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The ∞–categorical Eckmann–Hilton argument

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We define a reduced ∞–operad $\mathcal{P}$ to be d–connected if the spaces $\mathcal{P}(n)$ of n–ary operations are d–connected for all $n \geq 0$. Let $\mathcal{P}$ and $\mathcal{Q}$ be two reduced ∞–operads. We prove that if $\mathcal{P}$ is d_1–connected and $\mathcal{Q}$ is d_2–connected, then their Boardman–Vogt tensor product $\mathcal{P} \otimes \mathcal{Q}$ is (d_1+d_2+2)–connected. We consider this to be a natural ∞–categorical generalization of the classical Eckmann–Hilton argument.

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1 Introduction

Overview The classical Eckmann–Hilton argument (EHA), introduced in [10], states that given a set X with two unital (ie having a two-sided unit) binary operations

$$\circ, *\colon X \times X \to X,$$

if the two operations satisfy the "interchange law"

$$(a \circ b) * (c \circ d) = (a * c) \circ (b * d) \quad \text{for all } a, b, c, d \in X,$$

then they coincide and, moreover, this unique operation is associative and commutative. Even though it is easy to prove, the EHA is very useful. The most familiar applications are the commutativity of the higher homotopy groups of a space and the commutativity of the fundamental group of an H–space.

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A natural language for discussing different types of algebraic structures and the interactions between them is that of operads (by which, for now, we mean one-colored, symmetric operads in sets). For example, the data of a unital binary operation on a set X can be encoded as a structure of an algebra on X over a certain operad $\mathbf{Uni}$. Similarly, the data of a unital, associative and commutative binary operation on a set X (namely the structure of a commutative monoid) can be encoded as an algebra structure on X over the operad $\mathbf{Com}$. Furthermore, the category of operads is equipped with a tensor product operation, introduced by Boardman and Vogt [4], such that given two operads $\mathcal{P}$ and $\mathcal{Q}$, a $(\mathcal{P}\otimes\mathcal{Q})$–algebra structure on a set X is equivalent to a $\mathcal{P}$–algebra structure and a $\mathcal{Q}$–algebra structure on X, which satisfy a certain natural generalization of the interchange law defined above. Specializing to the case at hand, one can rephrase the EHA as

$$\mathbf{Uni}\otimes\mathbf{Uni}\simeq\mathbf{Com}.$$

Noting that $\mathbf{Com}$ is the terminal object in the category of operads (as all operation sets are singletons), this formulation looks perhaps a bit less surprising than the classical one. One can further observe that we can replace $\mathbf{Uni}$ by more general operads. We call an operad $\mathcal{P}$ *reduced* if both the set of nullary and the set of unary operations of $\mathcal{P}$ are singletons (ie there is a unique constant and it serves as a unit for all operations). The classical proof of the EHA can be easily modified[1] to show that given two reduced operads $\mathcal{P}$ and $\mathcal{Q}$ whose n–ary operation sets are nonempty for all n, we have

$$\mathcal{P}\otimes\mathcal{Q}\simeq\mathbf{Com}.$$

We call this the "operadic formulation of the EHA".

In many applications of the EHA, the two binary operations one starts with are actually known to be associative in advance. This version, which of course follows from the general EHA, can be stated as

$$\mathbf{Ass}\otimes\mathbf{Ass}\simeq\mathbf{Com},$$

where $\mathbf{Ass}$ is the operad that classifies the structure of a (unital, associative) monoid. For future reference, we call this "the associative EHA".

The language of operads already helps in organizing and systematizing the study of ordinary algebraic structures, but it is really indispensable for studying (and even defining) *enriched* and *homotopy coherent* algebraic structures. To start with, by

[1] See eg Proposition 3.8 of Fiedorowicz and Vogt [11].

replacing the *sets* of n–ary operations of an operad with *spaces* and requiring the various composition and permutation maps to be continuous, one obtains the notion of a topological operad. By further introducing an appropriate notion of a weak equivalence, one can study homotopy coherent algebraic structures. A fundamental example of such an object is the little n–cubes topological operad $\mathbb{E}_n$ for $0 \leq n \leq \infty$ (see eg J P May [18]). Loosely speaking, the structure of an $\mathbb{E}_n$–algebra on a space X can be thought of as a continuous unital multiplication map on X for which associativity holds up to a specified coherent homotopy and commutativity also holds up to a specified coherent homotopy, but only up to "level n".[2] On a more technical level, $\mathbb{E}_1$ and $\mathbb{E}_\infty$ can be interpreted as cofibrant models for **Ass** and **Com**, respectively, in a suitable model structure on the category of topological operads (see eg Vogt [21]). The sequence $\mathbb{E}_n$ serves as a kind of interpolation between them.

There are many approaches to modeling "homotopy coherent operads" (both one-colored and multicolored). Among them, the original approach of May via specific topological operads [18], via model structures (or partial versions thereof) on simplicial operads — see Berkger and Moerdijk [3], Vogt [21], Cisinski and Moerdijk [8] and Robertson [19] — or dendroidal sets/spaces — see Cisinski and Moerdijk [6; 7] — via "operator categories" of C Barwick [2] — or intrinsically to $(\infty, 1)$–categories via analytic monads — see Gepner, Haugseng and Kock [12] — or Day convolution; see Haugseng [14]. We have chosen to work with the notion of ∞–operads introduced and developed by J Lurie [17] based on the theory of ∞–categories introduced by A Joyal [15] and extensively developed in Lurie [16].[3] In this theory of ∞–operads (as in some of the others), there is a notion analogous to the Boardman–Vogt tensor product and it is natural to ask whether there is also an analogue of the EHA. For the associative EHA, one has the celebrated "additivity theorem", proved by G Dunn in the classical context [9] and by Lurie in the language of ∞–operads [17, Theorem 5.1.2.2], which states that for all integers $m, k \geq 0$, we have

$$\mathbb{E}_m \otimes \mathbb{E}_k \simeq \mathbb{E}_{m+k}.$$

The goal of this paper is to state and prove an ∞–categorical version of the classical (nonassociative) EHA. The key observation about the operadic formulation of the classical EHA is that both the hypothesis regarding the nonemptiness of the operation sets of $\mathcal{P}$ and $\mathcal{Q}$ and the characterization of **Com** as having singleton operation sets

[2] The situation is slightly different for $n = 0$ as an $\mathbb{E}_0$–algebra is just a pointed space.

[3] See Chu, Haugseng and Heuts [5] for a discussion of the comparison of the different models.

can be phrased in terms of *connectivity bounds*. For an integer $d \geq -2$, we say that a reduced ∞–operad $\mathcal{P}$ is d–connected if all of its operation spaces are d–connected. We prove:

Theorem 1.0.1 *Given integers $d_1, d_2 \geq -2$ and two reduced ∞–operads $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{P}$ is d_1–connected and $\mathcal{Q}$ is d_2–connected, the ∞–operad $\mathcal{P} \otimes \mathcal{Q}$ is (d_1+d_2+2)–connected.*

Unlike in the classical case, our result does not imply the additivity theorem (or vice versa), but the additivity theorem does demonstrate the sharpness of our result, since $\mathbb{E}_n$ is $(n-2)$–connected for all $n \geq 0$.

We shall deduce our ∞–categorical version of the EHA from a "relative" version, which might be of independent interest. For a reduced ∞–operad $\mathcal{P}$ and an integer $n \geq 0$, we denote by $\mathcal{P}(n)$ the space of n–ary operations of $\mathcal{P}$. We say that a map of spaces is a d–equivalence if it induces a homotopy equivalence on d–truncations, and that a map of reduced ∞–operads $\mathcal{P} \to \mathcal{Q}$ is a d–equivalence if for every integer $n \geq 0$, the map $\mathcal{P}(n) \to \mathcal{Q}(n)$ is a d–equivalence.

Theorem 1.0.2 *Let $\mathcal{P} \to \mathcal{Q}$ be a d–equivalence of reduced ∞–operads and let $\mathcal{R}$ be a k–connected reduced ∞–operad. The map $\mathcal{P} \otimes \mathcal{R} \to \mathcal{Q} \otimes \mathcal{R}$ is a $(d+k+2)$–equivalence.*

This behavior of the Boardman–Vogt tensor product on reduced ∞–operads is somewhat analogous to the behavior of the join operation on spaces. Given a map of spaces $X \to Y$ that is a d–equivalence and a k–connected space Z, the map $X \star Z \to Y \star Z$ is a $(d+k+2)$–equivalence. Incidentally, for the space of *binary* operations we have $(\mathcal{P} \otimes \mathcal{R})(2) \simeq \mathcal{P}(2) \star \mathcal{R}(2)$ (see [11, Proposition 4.8]), which relates the two phenomena.

Outline of the proof The proof of the classical EHA is straightforward. One simply uses repeatedly the unitality and interchange law to deduce the various equalities. For ∞–operads, the situation is considerably more complicated as all identities hold only up to a specified coherent homotopy and keeping track of this large amount of data is very difficult. Consequently, there is probably no hope of writing down an explicit formula for the operation spaces of $\mathcal{P} \otimes \mathcal{Q}$ in terms of those of $\mathcal{P}$ and $\mathcal{Q}$, except for low degrees. Therefore, as usual with ∞–categories, one has to adopt a less direct approach.

The proof of Theorem 1.0.2 proceeds by a sequence of reductions, which we now sketch in an informal way (we refer the reader to the end of this section for a list of notational conventions). An ∞–operad is called an essentially d–operad if all of its multimapping spaces are homotopically $(d-1)$–truncated. With every ∞–operad we can associate an essentially d–operad, called its d–homotopy operad, by $(d-1)$–truncating the multimapping spaces. This operation constitutes a left adjoint to the inclusion of the full subcategory on essentially d–operads into the ∞–category of ∞–operads. Using this adjunction and the Yoneda lemma, a map of ∞–operads $f\colon \mathcal{P} \to \mathcal{Q}$ is a d–equivalence if and only if the induced map

$$\mathrm{Map}(\mathcal{Q}, \mathcal{R}) \to \mathrm{Map}(\mathcal{P}, \mathcal{R})$$

is a homotopy equivalence for every essentially $(d+1)$–operad $\mathcal{R}$. Further analysis of the monad associated with a reduced ∞–operad shows that when $\mathcal{P}$ and $\mathcal{Q}$ are reduced, it is enough to check the above equivalence only for those $\mathcal{R}$ that are $(d+1)$–topoi endowed with the cartesian symmetric monoidal structure.

Now let $\mathcal{P} \to \mathcal{Q}$ be a d–equivalence of reduced ∞–operads and let $\mathcal{R}$ be a k–connected ∞–operad. We want to show that the map $\mathcal{P} \otimes \mathcal{R} \to \mathcal{Q} \otimes \mathcal{R}$ is a $(d+k+2)$–equivalence. By the above reductions, it is enough to show that for every $(d+k+3)$–topos $\mathcal{C}$ endowed with the cartesian symmetric monoidal structure, the induced map

$$\mathrm{Map}(\mathcal{Q} \otimes \mathcal{R}, \mathcal{C}) \to \mathrm{Map}(\mathcal{P} \otimes \mathcal{R}, \mathcal{C})$$

is a homotopy equivalence. A key property of the tensor product of ∞–operads is that it endows the ∞–category of ∞–operads with a symmetric monoidal structure that is *closed*. Namely, for every ∞–operad $\mathcal{O}$, there is an internal hom functor $\mathrm{Alg}_{\mathcal{O}}(-)$ that is right adjoint to the tensor product $-\otimes \mathcal{O}$.[4] It is therefore enough to show that the map

$$\mathrm{Map}(\mathcal{Q}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C})) \to \mathrm{Map}(\mathcal{P}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C}))$$

is a homotopy equivalence. Let **Triv** be the trivial operad. There are essentially unique maps $\mathbf{Triv} \to \mathcal{P}$ and $\mathbf{Triv} \to \mathcal{Q}$ that induce a commutative triangle

$$\begin{tikzcd}
\mathrm{Map}(\mathcal{Q}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C})) \arrow[rr] \arrow[dr] & & \mathrm{Map}(\mathcal{P}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C})) \arrow[dl] \\
& \mathrm{Map}(\mathbf{Triv}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C})) \simeq \mathrm{Alg}_{\mathcal{R}}(\mathcal{C})^{\simeq} &
\end{tikzcd}$$

[4]It is necessary to work here with multicolored operads as developed in [17], since even though the full subcategory of one-colored, or even reduced, ∞–operads is closed under the tensor product, the induced symmetric monoidal structure would not be closed.

and it is enough to show that the top map induces an equivalence on the fibers over each object X of $\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})$. Fixing such an X, the fiber of the left map consists of the space of ways to endow X with the structure of a $\mathcal{Q}$–algebra. Since $\mathcal{Q}$ is reduced, one can show that this is the space of maps from $\mathcal{Q}$ to the so-called "reduced endomorphism operad of X". This is a reduced ∞–operad $\mathrm{End}^{\mathrm{red}}(X)$ whose space of n–ary operations is roughly the space of maps $X^n \to X$ for which plugging the unique constant in all entries but one produces the identity map of X. More formally, we have a homotopy fiber sequence

$$\mathrm{End}^{\mathrm{red}}(X)(n) \to \mathrm{Map}(X^n, X) \to \mathrm{Map}(X^{\sqcup n}, X)$$

over the fold map $\nabla\colon X^{\sqcup n} \to X$. Consequently, by applying analogous reasoning to $\mathcal{P}$ and some naturality properties, we are reduced to showing that for all X in $\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})$, the induced map

$$\mathrm{Map}(\mathcal{Q}, \mathrm{End}^{\mathrm{red}}(X)) \to \mathrm{Map}(\mathcal{P}, \mathrm{End}^{\mathrm{red}}(X))$$

is a homotopy equivalence. Since $\mathcal{P} \to \mathcal{Q}$ is a d–equivalence, it will suffice to show that $\mathrm{End}^{\mathrm{red}}(X)$ is an essentially $(d+1)$–operad. Namely, we need only to show that the spaces $\mathrm{End}^{\mathrm{red}}(X)(n)$ are d–truncated. Using the homotopy fiber sequence above, we may present $\mathrm{End}^{\mathrm{red}}(X)(n)$ as the space of lifts in the commutative square

$$\begin{tikzcd} X^{\sqcup n} \arrow[r, "\nabla"] \arrow[d] & X \arrow[d] \\ X^n \arrow[r] \arrow[ur, dashed] & \mathrm{pt} \end{tikzcd}$$

The underlying ∞–category of $\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})$ is an essentially $(d+k+3)$–category (since $\mathcal{C}$ is); hence the right vertical map is $(d+k+2)$–truncated. We show that in a general presentable ∞–category, the space of lifts of an n–connected map against an m–truncated map is $(m-n-2)$–truncated. It is therefore enough to show that the map $X^{\sqcup n} \to X^n$ is k–connected in $\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})$. Under suitable conditions, which are satisfied in our situation, the k–connectedness of a map of algebras over an ∞–operad can be detected on the level of the underlying objects. Using the fact that $\mathcal{C}$ is an ∞*–topos* we are reduced to proving that the map $X^{\sqcup n} \to X^n$ has a section and that it becomes an equivalence after k–truncation in $\mathcal{C}$. For the first assertion, we show that one can construct a section rather easily using any n–ary operation of $\mathcal{R}$ for $n \geq 2$. The second assertion follows from the fact that $\mathcal{R}$ itself is k–connected, and so, roughly speaking, after k–truncation we can replace $\mathcal{R}$ with $\mathbb{E}_\infty$ and the coproduct of $\mathbb{E}_\infty$–algebras coincides with the product. The ∞–categorical EHA now follows easily from this by taking $\mathcal{Q} = \mathbb{E}_\infty$.

Organization The paper is organized as follows. In Section 2, we develop some general theory regarding reduced (and unital) ∞–operads. The first theme is the construction and analysis of the reduced endomorphism operad. The second is an explicit formula for the associated map of monads induced from a map of reduced ∞–operads.

In Section 3, we recall from Schlank and Yanovski [20] some basic definitions and properties of essentially d–categories (and operads), as well as the notion of a d–homotopy category (and operad). We then proceed to prove that a map of ∞–operads is a d–equivalence if and only if it induces an equivalence on the spaces of algebras in every $(d+1)$–topos endowed with the cartesian symmetric monoidal structure.

In Section 4, we prove some general results regarding the notions of d–connected and d–truncated morphisms in presentable ∞–categories.

In Section 5 we prove the main results of the paper. In particular we prove Theorem 1.0.2 and the ∞–categorical Eckmann–Hilton argument as a corollary. We also include a couple of simple applications.

For a more detailed outline we refer the reader to the introduction of each section.

Much of the length of the paper is due to the careful and detailed verification of many lemmas in ∞–category theory, whose proofs are arguably straightforward but nonetheless do not appear in the literature. This refers mainly to the material up to Section 4.3, from which the main theorems are Propositions 2.2.9, 3.1.8 and 3.2.6. Having said that, we believe that the theory and language of ∞–categories in general and ∞–operads in particular is still in an early enough stage of development to justify full detailed proofs of every claim that has no reference (known to the authors) in the literature. Hopefully, the added value in terms of rigor and accessibility to nonexperts compensates for the loss in brevity and elegance of exposition.

Conventions We work in the setting of ∞–categories (aka quasicategories) and ∞–operads, relying heavily on the results of [16; 17]. As a rule, we follow the notation of [16; 17] whenever possible. However, we supplement this notation and deviate from it in several cases in which we believe this enhances readability. In particular:

(1) We abuse notation by identifying an ordinary category $\mathcal{C}$ with its nerve $N(\mathcal{C})$.

(2) We use the symbol $\mathrm{pt}_{\mathcal{C}}$ to denote the terminal object of an ∞–category $\mathcal{C}$ (or just pt if $\mathcal{C}$ is clear from the context).

(3) We abbreviate the data of an ∞–operad $p\colon \mathcal{O}^{\otimes} \to \mathbf{Fin}_*$ by $\mathcal{O}$ and reserve the notation $\mathcal{O}^{\otimes}$ for the ∞–category that is the source of p. Similarly, given two ∞–operads $\mathcal{O}$ and $\mathcal{U}$, we write $f\colon \mathcal{O} \to \mathcal{U}$ for a map of ∞–operads from $\mathcal{O}$ to $\mathcal{U}$. The underlying ∞–category of $\mathcal{O}$, which in [17] is denoted by $\mathcal{O}^{\otimes}_{\langle 1\rangle}$, is here denoted by $\underline{\mathcal{O}}$.

(4) When the ∞–operad is a symmetric monoidal ∞–category, we usually denote it by $\mathcal{C}$ or $\mathcal{D}$. We will sometimes abuse notation and write $\mathcal{C}$ also for the underlying ∞–category $\underline{\mathcal{C}}$ when there is no chance of confusion.

(5) By a *presentably symmetric monoidal ∞–category* we mean a symmetric monoidal ∞–category $\mathcal{C}$ such that the underlying ∞–category $\underline{\mathcal{C}}$ is a presentable ∞–category and the tensor product preserves colimits separately in each variable.

