

Coarse homology theories and finite decomposition complexity

ULRICH BUNKE
ALEXANDER ENGEL
DANIEL KASPROWSKI
CHRISTOPH WINGES

Using the language of coarse homology theories, we provide an axiomatic account of vanishing results for the fibres of forget-control maps associated to spaces with equivariant finite decomposition complexity.

20F69; 20F65

1 Introduction

For a group G we consider the category $G\mathbf{BornCoarse}$ of G -bornological coarse spaces [2, Section 2]. In [2, Sections 3 and 4] we introduced the notion of an equivariant coarse homology theory and constructed the universal equivariant coarse homology theory

$$\mathrm{Yo}^s: G\mathbf{BornCoarse} \rightarrow G\mathbf{Sp}\mathcal{X},$$

whose target is the presentable stable ∞ -category of equivariant coarse motivic spectra. See Section 2A for an overview.

To every G -bornological coarse space X one can functorially associate the motivic forget-control map

$$(1-1) \quad \beta_X: F^\infty(X) \rightarrow \Sigma F^0(X),$$

which is a morphism in $G\mathbf{Sp}\mathcal{X}$ (see [2, Definition 11.10] and (1-3) below). Let \mathcal{C} be a cocomplete stable ∞ -category. If E is a \mathcal{C} -valued equivariant coarse homology theory, then it factors in an essentially unique way over $G\mathbf{Sp}\mathcal{X}$ and we get the induced forget-control map

$$E(\beta_X): E(F^\infty(X)) \rightarrow \Sigma E(F^0(X))$$

for E . The goal of the present paper is to show the following theorem:

Theorem 1.1 Assume:

- (1) E is weakly additive (see Definition 2.23).
- (2) E admits weak transfers (see Definition 2.16).
- (3) \mathcal{C} is compactly generated.
- (4) X has G -finite decomposition complexity (G -FDC) (see Definition 3.14).
- (5) G acts discontinuously on X (see Definition 3.16).

Then the forget-control map for E is an equivalence

$$E(\beta_X): E(F^\infty(X)) \rightarrow \Sigma E(F^0(X)).$$

In the case that X is the group G with the canonical bornological coarse structure and the action by left multiplication (we often denote this object by $G_{\text{can}, \min}$), the forget-control map is closely related to assembly maps which appears in Farrell–Jones-type conjectures; see [2, Section 11.2] for details.

Let $G\mathbf{Orb}$ denote the orbit category of G . A functor $M: G\mathbf{Orb} \rightarrow \mathcal{C}$ is called a CP-functor if it is obtained from an equivariant coarse homology theory with certain additional properties; see our [3, Definition 1.8] for a precise definition. Let \mathbf{FDC} denote the family of subgroups H of G such that $H_{\text{can}, \min}$ has $H_{\mathbf{Fin}}$ -FDC (Definition 3.15), where \mathbf{Fin} is the family of finite subgroups. Theorem 1.1 is used to prove the following result:

Theorem 1.2 [3, Theorem 1.11] Let M be a CP-functor. Assume that G admits a finite-dimensional model for the classifying space of proper actions $\underline{E}G$. Then the relative assembly map

$$\alpha_{\mathbf{Fin}, M}^{\mathbf{FDC}}: \operatorname{colim}_{G_{\mathbf{Fin}}\mathbf{Orb}} M \rightarrow \operatorname{colim}_{G_{\mathbf{FDC}}\mathbf{Orb}} M$$

is split injective.

Examples of CP-functors are algebraic K -theory [3, Example 1.10] and A -theory [5, Theorem 5.17].

In the following we give a more detailed description of Theorem 1.1. In [2, Section 9] we introduced the category of G -uniform bornological coarse space $G\mathbf{UBC}$. This category is related to $G\mathbf{BornCoarse}$ by a forgetful functor

$$\mathcal{F}_U: G\mathbf{UBC} \rightarrow G\mathbf{BornCoarse}$$

(which forgets the uniform structure) and a cone functor

$$\mathcal{O}: G\mathbf{UBC} \rightarrow G\mathbf{BornCoarse}.$$

These functors are connected by a natural transformation $\mathcal{F}_{\mathcal{U}} \rightarrow \mathcal{O}$, whose motivic cofibre

$$\mathcal{O}^{\infty} := \operatorname{Cofib}(\operatorname{Yo}^s \circ \mathcal{F}_{\mathcal{U}} \rightarrow \operatorname{Yo}^s \circ \mathcal{O}) \colon \mathbf{GUBC} \rightarrow \mathbf{GSp}\mathcal{X}$$

is called the germs-at- ∞ functor. By construction we have the cone fibre sequence

(1-2)

$$\operatorname{Yo}^s \circ \mathcal{F}_{\mathcal{U}} \rightarrow \operatorname{Yo}^s \circ \mathcal{O} \rightarrow \mathcal{O}^{\infty} \xrightarrow{\partial} \Sigma \operatorname{Yo}^s \circ \mathcal{F}_{\mathcal{U}}$$

of functors from \mathbf{GUBC} to $\mathbf{GSp}\mathcal{X}$.

The Rips complex construction provides a functor from $\mathbf{GBornCoarse}$ to \mathbf{GUBC} . We let $\mathbf{GBornCoarse}^{\mathcal{C}}$ be the category of pairs (X, U) of a G -bornological space X and an invariant entourage U of X . A morphism $(X, U) \rightarrow (X', U')$ in $\mathbf{GBornCoarse}^{\mathcal{C}}$ is a morphism $f \colon X \rightarrow X'$ in $\mathbf{GBornCoarse}$ such that $(f \times f)(U) \subseteq U'$. Formally, the category $\mathbf{GBornCoarse}^{\mathcal{C}}$ is the Grothendieck construction of the functor $\mathbf{GBornCoarse} \rightarrow \mathbf{Cat}$ which sends a G -bornological coarse space $(X, \mathcal{C}, \mathcal{B})$ to the poset \mathcal{C}^G of invariant entourages. In [2, Section 11.1] or Definition 3.18 here, we introduce the Rips complex functor

$$P \colon \mathbf{GBornCoarse}^{\mathcal{C}} \rightarrow \mathbf{GUBC}, \quad (X, U) \mapsto P_U(X).$$

We form the left Kan-extension of the precomposition of the cone sequence (1-2) with P along the forgetful functor

$$\mathbf{GBornCoarse}^{\mathcal{C}} \rightarrow \mathbf{GBornCoarse}, \quad (X, U) \mapsto X.$$

The result is a fibre sequence

(1-3)

$$F^0 \rightarrow F \rightarrow F^{\infty} \xrightarrow{\beta} \Sigma F^0$$

of functors from $\mathbf{GBornCoarse}$ to $\mathbf{GSp}\mathcal{X}$. If we apply this fibre sequence to a bornological coarse space X , then we get a fibre sequence in $\mathbf{GSp}\mathcal{X}$ whose third morphism is the motivic forget-control map (1-1).

Let X be a G -bornological coarse space and E be an equivariant coarse homology theory. In view of the fibre sequence (1-3), Theorem 1.1 is an immediate consequence of the following theorem:

Theorem 1.3 (Theorem 3.20) *Assume:*

- (1) E is weakly additive.
- (2) E admits weak transfers.

- (3) \mathcal{C} is compactly generated.
- (4) X has G -FDC.
- (5) G acts discontinuously on X .

Then $E(F(X)) \simeq 0$.

Note that the pointwise formula for the left Kan-extension gives

$$E(F(X)) \simeq \operatorname{colim}_{S \in \mathcal{C}^G} E(\mathcal{O}(P_S(X))),$$

whose right-hand side explicitly appears in [Theorem 3.20](#). To deduce [Theorem 1.3](#) from [Theorem 3.20](#), furthermore note that the condition that \mathcal{C} is compactly generated implies that phantom objects ([Definition 2.21](#)) are trivial.

The notion of finite decomposition complexity was introduced by Guentner, Tessera and Yu [\[7\]](#) as a generalization of finite asymptotic dimension. It was used by Guentner, Tessera and Yu [\[6\]](#) to prove instances of the stable Borel conjecture. Subsequently, Ramras, Tessera and Yu [\[10\]](#) employed this notion to prove instances of the K -theoretic Novikov conjecture. Kasprowski [\[8\]](#) introduced G -FDC as an equivariant version of finite decomposition complexity and proved that if X has G -FDC, then the forget-control map induces an equivalence $K\mathcal{A}\mathcal{X}^G(\beta_X)$ in equivariant coarse algebraic K -homology $K\mathcal{A}\mathcal{X}^G$ associated to an additive category \mathcal{A} .

The proofs in [\[10; 8\]](#) use properties of the coarse algebraic K -homology functor $K\mathcal{A}\mathcal{X}$ which go beyond the four general properties of an equivariant coarse homology listed in [Definition 2.10](#). The main new contribution of the present paper is the observation that in addition to the axioms of a coarse homology theory we just need the following two additional properties:

- (1) existence of weak transfers (see [Definition 2.16](#));
- (2) weak additivity (see [Definition 2.23](#)).

We show that equivariant coarse ordinary homology $H\mathcal{X}^G$ and equivariant coarse algebraic K -homology $K\mathcal{A}\mathcal{X}^G$ associated to an additive category \mathcal{A} with G -action both admit weak transfers and are weakly additive.

Acknowledgements

Bunke and Engel were supported by the SFB 1085 *Higher invariants* funded by the Deutsche Forschungsgemeinschaft DFG, and Engel was further supported by

the Research Fellowship EN 1163/1-1 *Mapping analysis to homology* of the DFG. Kasprowski and Winges acknowledge support by the Max Planck Society and the Hausdorff Center for Mathematics at the University of Bonn. Winges was further supported by Wolfgang Lück's ERC Advanced Grant *KL2MG-interactions* (no. 662400). Parts of the work presented here were obtained during the Junior Hausdorff Trimester Program *Topology* at the Hausdorff Research Institute for Mathematics (HIM) in Bonn. We also thank Mark Ullmann for helpful discussions.

2 Equivariant coarse homology theories

2A Basic notions

In this section we recall the definition of the category $G\mathbf{BornCoarse}$ of G -bornological coarse spaces. We further recall the notion of an equivariant coarse homology theory, in particular its universal version Yo^s with values in the stable ∞ -category $G\mathbf{Sp}\mathcal{X}$ of equivariant coarse motivic spectra. Most of this material has been developed in [2] (see also [1] for the nonequivariant case).

In the definitions below we will use the following notation.

- (1) For a set Z we let $\mathcal{P}(Z)$ denote the power set of Z .
- (2) If a group G acts on a set X , then it acts diagonally on $X \times X$ and therefore on $\mathcal{P}(X \times X)$. For U in $\mathcal{P}(X \times X)$ we set

$$GU := \bigcup_{g \in G} gU.$$

- (3) For U in $\mathcal{P}(X \times X)$ and B in $\mathcal{P}(X)$ we define the U -thickening $U[B]$ by

$$(2-1) \quad U[B] := \{x \in X \mid (x, y) \in U \text{ for some } y \in B\}.$$

- (4) For U in $\mathcal{P}(X \times X)$ we define the *inverse* by

$$U^{-1} := \{(y, x) \mid (x, y) \in U\}.$$

- (5) For U and V in $\mathcal{P}(X \times X)$ we define their *composition* by

$$(2-2) \quad U \circ V := \{(x, z) \mid (x, y) \in U \wedge (y, z) \in V \text{ for some } y \in X\}.$$

Let G be a group and let X be a G -set.

Definition 2.1 A G -coarse structure \mathcal{C} on X is a subset of $\mathcal{P}(X \times X)$ with the following properties:

- (1) \mathcal{C} is closed under composition, inversion and forming finite unions or subsets.
- (2) \mathcal{C} contains the diagonal $\text{diag}(X)$ of X .
- (3) For every U in \mathcal{C} , the set GU is also in \mathcal{C} .

The pair (X, \mathcal{C}) is called a G -coarse space, and the members of \mathcal{C} are called (coarse) entourages of X . \triangleleft

Let (X, \mathcal{C}) and (X', \mathcal{C}') be G -coarse spaces and let $f: X \rightarrow X'$ be an equivariant map between the underlying sets.

