an exact sequence of groups

$$[\Sigma^{k+1}\mathbb{R}P^{n-k-1}, \mathbb{R}P^n] \xrightarrow{(\Sigma^{k+1}p_{n-k-1})^*} \pi_n(\mathbb{R}P^n) \xrightarrow{(\Sigma^k q_{n-k})^*} [\Sigma^k\mathbb{R}P^{n-k}, \mathbb{R}P^n] \xrightarrow{(\Sigma^k i_{n-k-1})^*} [\Sigma^k\mathbb{R}P^{n-k-1}, \mathbb{R}P^n].$$

Since $\pi_n(\mathbb{R}P^n) = \mathbb{Z}\{p_n\}$, $q_k \circ p_k = 1 + (-1)^{k+1}$ and $[\Sigma^k\mathbb{R}P^{n-k-1}, \mathbb{R}P^n] = *$, we inductively get

$$[\Sigma^k\mathbb{R}P^{n-k}, \mathbb{R}P^n] \cong \begin{cases} \mathbb{Z} & \text{if } n-k \text{ is odd,} \\ \mathbb{Z}/2 & \text{if } n-k \text{ is even,} \end{cases}$$

where in both cases, $p_n \circ \Sigma^k q_{n-k}$ is a generator. We next consider the exact sequence of pointed sets

$$[\Sigma\mathbb{R}P^{n-1}, \mathbb{R}P^n] \xrightarrow{(\Sigma p_{n-1})^*} \pi_n(\mathbb{R}P^n) \xrightarrow{q_n^*} [\mathbb{R}P^n, \mathbb{R}P^n] \xrightarrow{i_{n-1}^*} [\mathbb{R}P^{n-1}, \mathbb{R}P^n] \xrightarrow{p_{n-1}^*} \pi_{n-1}(\mathbb{R}P^n),$$

where $\pi_{n-1}(\mathbb{R}P^n) = 0$ and $[\mathbb{R}P^{n-1}, \mathbb{R}P^n] = \{*, i_{n-1}\}$. Then, by the above calculation, we have

$$(i_{n-1}^*)^{-1}(*) = \{*, p_n \circ q_n\}.$$

On the other hand, by considering the action of the top cell, we see that

$$(i_{n-1}^*)^{-1}(i_{n-1}) = \{h_{2j-1} \mid j \in \mathbb{Z}\},$$

where h_m is the self-map of $\mathbb{R}P^n$ which lifts to the degree m self-map of S^n as in Lemma 6.7. Thus, we obtain that

$$[\mathbb{R}P^n, \mathbb{R}P^n] = \{*,\ p_n \circ q_n,\ h_{2j-1}\ (j \in \mathbb{Z})\}.$$

For $n > 2$, let $X(n)$ be the cofiber of the composite of maps

$$\bigvee_p S^{n+2p-3} \xrightarrow{\alpha_1} S^n \xrightarrow{\pi} \mathbb{R}P^n,$$

where p ranges over all odd primes and $\alpha_1|_{S^{n+2p-3}}$ is a generator $\alpha_1(p)$ of the homotopy group $\pi_{n+2p-3}(S^n) \cong \mathbb{Z}/p$ (see [13]). By definition, we have $H^1(X(n); \mathbb{Z}/2) \cong \mathbb{Z}/2$, and let $f\colon X(n) \to \mathbb{R}P^\infty$ be the generator of $H^1(X(n); \mathbb{Z}/2)$.

Proposition 6.8 *The map $f\colon X(n) \to \mathbb{R}P^\infty$ is a nontrivial relative phantom map to the inclusion $i_n\colon \mathbb{R}P^n \to \mathbb{R}P^\infty$.*

Proof Suppose that f is homotopic to a map $g\colon X(n) \to \mathbb{R}P^n$. Then, since g induces an isomorphism in π_1, $g|_{\mathbb{R}P^n}$ lifts to a degree k map of S^n for some odd integer k. By definition, the composite $g|_{\mathbb{R}P^n} \circ \pi \circ \alpha_1(p)$ must be null-homotopic for any odd prime p. Since $\alpha_1(p)$ is a co-H–map [1], we have

$$g|_{\mathbb{R}P^n} \circ \pi \circ \alpha_1(p) \simeq \pi \circ k \circ \alpha_1(p) \simeq \pi \circ (k\alpha_1(p)).$$

Then, since $\pi_*\colon \pi_*(S^n) \to \pi_*(\mathbb{R}P^n)$ is an isomorphism for $* \geq 2$, we get that $k\alpha_1(p)$ is null-homotopic. Thus, k is divisible by any odd prime, which is a contradiction because $k \neq 0$ since k is odd. Therefore f does not lift to $\mathbb{R}P^n$ through the inclusion $i_n\colon \mathbb{R}P^n \to \mathbb{R}P^\infty$, up to homotopy. So, if f is a relative phantom map, then it is nontrivial.

Fix an odd prime p. By Lemma 6.7, for any given odd integer k, there is a self-map $h_k\colon \mathbb{R}P^n \to \mathbb{R}P^n$ which lifts to a degree k self-map of S^n. Let $p_1, \dots, p_m$ be all the odd primes less than or equal to p. Then, by the above observation, we see that the map $h_k\colon \mathbb{R}P^n \to \mathbb{R}P^n$ extends to a map $\overline{h}_k\colon X(n) \to X(n)$ and, by looking at π_1, we have

$$f \simeq f \circ \overline{h}_{p_1} \circ \cdots \circ \overline{h}_{p_m}.$$

Since

$$h_{p_i} \circ \pi \circ \alpha_1(p) \simeq \pi \circ (p_i \alpha_1(p))$$

as above, we see that the restriction of $f \circ \overline{h}_{p_1} \circ \cdots \circ \overline{h}_{p_m}$ to $X(n)^{n+2p-2}$ lifts to $\mathbb{R}P^n$ through i_n, up to homotopy. Since the prime p can be arbitrary large, f is a relative phantom map to the inclusion $i_n\colon \mathbb{R}P^n \to \mathbb{R}P^\infty$. Therefore we obtain that f is a nontrivial relative phantom map to $i_n\colon \mathbb{R}P^n \to \mathbb{R}P^\infty$, completing the proof. $\square$

Remark It follows from Corollary 6.6 that if $H_{2n+1}(X;\mathbb{Q}) = 0$ then $\mathrm{Ph}(X, i_{2n+1})$ is trivial. On the other hand, $H_{2n}(X(2n);\mathbb{Q}) = 0$ but $\mathrm{Ph}(X, i_{2n})$ is nontrivial. In fact, there is no k such that $H_k(X;\mathbb{Q}) = 0$ implies that $\mathrm{Ph}(X, i_{2n})$ is trivial.

If such an integer k exists, $k = n + 2p - 2$ for some odd prime p by the rational homology of $X(2n)$. Let $X'(n)$ be the subcomplex of $X(n)$ after we delete the $n{+}2p{-}2$–cell from $X(n)$. Then the restriction $f|_{X'(n)}\colon X'(n) \to \mathbb{R}P^\infty$ is a nontrivial relative phantom map to i_n for the same reason.

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Department of Mathematical Sciences, Osaka Prefecture University
Sakai, Japan

Department of Mathematics, Kyoto University
Kyoto, Japan

Department of Mathematical Sciences, University of the Ryukyus
Nishihara, Japan

kiriye@mi.s.osakafu-u.ac.jp, kishi@math.kyoto-u.ac.jp,
mtst@sci.u-ryukyu.ac.jp

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