(6) Given two ∞–operads $\mathcal{O}$ and $\mathcal{U}$, we denote by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{U})$ the ∞–operad

$$\mathrm{Alg}_{\mathcal{O}}(\mathcal{U})^{\otimes} \to \mathbf{Fin}_*$$

from [17, Example 3.2.4.4]. This is the internal mapping object induced from the closed symmetric monoidal structure on $\mathbf{Op}_{\infty}$ (see [17, 2.2.5.13]). The underlying ∞–category $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{U})$ is the usual ∞–category of $\mathcal{O}$–algebras in $\mathcal{U}$ (which in [17] is denoted by $\mathrm{Alg}_{\mathcal{O}}(\mathcal{U})$). Moreover, the maximal Kan subcomplex $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{U})^{\simeq}$ is the space of morphisms $\mathrm{Map}_{\mathbf{Op}_{\infty}}(\mathcal{O},\mathcal{U})$ from $\mathcal{O}$ to $\mathcal{U}$ as objects of the ∞–category $\mathbf{Op}_{\infty}$. Recall from [17, 3.2.4.4] that for a symmetric monoidal ∞–category $\mathcal{C}$, the ∞–operad $\mathrm{Alg}_{\mathcal{O}}(\mathcal{C})$ is again symmetric monoidal and for every object $X \in \underline{\mathcal{O}}$, the evaluation functors $\mathrm{ev}_X\colon \mathrm{Alg}_{\mathcal{O}}(\mathcal{C}) \to \mathcal{C}$ are symmetric monoidal functors.

(7) Let $\mathcal{C}$ be an ∞–category. We denote the corresponding cocartesian ∞–operad $\mathcal{C}^{\sqcup} \to \mathbf{Fin}_*$ by $\mathcal{C}_{\sqcup}$ (see [17, Definition 2.4.3.7]). If $\mathcal{C}$ has all finite products, we denote the cartesian symmetric monoidal ∞–category $\mathcal{C}^{\times} \to \mathbf{Fin}_*$ by $\mathcal{C}_{\times}$ (see [17, Construction 2.4.1.4]).

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2 Reduced ∞–operads

In this section we develop some general theory of unital and reduced ∞–operads. In Section 2.1 we establish some formal results for adjunctions and under categories. In Section 2.2 we specialize the results of Section 2.1 to prove that the inclusion of reduced ∞–operads into pointed unital ∞–operads admits a right adjoint, and analyze it. More precisely, given a unital ∞–operad $\mathcal{O}$ and an object $X \in \underline{\mathcal{O}}$ we define a reduced ∞–operad $\mathrm{End}^{\mathrm{red}}_{\mathcal{O}}(X)$, which we call the reduced endomorphism operad of X, and show that it satisfies a universal property. Moreover, we give an explicit description of $\mathrm{End}^{\mathrm{red}}_{\mathcal{C}}(X)$, which will be fundamental in analyzing the truncatedness of its spaces of operations.

In Section 2.3 we discuss the underlying symmetric sequence of a reduced ∞–operad and in Section 2.4 we use it to write an explicit formula for the free algebra over an ∞–operad (this is essentially a reformulation of [17, 3.1.3]). The material of the last two subsections is well known in the 1–categorical setting and will come as no surprise to anyone familiar with the subject. We note that in [14], Haugseng develops a theory of ∞–operads using this approach and compares it with other models including Lurie's ∞–operads, though, as far as we know, the precise results for algebras have not been furnished yet. Thus, we take it upon ourselves to flesh out the details of the little part of this theory that is required for our purposes.

2.1 Adjunctions and under-categories

We begin with some formal general observations on adjunctions and under-categories.

Lemma 2.1.1 *Let $R\colon \mathcal{D} \leftrightarrows \mathcal{C} :L$ be an adjunction between ∞–categories. For every object $X \in \mathcal{C}$ there is a canonical equivalence of ∞–categories $\mathcal{D}_{L(X)/} \simeq \mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/}$.*

Proof We denote the ∞–category $\mathcal{D} \times_{\mathcal{C}} \mathcal{C}_{X/}$ by $\mathcal{D}_{X/}$. Let $\eta\colon X \to RL(X)$ be the X–component of the unit of the adjunction $L \dashv R$. By [16, 2.1.2.1], the projections $p_0\colon \mathcal{C}_{\eta/} \to \mathcal{C}_{X/}$ and $p_1\colon \mathcal{C}_{\eta/} \to \mathcal{C}_{RL(X)/}$ are left fibrations. Moreover, since $\Delta^{\{1\}} \hookrightarrow \Delta^1$ is right anodyne, the map p_1 is an equivalence of ∞–categories. By [16, 2.2.3.3], we can choose an inverse $p_1^{-1}\colon \mathcal{C}_{RL(X)/} \to \mathcal{C}_{\eta/}$ to p_1 that strictly commutes with the projections to $\mathcal{C}$. We obtain a commutative diagram of simplicial sets

$$\begin{tikzcd} \mathcal{D}_{L(X)/} \arrow[r] \arrow[d] & \mathcal{C}_{RL(X)/} \arrow[r, "p_0 p_1^{-1}"] \arrow[d] & \mathcal{C}_{X/} \arrow[d] \\ \mathcal{D} \arrow[r] & \mathcal{C} \arrow[r] & \mathcal{C} \end{tikzcd}$$

There is an induced map from the upper left corner to the pullback of the outer rectangle without the upper left corner, which is another commutative diagram of simplicial sets

$$\begin{tikzcd} \mathcal{D}_{L(X)/} \arrow[rr] \arrow[dr] & & \mathcal{D}_{X/} \arrow[dl] \\ & \mathcal{D} & \end{tikzcd}$$

Since left fibrations are closed under base change [16, 2.1.2.1], the vertical maps are left fibrations over $\mathcal{D}$. Hence, to show that the top map is an equivalence it is enough to show that the induced map on fibers is a homotopy equivalence [16, 2.2.3.3]. For every $Y \in \mathcal{D}$ we get a map

$$\mathrm{Map}^R_{\mathcal{D}}(L(X), Y) \to \mathrm{Map}^R_{\mathcal{C}}(X, R(Y)),$$

which is by construction obtained by applying the functor R and precomposing with the unit $\eta\colon X \to RL(X)$. By the universal property of the unit map this is a homotopy equivalence for all $Y \in \mathcal{D}$ and therefore the map $\mathcal{D}_{L(X)/} \to \mathcal{D}_{X/}$ is an equivalence of ∞–categories. $\square$

Lemma 2.1.2 *Let $L\colon \mathcal{C} \leftrightarrows \mathcal{D} :R$ be an adjunction of ∞–categories and let $X \in \mathcal{C}$. The induced functor*

$$L_X\colon \mathcal{C}_{X/} \to \mathcal{D}_{L(X)/}$$

has a right adjoint R_X. Moreover, if R is fully faithful, then R_X is also fully faithful.

Proof Let $p\colon \mathcal{M} \to \Delta^1$ be the cocartesian fibration associated with the functor L (which is also cartesian, since L has a right adjoint). We can assume that we have a commutative diagram

$$\begin{tikzcd} \Delta^1 \times \mathcal{C} \arrow[rr, "s"] \arrow[dr] & & \mathcal{M} \arrow[dl, "p"] \\ & \Delta^1 & \end{tikzcd}$$

such that $s|_{\Delta^{\{0\}}\times\mathcal{C}} = \mathrm{Id}$, $s|_{\Delta^{\{1\}}\times\mathcal{C}} = L$ and $s|_{\Delta^1\times\{X\}}$ is a cocartesian edge of $\mathcal{M}$ for every $X \in \mathcal{C}$ (combine [16, 5.2.1.1 and 5.2.1.3]). It is clear from [16, 1.2.9.2] that for any pair of ∞–categories with objects $X \in \mathcal{C}$ and $Y \in \mathcal{D}$ there is a canonical isomorphism

$$(\mathcal{C} \times \mathcal{D})_{(X,Y)/} \simeq \mathcal{C}_{X/} \times \mathcal{D}_{Y/}.$$

Hence, we get an induced commutative diagram

$$\begin{tikzcd}
\Delta^1 \times \mathcal{C}_{X/} \simeq (\Delta^1 \times \mathcal{C})_{(0,X)/} \arrow[rr, "s"] \arrow[dr] & & \mathcal{M}_{X/} \arrow[dl, "p_X"] \\
& \Delta^1 \simeq \Delta^1_{0/} &
\end{tikzcd}$$

The functor p_X is a cartesian and cocartesian fibration by the duals of [16, 2.4.3.1(1) and 2.4.3.2(1)]. Moreover, an edge in $\mathcal{M}_{X/}$ is (co)cartesian if and only if its projection to $\mathcal{M}$ is (co)cartesian by the duals of [16, 2.4.3.1(2) and 2.4.3.2(2)], which shows that the functor L_X is associated with p_X. It follows that L_X has a right adjoint R_X.

Assuming that R is fully faithful, we will show that R_X is fully faithful by showing that the counit of the adjunction $L_X \dashv R_X$ is an equivalence. For every object, the counit map is an edge of $\mathcal{M}_{X/}$. Since the projection $\mathcal{M}_{X/} \to \mathcal{M}$ is conservative, it is enough to show that the counit map of $L_X \dashv R_X$ is mapped to the counit map of $L \dashv R$. Indeed, for an object $Y \in \mathcal{D} \simeq \mathcal{M}|_{\Delta^{\{1\}}}$, we choose a cartesian edge $e\colon R(Y) \to Y$ and a cocartesian edge $d\colon R(Y) \to L(R(Y))$, and combine them into a commutative diagram of the form

$$\begin{tikzcd}
\Lambda^2_0 \arrow[r, "f"] \arrow[d] & \mathcal{M} \arrow[d, "p"] \\
\Delta^2 \arrow[r] & \Delta^1
\end{tikzcd}$$

where $f|_{\Delta^{\{0,1\}}} = d$ and $f|_{\Delta^{\{0,2\}}} = e$. Since d is cocartesian, there exists a lift $\bar{f}\colon \Delta^2 \to \mathcal{M}$ that gives an edge

$$\bar{f}|_{\Delta^{\{1,2\}}} = c\colon L(R(Y)) \to Y$$

that is isomorphic to the counit map of the adjunction $L \dashv R$ at Y in the homotopy category $h\mathcal{D}$. We can similarly construct the counit map for an object of $\mathcal{M}_{X/}$. The assertion now follows from the above characterization of (co)cartesian edges in $\mathcal{M}_{X/}$.

□

Lemma 2.1.3 *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor that preserves pullbacks; then $F_X\colon \mathcal{C}_{X/} \to \mathcal{D}_{F(X)/}$ also preserves pullbacks.*

Proof Consider the commutative square

$$\begin{tikzcd} \mathcal{C}_{X/} \arrow[r] \arrow[d] & \mathcal{D}_{F(X)/} \arrow[d] \\ \mathcal{C} \arrow[r] & \mathcal{D} \end{tikzcd}$$

The vertical functors and the bottom horizontal functor preserve pullbacks. The right vertical functor is conservative. It follows that the top horizontal functor preserves pullbacks as well. □

Definition 2.1.4 Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between ∞–categories. We say that an object Y is *reduced* if $F(Y)$ is initial in $\mathcal{D}$. We define $\mathcal{C}^{\mathrm{red}}$ to be the full subcategory of $\mathcal{C}$ spanned by the reduced objects (F will always be clear from the context when we employ this terminology).

Proposition 2.1.5 *Let $L\colon \mathcal{C} \leftrightarrows \mathcal{D} :R$ be an adjunction between ∞–categories. Assume that $\mathcal{C}$ admits and L preserves pullbacks, that $\mathcal{D}$ admits an initial object, and that R is fully faithful. For every object $Y \in \mathcal{C}$ we consider the pullback diagram*

$$\begin{tikzcd} Y^{\mathrm{red}} \arrow[r] \arrow[d] & Y \arrow[d] \\ R(\varnothing_{\mathcal{D}}) \arrow[r] & RL(Y) \end{tikzcd}$$

where the right vertical map is the unit map of Y and the bottom horizontal map is the image under R of the essentially unique map $\varnothing_{\mathcal{D}} \to L(Y)$. The top horizontal map $\rho\colon Y^{\mathrm{red}} \to Y$ exhibits Y^{red} as a colocalization of Y with respect to $\mathcal{C}^{\mathrm{red}}$ (dual to [16, 5.2.7.6]).

Proof First, we show that Y^{red} is in fact reduced. Applying L to the defining diagram of Y^{red} and using the fact that L preserves pullbacks, we see that the map $L(Y^{\mathrm{red}}) \to LR(\varnothing)$ is the pullback of the map $L(Y) \to LRL(Y)$, which is an equivalence (from the fact that the counit $LR(Y) \to Y$ is an equivalence, the zigzag identities and the 2-out-of-3 property). It follows that the map $L(Y^{\mathrm{red}}) \to LR(\varnothing)$ is an equivalence, but $LR(\varnothing) \to \varnothing$ is an equivalence as well (since R is fully faithful) and we are done.

Now, we show that ρ is a colocalization. Let Z be a reduced object. We have a homotopy pullback diagram of spaces

$$\begin{tikzcd} \mathrm{Map}(Z, Y^{\mathrm{red}}) \arrow[r] \arrow[d] & \mathrm{Map}(Z, Y) \arrow[d] \\ \mathrm{Map}(Z, R(\varnothing)) \arrow[r] & \mathrm{Map}(Z, RL(Y)) \end{tikzcd}$$

and we note that the space of maps from a reduced object to any object in the essential image of R is contractible. □

Corollary 2.1.6 *In the setting of Proposition 2.1.5, the inclusion $\mathcal{C}^{\mathrm{red}} \hookrightarrow \mathcal{C}$ admits a right adjoint and the colocalization map $Y^{\mathrm{red}} \to Y$ can be taken to be the counit of the adjunction at Y.*

Proof This follows from Proposition 2.1.5 and the dual of [16, 5.2.7.8]. □

2.2 Pointed unital and reduced ∞–operads

Recall from [17] the following definitions:

Definition 2.2.1 [17, 2.3.1.1 and 2.3.4.1] An ∞–operad $\mathcal{O}$ is called:

(1) *Unital* if for every object X of $\underline{\mathcal{O}}$, the space of constants $\mathrm{Mul}_{\mathcal{O}}(\varnothing, X)$ is contractible. We denote the full ∞–category spanned by the unital ∞–operads by $\mathbf{Op}_\infty^{\mathrm{un}}$.

(2) *Reduced* if it is unital and the underlying ∞–category is a contractible space. We denote the full ∞–category spanned by the reduced ∞–operads by $\mathbf{Op}_\infty^{\mathrm{red}}$.

Example 2.2.2 A symmetric monoidal ∞–category is unital if and only if the unit object is initial.

We proceed by listing the various adjunctions between the different ∞–categories of ∞–operads and ∞–categories. First, recall from [17, 2.1.4.10] that there is an underlying ∞–category functor $\underline{(-)}\colon \mathbf{Op}_\infty \to \mathbf{Cat}_\infty$ and that this functor has a left adjoint $\iota\colon \mathbf{Cat}_\infty \hookrightarrow \mathbf{Op}_\infty$, which is a fully faithful embedding. Informally, ι regards an ∞–category as an ∞–operad with empty higher (and nullary) multimapping spaces. On the other hand:

Lemma 2.2.3 *The restriction of the forgetful functor $\underline{(-)}\colon \mathbf{Op}_\infty^{\mathrm{un}} \to \mathbf{Cat}_\infty$ admits a right adjoint that takes every ∞–category $\mathcal{C}$ to the cocartesian ∞–operad $\mathcal{C}_\sqcup$ and the unit map of the adjunction $\underline{\mathcal{C}_\sqcup} \to \mathcal{C}$ is an equivalence (ie $(-)_\sqcup$ is fully faithful).*

Proof The first claim follows from [17, 2.4.3.9] by passing to maximal ∞–sub-groupoids. The second claim follows from [17, 2.4.3.11]. $\square$

By [17, 2.3.1.9], the fully faithful embedding $\mathbf{Op}_\infty^{\mathrm{un}} \hookrightarrow \mathbf{Op}_\infty$ has a left adjoint given by tensoring with $\mathbb{E}_0$ (which is a localization functor). From this follows:

Lemma 2.2.4 *$\mathbb{E}_0$ is the initial object of $\mathbf{Op}_\infty^{\mathrm{red}}$.*

Proof The composition of forgetful functors

$$\mathbf{Op}_\infty^{\mathrm{un}} \to \mathbf{Op}_\infty \to \mathbf{Cat}_\infty$$

has a left adjoint given as the composition of the corresponding left adjoints. The first one takes Δ^0 to $\mathbf{Triv}$ (by [17, 2.1.4.8]) and the second takes $\mathbf{Triv}$ to $\mathbf{Triv} \otimes \mathbb{E}_0 \simeq \mathbb{E}_0$ (by [17, 2.3.1.9]). Hence, for every reduced operad $\mathcal{P}$ (which is in particular unital), we get

$$\mathrm{Map}(\mathbb{E}_0, \mathcal{P}) \simeq \mathrm{Map}(\Delta^0, \underline{\mathcal{P}}) \simeq \underline{\mathcal{P}}^{\simeq} \simeq \Delta^0. \qquad \square$$

One source of unital symmetric monoidal ∞–categories is:

Lemma 2.2.5 *Let $\mathcal{Q}$ be a unital ∞–operad and let $\mathcal{C}$ be a symmetric monoidal ∞–category. The symmetric monoidal ∞–category $\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})$ is also unital.*

Proof By [17, 3.2.4.4], the ∞–operad $\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})^{\otimes} \to \mathbf{Fin}_*$ is also a symmetric monoidal ∞–category and so we only need to show that the unit object of $\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})$ is initial. Since $\mathcal{Q}$ is unital, the canonical map $\mathcal{Q} \to \mathbb{E}_0 \otimes \mathcal{Q}$ is an equivalence of ∞–operads (by [17, 2.3.1.9]) and therefore the forgetful functor

$$\underline{\mathrm{Alg}}_{\mathbb{E}_0 \otimes \mathcal{Q}}(\mathcal{C}) \simeq \underline{\mathrm{Alg}}_{\mathbb{E}_0}(\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})) \to \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{C})$$

is an equivalence of ∞–categories. On the other hand, by [17, 2.1.3.10] we have

$$\underline{\mathrm{Alg}}_{\mathbb{E}_0}(\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})) \simeq \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{C})_{1/},$$

where $1 \in \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{C})$ is the unit object and the projection $\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{C})_{1/} \to \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{C})$ is an equivalence of ∞–categories if and only if 1 is initial [16, 1.2.12.5]. $\square$

Definition 2.2.6 The ∞–category of pointed ∞–categories is denoted by $\mathbf{Cat}_{\infty,*} = (\mathbf{Cat}_\infty)_{\Delta^0/}$. The ∞–category of pointed ∞–operads is denoted by

$$\mathbf{Op}_{\infty,*} = \mathbf{Op}_\infty \times_{\mathbf{Cat}_\infty} \mathbf{Cat}_{\infty,*}.$$

We also denote by $\mathbf{Op}_{\infty,*}^{\mathrm{un}}$ and $\mathbf{Op}_{\infty,*}^{\mathrm{red}}$ the corresponding ∞–categories of pointed unital (resp. reduced) ∞–operads.