Definition 2.2 The map f is said to be *controlled* if for every U in \mathcal{C} we have $(f \times f)(U) \in \mathcal{C}'$. \triangleleft

We obtain a category $G\mathbf{Coarse}$ of G -coarse spaces and controlled equivariant maps.

Definition 2.3 A G -bornology \mathcal{B} on X is a subset of $\mathcal{P}(X)$ with the following properties:

- (1) \mathcal{B} is closed under forming finite unions and subsets.
- (2) \mathcal{B} contains all finite subsets of X .
- (3) \mathcal{B} is G -invariant.

The pair (X, \mathcal{B}) is called a G -bornological space, and the members of \mathcal{B} are called bounded subsets of X . \triangleleft

Let (X, \mathcal{B}) and (X', \mathcal{B}') be G -bornological spaces and let $f: X \rightarrow X'$ be an equivariant map between the underlying sets.

Definition 2.4 The map f is *proper* if for every B' in \mathcal{B}' we have $f^{-1}(B') \in \mathcal{B}$. \triangleleft

We obtain a category $G\mathbf{Born}$ of G -bornological spaces and proper equivariant maps.

Let X be a G -set with a G -coarse structure \mathcal{C} and a G -bornology \mathcal{B} .

Definition 2.5 The coarse structure \mathcal{C} and the bornology \mathcal{B} are said to be *compatible* if for every B in \mathcal{B} and U in \mathcal{C} the U -thickening $U[B]$ lies in \mathcal{B} . \triangleleft

Definition 2.6 A G -bornological coarse space is a triple $(X, \mathcal{C}, \mathcal{B})$ consisting of a G -set X , a G -coarse structure \mathcal{C} , and a G -bornology \mathcal{B} such that \mathcal{C} and \mathcal{B} are compatible. \triangleleft

Definition 2.7 A morphism $f: (X, \mathcal{C}, \mathcal{B}) \rightarrow (X', \mathcal{C}', \mathcal{B}')$ between G -bornological coarse spaces is an equivariant map $f: X \rightarrow X'$ of the underlying G -sets which is controlled and proper. \triangleleft

We obtain a category **$G\text{BornCoarse}$** of G -bornological coarse spaces and morphisms. If the structures are clear from the context, we will use the notation X instead of $(X, \mathcal{C}, \mathcal{B})$ in order to denote G -bornological coarse spaces.

We now introduce the notion of an equivariant coarse homology theory; see Section 3 of [2] for details.

Let X be a G -bornological coarse space.

Definition 2.8 An equivariant big family on X is a filtered family of G -invariant subsets $(Y_i)_{i \in I}$ of X such that for every entourage U of X and i in I there exists j in I such that $U[Y_i] \subseteq Y_j$.

An equivariant complementary pair (Z, \mathcal{Y}) on X is a pair of a G -invariant subset Z of X and an equivariant big family $\mathcal{Y} = (Y_i)_{i \in I}$ on X such that there exists i in I with $Z \cup Y_i = X$. \triangleleft

Let X be a G -bornological coarse space.

Definition 2.9 The space X is *flasque* if it admits a morphism $f: X \rightarrow X$ such that:

- (1) f is close to id_X , ie $(f, \text{id}_X)(\text{diag}(X))$ is an entourage of X .
- (2) For every entourage U of X the subset $\bigcup_{n \in \mathbb{N}} (f^n \times f^n)(U)$ is an entourage of X .
- (3) For every bounded subset B of X there is an integer n such that $B \cap f^n(X) = \emptyset$.

We say that flasqueness of X is *implemented* by f . \triangleleft

Let \mathcal{C} be a cocomplete stable ∞ -category and let

$$E: G\text{BornCoarse} \rightarrow \mathcal{C}$$

be a functor. If $\mathcal{Y} = (Y_i)_{i \in I}$ is a filtered family of G -invariant subsets of X , then we set

$$(2-3) \quad E(\mathcal{Y}) := \operatorname{colim}_{i \in I} E(Y_i).$$

In this formula we consider the subsets Y_i as G -bornological coarse spaces with the structures induced from X .

Let \mathcal{C} be a cocomplete stable ∞ -category and consider a functor

$$E: G\mathbf{BornCoarse} \rightarrow \mathcal{C}.$$

Definition 2.10 The functor E is called a G -equivariant \mathcal{C} -valued coarse homology theory if it satisfies the following conditions:

- (1) **Coarse invariance** For all X in $G\mathbf{BornCoarse}$ the functor E sends the projection $\{0, 1\}_{\max, \max} \otimes X \rightarrow X$ to an equivalence.
- (2) **Excision** $E(\emptyset) \simeq 0$ and for every equivariant complementary pair (Z, \mathcal{Y}) on a G -bornological coarse space X the square

$$\begin{array}{ccc} E(Z \cap \mathcal{Y}) & \longrightarrow & E(Z) \\ \downarrow & & \downarrow \\ E(\mathcal{Y}) & \longrightarrow & E(X) \end{array}$$

is a pushout.

- (3) **Flasqueness** If a G -bornological coarse space X is flasque, then $E(X) \simeq 0$.
- (4) **u -Continuity** For every G -bornological coarse space X the natural map

$$\operatorname{colim}_{U \in \mathcal{C}^G} E(X_U) \rightarrow E(X)$$

is an equivalence. Here X_U denotes the G -bornological coarse space X with the coarse structure replaced by the one generated by U .

If the group G is clear from the context, then we will often just speak of an equivariant coarse homology theory. \triangleleft

We have a universal equivariant coarse homology theory

$$\mathbf{Yo}^s: G\mathbf{BornCoarse} \rightarrow G\mathbf{Sp}\mathcal{X}$$

(see [2, Definition 4.9]), where $G\mathbf{Sp}\mathcal{X}$ is a stable presentable ∞ -category called the category of coarse motivic spectra. More precisely, for every cocomplete stable ∞ -category \mathcal{C} we have the following:

Proposition 2.11 [2, Corollary 4.10] *Restriction along Yo^s induces an equivalence between the ∞ -categories of colimit-preserving functors $G\mathbf{Sp}\mathcal{X} \rightarrow \mathcal{C}$ and \mathcal{C} -valued equivariant coarse homology theories.*

The category $G\mathbf{BornCoarse}$ has a symmetric monoidal structure \otimes ; see Example 2.17 of [2]. If X and Y are G -bornological coarse spaces, then $X \otimes Y$ has the following description:

- (1) The underlying G -coarse space of $X \otimes Y$ is the cartesian product in $G\mathbf{Coarse}$ of the underlying G -coarse spaces of X and Y .
- (2) The bornology on $X \otimes Y$ is generated by the products $A \times B$ for all bounded subsets A of X and B of Y .

Note that $X \otimes Y$ in general differs from the cartesian product $X \times Y$ in $G\mathbf{BornCoarse}$. The symmetric monoidal structure \otimes descends to $G\mathbf{Sp}\mathcal{X}$ such that Yo^s becomes a symmetric monoidal functor [2, Lemma 4.17].

Let $E: G\mathbf{BornCoarse} \rightarrow \mathcal{C}$ be a functor and let X be a G -bornological coarse space.

Definition 2.12 The *twist* E_X of E by X is the functor

$$E(X \otimes -): G\mathbf{BornCoarse} \rightarrow \mathcal{C}. \quad \triangleleft$$

Lemma 2.13 *If E is an equivariant coarse homology theory, then the twist E_X is also an equivariant coarse homology theory.*

Proof This follows from [2, Lemma 4.17]. □

2B Weak transfers

We consider a group G . The notion of the free union of a family $(X_i)_{i \in I}$ of G -bornological coarse spaces plays an important role in the present paper.

Definition 2.14 We define the *free union* $\coprod_{i \in I}^{\mathrm{free}} X_i$ in $G\mathbf{BornCoarse}$ as follows:

- (1) The underlying G -set of $\coprod_{i \in I}^{\mathrm{free}} X_i$ is the disjoint union $\bigsqcup_{i \in I} X_i$.

- (2) A set $B \subseteq \coprod_{i \in I}^{\text{free}} X_i$ is bounded if and only if:
- (a) The set $\{i \in I \mid B \cap X_i \neq \emptyset\}$ is finite.
 - (b) The set $B \cap X_i$ is bounded (as a subset of X_i) for all i in I .
- (3) The coarse structure of $\coprod_{i \in I}^{\text{free}} X_i$ is generated by the entourages $\coprod_{i \in I} U_i$ for all families $(U_i)_{i \in I}$, where U_i is an entourage of X_i . \triangleleft

Remark 2.15 Let $(X_i)_{i \in I}$ be a family of G -bornological coarse spaces. For every j in I we have a canonical morphism $X_j \rightarrow \coprod_{i \in I}^{\text{free}} X_i$. These morphisms induce a morphism

$$\coprod_{i \in I} X_i \rightarrow \coprod_{i \in I}^{\text{free}} X_i,$$

where $\coprod_i X_i$ denotes the coproduct of the family $(X_i)_{i \in I}$ in $\mathbf{GBornCoarse}$. The coproduct can be realized so that this morphism is the identity on the underlying sets. In general, it is not an isomorphism since the coarse structure on the free union is larger than the coarse structure on the categorical coproduct in $\mathbf{GBornCoarse}$, while the bornology of the coproduct is larger than that of the free union. \triangleleft

Let $E: \mathbf{GBornCoarse} \rightarrow \mathcal{C}$ be an equivariant \mathcal{C} -valued coarse homology theory.

We consider a family $(X_i)_{i \in I}$ of G -bornological coarse spaces and a point j in I . We set $I_j := I \setminus \{j\}$. Then $(X_j, \coprod_{i \in I_j}^{\text{free}} X_i)$ is a coarsely excisive decomposition (see [2, Definition 4.13]) of $\coprod_{i \in I}^{\text{free}} X_i$. Since E satisfies excision for coarsely excisive decompositions [2, Corollary 4.14] we can define a projection

$$(2-4) \quad p_j: E\left(\coprod_{i \in I}^{\text{free}} X_i\right) \simeq E(X_j) \oplus E\left(\coprod_{i \in I_j}^{\text{free}} X_i\right) \rightarrow E(X_j).$$

Let I be a set and $E: \mathbf{GBornCoarse} \rightarrow \mathcal{C}$ be an equivariant coarse homology theory. Then we define a functor

$$E^I: \mathbf{GBornCoarse} \rightarrow \mathcal{C}, \quad X \mapsto E\left(\coprod_{i \in I}^{\text{free}} X\right).$$

For every j in I the projection (2-4) then provides a natural transformation of functors

$$p_j: E^I \rightarrow E.$$

Definition 2.16 E has *weak transfers* for I if there exists a natural transformation

$$\mathrm{tr}_I\colon E \rightarrow E^I$$

such that

(2-5)

$$p_j \circ \mathrm{tr}_I \simeq \mathrm{id}_E$$

for every j in I . ◁

Example 2.17 Recall from [2, Definition 8.8] that equivariant coarse algebraic K –homology $K\mathcal{A}\mathcal{X}^G$ was constructed by assigning to a G –bornological coarse space X an additive category $V_A^G(X)$ of X –controlled objects and taking K –theory of this category. Let

$$(\widehat{K\mathcal{A}\mathcal{X}^G})^I := K\left(\prod_I V_A^G(X)\right).$$

The functor

(2-6)

$$V_A^G\left(\coprod_I^{\mathrm{free}} X\right) \rightarrow \prod_I V_A^G(X)$$

sending an object M to the family $(M|_{X_i})_{i \in I}$, where X_i is the i^{th} copy of X in the free union, yields a natural transformation

$$\tau\colon (K\mathcal{A}\mathcal{X}^G)^I \rightarrow (\widehat{K\mathcal{A}\mathcal{X}^G})^I.$$

From the definition of the free union, it follows that (2-6) and hence τ are natural equivalences. We then define

$$\mathrm{tr}_I\colon K\mathcal{A}\mathcal{X}^G \xrightarrow{K(\Delta)} (\widehat{K\mathcal{A}\mathcal{X}^G})^I \xrightarrow{\tau^{-1}} (K\mathcal{A}\mathcal{X}^G)^I,$$

where Δ is the diagonal. Property (2-5) for the weak transfer follows from Remark 8.16 of [2]. ◁

Remark 2.18 In [4] we introduce the notion of a coarse homology theory with transfers. More precisely, we extend the category $G\mathbf{BornCoarse}$ to an ∞ –category $G\mathbf{BornCoarse}_{\mathrm{tr}}^Q$ whose additional morphisms encode a more general kind of transfers. In [4, Lemma 2.59] we show that every coarse homology theory with transfers in particular has weak transfers. Furthermore, we construct extensions of equivariant coarse ordinary homology $H\mathcal{X}^G$ and equivariant coarse algebraic K –homology $K\mathcal{A}\mathcal{X}^G$ to coarse homology theories with transfers in [4, Proposition 3.8 and Lemma 3.10]. ◁

2C Weak additivity

Let $E: G\mathbf{BornCoarse} \rightarrow \mathcal{C}$ be an equivariant \mathcal{C} -valued coarse homology theory. The following definition is taken from [2, Definition 3.12]:

Definition 2.19 E is called *strongly additive* if for every family $(X_i)_{i \in I}$ of G -bornological coarse spaces the morphism

$$(2-7) \quad (p_j)_{j \in I}: E\left(\coprod_{i \in I}^{\text{free}} X_i\right) \rightarrow \prod_{j \in I} E(X_j)$$

is an equivalence. ◁

Many examples of equivariant coarse homology theories are strongly additive. The following has been shown in [2]:

Theorem 2.20 (1) [2, Lemma 7.11] *Equivariant coarse ordinary homology $H\mathcal{X}^G$ is strongly additive.*

(2) [2, Proposition 8.19] *Equivariant coarse algebraic K -homology $K\mathcal{A}\mathcal{X}^G$ with coefficients in an additive category \mathcal{A} with a strict G -action is strongly additive.*

The arguments of the present paper use a weaker form of additivity we will now introduce.