Remark 2.2.7 By Lemma 2.1.1, we have an equivalence of ∞–categories $\mathbf{Op}_{\infty,*} \simeq (\mathbf{Op}_{\infty})_{\mathbf{Triv}/}$. Since $\mathbf{Triv} \to \mathbb{E}_0$ is an equivalence after tensoring with $\mathbb{E}_0$, we get $\mathbf{Op}^{\mathrm{un}}_{\infty,*} \simeq (\mathbf{Op}^{\mathrm{un}}_{\infty})_{\mathbb{E}_0/}$ and therefore also $\mathbf{Op}^{\mathrm{red}}_{\infty,*} \simeq (\mathbf{Op}^{\mathrm{red}}_{\infty})_{\mathbb{E}_0/}$. We allow ourselves to pass freely between the two points of view on (unital, reduced) pointed ∞–operads.

Remark 2.2.8 By Lemma 2.2.4, the projection $\mathbf{Op}^{\mathrm{red}}_{\infty,*} \to \mathbf{Op}^{\mathrm{red}}_{\infty}$ is an equivalence. Hence, the inclusion $\mathbf{Op}^{\mathrm{red}}_{\infty} \hookrightarrow \mathbf{Op}^{\mathrm{un}}_{\infty}$ induces a functor

$$\mathbf{Op}^{\mathrm{red}}_{\infty} \simeq \mathbf{Op}^{\mathrm{red}}_{\infty,*} \hookrightarrow \mathbf{Op}^{\mathrm{un}}_{\infty,*}.$$

Moreover, it exhibits $\mathbf{Op}^{\mathrm{red}}_{\infty}$ as the full subcategory of $\mathbf{Op}^{\mathrm{un}}_{\infty,*}$ spanned by the reduced objects with respect to the underlying ∞–category functor $\underline{(-)}\colon \mathbf{Op}^{\mathrm{un}}_{\infty,*} \to \mathbf{Cat}_{\infty}$ in the sense of Definition 2.1.4. Thus, the two notions of "reduced ∞–operad" coincide.

We now apply the general observations from the previous subsection to deduce the following:

Proposition 2.2.9 *The inclusion*

$$\mathbf{Op}^{\mathrm{red}}_{\infty} \simeq \mathbf{Op}^{\mathrm{red}}_{\infty,*} \hookrightarrow \mathbf{Op}^{\mathrm{un}}_{\infty,*}$$

has a right adjoint $(-)^{\mathrm{red}}$. Moreover, for a pointed unital ∞–operad $\mathcal{Q}$ the value of the right adjoint is given by the pullback

$$\begin{tikzcd} \mathcal{Q}^{\mathrm{red}} \arrow[r] \arrow[d] & \mathcal{Q} \arrow[d] \\ \mathbb{E}_{\infty} \arrow[r] & \underline{\mathcal{Q}}_{\sqcup} \end{tikzcd}$$

in the ∞–category $\mathbf{Op}^{\mathrm{un}}_{\infty,*}$. Furthermore, the top map can be taken to be the counit of the adjunction at $\mathcal{Q}$.

Proof We need to verify the hypothesis of Proposition 2.1.5. The underlying ∞–category functor $L\colon \mathbf{Op}^{\mathrm{un}}_{\infty} \to \mathbf{Cat}_{\infty}$ is a composition of two functors $\mathbf{Op}^{\mathrm{un}}_{\infty} \to \mathbf{Op}_{\infty} \to \mathbf{Cat}_{\infty}$. The first is a right adjoint by [17, 2.3.1.9] and the second is a right adjoint by [17, 2.1.4.10]. Hence, the composition is a right adjoint as well and therefore preserves limits. Moreover, by Lemma 2.2.3, L is also a left adjoint and its right adjoint is fully faithful. Hence, the functor $\mathbf{Op}^{\mathrm{un}}_{\infty,*} \to \mathbf{Cat}_{\infty,*}$ also has a fully faithful right adjoint and we have $(\mathbf{Op}^{\mathrm{un}}_{\infty,*})^{\mathrm{red}} \simeq \mathbf{Op}^{\mathrm{red}}_{\infty,*}$. Finally, by Proposition 2.1.5, the inclusion $\mathbf{Op}^{\mathrm{red}}_{\infty} \simeq \mathbf{Op}^{\mathrm{red}}_{\infty,*} \hookrightarrow \mathbf{Op}^{\mathrm{un}}_{\infty,*}$ admits a right adjoint with the stated description. □

Definition 2.2.10 A unital ∞–operad $\mathcal{Q}$ and an object $X \in \underline{\mathcal{Q}}$ determine a pointed unital ∞–operad $\mathcal{Q}_X \in \mathbf{Op}^{\mathrm{un}}_{\infty,*}$. We write $\mathrm{End}^{\mathrm{red}}_{\mathcal{Q}}(X) := (\mathcal{Q}_X)^{\mathrm{red}}$ and call it the *reduced endomorphism ∞–operad* of X in $\mathcal{Q}$.

We can describe the reduced ∞–operad $\mathrm{End}^{\mathrm{red}}_{\mathcal{Q}}(X)$ informally as follows. For every $m \in \mathbb{N}$, denote by $X^{(m)}$ the m–tuple $(X, \dots, X)$. The space of m–ary operations is the "subspace" of $\mathrm{Mul}_{\mathcal{Q}}(X^{(m)}, X)$ of those maps that are reduced in the sense that plugging the unique constant in all arguments but one results in an identity morphism $X \to X$. We end this subsection by making the above description precise in a special case of a symmetric monoidal ∞–category. For this, we first need to analyze the way multimapping spaces interact with limits of ∞–operads.

For every integer m, there is a functor $hG^{(m)}\colon h\mathbf{Op}_{\infty,*} \to h\mathcal{S}$, that takes each ∞–operad $\mathcal{P}$ pointed by an object X to the space $\mathrm{Mul}_{\mathcal{P}}(X^{(m)}, X)$ and a map of pointed ∞–operads $f\colon \mathcal{P} \to \mathcal{Q}$ to the homotopy class of the induced map on multimapping spaces $\mathrm{Mul}_{\mathcal{P}}(X^{(m)}, X) \to \mathrm{Mul}_{\mathcal{Q}}(f(X)^{(m)}, f(X))$.

Lemma 2.2.11 *For every integer m, there is a limit-preserving functor*

$$G^{(m)}\colon \mathbf{Op}_{\infty,*} \to \mathcal{S}$$

that lifts the functor $hG^{(m)}\colon h\mathbf{Op}_{\infty,*} \to h\mathcal{S}$.

Proof Recall the combinatorial simplicial model category $\mathbf{POp}_\infty$ of ∞–preoperads, whose underlying ∞–category is $\mathbf{Op}_\infty$ (see [17, 2.1.4]). Let $\overline{\mathcal{Z}}_0 \subseteq \overline{\mathcal{Z}}_1 \subseteq \mathbf{Fin}_*$ be the following subcategories:

(1) The category $\overline{\mathcal{Z}}_0$ is discrete and contains only the objects $\langle 1\rangle$ and $\langle m\rangle$.

(2) The category $\overline{\mathcal{Z}}_1$ contains $\overline{\mathcal{Z}}_0$ together with a unique nonidentity morphism, which is the active map $\alpha\colon \langle m\rangle \to \langle 1\rangle$.

We endow $\overline{\mathcal{Z}}_0$ and $\overline{\mathcal{Z}}_1$ with the induced (trivial) marking. Unwinding the definition, for any ∞–operad $\mathcal{P}$, the simplicial set $\mathrm{Map}_{\mathbf{POp}_\infty}(\overline{\mathcal{Z}}_0, \mathcal{P}^\natural)$ is *isomorphic* to $\mathcal{P}^{\simeq}_{\langle m\rangle} \times \mathcal{P}^{\simeq}_{\langle 1\rangle}$. Moreover, given

$$\underline{X} = (X_1 \oplus \cdots \oplus X_m, Y) \in \mathcal{P}^{\simeq}_{\langle m\rangle} \times \mathcal{P}^{\simeq}_{\langle 1\rangle},$$

the fiber of the fibration (hence also the *homotopy fiber*)

$$\varphi_{\mathcal{P}}\colon \mathrm{Map}_{\mathbf{POp}_\infty}(\overline{\mathcal{Z}}_1, \mathcal{P}^\natural) \to \mathrm{Map}_{\mathbf{POp}_\infty}(\overline{\mathcal{Z}}_0, \mathcal{P}^\natural) \simeq \mathcal{P}^{\simeq}_{\langle m\rangle} \times \mathcal{P}^{\simeq}_{\langle 1\rangle}$$

over $\underline{X}$ is homotopy equivalent to the multimapping space $\mathrm{Mul}_{\mathcal{P}}(\{X_1,\dots,X_m\};Y)$. Let $\mathcal{Z}_0$ and $\mathcal{Z}_1$ be ∞–operads that are fibrant replacements of $\overline{\mathcal{Z}}_0$ and $\overline{\mathcal{Z}}_1$, respectively. Moreover, let $f\colon \mathcal{Z}_0\to\mathcal{Z}_1$ be a map corresponding to the inclusion $\overline{\mathcal{Z}}_0\hookrightarrow\overline{\mathcal{Z}}_1$. The functor $F\colon (\mathbf{Op}_\infty)_{\mathcal{Z}_0/}\to\mathcal{S}$, corepresented by $f\colon \mathcal{Z}_0\to\mathcal{Z}_1$, preserves limits. Furthermore, its value on $g\colon \mathcal{Z}_0\to\mathcal{P}$ fits by [16, 5.5.5.12] into a fiber sequence

$$\begin{tikzcd} F(\mathcal{P}) \arrow[r] \arrow[d] & \mathrm{Map}(\mathcal{Z}_1,\mathcal{P}) \arrow[d] \\ \Delta^0 \arrow[r, "{[g]}"] & \mathrm{Map}(\mathcal{Z}_0,\mathcal{P}) \end{tikzcd}$$

which therefore identifies $F(\mathcal{P})$ with $\mathrm{Mul}_{\mathcal{P}}(\{X_1,\dots,X_m\};Y)$ for the objects $X_1,\dots,X_m,Y\in\mathcal{P}$ determined by g.

Let $U\colon \mathbf{Op}_{\infty,*}\to(\mathbf{Op}_\infty)_{\mathcal{Z}_0/}$ be the functor induced from the map $\mathcal{Z}_0\to\mathbf{Triv}$ corresponding to the inclusion $\overline{\mathcal{Z}}_0\hookrightarrow\mathbf{Triv}$. By [16, 1.2.13.8], the functor U preserves limits. We define $G^{(m)}\colon \mathbf{Op}_{\infty,*}\to\mathcal{S}$ to be the composition of F and U, which is limit-preserving as a composition of limit-preserving functors. Unwinding the definitions, $G^{(m)}$ indeed lifts $hG^{(m)}$. □

Let $\mathcal{C}$ be a symmetric monoidal ∞–category that is unital as an ∞–operad (ie the unit is an initial object). For every $X\in\mathcal{C}$ and $m\in\mathbb{N}$ we have a canonical map $\sigma\colon X^{\sqcup m}\to X^{\otimes m}$ defined as follows. For $k=1,\dots,m$, on the k^{th} summand of $X^{\sqcup m}$ the map is the tensor product of m maps, where the k^{th} one is $X\xrightarrow{\mathrm{Id}}X$ and the rest are the unique map $1_{\mathcal{C}}\to X$.

Lemma 2.2.12 *Let $\mathcal{C}$ be a symmetric monoidal ∞–category that is unital as an ∞–operad and that admits finite coproducts. For every $X\in\mathcal{C}$ and every $m\in\mathbb{N}$, there is a fiber sequence*

$$\mathrm{End}^{\mathrm{red}}_{\mathcal{C}}(X)(m)\to\mathrm{Map}_{\mathcal{C}}(X^{\otimes m},X)\xrightarrow{\sigma^*}\mathrm{Map}_{\mathcal{C}}(X^{\sqcup m},X),$$

where the fiber is taken over the fold map $\nabla\colon X^{\sqcup m}\to X$.

Proof By Proposition 2.2.9 we have a pullback square of pointed unital ∞–operads

$$\begin{tikzcd} \mathrm{End}^{\mathrm{red}}_{\mathcal{C}}(X) \arrow[r] \arrow[d] & \mathcal{C} \arrow[d] \\ \mathbb{E}_\infty \arrow[r] & \underline{\mathcal{C}}_{\sqcup} \end{tikzcd}$$

which, by Lemma 2.2.11, induces a pullback square of multimapping spaces

$$\begin{tikzcd} \mathrm{End}^{\mathrm{red}}_{\mathcal{C}}(X)(m) \arrow[r] \arrow[d] & \mathrm{Mul}_{\mathcal{C}}(X^{(m)}, X) \arrow[d] \\ \mathbb{E}_\infty(m) \arrow[r] & \mathrm{Mul}_{\underline{\mathcal{C}}_\sqcup}(X^{(m)}, X) \end{tikzcd}$$

The bottom map is the map $\Delta^0 \to \mathrm{Map}_{\mathcal{C}}(X^{\sqcup m}, X)$ that chooses the fold map since it is induced from the map $\mathbb{E}_\infty = (\Delta^0)_\sqcup \to \underline{\mathcal{C}}_\sqcup$. The right vertical map is induced by precomposition with the map $\sigma\colon X^{\sqcup m} \to X^{\otimes m}$, since it is induced by the adjunction

$$(-)_\sqcup\colon \mathbf{Cat}_\infty \leftrightarrows \mathbf{Op}^{\mathrm{un}}_\infty :\underline{(-)}. \qquad \square$$

2.3 Symmetric sequences

There is another perspective on reduced ∞–operads provided by the notion of a symmetric sequence. Roughly speaking, a symmetric sequence is a sequence of Σ_n–spaces X_n for $n \geq 0$, where Σ_n is the symmetric group on n elements. From an ∞–operad $\mathcal{O}$ with an object $X \in \underline{\mathcal{O}}$ one can construct a symmetric sequence of spaces by

$$\mathcal{O}(n) = \mathrm{Mul}_{\mathcal{O}}(X^{(n)}; X),$$

where the action of Σ_n comes from permuting the inputs. For our purposes it is convenient to use the following model:

Definition 2.3.1 Let **Fin** denote the skeletal version of the category of finite sets, ie the full subcategory of **Set** spanned by the objects $[n] = \{1, \dots, n\}$ for each integer n. We define the ∞–category of symmetric sequences (in spaces), denoted by **SSeq**, to be $\mathcal{S}_{/\mathbf{Fin}^{\simeq}}$.

Remark 2.3.2 (1) The inclusion of the full subcategory $\mathcal{S}^{Kan}_{/\mathbf{Fin}^{\simeq}} \subseteq \mathcal{S}_{/\mathbf{Fin}^{\simeq}}$ spanned by Kan fibrations is an equivalence of ∞–categories and the straightening functor of [16] induces an equivalence of ∞–categories $\mathcal{S}^{Kan}_{/\mathbf{Fin}^{\simeq}} \simeq \mathrm{Fun}(\mathbf{Fin}^{\simeq}, \mathcal{S})$. Since $\mathbf{Fin}^{\simeq}$ is equivalent to the disjoint union of classifying spaces of the symmetric groups Σ_n, we get

$$\mathbf{SSeq} \simeq \mathrm{Fun}\Big(\coprod_{n\geq 0} B\Sigma_n, \mathcal{S}\Big) \simeq \prod_{n\geq 0} \mathrm{Fun}(B\Sigma_n, \mathcal{S}).$$

More explicitly, given a symmetric sequence $p\colon S \to \mathbf{Fin}^{\simeq}$, taking pullback along the map $\Delta^0 \to \mathbf{Fin}^{\simeq}$ that corresponds to the object $[n] \in \mathbf{Fin}^{\simeq}$, we obtain a space $S(n)$ that is the underlying space of the Σ_n–space on the right-hand side of the above equivalence.

(2) In relating ∞–operads to symmetric sequences it is useful to note that the functor $\mathbf{Fin} \to \mathbf{Fin}_*$, which adds a basepoint, induces an isomorphism of groupoids $\mathbf{Fin}^{\simeq} \xrightarrow{\sim} \mathbf{Fin}_*^{\simeq}$. Moreover, $\mathbf{Fin}_*^{\simeq}$ is isomorphic to $\mathbf{Triv}^{\otimes}_{\mathrm{act}}$ (see the notation in [16, 3.1.1.1]).

We next define the underlying symmetric sequence of a pointed ∞–operad $\mathcal{O}_X$, which is given by a map $\mathbf{Triv} \to \mathcal{O}$ such that X is the image of $\langle 1 \rangle \in \mathbf{Triv}$ (see Remark 2.2.7).

Definition 2.3.3 Given a pointed ∞–operad $\mathcal{O}_X$, we define its underlying symmetric sequence to be

$$p\colon \mathbf{Triv}^{\otimes}_{\mathrm{act}} \times_{\mathcal{O}^{\otimes}_{\mathrm{act}}} (\mathcal{O}^{\otimes}_{\mathrm{act}})_{/X} \to \mathbf{Triv}^{\otimes}_{\mathrm{act}} \simeq \mathbf{Fin}^{\simeq}$$

and denote it by $\mathcal{O}_{X,\mathbf{SSeq}}$. By analogy with ∞–operads, we denote by $\mathcal{O}^{\otimes}_{X,\mathbf{SSeq}}$ the source of p.

Lemma 2.3.4 *Given a pointed ∞–operad $\mathcal{O}_X$, the map*

$$p\colon \mathbf{Triv}^{\otimes}_{\mathrm{act}} \times_{\mathcal{O}^{\otimes}_{\mathrm{act}}} (\mathcal{O}^{\otimes}_{\mathrm{act}})_{/X} \to \mathbf{Triv}^{\otimes}_{\mathrm{act}} \simeq \mathbf{Fin}^{\simeq}$$

is a Kan fibration.

Proof Since p is a pullback of the right fibration $(\mathcal{O}^{\otimes}_{\mathrm{act}})_{/X} \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ it is itself a right fibration. The simplicial set $\mathbf{Triv}^{\otimes}_{\mathrm{act}}$ is isomorphic to $\mathbf{Fin}^{\simeq}$ and is in particular a Kan complex. By [16, 2.1.3.3] the map p is a Kan fibration. □

Definition 2.3.3 relates to the informal description at the beginning of the subsection by:

Lemma 2.3.5 *Given a pointed ∞–operad $\mathcal{O}_X$, there is a homotopy equivalence*

$$\mathcal{O}_{X,\mathbf{SSeq}}(n) \simeq \mathrm{Mul}_{\mathcal{O}}(X^{(n)}; X),$$

which is natural in $\mathcal{O}_X$.