Let \mathcal{C} be a stable ∞ -category. In the following an object K of \mathcal{C} is called *compact* if it is ω -compact, ie if the functor $\text{Map}_{\mathcal{C}}(K, -)$ preserves filtered colimits.

Definition 2.21 (1) An object C in \mathcal{C} is called a *phantom object* if $\text{Map}(K, C) \simeq *$ for every compact object K of \mathcal{C} .

(2) A morphism $f: C \rightarrow D$ in \mathcal{C} is called a *phantom monomorphism* if and only if for every compact object K and morphism $\phi: K \rightarrow C$ the condition $f \circ \phi \simeq 0$ implies $\phi \simeq 0$. ◁

Example 2.22 In general, if the fibre of a morphism in \mathcal{C} is a phantom object, then the morphism is a phantom monomorphism. The converse is not true.

If \mathcal{C} is compactly generated, then every phantom object is equivalent to the zero object. This applies eg to the stable ∞ -categories \mathbf{Ch}_{∞} of chain complexes and of spectra \mathbf{Sp} . ◁

Let $E: G\mathbf{BornCoarse} \rightarrow \mathcal{C}$ be an equivariant \mathcal{C} -valued coarse homology theory.

Definition 2.23 E is called *weakly additive* if the morphism

$$(p_i)_{i \in I}: E\left(\coprod_{i \in I}^{\text{free}} X_i\right) \rightarrow \prod_{i \in I} E(X_i)$$

is a phantom monomorphism. ◁

Since every equivalence is a phantom monomorphism it is obvious that a strongly additive equivariant coarse homology theory is weakly additive. Hence we have the following corollary of [Theorem 2.20](#):

Corollary 2.24 (1) *Equivariant coarse ordinary homology $H\mathcal{X}^G$ is weakly additive.*
 (2) *Equivariant coarse algebraic K -homology $KA\mathcal{X}^G$ with coefficients in an additive category \mathcal{A} with a strict G -action is weakly additive.*

3 Finite decomposition complexity

In this section we introduce the notion of G -equivariant finite decomposition complexity (G -FDC) for G -bornological coarse spaces. Finite decomposition complexity for metric spaces was introduced by Guentner, Tessera and Yu [\[6\]](#) and the equivariant version for metric spaces was defined in [\[8\]](#); see also [\[9\]](#). The main theorem of the section is [Theorem 3.20](#), which can be interpreted as a version of [\[10, Proposition 6.6\]](#) for general weakly additive coarse homology theories with weak transfers.

3A Finite decomposition complexity

Let G be a group. Recall that a G -coarse space is a coarse space (X, \mathcal{C}) such that the subset \mathcal{C}^G of invariant entourages is cofinal in \mathcal{C} .

Let T be an invariant entourage of a G -coarse space X . Furthermore, let U and V be G -invariant subsets of X . Recall the notation [\(2-1\)](#).

Definition 3.1 The subsets U and V are said to be T -disjoint if

$$T[U] \cap V = \emptyset \quad \text{and} \quad U \cap T[V] = \emptyset.$$

◁

Let X be a G -set.

Definition 3.2 An *equivariant family* of subsets of X is a family of subsets $(U_i)_{i \in I}$ indexed by a G -set I such that $U_{g(i)} = g(U_i)$ for every i in I and g in G . \triangleleft

Let Y be a G -set, X be a G -coarse space with coarse structure \mathcal{C} and $f: Y \rightarrow X$ be an equivariant map of sets.

Definition 3.3 The induced G -coarse structure $f^{-1}\mathcal{C}$ on Y is the maximal G -coarse structure on Y such that the map f is controlled. \triangleleft

Let X be a G -set and $\mathcal{X} := (X_i)_{i \in I}$ be a partition of X into G -invariant subsets. Then we form the G -invariant entourage

$$U(\mathcal{X}) := \bigsqcup_{i \in I} X_i \times X_i.$$

If \mathcal{C} is a G -coarse structure on X , then we define a new G -coarse structure

$$\mathcal{C}(\mathcal{X}) := \mathcal{C}(\{V \cap U(\mathcal{X}) \mid V \in \mathcal{C}\}).$$

Let X be a G -coarse space with coarse structure \mathcal{C} and $\mathcal{U} := (U_i)_{i \in I}$ be an equivariant family of subsets of X . Then we have a natural map of G -sets $f: \coprod_{i \in I} U_i \rightarrow X$.

Definition 3.4 We define the G -coarse space $\coprod_{i \in I}^{\text{sub}} U_i$ to be the G -set $\coprod_{i \in I} U_i$ with the G -coarse structure $(f^{-1}\mathcal{C})(\mathcal{U})$. \triangleleft

Note that the G -coarse structure $(f^{-1}\mathcal{C})(\mathcal{U})$ is generated by the entourages

$$\text{sub}^{\mathcal{U}}(V) := V \cap U(\mathcal{U})$$

for all entourages V of the induced coarse structure $f^{-1}\mathcal{C}$.

Remark 3.5 The set of entourages $\{\text{sub}^{\mathcal{U}}(V) \mid V \in \mathcal{C}^G\}$ is cofinal in the coarse structure of $\coprod_{i \in I}^{\text{sub}} U_i$. \triangleleft

Remark 3.6 The coarse space $\coprod_{i \in I}^{\text{sub}} U_i$ is a coarsely disjoint union of the subsets U_i , which all have the coarse structure induced from X via the inclusion $U_i \rightarrow X$. But the union is in general not free.

There is a coarse map $\coprod_{i \in I}^{\text{sub}} U_i \rightarrow X$, but the coarse structure on the domain is in general smaller than the one induced from X . \triangleleft

Let X be a G –bornological coarse space and $(U_i)_{i \in I}$ an equivariant family of subsets of X .

Definition 3.7 We say that the family $(U_i)_{i \in I}$ is *nice* if for every invariant entourage S of X containing the diagonal, the natural morphism

$$\coprod_{i \in I}^{\text{sub}} U_i \rightarrow \coprod_{i \in I}^{\text{sub}} S[U_i]$$

is an equivalence of G –bornological coarse spaces. ◁

Remark 3.8 In the nonequivariant case (ie if G is the trivial group) every family of subsets of a coarse space is nice. In contrast, for nontrivial groups the inclusion of an invariant subset of G –coarse space into its thickening need not be a coarse equivalence. ◁

Let X be a G –coarse space and let \mathcal{F} be a class of G –coarse spaces.

Definition 3.9 We say that X is *decomposable over \mathcal{F}* if for every invariant entourage T of X there exist nice equivariant families $(U_i^T)_{i \in I}$ and $(V_j^T)_{j \in J}$ of subsets of X such that:

- (1) Both $(U_i^T)_{i \in I}$ and $(V_j^T)_{j \in J}$ are pairwise T –disjoint.

(2) $X = U^T \cup V^T$ with $U^T := \bigcup_{i \in I} U_i^T$ and $V^T := \bigcup_{j \in J} V_j^T$.

(3) The G –coarse spaces $\coprod_{i \in I}^{\text{sub}} U_i^T$ and $\coprod_{j \in J}^{\text{sub}} V_j^T$ belong to \mathcal{F} .
- ◁

Let \mathcal{F} be a class of G –coarse spaces.

Definition 3.10 We say that the class \mathcal{F} is *closed under decomposition* if every G –coarse space that is decomposable over \mathcal{F} is contained in \mathcal{F} . ◁

Remark 3.11 Recall that the G –coarse structure \mathcal{C} of a G –coarse space X induces a G –invariant equivalence relation $\bigcup_{U \in \mathcal{C}} U$ on X . The equivalence classes for this relation are called coarse components. The group G acts on the set $\pi_0(X)$ of coarse components in the natural way.

Furthermore recall that a subset Y of a coarse space X is called T –bounded for an entourage T of X if $Y \times Y \subseteq T$. ◁

Let X be a G -coarse space.

Definition 3.12 X is called *semibounded* if there exists an invariant entourage T of X such that every coarse component of X is T -bounded. \triangleleft

We denote the class of semibounded G -coarse spaces by \mathcal{B} .

Definition 3.13 Let \mathcal{D} be the smallest class of G -coarse spaces that contains \mathcal{B} and is closed under decomposition. \triangleleft

Let X be a G -coarse space.

Definition 3.14 X has *G -equivariant finite decomposition complexity* (for short G -FDC) if it is contained in \mathcal{D} . \triangleleft

Let \mathbf{F} be a family of subgroups of G and let X be a G -coarse space.

Definition 3.15 X has $G_{\mathbf{F}}$ -equivariant finite decomposition complexity (abbreviated by $G_{\mathbf{F}}$ -FDC) if $S_{\min} \otimes X$ has G -FDC for every G -set S with stabilizers in \mathbf{F} . \triangleleft

Let X be a G -coarse space with coarse structure \mathcal{C} . Then X has a minimal bornology $\mathcal{B}_{\mathcal{C}}$ which is compatible with the coarse structure. It is generated by the subsets $S[x]$ for all x in X and S in \mathcal{C} .

Definition 3.16 The G -action on X is said to be *discontinuous* if for every point x of X and every B in $\mathcal{B}_{\mathcal{C}}$ the intersection $Gx \cap B$ is finite. \triangleleft

3B G -FDC for bornological coarse spaces and the main theorem

Let X be a G -bornological coarse space.

Definition 3.17 We say that X has G -FDC if its underlying G -coarse space has G -FDC. \triangleleft

We refer to [2, Definition 9.9] for the definition of the category of G -uniform bornological coarse spaces.

Let X be a G -bornological coarse space, Y be a G -invariant subset and S be an invariant entourage.

Definition 3.18 We define the G -uniform bornological coarse space $P_S^X(Y)$ as follows:

- (1) The underlying G -set of $P_S^X(Y)$ is the G -set of probability measures on X whose support is finite, S -bounded and contained in Y .
- (2) We consider $P_S^X(X)$ as a G -simplicial complex with the G -equivariant spherical quasimetric. This quasimetric induces the G -uniform structure on the subset $P_S^X(Y)$.
- (3) The quasimetric from (2) also induces the G -coarse structure on the subset $P_S^X(Y)$.
- (4) The bornology of $P_S^X(X)$ is generated by the subsets $P_S^X(B)$ for all bounded subsets B of X . It induces the bornology of $P_S^X(Y)$.

We call $P_S^X(Y)$ the *Rips complex*. ◁

In order to abbreviate the notation we will write

$$P_S(X) := P_S^X(X).$$

Remark 3.19 Let X be a G -bornological coarse space. If Y is a G -invariant subset, then we will use the notation Y also for the G -bornological coarse space obtained by equipping this subset with the coarse structure and the bornological structure induced from X . If we want to underline that the structures come from X , then we also use the more precise notation Y_X for this G -bornological coarse space.