Proof This follows directly from unwinding Definition 2.3.3. □

Let $f\colon \mathbf{Triv} \to \mathcal{O}$ be a pointed ∞–operad and let $p\colon \mathcal{O} \to \mathcal{U}$ be a map of ∞–operads. Consider $\mathcal{U}$ as pointed by the composition $p \circ f$. Let $X = f(\langle 1 \rangle)$ and $Y = p(f(\langle 1 \rangle))$.

The (1–categorical) functoriality of the formula in Lemma 2.3.4 induces a map of symmetric sequences

$$\mathbf{Triv}^{\otimes}_{\mathrm{act}} \times_{\mathcal{O}^{\otimes}_{\mathrm{act}}} (\mathcal{O}^{\otimes}_{\mathrm{act}})_{/X} \to \mathbf{Triv}^{\otimes}_{\mathrm{act}} \times_{\mathcal{U}^{\otimes}_{\mathrm{act}}} (\mathcal{U}^{\otimes}_{\mathrm{act}})_{/Y}.$$

One can verify that this yields a functor on the level of homotopy categories

$$(-)_{\mathbf{SSeq}}\colon h(\mathbf{Op}_{\infty})_{\mathbf{Triv}/} \to h\mathbf{SSeq}.$$

It will be important in what follows to know the following:

Proposition 2.3.6 *The functor* $(-)_{\mathbf{SSeq}}$ *is conservative.*

Proof Let $g\colon \mathcal{P} \to \mathcal{Q}$ be a map of reduced ∞–operads such that $g_{\mathbf{SSeq}}$ is an equivalence. The map g is defined by a commutative triangle

$$\begin{tikzcd} \mathcal{P}^{\otimes} \arrow[rr, "g^{\otimes}"] \arrow[dr] & & \mathcal{Q}^{\otimes} \arrow[dl] \\ & \mathbf{Fin}_{*} & \end{tikzcd}$$

To show that g is an equivalence of ∞–operads, we need to show that $g^{\otimes}$ is an equivalence of ∞–categories. Since $\mathcal{P}$ and $\mathcal{Q}$ are reduced, it is clear that $g^{\otimes}$ is essentially surjective. To show that $g^{\otimes}$ is fully faithful, we can use the Segal conditions to reduce this to showing that the map

$$\mathcal{P}(n) = \mathrm{Mul}_{\mathcal{P}}(*^{(n)}, *) \to \mathrm{Mul}_{\mathcal{Q}}(*^{(n)}, *) = \mathcal{Q}(n)$$

is a homotopy equivalence for all n. By Lemma 2.3.5, those maps are induced by the equivalence $g_{\mathbf{SSeq}}$ and therefore are equivalences. □

Remark 2.3.7 It is possible to lift $(-)_{\mathbf{SSeq}}$ to a functor of ∞–categories, but a bit tedious to do so. We shall be content with the above weaker version as it will suffice for our applications.

2.4 Free algebras

The symmetric sequence underlying a reduced ∞–operad $\mathcal{P}$ features in the construction of free $\mathcal{P}$–algebras. In what follows we briefly recall and summarize the material of [17, 3.1.3] specialized to the setting that is of interest to us. That is, let $\mathcal{P}$ be a reduced

∞–operad and let $p\colon \mathcal{C}^{\otimes} \to \mathbf{Fin}_*$ be a presentably symmetric monoidal ∞–category. By [17, 3.1.3.5] the forgetful functor

$$U_{\mathcal{P}}\colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbf{Triv}}(\mathcal{C}) \simeq \mathcal{C}$$

admits a left adjoint $F_{\mathcal{P}}$ (the free $\mathcal{P}$–algebra functor) that can be characterized as follows. By [17, Definition 3.1.3.1], for every object $X \in \mathcal{C}$ we get a diagram $\mathcal{P}_{\mathbf{SSeq}}(X)\colon \mathcal{P}^{\otimes}_{\mathbf{SSeq}} \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ that, loosely speaking, corresponds to a sequence of maps $\mathcal{P}_{\mathbf{SSeq}}(n) \to \mathcal{C}^{\otimes}_{\mathrm{act}}$ such that each map lands in the connected component of $X^{\otimes n}$ and is Σ_n–equivariant in the evident way. Furthermore, a map $f\colon X \to U_{\mathcal{P}}(A)$ in $\mathcal{C}$ gives a lift of $\mathcal{P}^{\otimes}_{\mathbf{SSeq}}(X)$ to a cone diagram

$$\mathcal{P}^{\otimes}_{\mathbf{SSeq}}(f)\colon \mathcal{P}^{\otimes}_{\mathbf{SSeq}} \to (\mathcal{C}^{\otimes}_{\mathrm{act}})_{/U_{\mathcal{P}}(A)}.$$

We say that f exhibits A as the free $\mathcal{P}$–algebra on X if $\mathcal{P}^{\otimes}_{\mathbf{SSeq}}(f)$ is an operadic p–colimit diagram. By [17, 3.1.3.2 and 3.1.3.5], such a map f exists for every X and can be taken as the X–component of a unit natural transformation for an adjunction $F_{\mathcal{P}} \dashv U_{\mathcal{P}}$.

Using our assumption on $\mathcal{C}$, we can reduce the *operadic* colimit in the above discussion to an *ordinary* colimit in $\mathcal{C}$. Consider the commutative diagram

$$\begin{tikzcd}
\Delta^{\{0\}} \times \mathcal{C}^{\otimes} \arrow[r, "\mathrm{Id}"] \arrow[d, hook] & \mathcal{C}^{\otimes} \arrow[d, "p"] \\
\Delta^{1} \times \mathcal{C}^{\otimes} \arrow[r, "\alpha"] \arrow[ur, "\bar{\alpha}"] & \mathbf{Fin}_*
\end{tikzcd}$$

where α is a natural transformation from p to the constant diagram on $\langle 1 \rangle$ that consists of active morphisms. Let $\bar{\alpha}$ be a cocartesian natural transformation that lifts α. The restricted functor $F = \bar{\alpha}|_{\Delta^{\{1\}} \times \mathcal{C}^{\otimes}_{\mathrm{act}}}$ lands in the fiber over $\langle 1 \rangle$ and is therefore a functor $F\colon \mathcal{C}^{\otimes}_{\mathrm{act}} \to \mathcal{C}$.

Remark 2.4.1 Informally speaking, F takes each multiobject $X_1 \oplus \cdots \oplus X_n$ to the tensor product $X_1 \otimes \cdots \otimes X_n$. There are two abstract characterizations of F (which we shall not use):

(1) It is the left adjoint of the inclusion $\underline{\mathcal{C}} \hookrightarrow \mathcal{C}^{\otimes}_{\mathrm{act}}$.

(2) The symmetric monoidal envelope is a left adjoint to the inclusion of symmetric monoidal ∞–categories into ∞–operads. The functor F is the induced functor on the underlying ∞–categories of the unit of this adjunction at the object $\mathcal{C}$.

By [17, 3.1.1.15 and 3.1.1.16], $\mathcal{P}^{\otimes}_{\mathbf{SSeq}}(f)$ is an operadic p–colimit diagram if and only if the diagram $\mathcal{P}_{\mathbf{SSeq}}(f) = F \circ \mathcal{P}^{\otimes}_{\mathbf{SSeq}}(f)$ is a colimit diagram in $\mathcal{C}$. In particular, we get:

Lemma 2.4.2 *Let $\mathcal{P}$ be a reduced ∞–operad and let $\mathcal{C}$ be a presentably symmetric monoidal ∞–category. The forgetful functor*

$$U_{\mathcal{P}}\colon \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathbf{Triv}}(\mathcal{C}) \simeq \mathcal{C}$$

admits a left adjoint $F_{\mathcal{P}}$ and the associated monad $T_{\mathcal{P}} = U_{\mathcal{P}} \circ F_{\mathcal{P}}$ acts on an object $X \in \mathcal{C}$ as follows:

$$T_{\mathcal{P}}(X) = U_{\mathcal{P}} F_{\mathcal{P}}(X) = \operatorname{colim} \mathcal{P}_{\mathbf{SSeq}}(X) = \coprod_{n \geq 0} (\mathcal{P}(n) \otimes X^{\otimes n})_{h\Sigma_n}$$

(where we let $\otimes$ denote the canonical enrichment of $\underline{\mathcal{C}}$ over $\mathcal{S}$ as well).

Our next goal is to articulate the functoriality of $T_{\mathcal{P}}$ in the ∞–operad $\mathcal{P}$.

Construction 2.4.3 Given a map of reduced ∞–operads $\mathcal{P} \to \mathcal{Q}$ we get a forgetful functor $G\colon \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C})$ such that $U_{\mathcal{P}} G = U_{\mathcal{Q}}$. The unit map

$$\mathrm{Id} \to U_{\mathcal{Q}} F_{\mathcal{Q}} = U_{\mathcal{P}} G F_{\mathcal{Q}}$$

has an adjunct $F_{\mathcal{P}} \to G F_{\mathcal{Q}}$ and by applying $U_{\mathcal{P}}$ we obtain an induced map of the associated monads (as endofunctors of $\mathcal{C}$),

$$\alpha_G\colon T_{\mathcal{P}} = U_{\mathcal{P}} F_{\mathcal{P}} \to U_{\mathcal{P}} G F_{\mathcal{Q}} = U_{\mathcal{Q}} F_{\mathcal{Q}} = T_{\mathcal{Q}},$$

which is well defined up to homotopy.

Lemma 2.4.4 *In the setting of Construction 2.4.3, if G is an equivalence of ∞–categories, then the map $\alpha_G\colon T_{\mathcal{P}} \to T_{\mathcal{Q}}$ is a natural equivalence of functors.*

Proof Since all the steps in the construction are invariant, we may assume without loss of generality that G is the identity functor and $U_{\mathcal{P}} = U_{\mathcal{Q}}$. In this case, the map α_G is given by applying $U_{\mathcal{Q}}$ to the composition

$$F_{\mathcal{Q}} \xrightarrow{F_{\mathcal{Q}} u} F_{\mathcal{Q}} U_{\mathcal{Q}} F_{\mathcal{Q}} \xrightarrow{c F_{\mathcal{Q}}} F_{\mathcal{Q}},$$

where u and c are the unit and counit of the adjunction $F_{\mathcal{Q}} \dashv U_{\mathcal{Q}}$. This composition is homotopic to the identity by the zigzag identities. $\square$

Our last task is to show that the map from Construction 2.4.3 is induced from the map of symmetric sequences $\mathcal{P}_{\mathbf{SSeq}} \to \mathcal{Q}_{\mathbf{SSeq}}$ by the functoriality of the explicit formula given in Lemma 2.4.2.

Lemma 2.4.5 *Given a map $f\colon X \to U_{\mathcal{P}}(A)$, the map $\operatorname{colim} \mathcal{P}_{\mathbf{SSeq}}(X) \to U_{\mathcal{P}}(A)$ induced by the diagram $\mathcal{P}_{\mathbf{SSeq}}(f)$ is equivalent to the canonical map $\tilde{f}\colon U_{\mathcal{P}}F_{\mathcal{P}}(X) \to U_{\mathcal{P}}(A)$ (ie $U_{\mathcal{P}}$ of the adjunct of f).*

Proof One only has to observe that the map $\tilde{f}\colon U_{\mathcal{P}}F_{\mathcal{P}}(X) \to U_{\mathcal{P}}(A)$ is a map of cones on $\mathcal{P}_{\mathbf{SSeq}}(X)$. Let $u_X\colon X \to U_{\mathcal{P}}F_{\mathcal{P}}(X)$ be the unit map of the free-forgetful adjunction at X. The adjunct map $F_{\mathcal{P}}(X) \to A$ induces a map $\tilde{f}^{\triangleright}\colon (\mathcal{C}^{\otimes}_{\mathrm{act}})_{/U_{\mathcal{P}}F_{\mathcal{P}}(X)} \to (\mathcal{C}^{\otimes}_{\mathrm{act}})_{/U_{\mathcal{P}}(A)}$. Inspecting [17, Construction 3.1.3.1], it can be seen that the cone diagram $\mathcal{P}_{\mathbf{SSeq}}(f)$ is equivalent to the composition of the universal cone diagram $\mathcal{P}_{\mathbf{SSeq}}(u_X)$ and $\tilde{f}^{\triangleright}$. □

From this we get:

Proposition 2.4.6 *Let $g\colon \mathcal{P} \to \mathcal{Q}$ be a map of reduced ∞–operads and let $\mathcal{C}$ be a presentably symmetric monoidal ∞–category. For every object $X \in \mathcal{C}$, the induced map of the associated monads*

$$T_{\mathcal{P}}(X) = \operatorname{colim} \mathcal{P}_{\mathbf{SSeq}}(X) \to \operatorname{colim} \mathcal{Q}_{\mathbf{SSeq}}(X) = T_{\mathcal{Q}}(X)$$

is equivalent to the canonical map on colimits that is induced by precomposition with

$$g_{\mathbf{SSeq}}\colon \mathcal{P}_{\mathbf{SSeq}} \to \mathcal{Q}_{\mathbf{SSeq}}.$$

Proof We denote by $G\colon \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C})$ the forgetful functor induced by the map g. Let

$$f\colon X \to U_{\mathcal{Q}}F_{\mathcal{Q}}(X) \simeq U_{\mathcal{P}}GF_{\mathcal{Q}}(X)$$

be the unit map. It induces a cone diagram

$$\mathcal{P}_{\mathbf{SSeq}}(f)\colon \mathcal{P}_{\mathbf{SSeq}} \to \mathcal{C}_{/U_{\mathcal{Q}}F_{\mathcal{Q}}(X)}$$

and, by Lemma 2.4.5, the associated map $\tilde{f}\colon U_{\mathcal{P}}F_{\mathcal{P}}(X) \to U_{\mathcal{Q}}F_{\mathcal{Q}}(X)$ is equivalent to the map $\operatorname{colim} \mathcal{P}_{\mathbf{SSeq}}(X) \to U_{\mathcal{Q}}F_{\mathcal{Q}}(X)$ specified by the cone diagram $\mathcal{P}_{\mathbf{SSeq}}(f)$. On the other hand, inspecting [17, Construction 3.1.3.1], it can be seen that the diagram $\mathcal{P}_{\mathbf{SSeq}}(f)$ is obtained from the diagram

$$\mathcal{Q}_{\mathbf{SSeq}}(f)\colon \mathcal{Q}_{\mathbf{SSeq}} \to \mathcal{C}_{/U_{\mathcal{Q}}F_{\mathcal{Q}}(X)}$$

by precomposition with $g_{\mathbf{SSeq}}\colon \mathcal{P}_{\mathbf{SSeq}} \to \mathcal{Q}_{\mathbf{SSeq}}$ and that $\mathcal{Q}_{\mathbf{SSeq}}(f)$ exhibits $U_{\mathcal{Q}}F_{\mathcal{Q}}(X)$ as the colimit of $\mathcal{Q}_{\mathbf{SSeq}}(X)$. Thus, we get the desired equivalence. □

3 d–Categories and d–operads

This section deals with *essentially d–categories*, ie ∞–categories all of whose mapping spaces are $(d-1)$–truncated (Definition 3.1.2), and with the analogous notion for ∞–operads (Definition 3.1.6).

In Section 3.1 we discuss the fact that the inclusion of the full subcategory spanned by the essentially d–categories (resp. d–operads) into $\mathbf{Cat}_\infty$ (resp. $\mathbf{Op}_\infty$) admits a left adjoint and that the unit of this adjunction consists of $(d-1)$–truncation of the (multi)mapping spaces. The proofs of these (very plausible) facts are rather technical, involving a combinatorial analysis of some strict models for the above constructions, and can be found in [20]. In Section 3.2 we use the results of Section 3.1 to characterize when a map of ∞–operads induces an equivalence on d–homotopy operads in terms of the induced functor on algebras in a d–topos (Proposition 3.2.6).

3.1 d–Homotopy categories and operads

Recall the following definition from classical homotopy theory.

Definition 3.1.1 For $d \geq 0$, a space $X \in \mathcal{S}$ is called *d–truncated* if $\pi_i(X, x) = 0$ for all $i > d$ and all $x \in X$. In addition, a space is called *(-2)–truncated* if and only if it is contractible and it is called *(-1)–truncated* if and only if it is either contractible or empty. We denote by $\mathcal{S}_{\leq d}$ the full subcategory of $\mathcal{S}$ spanned by the d–truncated spaces. The inclusion $\mathcal{S}_{\leq d} \hookrightarrow \mathcal{S}$ admits a left adjoint and we call the unit of the adjunction the *d–truncation map*.

This leads to the following definition in ∞–category theory.

Definition 3.1.2 Let $d \geq -1$ be an integer. An *essentially d–category* is an ∞–category $\mathcal{C}$ such that for all $X, Y \in \mathcal{C}$, the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is $(d-1)$–truncated. We denote by $\mathbf{Cat}_d$ the full subcategory of $\mathbf{Cat}_\infty$ spanned by essentially d–categories.

Remark 3.1.3 An ∞–category $\mathcal{C}$ is an essentially 1–category if and only if it lies in the essential image of the nerve functor $N\colon \mathbf{Cat} \to \mathbf{Cat}_\infty$ and it is an essentially 0–category if and only if it is equivalent to the nerve of a poset.

In [16, 2.3.4], Lurie develops the theory of d–categories (see [16, Definition 2.3.4.1]), which are a strict model for essentially d–categories. In particular, he associates with every ∞–category $\mathcal{C}$ a d–category $h_d\mathcal{C}$ (see [16, Proposition 2.3.4.12]), which we refer to as the *d–homotopy category* of $\mathcal{C}$. In [20] we make a further study of this theory and use it to prove the following:

Proposition 3.1.4 [20, Theorem 2.15] *The inclusion $\mathbf{Cat}_d \hookrightarrow \mathbf{Cat}_\infty$ admits a left adjoint h_d such that for every ∞–category $\mathcal{C}$, the value of h_d on $\mathcal{C}$ is the d–homotopy category of $\mathcal{C}$, the unit transformation $\theta_d\colon \mathcal{C} \to h_d\mathcal{C}$ is essentially surjective, and for all $X, Y \in \mathcal{C}$, the map of spaces*

$$\mathrm{Map}_{\mathcal{C}}(X,Y) \to \mathrm{Map}_{h_d\mathcal{C}}(\theta_d(X), \theta_d(Y))$$

is the $(d-1)$–truncation map.

Warning 3.1.5 An ∞–category $\mathcal{C}$ is an essentially d–category if and only if all objects of $\mathcal{C}$ are $(d-1)$–truncated in the sense of [16, 5.5.6.1]. Hence, another way to associate an essentially d–category with an ∞–category $\mathcal{C}$ is to consider the full subcategory spanned by the $(d-1)$–truncated objects. For a presentable ∞–category, this is denoted by $\tau_{\leq d-1}\mathcal{C}$ in [16, 5.5.6.1] and called the *$(d-1)$–truncation of $\mathcal{C}$*. We warn the reader that the two essentially d–categories $h_d\mathcal{C}$ and $\tau_{\leq d-1}\mathcal{C}$ are usually very different. For example, when $\mathcal{C} = \mathcal{S}$ is the ∞–category of spaces, $h_1\mathcal{S}$ is the ordinary homotopy category of spaces, while $\tau_{\leq 0}\mathcal{S}$ is equivalent to the ordinary category of sets. Both constructions will play a central role in the proof of the main result, and hopefully the distinction in notation and terminology will prevent confusion.