If S is an invariant entourage of X and we set

$$(3-1) \quad S_Y := (Y \times Y) \cap S,$$

then we have a natural morphism of G -uniform bornological coarse spaces

$$P_{S_Y}(Y) \rightarrow P_S^X(Y).$$

It is an isomorphism of the underlying bornological spaces. But the quasimetric on the left-hand side might be larger than the quasimetric on the right-hand side. ◁

The following is the main theorem of the present section. Let X be a G -bornological coarse space with coarse structure \mathcal{C} and let E be a weakly additive coarse homology theory with weak transfers.

Theorem 3.20 If X has G –FDC and a discontinuous G –action, then

$$\operatorname{colim}_{S \in \mathcal{C}^G} E(\mathcal{O}(P_S(X)))$$

is a phantom object.

Example 3.21 In view of [Example 2.17](#), [Remark 2.18](#) and [Theorem 2.20](#), [Theorem 3.20](#) applies to equivariant coarse ordinary homology $H\mathcal{X}^G$ and to equivariant coarse algebraic K –homology $K\mathcal{A}\mathcal{X}^G$ with coefficients in an additive category \mathcal{A} with a strict G –action. ◁

The remaining part of this section is devoted to the proof of [Theorem 3.20](#). We fix once and for all a \mathcal{C} –valued weakly additive (see [Definition 2.23](#)) coarse homology theory E with weak transfers (see [Definition 2.16](#)).

3C Structure of the proof

Let X be a G –bornological coarse space with G –coarse structure \mathcal{C} .

Definition 3.22 We call X E –vanishing if for every G –invariant subset Y of X ,

$$\operatorname{colim}_{S \in \mathcal{C}^G} E(\mathcal{O}(P_{S_Y}(Y)))$$

is a phantom object of \mathcal{C} . ◁

Let \mathcal{V}_E denote the class of E –vanishing G –bornological coarse spaces.

Remark 3.23 Let Y be a G –invariant subset of X and let $\mathcal{C}(Y)$ denote the coarse structure of Y_X . Then we have an equality of sets $\{S_Y \mid S \in \mathcal{C}^G\} = \mathcal{C}^G(Y)$ (see [\(3-1\)](#) for notation) of entourages of Y . Hence we have an equivalence of objects of \mathcal{C} ,

$$\operatorname{colim}_{S \in \mathcal{C}^G} E(\mathcal{O}(P_{S_Y}(Y))) \simeq \operatorname{colim}_{T \in \mathcal{C}^G(Y)} E(\mathcal{O}(P_T(Y))).$$

We could define a notion of a *weakly E –vanishing* G –bornological coarse space X by just requiring that

$$\operatorname{colim}_{S \in \mathcal{C}^G} E(\mathcal{O}(P_S(X)))$$

be a phantom object. Then X is E –vanishing if and only if all its G –invariant subsets are weakly E –vanishing.

The motivation to define the notion of E –vanishing G –bornological coarse spaces as above is that it better suits the induction arguments below. ◁

Theorem 3.20 follows from the next theorem. Since every equivariant subspace of a G –bornological coarse space with G –FDC has again G –FDC, both theorems are actually equivalent.

Theorem 3.24 *The class of G –bornological coarse spaces with G –FDC and discontinuous G –action is contained in \mathcal{V}_E .*

To prove **Theorem 3.24** it suffices to show that the class \mathcal{V}_E contains all semibounded G –bornological coarse spaces with discontinuous G –action (**Proposition 3.56**) and is closed under decomposition. We will proceed as follows:

In **Definition 3.35** we introduce the notion of an E –vanishing sequence of G –bornological coarse spaces. Furthermore, in **Definition 3.42** we define the concept of decomposability of sequences of G –bornological coarse spaces. We let \mathcal{VS}_E denote the class of E –vanishing sequences. The main steps of the proof are now as follows:

- (1) If a sequence of G –bornological coarse spaces is decomposable over \mathcal{VS}_E , then the sequence belongs to \mathcal{VS}_E (**Theorem 3.49**).
- (2) If a G –bornological coarse space X is decomposable over \mathcal{V}_E , then the constant sequence \underline{X} is decomposable over \mathcal{VS}_E (**Corollary 3.44**).
- (3) By (1) and (2), if X is decomposable over \mathcal{V}_E , then \underline{X} belongs to \mathcal{VS}_E .
- (4) If X is a G –bornological coarse space and \underline{X} belongs to \mathcal{VS}_E , then X belongs to \mathcal{V}_E (**Lemma 3.37**).
- (5) By (3) and (4), we can conclude that if X is decomposable over \mathcal{V}_E , then it belongs to \mathcal{V}_E .

3D Vanishing pairs and strong decomposability

Let $[0, \infty)_d$ denote the positive ray in \mathbb{R} considered as a G –bornological coarse space with the trivial G –action and the bornological coarse structure induced from the standard metric.

Let X be a G –uniform bornological coarse space with coarse structure \mathcal{C} . Recall that the cone $\mathcal{O}(X)$ is obtained from the G –bornological coarse space $[0, \infty)_d \otimes X$ by replacing the coarse structure \mathcal{C}_\otimes of this tensor product by the hybrid structure \mathcal{C}_h . In particular we have $\mathcal{C}_h \subseteq \mathcal{C}_\otimes$. It follows that the projection $\text{pr}: [0, \infty) \times X \rightarrow X$ is a morphism $([0, \infty) \times X, \mathcal{C}_h) \rightarrow (X, \mathcal{C})$ in the category of G –coarse spaces.

For t in $[0, \infty)$ and x in X we will denote the corresponding point in $\mathcal{O}(X)$ by (t, x) .

Let X be a G -uniform bornological coarse space whose coarse structure is induced by a metric d . We set

$$U_r := \{(x, y) \in X \times X \mid d(x, y) \leq r\}.$$

Let R be an entourage of $\mathcal{O}(X)$.

Definition 3.25 We define the *propagation* $l(R)$ in $[0, \infty)$ by

$$l(R) := \inf\{r \in [0, \infty] \mid (\text{pr} \times \text{pr})(R) \subseteq U_r\}. \quad \triangleleft$$

We consider a family $(X_i)_{i \in I}$ of G -uniform bornological coarse spaces such that the coarse structure of X_i is induced by an invariant metric. Here we consider the metric as specified, but we will not introduce special notation for it. Recall that every entourage R of the free union $\coprod_{i \in I}^{\text{free}} \mathcal{O}(X_i)$ is given by $\coprod_{i \in I} R_i$ for a uniquely determined family $(R_i)_{i \in I}$, where R_i is an entourage of X_i . We define the *propagation* of R to be the element

$$r(R) := \sup_{i \in I} l(R_i)$$

in $[0, \infty]$.

Definition 3.26 For r in $(0, \infty)$ we define $\coprod_{i \in I}^{\text{semi}(r)} \mathcal{O}(X_i)$ to be the G -bornological coarse space given as follows:

- (1) The underlying G -bornological space is the one of $\coprod_{i \in I}^{\text{free}} \mathcal{O}(X_i)$.
- (2) The G -coarse structure is generated by the entourages R of $\coprod_{i \in I}^{\text{free}} \mathcal{O}(X_i)$ with propagation satisfying $r(R) \leq r$.

We further set

$$\coprod_{i \in I}^{\text{semi}} \mathcal{O}(X_i) := \operatorname{colim}_{r \in (0, \infty)} \coprod_{i \in I}^{\text{semi}(r)} \mathcal{O}(X_i). \quad \triangleleft$$

Note that the structure maps of the colimit above are given by the identity of the underlying set.

Remark 3.27 In other words, the coarse structure of $\coprod_{i \in I}^{\text{semi}} \mathcal{O}(X_i)$ is generated by those entourages of the free union which in addition have finite propagation. \triangleleft

We have canonical morphisms

$$\coprod_{i \in I}^{\text{semi}(r)} \mathcal{O}(X_i) \rightarrow \coprod_{i \in I}^{\text{semi}} \mathcal{O}(X_i) \rightarrow \coprod_{i \in I}^{\text{free}} \mathcal{O}(X_i).$$

Example 3.28 Let X be a G -bornological coarse space, Y be an invariant subset of X and S be an invariant entourage of X . By Definition 3.18, the Rips complex $P_S^X(Y)$ is a G -uniform bornological coarse space with a specified invariant metric. This is our main source of examples. \triangleleft

In the following we will use the abbreviated notation (X_n) for sequences $(X_n)_{n \in \mathbb{N}}$ of bornological coarse spaces indexed by \mathbb{N} . If not said differently, by \mathcal{C}_n we denote the coarse structure of X_n .

Let (X_n) be a sequence of G -bornological coarse spaces and let (Y_n) be a sequence of G -invariant subsets, ie we have $Y_n \subseteq X_n$. Let \mathcal{C}_n^G denote the partially ordered set of G -invariant entourages of X_n and consider a family (S_n) in $\prod_{n \in \mathbb{N}} \mathcal{C}_n^G$.

Lemma 3.29 For d in \mathbb{N} the inclusion of the underlying G -bornological spaces defines a morphism of G -bornological coarse spaces

$$(3-2) \quad \coprod_{n \in \mathbb{N}}^{\text{semi}(d)} \mathcal{O}(P_{S_n}^{X_n}(Y_n)) \rightarrow \coprod_{n \in \mathbb{N}}^{\text{semi}} \mathcal{O}(P_{(S_n^d)_{Y_n}}(Y_n)).$$

Furthermore, the canonical map

$$(3-3) \quad \text{colim}_{(S_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}_n^G} E \left(\coprod_{n \in \mathbb{N}}^{\text{semi}} \mathcal{O}(P_{(S_n)_{Y_n}}(Y_n)) \right) \rightarrow \text{colim}_{(S_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}_n^G} E \left(\coprod_{n \in \mathbb{N}}^{\text{semi}} \mathcal{O}(P_{S_n}^{X_n}(Y_n)) \right)$$

is an equivalence.

Proof Assume that the sequence (R_n) defines an entourage of the free union

$$\coprod_{n \in \mathbb{N}}^{\text{free}} \mathcal{O}(P_{S_n}^{X_n}(Y_n))$$

with propagation bounded by d . Fix a natural number n and consider (t, y) and (t', y') in $\mathcal{O}(P_{S_n}^{X_n}(Y_n))$. If $((t, y), (t', y'))$ belongs to R_n , then the distance between y and y' in $P_{S_n}^{X_n}(Y_n)$ is bounded by d . There exists a family of points $(x_i)_{i=0}^d$ in X_n

such that x_0 is a vertex of the simplex containing y and x_d is a vertex of a simplex containing y' , and for every i in $\{0, \dots, d-1\}$ we have $(x_i, x_{i+1}) \in S_n$. Then x_0 and x_n belong to Y_n , and the distance between x_0 and x_d in $P_{(S_n^d)_{Y_n}}(Y_n)$ is bounded by 1. Consequently, the distance between y and y' in $P_{(S_n^d)_{Y_n}}(Y_n)$ is bounded by 3.

By u -continuity we have

$$E\left(\coprod_{n \in \mathbb{N}}^{\text{semi}} \mathcal{O}(P_{S_n}^{X_n}(Y_n))\right) \simeq \operatorname{colim}_{d \in \mathbb{N}} E\left(\coprod_{n \in \mathbb{N}}^{\text{semi}(d)} \mathcal{O}(P_{S_n}^{X_n}(Y_n))\right)$$

and hence the maps from (3-2) induce an inverse to (3-3), as claimed. \square

Remark 3.30 The proof shows that we actually have a morphism of G -bornological coarse spaces

$$\coprod_{n \in \mathbb{N}}^{\text{semi}(d)} \mathcal{O}(P_{S_n}^{X_n}(Y_n)) \rightarrow \coprod_{n \in \mathbb{N}}^{\text{semi}(3)} \mathcal{O}(P_{(S_n^d)_{Y_n}}(Y_n)). \quad \triangleleft$$

Let (X_n) be a sequence of G -bornological coarse spaces and (S_n) and (S'_n) be families in $\prod_{n \in \mathbb{N}} \mathcal{C}_n^G$ such that $(S_n) \leq (S'_n)$. Then we have a commuting square of morphisms of G -bornological coarse spaces,

$$\begin{array}{ccc} \coprod_{n \in \mathbb{N}}^{\text{semi}} \mathcal{O}(P_{S_n}(X_n)) & \longrightarrow & \coprod_{n \in \mathbb{N}}^{\text{free}} \mathcal{O}(P_{S_n}(X_n)) \\ \downarrow & & \downarrow \\ \coprod_{n \in \mathbb{N}}^{\text{semi}} \mathcal{O}(P_{S'_n}(X_n)) & \longrightarrow & \coprod_{n \in \mathbb{N}}^{\text{free}} \mathcal{O}(P_{S'_n}(X_n)) \end{array}$$

In the following lemma we consider the colimit in the vertical direction.

Lemma 3.31 *We have an equivalence*

$$\operatorname{colim}_{(S_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}_n^G} E\left(\coprod_{n \in \mathbb{N}}^{\text{semi}} \mathcal{O}(P_{S_n}(X_n))\right) \rightarrow \operatorname{colim}_{(S_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}_n^G} E\left(\coprod_{n \in \mathbb{N}}^{\text{free}} \mathcal{O}(P_{S_n}(X_n))\right).$$

Proof We produce an inverse equivalence. Let R be an entourage of the free union associated to the family (R_n) . Then by the same argument as in the proof of Lemma 3.29

the inclusion of the underlying sets induces a morphism of bornological coarse spaces,

$$\left(\coprod_{n \in \mathbb{N}}^{\text{free}} \mathcal{O}(P_{S_n}(X_n)) \right)_R \rightarrow \coprod_{n \in \mathbb{N}}^{\text{semi}(3)} \mathcal{O}(P_{S_n^{l(R_n)}}(X_n)).$$

By u -continuity of E applied to the colimit over the entourages R of the free union we get the desired inverse. \square

Let (X_n) be a sequence of G -bornological coarse spaces and let (Y_n) be a sequence of G -invariant subsets. Then we consider the object $C^\infty((Y_n))$ of \mathcal{C} given by

$$(3-4) \quad C^\infty((Y_n)) := \operatorname{colim}_{N \in \mathbb{N}} \operatorname{colim}_{(S_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}_n^G} E \left(\coprod_{N \leq n}^{\text{semi}} \mathcal{O}(P_{(S_n)_{Y_n}}(Y_n)) \right).$$

Remark 3.32 The connecting map for the outer colimit involves the projection

$$E \left(\coprod_{N \leq n}^{\text{semi}} \mathcal{O}(P_{(S_n)_{Y_n}}(Y_n)) \right) \rightarrow E \left(\coprod_{N+1 \leq n}^{\text{semi}} \mathcal{O}(P_{(S_n)_{Y_n}}(Y_n)) \right)$$

given by excision for E along the coarsely excisive pair

$$\left(\mathcal{O}(P_{(S_N)_{Y_N}}(Y_N)), \coprod_{N+1 \leq n}^{\text{semi}} \mathcal{O}(P_{(S_n)_{Y_n}}(Y_n)) \right). \quad \triangleleft$$

Remark 3.33 In view of the first part of [Remark 3.23](#) we could rewrite the definition of $C^\infty((Y_n))$ as

$$C^\infty((Y_n)) := \operatorname{colim}_{N \in \mathbb{N}} \operatorname{colim}_{(T_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}_n^G(Y_n)} E \left(\coprod_{N \leq n}^{\text{semi}} \mathcal{O}(P_{T_n}(Y_n)) \right).$$

In particular, the object $C^\infty((Y_n))$ of \mathcal{C} is an intrinsic invariant of the sequence of G -bornological coarse spaces (Y_n) . \triangleleft

We furthermore define the free version

$$C_{\text{free}}^\infty((Y_n)) := \operatorname{colim}_{N \in \mathbb{N}} \operatorname{colim}_{(T_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}^G(Y_n)} E \left(\coprod_{N \leq n}^{\text{free}} \mathcal{O}(P_{T_n}(Y_n)) \right).$$

Then [Lemma 3.31](#) has the following consequence:

Corollary 3.34 *The natural morphism $C^\infty((Y_n)) \rightarrow C_{\text{free}}^\infty((Y_n))$ is an equivalence.*

Let (X_n) be a sequence of G -bornological coarse spaces.

Definition 3.35 The sequence (X_n) is called E -vanishing if for every sequence (Y_n) of G -invariant subsets the object $\mathcal{C}^\infty((Y_n))$ is a phantom object. \triangleleft

We let \mathcal{VS}_E denote the class of E -vanishing sequences.

Let (X_n) and (Y_n) be sequences of G -bornological coarse spaces.

Lemma 3.36 Assume that for every natural number n the bornological coarse space X_n is equivalent to Y_n . Then the sequence (X_n) is E -vanishing if and only if the sequence (Y_n) is E -vanishing.

Proof By the symmetry of the assertion it suffices to show that if (Y_n) is E -vanishing, then also (X_n) is E -vanishing.

By assumption, for every natural number n we can find morphisms of G -bornological coarse spaces $f_n: X_n \rightarrow Y_n$ and $g_n: Y_n \rightarrow X_n$ and entourages U_n of X_n and V_n of Y_n such that $g_n \circ f_n$ is U_n -close to id_{X_n} and $f_n \circ g_n$ is V_n -close to id_{Y_n} .

Assume that S_n in \mathcal{C}_n^G is given. If we choose S'_n in $\mathcal{C}^G(X_n)$ such that

$$((g_n \circ f_n) \times (g_n \circ f_n))(S_n) \cup U_n \subseteq S'_n,$$

then $g_n \circ f_n$ induces a map $P_{S_n}(X_n) \rightarrow P_{S'_n}(X_n)$ which is homotopic to id_{X_n} by a unit-speed homotopy. We have a similar statement for the composition $f_n \circ g_n$. By homotopy invariance of the cone functor [2, Corollary 9.38] and a cofinality consideration, this implies that f_n induces an equivalence

$$(3-5) \quad \text{colim}_{S \in \mathcal{C}_n^G} E(\mathcal{O}(P_S(X_n))) \simeq \text{colim}_{T \in \mathcal{C}^G(Y_n)} E(\mathcal{O}(P_T(Y_n)))$$

in \mathcal{C} for every natural number n .

We assume that (Y_n) is E -vanishing. Then, by Definition 3.35, the object $\mathcal{C}^\infty((Y_n))$ is a phantom object. We show that $\mathcal{C}^\infty((X_n))$ is a phantom object, too.

Let K be a compact object in \mathcal{C} and $\phi: K \rightarrow \mathcal{C}^\infty((X_n))$ be a morphism. We must show that ϕ is equivalent to zero. By Corollary 3.34 we can replace \mathcal{C}^∞ by $\mathcal{C}_{\text{free}}^\infty$.

By compactness of K there is a factorization $\tilde{\phi}$ as in the diagram

$$\begin{array}{ccccc} & E\left(\coprod_{N\leq n}^{\text{free}} \mathcal{O}(P_{S_n}(X_n))\right) & \dashrightarrow & E\left(\coprod_{N\leq n}^{\text{free}} \mathcal{O}(P_{T_n}(Y_n))\right) & \\ & \uparrow \tilde{\phi} & \nearrow !! & \searrow & \\ K & \xrightarrow{\phi} \mathbf{C}_{\text{free}}^{\infty}((X_n)) & & \prod_{N\leq n} E(\mathcal{O}(P_{S_n}(X_n))) & \dashrightarrow \prod_{N\leq n} E(\mathcal{O}(P_{T_n}(Y_n))) \\ & \searrow \phi & \nwarrow \text{!} & & \end{array}$$

If we choose the sequence of entourages (T_n) sufficiently large (depending on the choice of (S_n)), then the dashed arrows exist. Note that by [Corollary 3.34](#) and our assumption we know that $\mathbf{C}_{\text{free}}^{\infty}((Y_n))$ is a phantom object. Since K is compact, after increasing N further and choosing (T_n) sufficiently large the dotted arrow becomes equivalent to zero. Now we use [\(3-5\)](#) again in order to see that if we choose (S_n) and (T_n) sufficiently large, then the arrow marked by $!$ is equivalent to zero. But then $\tilde{\phi}$ is equivalent to zero since we assume that E is weakly additive and hence the arrow marked by $!!$ is a phantom monomorphism. This finally implies that ϕ is equivalent to zero. \square

If X is a G –bornological coarse space, then we can consider the constant sequence \underline{X} of G –bornological spaces indexed by \mathbb{N} .

Let X be a G –bornological coarse space with coarse structure \mathcal{C} .

Lemma 3.37 *If \underline{X} is an E –vanishing sequence, then X is E –vanishing.*

Proof We must show that for every G –invariant subset Y of X the object

(3-6)
$$\operatorname{colim}_{S \in \mathcal{C}^G} E(\mathcal{O}(P_{S_Y}(Y)))$$

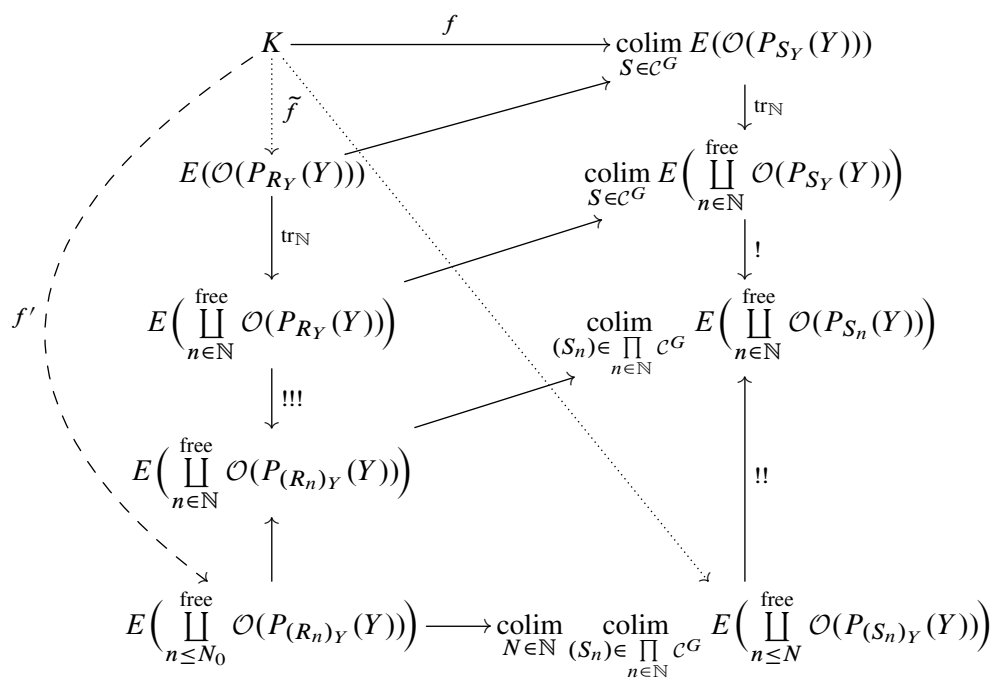
is a phantom object.

Let K be a compact object of \mathbf{C} and consider a morphism

$$f: K \rightarrow \operatorname{colim}_{S \in \mathcal{C}^G} E(\mathcal{O}(P_{S_Y}(Y))).$$

We must show that f is equivalent to zero.

In the following we build step by step the diagram



First of all, by compactness of K there exists an invariant entourage R of X such that we have the factorization \tilde{f} . The next square expresses the naturality of the transfer $\operatorname{tr}_{\mathbb{N}}$ (see [Definition 2.16](#)). The map marked by $!$ is the canonical map induced by the inclusion of the index set of the colimit in the domain into the index set of the colimit of its target.

By [Corollary 3.34](#), for every G –invariant subset Y of X we have an equivalence

(3-7)
$$C^\infty(\underline{Y}) \simeq C^\infty_{\text{free}}(\underline{Y}).$$

By assumption, $C^\infty(\underline{Y})$, and hence by (3-7) also $C^\infty_{\text{free}}(\underline{Y})$, is a phantom object. Since the latter is the cofibre of the lower right vertical map marked by $!!$, and since K is compact, we get the diagonal dotted factorization of the composition $!! \circ \operatorname{tr}_{\mathbb{N}} \circ f$. Using again compactness of K we can choose N_0 in \mathbb{N} and (R_n) in $\prod_{N_0 \leq n} \mathcal{C}^G$ sufficiently large that the factorization f' exists and the diagram commutes.