With these ideas in mind, one might hope that for an ∞–category $\mathcal{C}$, the condition of being an essentially $(d+1)$–category would coincide with the condition of begin a d–truncated object of the presentable ∞–category $\mathbf{Cat}_\infty$. This turns out to be *false*. More precisely, it can be shown that a d–truncated object of $\mathbf{Cat}_\infty$ is an essentially $(d+1)$–category and that an essentially $(d+1)$–category is a $(d+1)$–truncated object of $\mathbf{Cat}_\infty$, but neither of the converses hold (see [20, Remark 2.10]).

By analogy with the above, we also have a natural notion of an essentially d–operad.

Definition 3.1.6 Let $d \geq -1$. An *essentially d–operad* is an ∞–operad $\mathcal{O}$ such that for all $X_1, \dots, X_n, Y \in \underline{\mathcal{O}}$, the multimapping space $\mathrm{Mul}_{\mathcal{O}}(\{X_1, \dots, X_n\}; Y)$ is $(d-1)$–truncated. We denote by $\mathbf{Op}_d$ the full subcategory of $\mathbf{Op}_\infty$ spanned by essentially d–operads.

Example 3.1.7 Two important special cases are:

(1) A symmetric monoidal ∞–category $\mathcal{C}$ is an essentially d–operad if and only if the underlying ∞–category $\underline{\mathcal{C}}$ is an essentially d–category.

(2) A reduced ∞–operad $\mathcal{P}$ is an essentially d–operad if and only if the symmetric sequence $\{\mathcal{P}(n)\}_{n\geq 0}$ consists of $(d-1)$–truncated spaces.

In [20] we develop a parallel notion of a *d–operad*, which bears the same relation to an essentially d–operad as a d–category does to an essentially d–category, ie it is a strict model for an essentially d–operad. Using this theory we show the following:

Proposition 3.1.8 [20, Theorem 3.12] *The inclusion $\mathbf{Op}_d \hookrightarrow \mathbf{Op}_\infty$ admits a left adjoint h_d such that for every ∞–operad $\mathcal{O}$, the unit transformation $\theta_d\colon \mathcal{O} \to h_d\mathcal{O}$ is essentially surjective and for all $X_1, \dots, X_n, Y \in \underline{\mathcal{O}}$, the map of spaces*

$$\mathrm{Mul}_{\mathcal{O}}(\{X_1, \dots, X_n\}; Y) \to \mathrm{Mul}_{h_d\mathcal{O}}(\{\theta_d(X_1), \dots, \theta_d(X_n)\}; \theta_d(Y))$$

is the $(d-1)$–truncation map.

Definition 3.1.9 Given an ∞–operad $\mathcal{O}$, we refer to $h_d\mathcal{O}$ as the *d–homotopy operad* of $\mathcal{O}$.

For future use, we record the following fact:

Proposition 3.1.10 [20, Proposition 3.13] *Let $\mathcal{O}$ be an ∞–operad and let $\mathcal{U}$ be an essentially d–operad. The ∞–category $\underline{\mathrm{Alg}}_{\mathcal{O}}(\mathcal{U})$ is an essentially d–category.*

3.2 d–Equivalences and d–topoi

Definition 3.2.1 For $d \geq -2$, a map of ∞–operads $f\colon \mathcal{O} \to \mathcal{U}$ is called a *d–equivalence* if the induced map $h_{d+1}(f)\colon h_{d+1}\mathcal{O} \to h_{d+1}\mathcal{U}$ is an equivalence of ∞–operads, ie if it is essentially surjective on the underlying categories and induces an equivalence on the d–truncations of all the multimapping spaces.

An important special case is:

Definition 3.2.2 For $d \geq -2$, an ∞–operad $\mathcal{O}$ is called *d–connected* if the unique map from $\mathcal{O}$ to the terminal ∞–operad $\mathbb{E}_\infty$ is a d–equivalence, ie if all the multimapping spaces in $\mathcal{O}$ are d–connected.

Remark 3.2.3 Let $\mathcal{P}$ be a reduced ∞–operad. It is d–connected if and only if all the spaces $\mathcal{P}(n)$ in the underlying symmetric sequence of $\mathcal{P}$ are d–connected. If $\mathcal{P}$ is not equivalent to $\mathbb{E}_0$, then for some $n \geq 2$ we have $\mathcal{P}(n) \neq \varnothing$, and so there exists an n–ary operation $\mu \in \mathcal{P}(n)$ for $n \geq 2$. By composing μ with itself, we can obtain an operation in $\mathcal{P}$ of arbitrarily high arity and, by composition with the unique nullary operation, we can obtain an operation of arbitrary arity. It follows that $\mathcal{P} \not\simeq \mathbb{E}_0$ if and only if $\mathcal{P}$ is (-1)–connected.

The main result of this section is a characterization of d–equivalences of reduced ∞–operads. But first, we need some preliminary observations about cartesian symmetric monoidal structures.

Lemma 3.2.4 *Let $f_\alpha\colon \mathcal{D} \to \mathcal{C}_\alpha$ be a collection of jointly conservative, symmetric monoidal functors between symmetric monoidal ∞–categories.*

(1) *If $\mathcal{C}_\alpha$ is cartesian and f_α preserves finite products for all α and $\underline{\mathcal{D}}$ has all finite products, then $\mathcal{D}$ is cartesian.*

(2) *If $\mathcal{C}_\alpha$ is cocartesian and f_α preserves finite coproducts for all α and $\underline{\mathcal{D}}$ has all finite coproducts, then $\mathcal{D}$ is cocartesian.*

Proof By [17, 2.4.2.7], the opposite of a symmetric monoidal ∞–category acquires a symmetric monoidal structure, which is cartesian if and only if the original symmetric monoidal ∞–category is cocartesian. Hence, it is enough to prove (2). The unit object $1 \in \mathcal{D}$ has a unique map from the initial object $\varnothing \to 1$. Since f_α is both symmetric monoidal and preserves finite coproducts, $f_\alpha(\varnothing \to 1)$ is the unique map from the initial object to the unit object of $\mathcal{C}_\alpha$, which is an equivalence by assumption. Since the collection of f_α is jointly conservative, it follows that the unit of $\mathcal{D}$ is initial in $\underline{\mathcal{D}}$ as well. Namely, $\mathcal{D}$ is unital as an ∞–operad. Using Lemma 2.2.3 we have a map of ∞–operads $G\colon \mathcal{D} \to \mathcal{D}_\sqcup$, which is an equivalence on the underlying ∞–categories. We need to show that this map is symmetric monoidal, namely that it maps cocartesian edges (over $\mathbf{Fin}_*$) to cocartesian edges. Since we already know that it is a map of ∞–operads and hence preserves inert morphisms, we only need to show that active cocartesian edges map to cocartesian edges. Using the Segal conditions, we are further reduced to considering only cocartesian lifts of the unique active morphism $\mu\colon \langle n\rangle \to \langle 1\rangle$. For every collection of objects $X_1, X_2, \dots, X_n \in \underline{\mathcal{D}}$, let

$$\mu_\otimes\colon X_1 \oplus \cdots \oplus X_n \to X_1 \otimes X_2 \otimes \cdots \otimes X_n$$

be a cocartesian lift of μ to $\mathcal{D}^\otimes$. Since G is an equivalence on the underlying ∞–categories, $G(\mu_\otimes)$ can be considered as a map

$$X_1 \oplus \cdots \oplus X_n \to X_1 \otimes X_2 \otimes \cdots \otimes X_n$$

in $\mathcal{D}_\sqcup$. Now, let

$$\mu_\sqcup\colon X_1 \oplus \cdots \oplus X_n \to X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n$$

be a cocartesian lift of μ to $\mathcal{D}^{\sqcup}$. There exists a unique (up to homotopy) map

$$g_{X_1,\dots,X_n}\colon X_1 \sqcup X_2 \sqcup \cdots \sqcup X_n \to X_1 \otimes X_2 \otimes \cdots \otimes X_n$$

such that $G(\mu_{\otimes}) = g_{X_1,\dots,X_n} \circ \mu_{\sqcup}$. We need to show that $g_{X_1,\dots,X_n}$ is an equivalence in $\underline{\mathcal{D}}$ for all $X_1,\dots,X_n \in \underline{\mathcal{D}}$. For every α, we have a homotopy commutative diagram

$$\begin{tikzcd} \mathcal{D}^{\otimes} \arrow[r] \arrow[d, "f_\alpha"] & \mathcal{D}^{\sqcup} \arrow[d, "f_\alpha"] \\ \mathcal{C}_\alpha^{\otimes} \arrow[r] & \mathcal{C}_\alpha^{\sqcup} \end{tikzcd}$$

in which the vertical and bottom maps are symmetric monoidal. It follows that $f_\alpha(g_{X_1,\dots,X_n})$ is an equivalence in $\mathcal{C}_\alpha$ for all α. By joint conservativity, $g_{X_1,\dots,X_n}$ is an equivalence as well. □

Lemma 3.2.5 *Let $\mathcal{C}_\times$ be a cartesian symmetric monoidal ∞–category. For every ∞–operad $\mathcal{D}$, the ∞–operad* $\mathrm{Alg}_{\mathcal{D}}(\mathcal{C})$ (see [17, 2.2.5.4]) *is also cartesian.*

Proof By [17, 2.2.5.4], since $\mathcal{C}_\times$ is symmetric monoidal, so is $\mathrm{Alg}_{\mathcal{D}}(\mathcal{C})$ and, for every $X \in \mathcal{D}$, the evaluation functor $e_X\colon \mathrm{Alg}_{\mathcal{D}}(\mathcal{C}) \to \mathcal{C}_\times$ is a symmetric monoidal functor. On the underlying ∞–categories, e_X also preserves finite products since it preserves all limits. Finally, we show that the collection of evaluation functors is jointly conservative since they can be presented as the composition of the conservative restriction functor

$$\mathrm{Alg}_{\mathcal{D}}(\mathcal{C}) \to \mathrm{Fun}(\mathcal{D},\mathcal{C})$$

and the collection of evaluation functors

$$e_X\colon \mathrm{Fun}(\mathcal{D},\mathcal{C}) \to \mathcal{C},$$

which are jointly conservative by [16, 3.1.2.1]. Now, by Lemma 3.2.4(1), $\mathrm{Alg}_{\mathcal{D}}(\mathcal{C})$ is cartesian. □

We are now ready for the main proposition.

Proposition 3.2.6 *Let $d \geq -1$. Given a map of reduced ∞–operads $f\colon \mathcal{P} \to \mathcal{Q}$, the following are equivalent:*

(1) *The map f is a d–equivalence.*

(2) *For every $(d+1)$–topos $\mathcal{C}$, the induced map*

$$\mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{Q},\mathcal{C}_\times) \to \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{P},\mathcal{C}_\times)$$

is a homotopy equivalence.

(3) *For every simplicial set K, the induced map*

$$\mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{Q}, \mathcal{S}_{\leq d}^K) \to \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{P}, \mathcal{S}_{\leq d}^K)$$

is a homotopy equivalence, where $\mathcal{S}_{\leq d}^K$ is given the cartesian symmetric monoidal structure.

(4) *The induced map*

$$\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{S}_{\leq d}) \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{S}_{\leq d})$$

is an equivalence of ∞–categories, where $\mathcal{S}_{\leq d}$ is given the cartesian symmetric monoidal structure.

Proof (1) $\Longrightarrow$ (2) Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{Q}, \mathcal{C}) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{P}, \mathcal{C}) \arrow[d] \\
\mathrm{Map}_{\mathbf{Op}_\infty}(h_{d+1}\mathcal{Q}, \mathcal{C}) \arrow[r] & \mathrm{Map}_{\mathbf{Op}_\infty}(h_{d+1}\mathcal{P}, \mathcal{C})
\end{tikzcd}$$

Since $h_{d+1}(\mathcal{P}) \to h_{d+1}(\mathcal{Q})$ is an equivalence of ∞–operads, the bottom map is a homotopy equivalence. By Proposition 3.1.8, the vertical maps are equivalences as well, and so, by the 2-out-of-3 property, the top map is an equivalence.

(2) $\Longrightarrow$ (3) Since $\mathcal{S}_{\leq d}^K$ is a $(d+1)$–topos, this is just a special case.

(3) $\Longrightarrow$ (4) By Yoneda's lemma applied to $\mathbf{Cat}_\infty$, the map

$$\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{S}_{\leq d}) \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{S}_{\leq d})$$

is an equivalence of ∞–categories if for every ∞–category $\mathcal{E}$, the map

$$\mathrm{Map}_{\mathbf{Cat}_\infty}(\mathcal{E}, \underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{S}_{\leq d})) \to \mathrm{Map}_{\mathbf{Cat}_\infty}(\mathcal{E}, \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{S}_{\leq d}))$$

is a homotopy equivalence. Using the fully faithful embedding $\mathbf{Cat}_\infty \hookrightarrow \mathbf{Op}_\infty$, which is left adjoint to the underlying category functor $\mathbf{Op}_\infty \to \mathbf{Cat}_\infty$ (see [17, 2.1.4.11]), this map is equivalent to

$$\mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{E}, \mathrm{Alg}_{\mathcal{Q}}(\mathcal{S}_{\leq d})) \to \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{E}, \mathrm{Alg}_{\mathcal{P}}(\mathcal{S}_{\leq d})).$$

By adjointness with the Boardman–Vogt tensor product and the fact that it is symmetric, the map is equivalent to

$$\mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{Q}, \mathrm{Alg}_{\mathcal{E}}(\mathcal{S}_{\leq d})) \to \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{P}, \mathrm{Alg}_{\mathcal{E}}(\mathcal{S}_{\leq d})).$$

Since $\mathcal{E}$ is an ∞–category, by Lemma 3.2.5 the ∞–operad $\mathrm{Alg}_{\mathcal{E}}(\mathcal{S}_{\le d})$ is just the ∞–category of functors $(\mathcal{S}_{\le d})^{\mathcal{E}}$ endowed with the cartesian symmetric monoidal structure. Since the functor category is invariant under Joyal equivalences, we can replace $\mathcal{E}$ with any simplicial set K.

(4) $\Longrightarrow$(1) Consider the commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{S}_{\le d}) \arrow[r] \arrow[d, "\wr"] & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{S}_{\le d}) \arrow[d, "\wr"] \\
\underline{\mathrm{Alg}}_{h_d\mathcal{Q}}(\mathcal{S}_{\le d}) \arrow[r] & \underline{\mathrm{Alg}}_{h_d\mathcal{P}}(\mathcal{S}_{\le d})
\end{tikzcd}$$

By Proposition 3.1.8, the vertical maps are equivalences; hence, by 2-out-of-3, the top map is an equivalence if and only if the bottom map is. We can therefore assume without loss of generality that $\mathcal{P}$ and $\mathcal{Q}$ are themselves essentially d–operads. This implies that $\mathcal{P}(n)$ and $\mathcal{Q}(n)$ are d–truncated spaces for all $n \ge 0$. Now, consider the commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Alg}}_{\mathcal{Q}}(\mathcal{S}_{\le d}) \arrow[rr, "f^*"] \arrow[dr, "U_{\mathcal{Q}}"'] & & \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{S}_{\le d}) \arrow[dl, "U_{\mathcal{P}}"] \\
& \mathcal{S}_{\le d} &
\end{tikzcd}$$

where $U_{\mathcal{P}}$ and $U_{\mathcal{Q}}$ are the corresponding forgetful functors. By Lemma 2.4.4, the associated map

$$T_{\mathcal{P}} = \coprod_n (\mathcal{P}(n) \times X^n)_{h\Sigma_n} \xrightarrow{\sim} \coprod_n (\mathcal{Q}(n) \times X^n)_{h\Sigma_n} = T_{\mathcal{Q}}$$

of Construction 2.4.3 is a natural equivalence of functors. On the other hand, by Proposition 2.4.6, this map is induced from a map $f_{\mathbf{SSeq}}\colon \{\mathcal{P}(n)\} \to \{\mathcal{Q}(n)\}$ of symmetric sequences. We want to deduce that $f_{\mathbf{SSeq}}$ is an equivalence. For $d = -1$, there is nothing to prove and so we assume that $d \ge 0$. Taking $X = [n]$, there is a coproduct decomposition

$$(\mathcal{P}(n) \times X^n)_{h\Sigma_n} = \mathcal{P}(n) \sqcup J,$$

where the summand $\mathcal{P}(n)$ corresponds to orbits of points whose X^n component is a permutation (note that when $d = 0$, the homotopy orbits in $\mathcal{S}_{\le 0}$ are just the orbits as a set). This characterization implies that $f_{\mathbf{SSeq}}\colon \mathcal{P}(n) \to \mathcal{Q}(n)$ is an equivalence. Finally, since $(-)_{\mathbf{SSeq}}$ is conservative, by Proposition 2.3.6 we deduce that f is an equivalence. $\square$

4 Truncatedness and connectedness

This section deals with properties of truncated and connected morphisms in a presentable ∞–category. We begin in Section 4.1 with some basic facts about the space of lifts in a commutative square. The key result is Proposition 4.1.5, which expresses the homotopy fiber of the diagonal of the space of lifts as the space of lifts in a closely related square. In Section 4.2 we expand on the notions of n–truncated and n–connected morphisms. The main result is Proposition 4.2.8, which is a quantitative version of the defining orthogonality relation between n–connected and n–truncated morphisms. In Section 4.3 we introduce an auxiliary notion of an $\left(n-\frac{1}{2}\right)$–connected morphism and compare it with the notion of an n–connected morphism under some assumptions on the ambient ∞–category. We conclude with Section 4.4, in which we study the notion of n–connectedness for the ∞–category of algebras over a reduced ∞–operad. In particular, we show that under some reasonably general conditions, a map of algebras is n–connected if the map between the underlying objects is n–connected (Proposition 4.4.5).

We rely on [16, 5.5.6] for the basic theory of truncated morphisms and objects, but we note that the properties of connected morphisms are studied in [16] only in the context of ∞–topoi. Some further results, still in the context of ∞–topoi, can be found in [1]. For example, our Proposition 4.2.8 is a generalization of Proposition 3.15 of [1] from ∞–topoi to general presentable ∞–categories (such as the ∞–category of algebras over an ∞–operad). Some results on truncatedness and connectedness for general presentable ∞–categories can also be found in [13]. In fact, Lemmas 4.2.5 and 4.2.6 (with its corollary) already appear in [13], yet we have chosen to include detailed proofs for completeness. Though we shall not use it, it is worthwhile to mention another result from [13], namely, that the pair of classes of n–connected and n–truncated morphisms form a factorization system for every presentable ∞–category $\mathcal{C}$ (generalizing [16, 5.2.8.16] from ∞–topoi).

We reiterate that, especially in this section, some of the facts that we state as lemmas might appear obvious or well known. Nonetheless, we have chosen to include detailed proofs where those are not to be found in the literature (to the best of our knowledge).

4.1 Space of lifts

Definition 4.1.1 [16, 5.2.8.1] A commutative square in an ∞–category $\mathcal{C}$ is a map $q\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$, which we write somewhat informally as

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

suppressing the homotopies. The space of lifts for q is defined as follows. Restricting to the diagonal $\Delta^1 \to \Delta^1 \times \Delta^1$, we get a morphism $h\colon A \to Y$ in $\mathcal{C}$, which can be viewed as an object $\overline{Y}$ in the ∞–category $\mathcal{C}_{A/}$. The diagram q can be encoded as a pair of objects $B, X \in \mathcal{C}_{A//\overline{Y}}$ and the space of lifts for q is given as the mapping space

$$L(q) = \mathrm{Map}_{\mathcal{C}_{A//\overline{Y}}}(\overline{B}, \overline{X}).$$

Remark 4.1.2 Let us denote the horizontal morphisms in the above diagram by $f\colon A \to X$ and $g\colon B \to Y$. By the dual of [16, 5.5.5.12] we have a homotopy fiber sequence

$$\mathrm{Map}_{\mathcal{C}_{A//\overline{Y}}}(B, X) \to \mathrm{Map}_{\mathcal{C}_{A/}}(B, X) \to \mathrm{Map}_{\mathcal{C}_{A/}}(B, Y)$$

over $g \in \mathrm{Map}_{\mathcal{C}_{A/}}(B, Y)$. Using [16, 5.5.5.12] again for the middle and the right term we obtain a presentation of $\mathrm{Map}_{\mathcal{C}_{A//\overline{Y}}}(B, X)$ as the total fiber of the square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}}(B, X) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(B, Y) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}}(A, X) \arrow[r] & \mathrm{Map}_{\mathcal{C}}(A, Y) \end{tikzcd}$$

In other words, we have a homotopy fiber sequence

$$L(q) \to \mathrm{Map}_{\mathcal{C}}(B, X) \to \mathrm{Map}_{\mathcal{C}}(A, X) \times^{h}_{\mathrm{Map}_{\mathcal{C}}(A,Y)} \mathrm{Map}_{\mathcal{C}}(B, Y)$$

over the point determined by the diagram q.

Another reasonable definition of the space of lifts is as follows. The inclusion

$$\Delta^{\{0,1\}} \times \Delta^{\{0,2\}} \hookrightarrow \Delta^3$$

induces a restriction map $\mathcal{C}^{\Delta^3} \to \mathcal{C}^{\Delta^1 \times \Delta^1}$ and we can consider the (automatically homotopy) fiber over the vertex $q \in \mathcal{C}^{\Delta^1 \times \Delta^1}$, which is an ∞–category. In [16, 5.2.8.22] it is proved that this ∞–category is categorically equivalent to $L(q)$ (and in particular a Kan complex).

The next lemma shows that the space of lifts behaves well with respect to pullback and pushout.

Lemma 4.1.3 *Given a commutative rectangle $\Delta^1 \times \Delta^2 \to \mathcal{C}$, depicted as*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & Z \arrow[d] \\ B \arrow[r] & Y \arrow[r] & W \end{tikzcd}$$

with left square q_l, right square q_r and outer square q,

(1) *if q_r is a pullback square, then we have a canonical equivalence $L(q) \simeq L(q_l)$;*

(2) *if q_l is a pushout square, then we have a canonical equivalence $L(q) \simeq L(q_r)$.*

Proof By symmetry, it is enough to prove (1). Observe that the prism $\Delta^1 \times \Delta^2$ is a left cone on the simplicial set obtained by removing the initial vertex. Formally,

$$\Delta^1 \times \Delta^2 \simeq (\Delta^2 \times \Delta^{\{1\}} \sqcup_{\Delta^{\{1,2\}} \times \Delta^{\{1\}}} \Delta^{\{1,2\}} \times \Delta^1)^{\triangleleft}.$$

We can therefore interpret the rectangle as a diagram in $\mathcal{C}_{A/}$ (and hence ignore A). Since the projection $\mathcal{C}_{A/} \to \mathcal{C}$ preserves and reflects limits (dual of [16, 1.2.13.8]), the square q_r is a pullback square in $\mathcal{C}_{A/}$. The universal property of the pullback implies that we have a homotopy cartesian square

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}_{A/}}(B, X) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}_{A/}}(B, Z) \arrow[d] \\ \mathrm{Map}_{\mathcal{C}_{A/}}(B, Y) \arrow[r] & \mathrm{Map}_{\mathcal{C}_{A/}}(B, W) \end{tikzcd}$$

which in turn induces a homotopy equivalence of homotopy fibers of the vertical maps. Considering the given map $B \to Y$ as a point in $\mathrm{Map}_{\mathcal{C}_{A/}}(B, Y)$ and considering the induced equivalence on the homotopy fibers of the vertical maps, we obtain by [16, 5.5.5.12] an equivalence

$$\mathrm{Map}_{\mathcal{C}_{A/\!/Y}}(\overline{B}, \overline{X}) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}_{A/\!/W}}(\overline{\overline{B}}, \overline{\overline{Z}}),$$

where $\overline{B}$ and $\overline{X}$ are $A \to B \to Y$ and $A \to X \to Y$ viewed as objects of $\mathcal{C}_{A/\!/Y}$ and $\overline{\overline{B}}$ and $\overline{\overline{Z}}$ are $A \to B \to W$ and $A \to Z \to W$ viewed as objects of $\mathcal{C}_{A/\!/W}$. By the definition of the space of lifts, this is precisely the equivalence $L(q_l) \simeq L(q)$. $\square$

The following lemma expands on [16, Remark 5.2.8.7]:

Lemma 4.1.4 *Let $F\colon \mathcal{C} \leftrightarrows \mathcal{D} :G$ be an adjunction of ∞–categories. For every commutative square $q\colon \Delta^1\times\Delta^1\to\mathcal{D}$ of the form*

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d, "F(f)"] & X \arrow[d, "g"] \\ F(B) \arrow[r] & Y \end{tikzcd}$$

there is an adjoint square $p\colon \Delta^1\times\Delta^1\to\mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & G(X) \arrow[d, "G(g)"] \\ B \arrow[r] & G(Y) \end{tikzcd}$$

and a canonical homotopy equivalence $L(q)\simeq L(p)$.

Proof Let $\mathcal{M}\to\Delta^1$ be the cartesian–cocartesian fibration associated with the adjunction $F\dashv G$. Since $\mathcal{C}$ and $\mathcal{D}$ are full subcategories of $\mathcal{M}$ we can think of the square q as taking values in $\mathcal{M}$ and it does not change the space of lifts. Consider the diagram in $\mathcal{M}$ given by

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & F(A) \arrow[r] \arrow[d, "F(f)"] & X \arrow[d, "g"] \\ B \arrow[r] & F(B) \arrow[r] & Y \end{tikzcd}$$

where in the left square q_l the horizontal arrows are cocartesian and the rest of the data is given by the lifting property of cocartesian edges. Since the inclusion of the spine $\Lambda^2_1\hookrightarrow\Delta^2$ is inner anodyne, so is $\Delta^1\times\Lambda^2_1\hookrightarrow\Delta^1\times\Delta^2$ (by [16, 2.3.2.4]) and since $\mathcal{M}\to\Delta^1$ is an inner fibration, the diagram can be extended to $\Delta^1\times\Delta^2\to\mathcal{M}$ and we can denote the outer square by $r\colon\Delta^1\times\Delta^1\to\mathcal{M}$. We now claim that q_l is a pushout square in $\mathcal{M}$. For every $Z\in\mathcal{M}$, consider the induced diagram

$$\begin{tikzcd} \mathrm{Map}(F(B),Z) \arrow[r] \arrow[d] & \mathrm{Map}(B,Z) \arrow[d] \\ \mathrm{Map}(F(A),Z) \arrow[r] & \mathrm{Map}(A,Z) \end{tikzcd}$$

If $Z\in\mathcal{M}_0\simeq\mathcal{C}$, then the spaces on both left corners are empty and if $Z\in\mathcal{M}_1\simeq\mathcal{D}$, then both horizontal arrows are equivalences. Either way, this is a pullback square and hence q_l is a pushout square. By Lemma 4.1.3 we get $L(q)\simeq L(r)$.

We can now factor the outer square $r\colon \Delta^1 \times \Delta^1 \to \mathcal{M}$ as

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & G(X) \arrow[r] \arrow[d, "G(g)"] & X \arrow[d, "g"] \\ B \arrow[r] & G(Y) \arrow[r] & Y \end{tikzcd}$$

where the left square is p and in the right square q_r the horizontal arrows are cartesian and the square is determined by the lifting property of cartesian edges. Repeating the argument in the dual form we get that q_r is a pullback square and using Lemma 4.1.3 again we get $L(p) \simeq L(r)$ and therefore $L(p) \simeq L(q)$. □

Proposition 4.1.5 *Let $\mathcal{C}$ be an ∞–category. Let $q\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ be a commutative square*

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "\beta"] & Y \end{tikzcd}$$

with space of lifts $L(q)$. Given a point $(s_0, s_1) \in L(q) \times L(q)$, the homotopy fiber of the diagonal

$$\delta_{L(q)}\colon L(q) \to L(q) \times L(q)$$

over (s_0, s_1) is homotopy equivalent to the space of lifts for a square $p\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "f"'] & X \arrow[d, "\delta g"] \\ B \arrow[r, "{(s_0,s_1)}"] & X \times_Y X \end{tikzcd}$$

Proof For ease of notation, set $\mathcal{D} = \mathcal{C}_{A/}$. Recall that

$$L(q) = \mathrm{Map}_{\mathcal{D}_{/\overline{Y}}}(\overline{B}, \overline{X})$$

and therefore

$$L(q) \times L(q) = \mathrm{Map}_{\mathcal{D}_{/\overline{Y}}}(\overline{B}, \overline{X}) \times \mathrm{Map}_{\mathcal{D}_{/\overline{Y}}}(\overline{B}, \overline{X}) \simeq \mathrm{Map}_{\mathcal{D}_{/\overline{Y}}}(\overline{B}, \overline{X} \times \overline{X}).$$

Products in the over-category are fibered products and products in the under-category are just ordinary products (dual of [16, 1.2.13.8]). Hence, $\overline{X} \times \overline{X}$ is the diagram $A \to X \times_Y X \to Y$, which we denote by $\overline{X \times_Y X}$. Thus, a point $s = (s_0, s_1) \in L(q) \times L(q)$

corresponds to a lift in the diagram

$$\begin{tikzcd} & \overline{X \times_Y X} \arrow[d] \\ \overline{B} \arrow[r] \arrow[ur, dashed] & \overline{Y} \end{tikzcd}$$

in the category $\mathcal{D}$. Furthermore, the diagonal map $\delta_{L(q)}\colon L(q) \to L(q) \times L(q)$ is induced from the diagonal map $\delta_{\overline{X}}\colon \overline{X} \to \overline{X} \times \overline{X}$. Namely, $\delta_{L(q)} = (\delta_{\overline{X}})_*$. Our goal is therefore to compute the homotopy fiber of $(\delta_{\overline{X}})_*$ over a given point

$$s = (s_0, s_1) \simeq \mathrm{Map}_{\mathcal{D}_{/\overline{Y}}}(\overline{B}, \overline{X} \times \overline{X}).$$

The projection $\mathcal{D}_{/\overline{Y}} \to \mathcal{D}$ induces an equivalence

$$(\mathcal{D}_{/\overline{Y}})_{/\overline{X \times_Y X}} \simeq \mathcal{D}_{/\overline{X \times_Y X}}.$$

It follows that the fiber is the space of lifts in the diagram

$$\begin{tikzcd} & \overline{X} \arrow[d] \\ \overline{B} \arrow[r, "s"] \arrow[ur, dashed] & \overline{X \times_Y X} \end{tikzcd}$$

in $\mathcal{D}$. By (the dual of) [16, 5.5.5.12], this space of lifts is homotopy equivalent to the mapping space $\mathrm{Map}_{\mathcal{D}_{/\overline{X \times_Y X}}}(\overline{B}, \overline{X})$. Recalling that $\mathcal{D} = \mathcal{C}_{A/}$, we see that this is none other than the space of lifts for p. $\square$

4.2 Truncatedness and connectedness

We recall the following definition from classical homotopy theory:

Definition 4.2.1 For $d \geq -2$, a map $f\colon X \to Y$ of spaces is called *d–truncated* if all of its homotopy fibers are d–truncated spaces (Definition 3.1.1).

Using this definition, one can define a general notion of d–truncatedness in an ∞–category.

Definition 4.2.2 [16, 5.5.6.1] For $d \geq -2$, a map $f\colon X \to Y$ in an ∞–category $\mathcal{C}$ is called *d–truncated* if for every $Z \in \mathcal{C}$ the induced map

$$\mathrm{Map}(Z, X) \to \mathrm{Map}(Z, Y)$$

is a d–truncated map of spaces. An object X is d–truncated if the map $X \to \mathrm{pt}_{\mathcal{C}}$ is d–truncated. We denote by $\tau_{\le d}\mathcal{C}$ the full subcategory of $\mathcal{C}$ spanned by the d–truncated objects. When $\mathcal{C}$ is presentable, by [16, 5.5.6.21] the ∞–category $\tau_{\le d}\mathcal{C}$ is itself presentable and, by [16, 5.5.6.18], the inclusion $\tau_{\le d}\mathcal{C} \hookrightarrow \mathcal{C}$ has a left adjoint $\tau^{\mathcal{C}}_{\le d}\colon \mathcal{C} \to \tau_{\le d}\mathcal{C}$.

Remark 4.2.3 It is not difficult to show that $\tau_{\le d}$ extends to a functor from the ∞–category of presentable ∞–categories to the full subcategory spanned by presentable essentially $(d+1)$–categories and that it is left adjoint to the inclusion. The maps $\tau^{\mathcal{C}}_{\le d}$ can be taken to be the components of the unit transformation (this essentially follows from [16, 5.5.6.22]), but we shall not need this.

We now turn to discuss the dual notion of n–connectedness.

Definition 4.2.4 For $n \ge -2$, a map $f\colon A \to B$ in an ∞–category $\mathcal{C}$ is *n–connected* if it is left orthogonal to every n–truncated map; ie for every commutative square $q\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$,

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] & Y \end{tikzcd}$$

in which g is n–truncated, $L(q)$ is contractible. An object $A \in \mathcal{C}$ is called n–connected if $A \to \mathrm{pt}_{\mathcal{C}}$ is n–connected.

Lemma 4.2.5 *Let $\mathcal{C}$ and $\mathcal{D}$ be ∞–categories that admit finite limits and let $F\colon \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjunction with $F \dashv G$.*

(1) *For every $d \ge -2$ and a d–truncated morphism g in $\mathcal{D}$, the morphism $G(g)$ is a d–truncated morphism in $\mathcal{C}$.*

(2) *For every $n \ge -2$ and an n–connected morphism f in $\mathcal{C}$, the morphism $F(f)$ is an n–connected morphism in $\mathcal{D}$.*

Proof As a right adjoint, G is left exact and therefore preserves d–truncated morphisms by [16, 5.5.6.16]. Since G preserves n–truncated morphisms and the space of lifts in the square

$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & X \arrow[d] \\ F(B) \arrow[r] & Y \end{tikzcd}$$

is homotopy equivalent to the space of lifts in the adjoint square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & G(X) \arrow[d] \\ B \arrow[r] & G(Y) \end{tikzcd}$$

given by Lemma 4.1.4, we see that if f is left orthogonal to all n–truncated morphisms then so is $F(f)$. □

Lemma 4.2.6 *Let $\mathcal{C}$ be a presentable ∞–category, let $f\colon A \to B$ be a morphism in $\mathcal{C}$, and let $n \geq -2$ be an integer. The map f is n–connected if and only if, viewed as an object $\overline{A}$ of $\mathcal{C}_{/B}$, its n–truncation $\tau_{\leq n}^{\mathcal{C}_{/B}}(\overline{A})$ is the terminal object (ie $\mathrm{Id}_B\colon B \to B$).*

Proof Since $\mathcal{C}$ has all pullbacks, every commutative square $q\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

can be factored as

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & B \arrow[r] & Y \end{tikzcd}$$

By Lemma 4.1.3, the space of lifts for the original square q is equivalent to the space of lifts in the left square of the above rectangle. Moreover, n–truncated morphisms are closed under base change and so to check that f is n–connected, we can equivalently restrict ourselves to checking the left orthogonality condition only for squares q in which the map $B \to Y$ is the identity on B. Writing $\overline{A}$, $\overline{X}$ and $\overline{B}$ for $A \to B$, $X \to B$ and $\mathrm{Id}\colon B \to B$ as objects of $\mathcal{C}_{/B}$, respectively, we see that by the dual of [16, 5.5.5.12] the space of lifts fits into a fiber sequence

$$L(q) = \mathrm{Map}_{\mathcal{C}_{A//B}}(\overline{B}, \overline{X}) \to \mathrm{Map}_{\mathcal{C}_{/B}}(\overline{B}, \overline{X}) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}_{/B}}(\overline{A}, \overline{X}).$$

Hence, f is n–connected if and only if f^* is an equivalence for every n–truncated morphism $X \to B$. By [16, 5.5.6.10], a morphism $X \to B$ is n–truncated if and only if $\overline{X}$ is an n–truncated object of $\mathcal{C}_{/B}$. Hence, we need the above map to be an equivalence for every n–truncated object $\overline{X} \in \mathcal{C}_{/B}$. This precisely means that the map $\overline{A} \to \overline{B}$ exhibits $\overline{B}$, the terminal object of $\mathcal{C}_{/B}$, as the n–truncation of $\overline{A}$. □

Corollary 4.2.7 *In a presentable ∞–category $\mathcal{C}$, an object X is n–connected for some $n \geq -2$ if and only if its n–truncation $\tau^{\mathcal{C}}_{\leq n} X$ is a terminal object of $\mathcal{C}$.*

The following is a quantitative generalization of the defining property of an n–connected morphism.

Proposition 4.2.8 *Let $\mathcal{C}$ be a presentable ∞–category. Fix integers $d \geq n \geq -2$. For every square $q\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

in which $f\colon A \to B$ is n–connected and $g\colon X \to Y$ is d–truncated, the space of lifts $L(q)$ is $(d-n-2)$–truncated.