We now consider the projection (see (2-4))

$$p: E\left(\coprod_{n \in \mathbb{N}}^{\text{free}} \mathcal{O}(P_{(R_n)_Y}(Y))\right) \rightarrow E(\mathcal{O}(P_{(R_{N_0+1})_Y}(Y)))$$

arising from excision and the inclusion of the summand with index $N_0 + 1$. The composition of f' with the canonical map

$$E\left(\coprod_{n \leq N_0}^{\text{free}} \mathcal{O}(P_{R_Y}(Y))\right) \rightarrow E\left(\coprod_{n \in \mathbb{N}}^{\text{free}} \mathcal{O}(P_{R_Y}(Y))\right)$$

and p is equivalent to zero. Hence the composition $p \circ \text{tr}_{\mathbb{N}} \circ \tilde{f}$ is equivalent to zero. On the other hand, by property (2-5) of the transfer, the latter is equivalent to

$$K \xrightarrow{\tilde{f}} E(\mathcal{O}(P_{R_Y}(Y))) \rightarrow E(\mathcal{O}(P_{R_{N_0+1}}(Y))).$$

This implies that f is equivalent to zero. □

We now consider a sequence (X_n) of G -bornological coarse spaces.

Lemma 3.38 *If the G -bornological coarse space X_n is E -vanishing for every n in \mathbb{N} , then the sequence of G -bornological coarse spaces (X_n) is E -vanishing.*

Proof We consider a sequence (Y_n) of invariant subspaces of (X_n) . We must show that $C^\infty((Y_n))$ is a phantom object.

By Corollary 3.34 it suffices to show that $C^\infty_{\text{free}}((Y_n))$ is a phantom object.

Let K be a compact object of \mathcal{C} and

$$f: K \rightarrow \operatorname{colim}_{N \in \mathbb{N}} \operatorname{colim}_{(S_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}^G(X_n)} E\left(\coprod_{N \leq n}^{\text{free}} \mathcal{O}(P_{(S_n)_{Y_n}}(Y_n))\right)$$

be some morphism. We must show that f is equivalent to zero.

Since K is compact there exists a factorization

$$\begin{array}{ccc} K & \xrightarrow{f} & \operatorname{colim}_{N \in \mathbb{N}} \operatorname{colim}_{(S_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}^G(X_n)} E\left(\coprod_{N \leq n}^{\text{free}} \mathcal{O}(P_{(S_n)_{Y_n}}(Y_n))\right) \\ & \searrow \tilde{f} & \nearrow \\ & E\left(\coprod_{N_0 \leq n}^{\text{free}} \mathcal{O}(P_{(R_n)_{Y_n}}(Y_n))\right) & \end{array}$$

for some sufficiently large choices of N_0 in \mathbb{N} and (R_n) in $\prod_{n \in \mathbb{N}} \mathcal{C}^G(X_n)$.

By our assumption on the G -bornological coarse spaces X_k for each k in \mathbb{N} with $N_0 \leq k$ there exists R'_k in $\mathcal{C}^G(X_k)$ such that the composition of \tilde{f} with the projection

$$E\left(\coprod_{N_0 \leq n}^{\text{free}} \mathcal{O}(P_{(R_n)_Y}(Y_n))\right) \rightarrow E(\mathcal{O}(P_{(R_k)_Y}(Y_k))) \rightarrow E(\mathcal{O}(P_{(R'_k)_Y}(Y_k)))$$

is equivalent to zero, where the first morphism is the projection onto the k^{th} summand. By weak additivity of E , it follows that the map

$$K \xrightarrow{\tilde{f}} E\left(\coprod_{N_0 \leq n}^{\text{free}} \mathcal{O}(P_{(R_n)_Y}(Y_n))\right) \rightarrow E\left(\coprod_{N_0 \leq n}^{\text{free}} \mathcal{O}(P_{(R'_n)_Y}(Y_n))\right)$$

is also trivial. This implies that f is equivalent to zero. \square

Let X be a G -bornological space with bornology \mathcal{B} , Y be a G -set and $f: Y \rightarrow X$ be an equivariant map of sets.

Definition 3.39 The *induced bornology* $f^{-1}\mathcal{B}$ on Y is defined to be the minimal bornology on Y such that the map f is proper. \triangleleft

Let X be a G -bornological coarse space and $(U_i)_{i \in I}$ be an equivariant family of subsets; see [Definition 3.2](#). The following generalizes [Definition 3.4](#) to the bornological coarse case.

Definition 3.40 We define the G -bornological coarse space $\coprod_{i \in I}^{\text{sub}} U_i$ as follows:

- (1) The underlying G -set is $\coprod_{i \in I} U_i$.
- (2) The coarse structure is the one defined in [Definition 3.4](#).
- (3) The bornology is induced from X via the canonical map $\coprod_{i \in I} U_i \rightarrow X$. \triangleleft

Remark 3.41 In the situation of [Definition 3.40](#) we have a morphism of G -bornological coarse spaces $\coprod_{i \in I}^{\text{sub}} U_i \rightarrow X$. The bornology on the domain of that map is induced from X , but the coarse structure is in general smaller than the induced coarse structure. \triangleleft

We consider a class \mathcal{FS} of sequences of G -bornological coarse spaces. Let (X_n) be a sequence of G -bornological coarse spaces.

Definition 3.42 The sequence (X_n) is *decomposable over \mathcal{FS}* if for every sequence (T_n) in $\prod_{n \in \mathbb{N}} \mathcal{C}_n^G$ and for all natural numbers n there exist pairwise T_n -disjoint and nice equivariant families $\mathcal{U}^{T_n} := (U_i^{T_n})_{i \in I_n}$ and $\mathcal{V}^{T_n} := (V_j^{T_n})_{j \in J_n}$ of subsets of X such that:

- (1) $X_n = U^{T_n} \cup V^{T_n}$ with $U^{T_n} := \bigcup_{i \in I_n} U_i^{T_n}$ and $V^{T_n} := \bigcup_{j \in J_n} V_j^{T_n}$.
- (2) The sequences of G -bornological coarse spaces

$$\left(\prod_{i \in I_n}^{\text{sub}} U_i^{T_n} \right), \quad \left(\prod_{j \in J_n}^{\text{sub}} V_j^{T_n} \right)$$

belong to \mathcal{FS} . ◁

Let (X_n) be a sequence of G -bornological coarse spaces and (Y_n) be a sequence of subspaces.

Lemma 3.43 *If \mathcal{FS} is closed under taking sequences of subspaces and the sequence (X_n) is decomposable over \mathcal{FS} , then (Y_n) is decomposable over \mathcal{FS} .*

Proof We use the notation appearing in Definition 3.42. For every natural number n we can interpret the invariant entourages T_n of Y_n as invariant entourages of X_n . We set

$$U_{Y,i}^{T_n} := U_i^{T_n} \cap Y_n, \quad V_{Y,j}^{T_n} := V_j^{T_n} \cap Y.$$

In the following we use the notation $\mathcal{U}_Y^{T_n} := (U_{Y,i}^{T_n})_{i \in I}$. We observe by an inspection of the definitions that we have an isomorphism

$$\prod_{i \in I_n}^{\text{sub}_Y} U_{Y,i}^{T_n} \cong \prod_{i \in I_n}^{\text{sub}_X} U_{Y,i}^{T_n}$$

of G -bornological coarse spaces. Here the subscript Y or X at the sub-symbol indicates that the coarse structures are generated by the entourages $\text{sub}^{\mathcal{U}_Y^{T_n}}((S_n)_Y)$ (or $\text{sub}^{\mathcal{U}_Y^{T_n}}(S_n)$, respectively) for all entourages S_n of X_n , and that the bornologies are induced from the bornologies of Y_n (or X_n , respectively). We conclude that $(\prod_{i \in I_n}^{\text{sub}_Y} U_{Y,i}^{T_n})$ is a sequence of subspaces of the sequence $(\prod_{i \in I_n}^{\text{sub}_X} U_i^{T_n})$ and hence belongs to \mathcal{FS} by assumption.

Similar reasoning applies to the V -sequences. ◻

Recall that \mathcal{V}_E and \mathcal{VS}_E denote the classes of E -vanishing G -bornological coarse spaces (Definition 3.22) and of E -vanishing sequences of G -bornological coarse spaces (Definition 3.35).

Let X be a G -bornological coarse space with coarse structure \mathcal{C} , and let \underline{X} be the corresponding constant sequence of G -coarse spaces.

Corollary 3.44 *If X is decomposable over \mathcal{V}_E , then the constant sequence \underline{X} is decomposable over \mathcal{VS}_E .*

Proof Let (T_n) in $\prod_{n \in \mathbb{N}} \mathcal{C}^G$ be given. Since X is decomposable over \mathcal{V}_E , for each natural number n there exist pairwise T_n -disjoint nice equivariant families $(U_i^{T_n})_{i \in I_n}$ and $(V_j^{T_n})_{j \in J_n}$ of subsets of X such that $X = U^{T_n} \cup V^{T_n}$ with

$$U^{T_n} := \bigcup_{i \in I_n} U_i^{T_n}, \quad V^{T_n} := \bigcup_{j \in J_n} V_j^{T_n}$$

and such that the G -bornological coarse spaces $\coprod_{i \in I_n}^{\text{sub}} U_i^{T_n}$ and $\coprod_{j \in J_n}^{\text{sub}} V_j^{T_n}$ are E -vanishing. By Lemma 3.38, the sequences of G -bornological coarse spaces $(\coprod_{i \in I_n}^{\text{sub}} U_i^{T_n})$ and $(\coprod_{j \in J_n}^{\text{sub}} V_j^{T_n})$ are E -vanishing sequences. We therefore conclude that the sequence \underline{X} is decomposable over \mathcal{VS}_E . \square

3E Properties of the Rips complex

Remark 3.45 As a preparation for what follows we recall the following conventions.

If Y is a G -set with a G -invariant quasimetric d , then for every r in \mathbb{R} we define the invariant entourage

$$U_r := \{(x, y) \in Y \times Y \mid d(x, y) \leq r\}.$$

The G -coarse structure \mathcal{C}_d on Y induced by the metric is the coarse structure generated by the entourages U_r for all r in \mathbb{R} .

All this applies in particular to the quasimetric G -space $P_S^X(Y)$ for a G -bornological coarse space X and invariant subset Y of X and invariant entourage S of X . \triangleleft

Let X be a G -bornological coarse space, Y be an invariant subset of X and S be an invariant entourage of X such that $\text{diag}(X) \subseteq S$. This implies that $S^k \subseteq S^{k+1}$ for all integers k and that $(S^n[Y])_{n \in \mathbb{N}}$ is an increasing family of invariant subsets of X .

Lemma 3.46 For every integer k we have the inclusion

$$U_{k-2}[P_S^X(Y)] \subseteq P_S^X(S^k[Y]) \subseteq U_{k+1}[P_S^X(Y)].$$

In particular, $(P_S^X(S^n[Y]))_{n \in \mathbb{N}}$ is a big family in the G -bornological coarse space $P_S(X)$.

Proof In the following argument we identify the zero skeleton of $P_S(X)$ with X . This is possible since by assumption S contains the diagonal.

Let μ be a point in $U_{k-2}[P_S^X(Y)]$. Then it is contained in some simplex. We choose some vertex x of that simplex. Then $x \in U_{k-1}[P_S^X(Y)]$. The shortest path which connects x with a point in $P_S^X(Y)$ is contained in the one-skeleton. Hence we can conclude that $x \in S^{k-1}[Y]$. But then all vertices of the simplex containing μ are contained in $S^k[Y]$. Hence $\mu \in P_S^X(S^k[Y])$.

Now let μ be a point in $P_S^X(S^k[Y])$. We again choose a vertex x of the simplex containing μ . Then the distance of x from $P_S^X(Y)$ is bounded by k . Hence the distance of μ from $P_S^X(Y)$ is bounded by $k+1$, ie we have $\mu \in U_{k+1}[P_S^X(Y)]$. \square

Let X be a G -bornological coarse space and $\mathcal{V} := (V_i)_{i \in I}$ be an equivariant pairwise disjoint family of subsets. Let S and T be invariant entourages of X such that S contains the diagonal, and set $S' := S \cup \text{sub}^{\mathcal{V}}(T)$ (Definition 3.4).

Let n be a natural number.

Lemma 3.47 If $(P_S^X(V_i))_{i \in I}$ is U_n -disjoint in $P_S^X(X)$, then $(P_{S'}^X(V_i))_{i \in I}$ is U_{n-2} -disjoint in $P_{S'}^X(X)$.

Proof Since S' contains the diagonal we can identify X with the zero skeleton of the Rips complex $P_{S'}^X(X)$. Since every point in the Rips complex has distance at most one to a point in the 0-skeleton, it suffices to show that the family $(V_i)_{i \in I}$ is U_n -disjoint in $P_{S'}^X(X)$.

We argue by contradiction. Let d denote the distance in $P_{S'}^X(X)$. Assume that i and j belong to I such that $i \neq j$ and $d(V_i, V_j) < n$. Then there exists a sequence $(x_k)_{k=0, \dots, n-1}$ in X with $x_0 \in V_i$, $x_{n-1} \in V_j$ and $(x_k, x_{k+1}) \in S'$ for all k in $\{0, \dots, n-2\}$.

We let k_0 in $\{0, \dots, n-1\}$ be the minimal element such that $x_{k_0} \in V_i$ and $x_{k_0+1} \notin V_i$. Such an element k_0 exists since $V_i \cap V_j = \emptyset$.

We let k_1 in $\{0, \dots, n-1\}$ be the maximal element such that for all m in $\{k_0, \dots, k_1-1\}$ we have $(x_m, x_{m+1}) \notin \bigcup_{h \in I} (T \cap (V_h \times V_h))$. We have $k_0 < k_1 \leq n-1$ again since the family $(V_h)_{h \in I}$ is pairwise disjoint. There exists l in I such that $x_{k_1} \in V_l$. Hence $d(V_i, V_l) \leq k_1 - k_0 \leq n-1$. This contradicts the assumption that the family $(V_i)_{i \in I}$ is U_n -disjoint. \square

Let X be a G -bornological coarse space, let U be an invariant subspace and let k be a natural number. We consider again invariant entourages S and T of X containing the diagonal and form the invariant entourage

$$S' := S \cup (T \cap (S^k[U] \times S^k[U])).$$

Lemma 3.48 $(S')^m[U] \subseteq S^{k+m}[U]$.

Proof We consider a point x in $(S')^m[U]$. Then there exist a sequence (x_0, \dots, x_m) in X with $x_0 \in U$, $x_m = x$ and $(x_i, x_{i+1}) \in S'$ for all i in $\{0, \dots, m-1\}$. Let l in $\{0, \dots, m-1\}$ be maximal with $x_l \in S^k[U]$. By definition of S' we have $(x_i, x_{i+1}) \in S$ for all i in $\{l+1, \dots, m-1\}$. Thus $x \in S^{m-l}[S^k[U]] \subseteq S^{m+k}[U]$. \square

3F Vanishing is closed under decomposition

Recall that \mathcal{VS}_E denotes the class of E -vanishing sequences (Definition 3.35).

Theorem 3.49 *The class \mathcal{VS}_E is closed under decomposition.*

This subsection is devoted to the proof of this theorem.

Suppose that the sequence (X_n) is a sequence of G -bornological coarse spaces which is decomposable over \mathcal{VS}_E (see Definition 3.42). In view of Definition 3.35 we have to show for every sequence (Y_n) of G -invariant subsets that $\mathcal{C}^\infty((Y_n))$ defined in (3-4) is a phantom object.

It immediately follows from Definition 3.35 that the class \mathcal{VS}_E is closed under taking sequences of G -invariant subspaces. By Lemma 3.43 every sequence of G -invariant subspaces (Y_n) is also decomposable over \mathcal{VS}_E . So we must see that the decomposability of the sequence (Y_n) implies that it is E -vanishing.

It therefore suffices to show that $\mathcal{C}^\infty((X_n))$ is a phantom. Note that

$$\mathcal{C}^\infty((X_n)) \simeq \operatorname{colim}_{r \in (0, \infty)} \operatorname{colim}_{N \in \mathbb{N}} \operatorname{colim}_{(S_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}_n^G} E \left(\prod_{N \leq n}^{\operatorname{semi}(r)} \mathcal{O}(P_{S_n}(X_n)) \right).$$

Let K be a compact object in \mathcal{C} and

$$f\colon K \rightarrow \operatorname{colim}_{r \in (0,\infty)} \operatorname{colim}_{N \in \mathbb{N}} \operatorname{colim}_{(S_n) \in \prod_{n \in \mathbb{N}} \mathcal{C}_n^G} E\left(\coprod_{N \leq n}^{\operatorname{semi}(r)} \mathcal{O}(P_{S_n}(X_n))\right)$$

be a morphism. By compactness of K it factorizes over a morphism

$$\tilde{f}\colon K \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\operatorname{semi}(r)} \mathcal{O}(P_{S_n}(X_n))\right)$$

for some real number r , and a sequence (S_n) (we will assume that S_n contains the diagonal of X_n for every integer n) in $\prod_{n \in \mathbb{N}} \mathcal{C}_n^G$. It suffices to show that there exists a real number r'' with $r \leq r''$ and a sequence (S''_n) in $\prod_{n \in \mathbb{N}} \mathcal{C}_n^G$ with $(S_n) \leq (S''_n)$ such that the induced (by the inclusion of Rips complexes) map

(3-8)

$$\tilde{f}'': K \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\operatorname{semi}(r'')} \mathcal{O}(P_{S''_n}(X_n))\right)$$

is equivalent to zero.

Let (U_n) and (V_n) be sequences of invariant subsets of the sequence (X_n) such that for every natural number n we have $X_n = U_n \cup V_n$. Using [Lemma 3.46](#) and excision for E we obtain a pushout

(3-9)

$$\begin{array}{ccc} \operatorname{colim}_{N,k \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\operatorname{semi}(r)} \mathcal{O}(P_{S_n}^{X_n}(S_n^k[U_n] \cap S_n^k[V_n]))\right) & \longrightarrow & \operatorname{colim}_{N,k \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\operatorname{semi}(r)} \mathcal{O}(P_{S_n}^{X_n}(S_n^k[U_n]))\right) \\ \downarrow & & \downarrow \\ \operatorname{colim}_{N,k \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\operatorname{semi}(r)} \mathcal{O}(P_{S_n}^{X_n}(S_n^k[V_n]))\right) & \longrightarrow & \operatorname{colim}_{N \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\operatorname{semi}(r)} \mathcal{O}(P_{S_n}(X_n))\right) \end{array}$$

Remark 3.50 At this point it is important to work with the semifree union (see [Definition 3.26](#)). Indeed, in general the family $(\coprod_{N \leq n} \mathcal{O}(P_{S_n}^{X_n}(S_n^k[V_n])))_{k \in \mathbb{N}}$ of invariant subsets is not a big family in the G –bornological coarse space $\coprod_{N \leq n}^{\operatorname{free}} \mathcal{O}(P_{S_n}(X_n))$. ◁

By the decomposability assumption on (X_n) for every natural number n we can choose nice equivariant S_n^n –disjoint families $(U_{n,i})_{i \in I_n}$ and $(V_{n,j})_{j \in J_n}$ of subspaces of X_n such that $X_n = U_n \cup V_n$ with

$$U_n := \bigcup_{i \in I_n} U_{n,i}, \quad V_n := \bigcup_{j \in J_n} V_{n,j}$$

and such that the sequences

$$\left(\coprod_{i \in I_n}^{\text{sub}} U_{n,i} \right), \quad \left(\coprod_{j \in J_n}^{\text{sub}} V_{n,j} \right)$$

belong to $\mathcal{V}S_E$. This choice will be fixed for the rest of the subsection.

Since above we have chosen S_n^n -disjoint families (the n^{th} power is important), by the first assertion of [Lemma 3.46](#) for every k in \mathbb{N} there exists an $N_1(k)$ in \mathbb{N} such that for every integer n with $N_1(k) \leq n$ the families

$$(3-10) \quad (S_n^k[U_{n,i}])_{i \in I_n}, \quad (S_n^k[V_{n,j}])_{j \in J_n}$$

are S_n -disjoint, and the families

$$(3-11) \quad (P_{S_n}^{X_n}(S_n^k[U_{n,i}]))_{i \in I_n}, \quad (P_{S_n}^{X_n}(S_n^k[V_{n,j}]))_{j \in J_n}$$

are U_r -disjoint. Note that $N_1(k)$ also depends on r , but we will not indicate this in the notation.

For a G -uniform bornological coarse space A , whose uniform structure is induced by an invariant metric and a real number r , we let $\mathcal{O}^r(A)$ denote the G -bornological coarse space obtained from $\mathcal{O}(A)$ by replacing the coarse structure by the coarse structure generated by all entourages of propagation ([Definition 3.25](#)) bounded by r .

We abbreviate

$$Y_{n,i,j}^k := S_n^k[U_{n,i}] \cap S_n^k[V_{n,j}]$$

and consider the family

$$\mathcal{Y}_n^k := (Y_{n,i,j}^k)_{i \in I_n, j \in J_n}.$$

Below, we use the abbreviation

$$(3-12) \quad \text{sub}^k(S_n) := \text{sub}^{\mathcal{Y}_n^k}(S_n);$$

see [Definition 3.4](#).

Lemma 3.51 *For all natural numbers n and k with $N_1(k) \leq n$ we have an isomorphism of G -bornological coarse spaces*

$$(3-13) \quad \mathcal{O}^r(P_{S_n}^{X_n}(S_n^k[U_n] \cap S_n^k[V_n])) \cong \mathcal{O}^r\left(P_{\text{sub}^k(S_n)}\left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k\right)\right).$$

Proof First of all the underlying G –simplicial complexes $P_{S_n}^{X_n}(S_n^k[U_n] \cap S_n^k[V_n])$ and $P_{\text{sub}^k(S_n)}(\bigsqcup_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k)$ are isomorphic since the families (3-10) are S_n –disjoint. The distance of the different components on the right-hand side in the metric of $P_{S_n}^{X_n}(X_n)$ is bigger than r since the families (3-11) are U_r –disjoint. Hence the entourages on $P_{S_n}^{X_n}(S_n^k[U_n] \cap S_n^k[V_n])$ and on $P_{\text{sub}^k(S_n)}(\bigsqcup_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k)$ of propagation less than or equal to r (measured in the respective quasimetric) are equal (under the natural identification). The bornologies of both spaces coincide by construction.

Since we consider the cone \mathcal{O}^r , we can conclude that the natural map (3-13) induces an isomorphism of G –bornological coarse spaces. □

Using Lemma 3.51 we can rewrite the pushout square (3-9) in the form

$$\begin{array}{ccc} \text{colim}_{N,k \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\text{semi}(r)} \mathcal{O}\left(P_{\text{sub}^k(S_n)}\left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k\right)\right)\right) & \longrightarrow & \text{colim}_{N,k \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\text{semi}(r)} \mathcal{O}(P_{S_n}^{X_n}(S_n^k[U_n]))\right) \\ \downarrow & & \downarrow \\ \text{colim}_{N,k \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\text{semi}(r)} \mathcal{O}(P_{S_n}^{X_n}(S_n^k[V_n]))\right) & \longrightarrow & \text{colim}_{N \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\text{semi}(r)} \mathcal{O}(P_{S_n}(X_n))\right) \end{array}$$

Since we take the colimit over N the fact that the isomorphism (3-13) exists only for sufficiently large n (and the condition of being sufficiently large depends on k) does not cause any problem.

Let

$$\partial: \text{colim}_{N \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\text{semi}(r)} \mathcal{O}(P_{S_n}(X_n))\right) \rightarrow \text{colim}_{N,k \in \mathbb{N}} \Sigma E\left(\coprod_{N \leq n}^{\text{semi}(r)} \mathcal{O}\left(P_{\text{sub}^k(S_n)}\left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k\right)\right)\right)$$

denote the boundary map of the Mayer–Vietoris sequence. Using compactness of K , the morphism $\partial \circ \tilde{f}$ factorizes over a morphism

$$g: K \rightarrow \text{colim}_{N \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\text{semi}(r)} \mathcal{O}\left(P_{\text{sub}^k(S_n)}\left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k\right)\right)\right)$$

for some k in \mathbb{N} which will be fixed from now on.