Proof We prove this by induction on d. For $d = n$, the claim follows from the definition of an n–connected morphism and the fact that a space is (-2)–connected if and only if it is contractible. We now assume that this is true for $d-1$, and prove it for d. Denote the space of lifts by $L(q)$. By [16, 5.5.6.15], it suffices to show that the diagonal map $\delta\colon L(q) \to L(q) \times L(q)$ is $(d-n-3)$–truncated. By Proposition 4.1.5, the homotopy fiber over a point $(s_0, s_1) \in L(q) \times L(q)$ is equivalent to the space of lifts in the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & X \times_Y X \end{tikzcd}$$

where the bottom map is (s_0, s_1). By [16, 5.5.6.15], since $X \to Y$ is d–truncated, $X \to X \times_Y X$ is $(d-1)$–truncated and, therefore, by induction, the space of lifts is $((d-1)-n-2)$–truncated and we are done. □

4.3 $(n-\frac{1}{2})$–Connectedness

We begin by introducing an auxiliary notion that will be helpful in the study of n–connectedness.

Definition 4.3.1 For every $n \geq -2$, a morphism $f\colon X \to Y$ is called $(n-\frac{1}{2})$–*connected* if the induced map $\tau^{\mathcal{C}}_{\leq n}(f)\colon \tau^{\mathcal{C}}_{\leq n} X \to \tau^{\mathcal{C}}_{\leq n} Y$ is an equivalence.

To justify the terminology we need to show that it indeed sits between n– and $(n-1)$–connectedness, at least under some reasonable conditions. One direction is completely general:

Lemma 4.3.2 *Let $n \geq -2$ and let $\mathcal{C}$ be a presentable ∞–category. If a morphism $f\colon A \to B$ is n–connected, then it is $\left(n-\frac{1}{2}\right)$–connected.*

Proof By the Yoneda lemma it is enough to show that for every n–truncated object Z in $\mathcal{C}$ the induced map

$$f_*\colon \mathrm{Map}(B, Z) \to \mathrm{Map}(A, Z)$$

is an equivalence. For this, it is enough to show that for every $g\colon A \to Z$, the fiber of f_* over g is contractible. By [16, 5.5.5.12], the fiber is equivalent to the space of lifts for the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & Z \arrow[d] \\ B \arrow[r] & \mathrm{pt} \end{tikzcd}$$

which is contractible by definition as $f\colon A \to B$ was assumed to be n–connected. □

For the other direction, we need to assume that our ∞–category is an m–topos. First:

Lemma 4.3.3 *Let $\mathcal{C}$ be an m–topos for some $-1 \leq m \leq \infty$. For every d–truncated morphism $g\colon X \to Y$, the diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \tau^{\mathcal{C}}_{\leq d+1} X \arrow[d] \\ Y \arrow[r] & \tau^{\mathcal{C}}_{\leq d+1} Y \end{tikzcd}$$

is a pullback square.

Proof For $\mathcal{C} = \mathcal{S}$, this follows from inspecting the induced map between the long exact sequences of homotopy groups associated with the vertical maps. For $\mathcal{C} = \mathcal{S}^K$, this follows from the claim for $\mathcal{S}$, since both truncation and pullbacks are computed levelwise. A general ∞–topos is a left exact localization of $\mathcal{S}^K$ for some K, and left exact colimit-preserving functors between presentable ∞–categories commute with truncation by [16, 5.5.6.28] and with pullbacks by assumption. Finally, by [16, 6.4.1.5] every m–topos is the full subcategory on $(m-1)$–truncated objects in an ∞–topos and this full subcategory is closed under limits. □

From this we deduce:

Lemma 4.3.4 *Let $n \geq -2$ and let $\mathcal{C}$ be an m–topos for some $-1 \leq m \leq \infty$. If a morphism $f\colon A \to B$ is $\left(n+\frac{1}{2}\right)$–connected then it is n–connected.*

Proof To show that $f\colon A \to B$ is n–connected, we need to show that the space of lifts for every square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

in which the right vertical arrow is n–truncated is contractible. Applying Lemmas 4.3.3 and 4.1.3, we see that this space is equivalent to the space of lifts in the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \tau^{\mathcal{C}}_{\leq n+1} X \arrow[d] \\ B \arrow[r] & \tau^{\mathcal{C}}_{\leq n+1} Y \end{tikzcd}$$

which, by Lemma 4.1.4, is equivalent to the space of lifts in the adjoint square

$$\begin{tikzcd} \tau^{\mathcal{C}}_{\leq n+1} A \arrow[r] \arrow[d] & \tau^{\mathcal{C}}_{\leq n+1} X \arrow[d] \\ \tau^{\mathcal{C}}_{\leq n+1} B \arrow[r] & \tau^{\mathcal{C}}_{\leq n+1} Y \end{tikzcd}$$

which is contractible since the left vertical arrow is an equivalence. □

As a consequence, we obtain another sense in which $\left(n-\frac{1}{2}\right)$–connected morphisms are "close" to being n–connected:

Proposition 4.3.5 *Let $n \geq -2$ and let $\mathcal{C}$ be an m–topos for some $-1 \leq m \leq \infty$. If a morphism $f\colon A \to B$ in $\mathcal{C}$ is $\left(n-\frac{1}{2}\right)$–connected and has a section (ie there exists $s\colon B \to A$ such that $f \circ s \sim \mathrm{Id}_B$), then f is n–connected.*

Proof We first prove the case of $m = \infty$. For $n = -2$, there is nothing to prove, and so we assume that $n \geq -1$. Since $f \circ s = \mathrm{Id}_B$ we get $\tau^{\mathcal{C}}_{\leq n}(f) \circ \tau^{\mathcal{C}}_{\leq n}(s) = \mathrm{Id}_B$ and, since $\tau_{\leq n}(f)$ is an equivalence, so is $\tau_{\leq n}(s)$ and hence s is $\left(n-\frac{1}{2}\right)$–connected. By Lemma 4.3.4, s is $(n-1)$–connected and hence, by [16, 6.5.1.20], the map f is n–connected (note that n*–connective* means $(n-1)$–connected).

For a general m, by [16, 6.4.1.5] there exists an ∞–topos $\mathcal{D}$ and an equivalence $\mathcal{C} \simeq \tau_{\leq m-1}\mathcal{D}$, and so we may identify $\mathcal{C}$ with the full subcategory of $(m-1)$–truncated objects of $\mathcal{D}$. If $f\colon A \to B$ is $\left(n-\frac{1}{2}\right)$–connected in $\mathcal{C}$, then it is also $\left(n-\frac{1}{2}\right)$–connected in $\mathcal{D}$, since the restriction of $\tau^{\mathcal{D}}_{\leq n}$ to $\mathcal{C}$ is equivalent to $\tau^{\mathcal{C}}_{\leq n}$. It follows from the case of $m = \infty$ that f is n–connected in $\mathcal{D}$. Since $f = \tau^{\mathcal{D}}_{\leq m-1} f$ and $\tau^{\mathcal{D}}_{\leq m-1}$ is a left adjoint functor, by Lemma 4.2.5 the map f is also n–connected as a map in $\mathcal{C}$. □

4.4 Connectedness in algebras

We begin with the following general fact:

Lemma 4.4.1 *Let $F\colon \mathcal{C} \leftrightarrows \mathcal{D} :U$ be a monadic adjunction between presentable ∞–categories. If the monad $T = U \circ F$ preserves n–connected morphisms, then U detects n–connected morphisms. Namely, given a morphism $f\colon A \to B$ in $\mathcal{D}$, if $U(f)$ is n–connected for some $n \geq -2$, then f is n–connected.*

Proof Given a morphism $f\colon A \to B$ in $\mathcal{D}$, using the canonical simplicial resolution provided by the proof of [17, 4.7.3.13], we can express it as a colimit of the simplicial diagram of morphisms,

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}}(T^{n+1}(A) \to T^{n+1}(B)),$$

which one can write as

$$\operatorname*{colim}_{\Delta^{\mathrm{op}}}(F T^n U(A) \to F T^n U(B)).$$

If $U(f)$ is n–connected as in the statement, then since T preserves n–connected morphisms by assumption and F preserves n–connected morphisms by being left adjoint, it follows that all the maps in the diagram are n–connected. By [16, 5.2.8.6(7)], the map f is also n–connected. □

We want to apply the above to the free-forgetful adjunction between a symmetric monoidal ∞–category $\mathcal{C}$ and the category of $\mathcal{P}$–algebras in $\mathcal{C}$, where $\mathcal{P}$ is a reduced ∞–operad. For this, we need some compatibility between the notion of n–connectedness and the symmetric monoidal structure:

Lemma 4.4.2 *Let $\mathcal{C}$ be a presentably symmetric monoidal ∞–category. For every integer $n \geq -2$, the class of n–connected morphisms in $\mathcal{C}$ is closed under tensor products.*

Proof Since $\mathcal{C}$ is presentable and the tensor product commutes with colimits separately in each variable, for each object $X \in \mathcal{C}$ the functor $Y \mapsto X \otimes Y$ is a left adjoint and therefore preserves n–connected morphisms by Lemma 4.1.4. Hence, given two n–connected morphisms $f\colon A_1 \to B_1$ and $g\colon A_2 \to B_2$, the composition

$$A_1 \otimes B_1 \xrightarrow{A_1 \otimes g} A_1 \otimes B_2 \xrightarrow{f \otimes B_2} A_2 \otimes B_2$$

is n–connected as a composition of two n–connected morphisms. □

Example 4.4.3 For every m–topos (with $-1 \le m \le \infty$) and $n \ge -2$, the class of n–connected morphisms is closed under cartesian products. In particular, this applies to $\mathcal{S}^K_{\le m}$ for every simplicial set K.

Lemma 4.4.4 *Let $\mathcal{P}$ be a reduced ∞–operad and let $\mathcal{C}$ be a presentably symmetric monoidal ∞–category. The free-forgetful adjunction*

$$F\colon \mathcal{C} \leftrightarrows \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) :U$$

is monadic and the associated monad $T = U \circ F$ preserves n–connected morphisms.

Proof By [17, 4.7.3.11], the adjunction $F \dashv U$ is monadic. Hence, given a morphism $A \to B$ in $\mathcal{C}$, by Proposition 2.4.6 the morphism $T(A) \to T(B)$ can be expressed as

$$\coprod_{n \ge 0} (P(n) \otimes A^{\otimes n})_{h\Sigma_n} \to \coprod_{n \ge 0} (P(n) \otimes B^{\otimes n})_{h\Sigma_n}.$$

By Lemma 4.4.2, n–connected morphisms are closed under $\otimes$ and, by [16, 5.2.8.6], they are closed under colimits. Hence, we obtain that $T(A) \to T(B)$ is n–connected as well. □

Proposition 4.4.5 *Let $\mathcal{P}$ be a reduced ∞–operad and let $\mathcal{C}$ be a presentably symmetric monoidal ∞–category. Given a morphism $f\colon A \to B$ in $\mathrm{Alg}_{\mathcal{P}}(\mathcal{C})$, if the underlying map $U(f)$ is n–connected for some $n \ge -2$, then f is n–connected.*

Proof Combine Lemmas 4.4.4 and 4.4.1. □

5 The ∞–categorical Eckmann–Hilton argument

In this final section we prove our main results. In Section 5.1 we analyze the canonical map from the coproduct to the tensor product of two algebras over a reduced ∞–operad.

The main result is that under suitable assumptions, if the ∞–operad is highly connected, then this map is also highly connected (Proposition 5.1.3). In Section 5.2 we use the connectivity bound established in Section 5.1 to analyze the reduced endomorphism operad of an object in an ∞–topos. This analysis recovers and expands on classical results on deloopings of spaces with nonvanishing homotopy groups in a bounded region. In Section 5.3 we prove our main theorem, Theorem 5.3.1, and its main corollary, the ∞–categorical Eckmann–Hilton argument (Corollary 5.3.3). We conclude with some curious applications of the main theorem to some questions regarding tensor products of reduced ∞–operads.

5.1 Coproducts of algebras

Let $\mathcal{C}$ be a symmetric monoidal ∞–category and let $\mathcal{P}$ be a reduced ∞–operad. For every two algebras $A, B \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{C})$, there is a canonical map of algebras

$$f_{A,B}\colon A \sqcup B \to A \otimes B$$

formally given by

$$f_{A,B} = \mathrm{Id}_A \otimes 1_B \sqcup 1_A \otimes \mathrm{Id}_B,$$

where $1_A\colon 1 \to A$ and $1_B\colon 1 \to B$ are the respective unit maps viewed as maps of algebras (see [17, 3.2.1]).

Lemma 5.1.1 *Let $\mathcal{C}$ be a symmetric monoidal ∞–category and let $\mathcal{P}$ be a reduced ∞–operad. If $\mathcal{P} \not\simeq \mathbb{E}_0$, then for every pair of algebras $A, B \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{C})$, the canonical map*

$$f_{A,B}\colon A \sqcup B \to A \otimes B$$

has a section after we apply the forgetful functor $\underline{(-)}\colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathcal{C}$.

Proof By Remark 3.2.3, if $\mathcal{P} \not\simeq \mathbb{E}_0$, then it is (-1)–connected and in particular $\mathcal{P}(2) \neq \varnothing$. We shall construct a section to $\underline{f_{A,B}}$ using any binary operation $\mu \in \mathcal{P}(2)$. Let $i_A\colon A \to A \sqcup B$ and $i_B\colon B \to A \sqcup B$ be the canonical maps of the coproduct. Define s to be the composition of the maps

$$\underline{A} \otimes \underline{B} \xrightarrow{i_A \otimes i_B} (\underline{A \sqcup B}) \otimes (\underline{A \sqcup B}) \xrightarrow{\mu_{A \sqcup B}} \underline{A \sqcup B}.$$

Now, consider the diagram in the homotopy category of $\mathcal{C}$,

$$\begin{tikzcd}
& (\underline{A \sqcup B}) \otimes (\underline{A \sqcup B}) \arrow[r, "\mu_{A \sqcup B}"] \arrow[d, "\underline{f_{A,B}} \otimes \underline{f_{A,B}}"] & \underline{A \sqcup B} \arrow[d, "\underline{f_{A,B}}"] \\
\underline{A \otimes B} \arrow[ur, "\underline{i_A} \otimes \underline{i_B}"] \arrow[r, "\underline{(\mathrm{Id}_A \otimes 1_B) \otimes (1_A \otimes \mathrm{Id}_B)}"] \arrow[dr, "\underline{(\mathrm{Id}_A \otimes 1_A) \otimes (1_B \otimes \mathrm{Id}_B)}"'] & (\underline{A \otimes B}) \otimes (\underline{A \otimes B}) \arrow[r, "\mu_{A \otimes B}"] \arrow[d, "\mathrm{Id}_{\underline{A}} \otimes \sigma_{\underline{A},\underline{B}} \otimes \mathrm{Id}_{\underline{B}}"] & \underline{A \otimes B} \arrow[d, equal] \\
& (\underline{A \otimes A}) \otimes (\underline{B \otimes B}) \arrow[r, "\mu_A \otimes \mu_B"] & \underline{A \otimes B}
\end{tikzcd}$$

The upper square commutes since $f_{A,B}$ is a map of algebras. The upper triangle commutes since it is the tensor product of two triangles, which commute by the very definition of $f_{A,B}$. The lower square commutes by the definition of the algebra structure on $A \otimes B$ and the lower triangle also clearly commutes. The composition of the bottom diagonal map and the bottom right map is the identity, since the restriction of μ to the unit in one of the arguments is homotopic to the identity map of the other argument. The composition of the top diagonal map with the top right map is s. It follows that $\underline{f_{A,B}} \circ s \sim \mathrm{Id}_{\underline{A \otimes B}}$. □

Lemma 5.1.2 *Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal ∞–categories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor. If $\underline{F}\colon \underline{\mathcal{C}} \to \underline{\mathcal{D}}$ is a left adjoint, then the induced functor $F^{\otimes}\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a left adjoint relative to $\mathbf{Fin}_*$ and for every ∞–operad $\mathcal{P}$ the induced functor $\underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) \to \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{D})$ is a left adjoint.*

Proof For every $\langle n \rangle \in \mathbf{Fin}_*$, the restriction of $F^{\otimes}$ to the fiber over $\langle n \rangle$ is just $F^n\colon \mathcal{C}^n \to \mathcal{D}^n$, which is clearly a left adjoint. Hence, by [17, 7.3.2.7], the functor $F^{\otimes}$ is a left adjoint relative to $\mathbf{Fin}_*$. Let $G^{\otimes}$ be the right adjoint of $F^{\otimes}$. Applying [17, 7.3.2.13], we obtain that $F^{\otimes}$ and $G^{\otimes}$ induce an adjunction

$$\underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{C}) \leftrightarrows \underline{\mathrm{Alg}}_{\mathcal{P}}(\mathcal{D}). \qquad □$$

In what follows we are going to restrict ourselves to the case of a cartesian monoidal structure. The next proposition is the key connectivity bound on which the main theorems of this paper rest.

Proposition 5.1.3 *Let $\mathcal{C}$ be an m–topos for some $-1 \leq m \leq \infty$ with the cartesian symmetric monoidal structure and let $\mathcal{P}$ be a reduced d–connected ∞–operad for some $d \geq -2$. For every pair of algebras $A, B \in \mathrm{Alg}_{\mathcal{P}}(\mathcal{C})$, the canonical map*

$$f_{A,B}\colon A \sqcup B \to A \times B$$

is d–connected.

Proof For $d = -2$ there is nothing to prove and so we assume that $d \geq -1$. By Proposition 4.4.5, it is enough to show that $\underline{f_{A,B}}$ is d–connected, where

$$\underline{(-)}\colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathcal{C}$$

is the forgetful functor. By Proposition 4.3.5, it is enough to show that $\underline{f_{A,B}}$ has a section and is $\left(d-\frac{1}{2}\right)$–connected. Since $d \geq -1$, we have $\mathcal{P} \neq \mathbb{E}_0$ and, therefore, by Lemma 5.1.1, $\underline{f_{A,B}}$ has a section. Thus, we are reduced to showing that the image of $\underline{f_{A,B}}$ under the functor $\tau^{\mathcal{C}}_{\leq d}\colon \mathcal{C} \to \tau_{\leq d}\mathcal{C}$ is an equivalence. First, we show that $\tau^{\mathcal{C}}_{\leq d}$ preserves binary products. For $m = \infty$, this follows from [16, 6.5.1.2]. The general case reduces to $m = \infty$ as by [16, 6.4.1.5] we can embed $\mathcal{C}$ as a full subcategory of an ∞–topos spanned by the $(m-1)$–truncated objects. It follows that we get a symmetric monoidal functor $\tau^{\times}_{\leq d}\colon \mathcal{C}^{\times} \to (\tau_{\leq d}\mathcal{C})^{\times}$. By Lemma 5.1.2, the functor

$$F\colon \mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \to \mathrm{Alg}_{\mathcal{P}}(\tau_{\leq d}\mathcal{C})$$

induced by $\tau^{\times}_{\leq n}$ is a left adjoint. Consider the (solid) commutative diagram in the homotopy category of $\mathbf{Cat}_{\infty}$,

$$\begin{tikzcd}
\mathrm{Alg}_{\mathcal{P}}(\mathcal{C}) \arrow[r, "F"] \arrow[d] & \mathrm{Alg}_{\mathcal{P}}(\tau_{\leq d}\mathcal{C}) \arrow[r, dashed, shift left, "G'"] \arrow[d] & \mathrm{Alg}_{\mathbb{E}_\infty}(\tau_{\leq d}\mathcal{C}) \arrow[l, shift left, "G"] \arrow[d] \\
\mathcal{C} \arrow[r, "\tau^{\mathcal{C}}_{\leq d}"] & \tau_{\leq d}\mathcal{C} \arrow[r, equal] & \tau_{\leq d}\mathcal{C}
\end{tikzcd}$$

where the vertical maps are the forgetful functors and G is induced by restriction along the essentially unique map $\mathcal{P} \to \mathbb{E}_\infty$. Since $\tau_{\leq d}\mathcal{C}$ is an essentially $(d+1)$–category, it follows from Proposition 3.1.8 that G is an equivalence. Taking G' to be an inverse of G up to homotopy, the outer rectangle is a commutative square in the homotopy category of $\mathbf{Cat}_{\infty}$. Therefore, to show that $\tau^{\mathcal{C}}_{\leq d}(\underline{f_{A,B}})$ is an equivalence, it is enough to show that $\underline{G'(F(f_{A,B}))}$ is an equivalence. In fact, we shall show that $G'(F(f_{A,B}))$ is an equivalence. Note that the composition of the left and then bottom functors preserves binary products and since the right vertical functor preserves products and is conservative, it follows that the top functor $G' \circ F$ also preserves binary products. On the other hand, $G' \circ F$ also preserves coproducts, since F is left adjoint (by the above discussion) and G is an equivalence. Finally, in $\mathrm{Alg}_{\mathbb{E}_\infty}(\tau_{\leq d}\mathcal{C})$, the canonical map from the coproduct to the product is an equivalence by [17, 3.2.4.7]. $\square$

We now apply the above results to the study of reduced endomorphism operads. For every unital ∞–operad $\mathcal{Q}$ and a symmetric monoidal ∞–category $\mathcal{C}$, the symmetric monoidal ∞–category $\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})$ is unital by Lemma 2.2.5. Hence, for every $X \in \mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})$ we can consider the reduced endomorphism ∞–operad $\mathrm{End}^{\mathrm{red}}_{\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})}(X)$.

Corollary 5.1.4 *Let $\mathcal{Q}$ be a reduced n–connected ∞–operad for some $n \geq -2$ and let $\mathcal{C}$ be a $(d+1)$–topos with the cartesian symmetric monoidal structure for some $d \geq -2$. For every object $X \in \mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})$, the reduced endomorphism operad $\mathrm{End}^{\mathrm{red}}_{\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})}(X)$ is an essentially $(d-n-1)$–operad (ie all multimapping spaces are $(d-n-2)$–truncated).*

Proof The ∞–operad $\mathcal{E} = \mathrm{End}^{\mathrm{red}}_{\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})}(X)$ has a unique object, which we call X. We need to show that for every $m \in \mathbb{N}$, the multimapping space $\mathrm{Mul}_{\mathcal{E}}(X^{(m)}, X)$ is $(d-n-2)$–truncated. By Lemma 2.2.12, we have a fiber sequence

$$\mathrm{Mul}_{\mathcal{E}}(X^{(m)}, X) \to \mathrm{Mul}_{\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})}(X^m, X) \to \mathrm{Map}_{\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})}(X^{\sqcup m}, X),$$

where the fiber is taken over the fold map $\nabla\colon X^{\sqcup m} \to X$. The fiber is equivalent to the space of lifts for the square

$$\begin{tikzcd} X^{\sqcup m} \arrow[r, "\nabla"] \arrow[d] & X \arrow[d] \\ X^m \arrow[r] & \mathrm{pt} \end{tikzcd}$$

Since $\mathcal{C}$ is an essentially $(d+1)$–category, so is the cartesian ∞–operad $\mathcal{C}_\times$ and, therefore, by Proposition 3.1.10, so is $\mathrm{Alg}_{\mathcal{Q}}(\mathcal{C})$. In particular, X is d–truncated. Hence, by Proposition 4.2.8, it is enough to show that the canonical map $X^{\sqcup m} \to X^m$ is n–connected. Since $\mathcal{Q}$ is n–connected, this follows from repeated application of Proposition 5.1.3. □

5.2 Topoi and the reduced endomorphism operad

In this subsection we describe a simple application of Corollary 5.1.4. Let $\mathcal{C}$ be an ∞–topos and let $\mathcal{C}_*$ be the ∞–category of pointed objects in $\mathcal{C}$ with the cartesian symmetric monoidal structure.

Definition 5.2.1 For a pair of integers $m, k \geq -2$, we denote by $\mathcal{C}_*^{[k,m]} \subseteq \mathcal{C}_*$ the full subcategory spanned by objects which are simultaneously $(k-1)$–connected (ie *k–connective*) and m–truncated.

Theorem 5.2.2 *Let $\mathcal{C}$ be an ∞–topos and let $k, d \geq -2$. For every $X \in \mathcal{C}_*^{[k,2k+d]}$ the ∞–operad $\mathrm{End}^{\mathrm{red}}_{\mathcal{C}_*}(X)$ is an essentially $(d+1)$–operad. In particular, for $d=-1$, the ∞–operad $\mathrm{End}^{\mathrm{red}}_{\mathcal{C}_*}(X)$ is either $\mathbb{E}_0$ or $\mathbb{E}_\infty$ and for $d=-2$, it is $\mathbb{E}_\infty$.*

Proof By [17, 5.2.6.10 and 5.2.6.12], we have a commutative diagram of ∞–categories

$$\begin{tikzcd} \mathcal{C}_*^{\geq k} \arrow[r, "\sim"] \arrow[dr, "\Omega^k"'] & \underline{\mathrm{Alg}}^{\mathrm{grp}}_{\mathbb{E}_k}(\mathcal{C}_*) \arrow[d, "U"] \\ & \mathcal{C}_* \end{tikzcd}$$

in which U is the forgetful functor. Since the k–fold loop space functor restricts to a functor $\mathcal{C}_*^{[k,2k+d]} \to \tau_{\leq k+d}\mathcal{C}_*$, we can restrict the above diagram to

$$\begin{tikzcd} \mathcal{C}_*^{[k,2k+d]} \arrow[r, "\sim"] \arrow[dr, "\Omega^k"'] & \underline{\mathrm{Alg}}^{\mathrm{grp}}_{\mathbb{E}_k}(\tau_{\leq k+d}\mathcal{C}_*) \arrow[d, "U"] \\ & \tau_{\leq k+d}\mathcal{C}_* \end{tikzcd}$$

The ∞–category $\underline{\mathrm{Alg}}^{\mathrm{grp}}_{\mathbb{E}_k}(\tau_{\leq k+d}\mathcal{C}_*)$ is a full subcategory of $\underline{\mathrm{Alg}}_{\mathbb{E}_k}(\tau_{\leq k+d}\mathcal{C}_*)$, which is itself equivalent to $\underline{\mathrm{Alg}}_{\mathbb{E}_k}(\tau_{\leq k+d}\mathcal{C})$. The ∞–category $\tau_{\leq k+d}\mathcal{C}$ is a $(k+d+1)$–topos (with the cartesian symmetric monoidal structure) and $\mathbb{E}_k$ is $(k-2)$–connected. Thus, Corollary 5.1.4 implies that for every X in $\underline{\mathrm{Alg}}^{\mathrm{grp}}_{\mathbb{E}_k}(\tau_{\leq k+d}\mathcal{C}_*)$, the reduced endomorphism operad of X is an essentially $(d+1)$–operad.

Let $d=-1$. We recall from Remark 3.2.3 that if $\mathcal{P} \not\simeq \mathbb{E}_0$, then it is (-1)–connected. Therefore, if $\mathcal{P}$ is an essentially 0–operad, then $\mathcal{P} \simeq \mathbb{E}_\infty$. Hence, $\mathcal{P}$ is either $\mathbb{E}_0$ or $\mathbb{E}_\infty$.

Let $d=-2$. We get that $\mathcal{P}$ is an essentially (-1)–operad and hence equivalent to $\mathbb{E}_\infty$. □

For every reduced ∞–operad $\mathcal{P}$, the structure of a $\mathcal{P}$–algebra on an object $X \in \mathcal{C}_*$ is equivalent to the data of a map $\mathcal{P} \to \mathrm{End}^{\mathrm{red}}_{\mathcal{C}_*}(X)$. Thus, if $X \in \mathcal{C}_*^{[k,2k-2]}$, then X has a unique $\mathcal{P}$–algebra structure for every reduced ∞–operad $\mathcal{P}$. Combining this with the fact that for a pointed connected object in an ∞–topos, a structure of an $\mathbb{E}_\infty$–algebra is equivalent to an ∞–delooping, we get the following classical fact:

Corollary 5.2.3 *Let $\mathcal{C}$ be an ∞–topos and let $k \geq 1$ be an integer. Every $X \in \mathcal{C}_*^{[k,2k-2]}$ admits a unique ∞–delooping.*

In fact, we can get slightly more from Theorem 5.2.2. For example:

Corollary 5.2.4 *Let $\mathcal{C}$ be an ∞–topos, let $k \geq 1$ be an integer and let $X \in \mathcal{C}_*^{[k,2k-1]}$. If X admits an H–structure, then it admits a unique ∞–delooping.*

Proof By Theorem 5.2.2, the ∞–operad $\mathrm{End}^{\mathrm{red}}_{\mathcal{C}_*}(X)$ is either $\mathbb{E}_0$ or $\mathbb{E}_\infty$. On the other hand, the existence of an H–structure is equivalent to $\mathrm{End}^{\mathrm{red}}_{\mathcal{C}_*}(X)(2) \neq \varnothing$. Thus, X admits an H–structure if and only if $\mathrm{End}^{\mathrm{red}}_{\mathcal{C}_*}(X) \simeq \mathbb{E}_\infty$ if and only if X admits a unique ∞–delooping. □

5.3 The ∞–categorical Eckmann–Hilton argument

The main theorem of this paper is:

Theorem 5.3.1 *For all integers $d_1, d_2 \geq -2$, given a d_1–equivalence $\mathcal{P} \to \mathcal{Q}$ between two reduced ∞–operads and a reduced d_2–connected ∞–operad $\mathcal{R}$, the induced map $\mathcal{P} \otimes \mathcal{R} \to \mathcal{Q} \otimes \mathcal{R}$ is a (d_1+d_2+2)–equivalence.*

Proof Set $d = d_1 + d_2 + 2$. By Proposition 3.2.6, it is enough to show that for every $(d+1)$–topos $\mathcal{C}$ with the cartesian symmetric monoidal structure, the map

$$\mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{Q} \otimes \mathcal{R}, \mathcal{C}) \to \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{P} \otimes \mathcal{R}, \mathcal{C}),$$

induced by precomposition with f, is a homotopy equivalence. Using the tensor–hom adjunction, it is the same as showing that the map

$$\mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{Q}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C})) \to \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{P}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C}))$$

is an equivalence. The underlying category functor gives a commutative diagram

$$(*) \qquad \begin{tikzcd} \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{Q}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C})) \arrow[r] \arrow[d] & \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{P}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C})) \arrow[d] \\ \mathrm{Map}_{\mathbf{Cat}_\infty}(\underline{\mathcal{Q}}, \underline{\mathrm{Alg}}_{\mathcal{R}}(\mathcal{C})) \arrow[r] & \mathrm{Map}_{\mathbf{Cat}_\infty}(\underline{\mathcal{P}}, \underline{\mathrm{Alg}}_{\mathcal{R}}(\mathcal{C})) \end{tikzcd}$$

As $\underline{P} \to \underline{\mathcal{Q}}$ is an equivalence of ∞–categories (both are equivalent to Δ^0), the bottom map is a homotopy equivalence. Hence, it suffices to show that the induced map on the homotopy fibers is a homotopy equivalence for each choice of a basepoint. A point in the space $\mathrm{Map}(\Delta^0, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C}))$ is just an $\mathcal{R}$–algebra X in $\mathcal{C}$. We denote by $\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})_X$ the ∞–operad $\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})$ pointed by X viewed as an object of $\mathbf{Op}^{\mathrm{un}}_{\infty,*}$. With this notation, we see that the homotopy fiber of the right vertical map is equivalent to

$$\mathrm{Map}_{\mathbf{Op}^{\mathrm{un}}_{\infty,*}}(\mathcal{P}, \mathrm{Alg}_{\mathcal{R}}(\mathcal{C})_X).$$

By Lemma 2.2.5, the ∞–operad $\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})$ is unital. Therefore, by the adjunction

$$\iota\colon \mathbf{Op}^{\mathrm{red}}_\infty \leftrightarrows \mathbf{Op}^{\mathrm{un}}_{\infty,*} :(-)^{\mathrm{red}},$$

the above mapping space is also equivalent to

$$\mathrm{Map}_{\mathbf{Op}_\infty^{\mathrm{red}}}(\mathcal{P}, \mathrm{End}^{\mathrm{red}}_{\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})}(X)) \simeq \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{P}, \mathrm{End}^{\mathrm{red}}_{\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})}(X)),$$

since $\mathbf{Op}_\infty^{\mathrm{red}} \subseteq \mathbf{Op}_\infty$ is a full subcategory. The induced map on the fibers of the vertical maps in $(*)$ over X is therefore equivalent to

$$\mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{Q}, \mathrm{End}^{\mathrm{red}}_{\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})}(X)) \to \mathrm{Map}_{\mathbf{Op}_\infty}(\mathcal{P}, \mathrm{End}^{\mathrm{red}}_{\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})}(X)).$$

Finally, since $\mathcal{C}$ is a $(d+1)$–topos and $\mathcal{R}$ is d_2–connected, Corollary 5.1.4 implies that the ∞–operad $\mathrm{End}^{\mathrm{red}}_{\mathrm{Alg}_{\mathcal{R}}(\mathcal{C})}(X)$ is an essentially $(d-d_2-1=d_1+1)$–operad. Since $\mathcal{P} \to \mathcal{Q}$ is a d_1–equivalence, by Proposition 3.1.8 the above map is a homotopy equivalence and this completes the proof. □

Example 5.3.2 Let $\mathcal{P} \to \mathcal{Q}$ be a d–equivalence of reduced ∞–operads. For every integer $k \geq 0$, the induced map $\mathcal{P} \otimes \mathbb{E}_k \to \mathcal{Q} \otimes \mathbb{E}_k$ is a $(d+k)$–equivalence.

The ∞–categorical Eckmann–Hilton argument is now an immediate consequence of Theorem 5.3.1.

Corollary 5.3.3 *For all integers $d_1, d_2 \geq -2$, given two reduced ∞–operads $\mathcal{P}$ and $\mathcal{Q}$, if $\mathcal{P}$ is d_1–connected and $\mathcal{Q}$ is d_2–connected, then $\mathcal{P} \otimes \mathcal{Q}$ is (d_1+d_2+2)–connected.*

Proof Since $\mathcal{P}$ is d_1–connected, the essentially unique map $\mathcal{P} \to \mathbb{E}_\infty$ is a d_1–equivalence. Hence, by Theorem 5.3.1, the map $\mathcal{P} \otimes \mathcal{Q} \to \mathcal{P} \otimes \mathbb{E}_\infty$ is a (d_1+d_2+2)–equivalence. Since $\mathbb{E}_\infty$ is also d_2–connected, by the same argument the induced map

$$\mathcal{P} \otimes \mathbb{E}_\infty \to \mathbb{E}_\infty \otimes \mathbb{E}_\infty \simeq \mathbb{E}_\infty$$

is also a (d_1+d_2+2)–equivalence. The (d_1+d_2+2)–equivalences are closed under composition, and so the result follows (in fact, we know a posteriori that the map above is actually an equivalence of ∞–operads). □

We conclude this section (and this paper) with a couple of curious applications of the ∞–categorical Eckmann–Hilton argument. The first is the classification of idempotent reduced ∞–operads.

Corollary 5.3.4 *Let $\mathcal{P}$ be a reduced ∞–operad. If $\mathcal{P} \otimes \mathcal{P} \simeq \mathcal{P}$, then $\mathcal{P} \simeq \mathbb{E}_0$ or $\mathcal{P} \simeq \mathbb{E}_\infty$.*

Proof If $\mathcal{P} \not\simeq \mathbb{E}_0$, then, by Remark 3.2.3, $\mathcal{P}$ is d–connected for some $d \geq -1$. Therefore, by Theorem 5.2.2, $\mathcal{P} \otimes \mathcal{P}$ is $(2d+2)$–connected, and $2d+2 > d$. Since

$\mathcal{P} \simeq \mathcal{P} \otimes \mathcal{P}$, we can continue by induction and deduce that $\mathcal{P}$ is ∞–connected; hence $\mathcal{P} \simeq \mathbb{E}_\infty$. □

The second application is to a tensor product of a sequence of reduced ∞–operads. Given a sequence of reduced ∞–operads $(\mathcal{P}_i)_{i=1}^\infty$, we can define the tensor product of them all, $\bigotimes_{i=1}^\infty \mathcal{P}_i$, as the colimit of the sequence

$$\mathbb{E}_0 \to \mathcal{P}_1 \to \mathcal{P}_1 \otimes \mathcal{P}_2 \to \mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \mathcal{P}_3 \to \cdots,$$

where the i^{th} map is obtained by tensoring the essentially unique map $\mathbb{E}_0 \to \mathcal{P}_i$ with $\mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_{i-1}$.

Example 5.3.5 If we take $\mathcal{P}_i = \mathbb{E}_1$ for all i, then the additivity theorem [17, 5.1.2.2] implies that $\bigotimes_{i=1}^\infty \mathbb{E}_1$ is the colimit of the sequence of ∞–operads

$$\mathbb{E}_0 \to \mathbb{E}_1 \to \mathbb{E}_2 \to \mathbb{E}_3 \to \cdots,$$

which is $\mathbb{E}_\infty$.

We offer the following generalization:

Corollary 5.3.6 *Let $(\mathcal{P}_i)_{i=1}^\infty$ be a sequence of reduced ∞–operads not equivalent to $\mathbb{E}_0$. There is an equivalence of ∞–operads $\bigotimes_{i=1}^\infty \mathcal{P}_i \simeq \mathbb{E}_\infty$.*

Proof By Remark 3.2.3, all $\mathcal{P}_i$ are (-1)–connected. By induction on k and Corollary 5.3.3, the ∞–operad $\mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_k$ is $(k-2)$–connected. For every $n \in \mathbb{N}$ we get

$$\left(\bigotimes_{i=1}^\infty \mathcal{P}_i\right)(n) \simeq \operatorname*{colim}_k (\mathcal{P}_1 \otimes \cdots \otimes \mathcal{P}_k)(n) \simeq \mathrm{pt}$$

and therefore $\bigotimes_{i=1}^\infty \mathcal{P}_i \simeq \mathbb{E}_\infty$. □

For example, this implies that putting countably many compatible H–space structures on a pointed connected space X is the same as putting an ∞–loop space structure on X.

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