Lemma 3.52 *There exist a real number r' such that $r' \geq r$ and a sequence (T_n) in $\prod_{n \in \mathbb{N}} \mathcal{C}^G$ such that with $(S'_n) := (S_n \cup \text{sub}^k(T_n))$ the map*

$$K \rightarrow \text{colim}_{N \in \mathbb{N}} E\left(\coprod_{N \leq n}^{\text{semi}(r')} \mathcal{O}\left(P_{\text{sub}^k(S'_n)}\left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k\right)\right)\right)$$

induced by g is equivalent to zero.

Proof As in [Lemma 3.51](#), for all integers n with $N_1(k) \leq n$ we also have an isomorphism

$$(3-14) \quad \mathcal{O}^r \left(P_{\text{sub}^{k, \prime}(S_n)}^{\coprod_{i \in I_n}^{\text{sub}} S_n^k[U_{n,i}]} \left(\coprod_{i \in I_n} S_n^k[U_{n,i}] \cap S_n^k[V_n] \right) \right) \cong \mathcal{O}^r \left(P_{\text{sub}^k(S_n)} \left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k \right) \right),$$

where we use the abbreviation

$$(3-15) \quad \text{sub}^{k, \prime}(S_n) := \text{sub}^{\mathcal{U}_n^k}(S_n),$$

with $\mathcal{U}_n^k := (S_n^k[U_{n,i}])_{i \in I_n}$, and we use the equality

$$S_n^k[V_n] = \bigcup_{j \in J_n} S_n^k[V_{n,j}]$$

of subsets of X .

The sequence $(\coprod_{i \in I_n}^{\text{sub}} S_n^k[U_{n,i}] \cap S_n^k[V_n])_{N_1(k) \leq n}$ of G -bornological coarse spaces is a sequence of subspaces of $(\coprod_{i \in I_n}^{\text{sub}} S_n^k[U_{n,i}])_{N_1(k) \leq n}$. Since we assume that the family $(U_{n,i})_{i \in I_n}$ is nice, the bornological coarse space $\coprod_{i \in I_n}^{\text{sub}} S_n^k[U_{n,i}]$ is equivalent to $\coprod_{i \in I_n}^{\text{sub}} U_{n,i}$.

By assumption, the sequence $(\coprod_{i \in I_n}^{\text{sub}} [U_{n,i}])$ is E -vanishing. By [Lemma 3.36](#) we can now conclude that the sequence $(\coprod_{i \in I_n}^{\text{sub}} S_n^k[U_{n,i}])$ is E -vanishing. This implies that the sequence $(\coprod_{i \in I_n}^{\text{sub}} S_n^k[U_{n,i}] \cap S_n^k[V_n])$ is E -vanishing. Hence, using [Remark 3.5](#), there exist a real number r' such that $r \leq r'$ and a sequence (T_n) in $\prod_{n \in \mathbb{N}} \mathcal{C}_n^G$ with $(S_n) \leq (T_n)$ such that

$$\begin{aligned} K &\xrightarrow{g} \text{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\text{semi}(r)} \mathcal{O} \left(P_{\text{sub}^k(S_n)} \left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k \right) \right) \right) \\ &\rightarrow \text{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\text{semi}(r')} \mathcal{O} \left(P_{\text{sub}^k(T_n)} \left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k \right) \right) \right) \end{aligned}$$

is equivalent to zero. We now note that $\text{sub}^k(T_n) = \text{sub}^k(S'_n)$. □

It follows from [Lemma 3.52](#) that

$$\begin{aligned} K &\xrightarrow{\partial \circ \tilde{f}} \text{colim}_{N, k' \in \mathbb{N}} \Sigma E \left(\coprod_{N \leq n}^{\text{semi}(r)} \mathcal{O} \left(P_{\text{sub}^k(S_n)} \left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^k \right) \right) \right) \\ &\rightarrow \text{colim}_{N, k' \in \mathbb{N}} \Sigma E \left(\coprod_{N \leq n}^{\text{semi}(r')} \mathcal{O} \left(P_{\text{sub}^{k'}(S'_n)} \left(\coprod_{i \in I_n, j \in J_n}^{\text{sub}} Y_{n,i,j}^{k'} \right) \right) \right) \end{aligned}$$

is equivalent to zero. Consequently, using the naturality of the Mayer–Vietoris sequences and the compactness of K (in order to be able to choose k'), the composition

$$K \xrightarrow{\tilde{f}'} \operatorname{colim}_{N \in \mathbb{N}} E \left(\prod_{N \leq n}^{\operatorname{semi}(r)} \mathcal{O}(P_{S_n}(X_n)) \right) \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E \left(\prod_{N \leq n}^{\operatorname{semi}(r')} \mathcal{O}(P_{S'_n}(X_n)) \right)$$

(where r' and (S'_n) are as in [Lemma 3.52](#)) lifts to a morphism

$$(\hat{f}_U, \hat{f}_V): K \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E \left(\prod_{N \leq n}^{\operatorname{semi}(r')} \mathcal{O}(P_{S'_n}^{X_n}(S_n^{k'}[U_n])) \right) \oplus \operatorname{colim}_{N \in \mathbb{N}} E \left(\prod_{N \leq n}^{\operatorname{semi}(r')} (P_{S'_n}^{X_n}(S_n^{k'}[V_n])) \right)$$

for some integer k' such that $k' \geq k$.

The notation $\operatorname{sub}^{k','}(T_n)$ is as in [\(3-15\)](#), while $\operatorname{sub}^k(T_n)$ is as in [\(3-12\)](#) for the specific choice of the integer k made just before the statement of [Lemma 3.52](#).

Let (T''_n) be a sequence in $\prod_{n \in \mathbb{N}} \mathcal{C}_n^G$ such that $(T_n) \leq (T''_n)$. We define the sequence

$$(3-16) \quad (S''_n) := (S'_n \cup \operatorname{sub}^{k','}(T''_n)).$$

By [Lemmas 3.48](#) and [3.47](#) there exists an integer N_2 such that $N_1(k) \leq N_2$ and, for all integers n with $N_2 \leq n$, the family $(S_n^{k'}[U_{n,i}])_{i \in I_n}$ is \tilde{S}''_n -disjoint and the family $(P_{S''_n}(S_n^{k'}[U_{n,i}]))_{i \in I_n}$ is r' -disjoint.

The following is similar to [Lemma 3.51](#).

Lemma 3.53 *For all integers n with $N_2 \leq n$ we have an isomorphism of G -bornological coarse spaces*

$$\mathcal{O}^{r'}(P_{S''_n}^{X_n}(S_n^{k'}[U_n])) \cong \mathcal{O}^{r'} \left(P_{\operatorname{sub}^{k','}(S''_n)} \left(\prod_{i \in I}^{\operatorname{sub}} S_n^{k'}[U_{n,i}] \right) \right).$$

Proof First of all the underlying G -simplicial complexes

$$P_{S''_n}^{X_n}(S_n^{k'}[U_n]), \quad P_{\operatorname{sub}^{k','}(S''_n)} \left(\prod_{i \in I}^{\operatorname{sub}} S_n^{k'}[U_{n,i}] \right)$$

are isomorphic since the family $(S_n^{k'}[U_{n,i}])_{i \in I_n}$ is S''_n -disjoint. The distance of the different components on the right-hand side in the metric of $P_{S''_n}^{X_n}(X_n)$ is bigger than r' . Since we consider the cone $\mathcal{O}^{r'}$, we now can conclude that the G -coarse structures on both sides coincide. The bornologies coincide by construction. \square

We get an equivalence

$$(3-17) \quad \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r')} \mathcal{O}(P_{S''_n}^{X_n}(S_n^{k'}[U_n])) \right) \\ \simeq \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r')} \mathcal{O} \left(P_{\operatorname{sub}^{k', \cdot'}(S''_n)} \left(\coprod_{i \in I}^{\operatorname{sub}} S_n^{k'}[U_{n,i}] \right) \right) \right).$$

Lemma 3.54 *There exist a real number r'' such that $r'' \geq r'$ and a sequence (S''_n) in $\prod_{n \in \mathbb{N}} \mathcal{C}_n^G$ with $(S''_n) \geq (S'_n)$ such that the map*

$$K \xrightarrow{\hat{f}_U} \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r')} \mathcal{O}(P_{S''_n}^{X_n}(S_n^{k'}[U_n])) \right) \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r'')} \mathcal{O}(P_{S''_n}^{X_n}(S_n^{k'}[U_n])) \right)$$

is equivalent to zero.

Proof In view of the equivalence (3-17) we must show that for an appropriate choice of the sequence (T''_n) (which determines the sequence (S''_n) by (3-16)) the morphism

$$K \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r')} \mathcal{O} \left(P_{\operatorname{sub}^{k', \cdot'}(S''_n)} \left(\coprod_{i \in I}^{\operatorname{sub}} S_n^{k'}[U_{n,i}] \right) \right) \right) \\ \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r'')} \mathcal{O} \left(P_{\operatorname{sub}^{k', \cdot'}(S''_n)} \left(\coprod_{i \in I}^{\operatorname{sub}} S_n^{k'}[U_{n,i}] \right) \right) \right)$$

is equivalent to zero. As in the proof of Lemma 3.52 we argue that the sequence of G -bornological coarse spaces $(\coprod_{i \in I}^{\operatorname{sub}} S_n^{k'}[U_{n,i}])$ is E -vanishing. Using Remark 3.5 and compactness of K we find the desired real number r'' and sequence (S''_n) . \square

Using Lemma 3.54, we find a real number r'' with $r'' \geq r'$ and a sequence (S''_n) in $\prod_{n \in \mathbb{N}} \mathcal{C}_n$ with $(S''_n) \geq (S'_n)$ such that the map

$$K \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r'')} \mathcal{O}(P_{S''_n}^{X_n}(S_n^{k'}[U_n])) \right) \oplus \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r'')} \mathcal{O}(P_{S''_n}^{X_n}(S_n^{k'}[V_n])) \right)$$

induced by (\hat{f}_U, \hat{f}_V) and the natural inclusions is equivalent to zero.

Composition of the induced map with the morphism

$$\begin{aligned} \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r'')} \mathcal{O}(P_{S_n''}^{X_n}(S_n^{k'}[U_n])) \right) \oplus \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r'')} \mathcal{O}(P_{S_n''}^{X_n}(S_n^{k'}[V_n])) \right) \\ \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r'')} \mathcal{O}(P_{S_n''}(X_n)) \right) \end{aligned}$$

from the Mayer–Vietoris sequence gives the map (3-8),

$$\tilde{f}': K \xrightarrow{\tilde{f}} \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r)} \mathcal{O}(P_{S_n}(X_n)) \right) \rightarrow \operatorname{colim}_{N \in \mathbb{N}} E \left(\coprod_{N \leq n}^{\operatorname{semi}(r'')} \mathcal{O}(P_{S_n''}(X_n)) \right),$$

which is then also equivalent to zero.

This finishes the proof of [Theorem 3.49](#).

3G Proof of [Theorem 3.20](#)

As already observed in [Section 3C](#) the results shown so far imply:

Corollary 3.55 *The class \mathcal{V}_E of E –vanishing G –bornological coarse spaces is closed under decomposition.*

Recall [Definition 3.12](#) of the notion of a semibounded G –coarse space. Furthermore recall [Definition 3.16](#) of a discontinuous G –action on a G –bornological space.

Proposition 3.56 *The class of G –bornological coarse spaces whose underlying G –coarse space is semibounded and has a discontinuous G –action is contained in the class of E –vanishing spaces \mathcal{V}_E .*

Proof Let X be a G –bornological coarse spaces whose underlying space is semibounded and has a discontinuous G –action. Then every invariant subset with the induced G –bornological coarse structure has the same properties. So it suffices to show that

$$E(\mathcal{O}(P_S(X))) \simeq 0$$

for all sufficiently large invariant entourages S of X .

By assumption we can choose an invariant entourage T of X such that every coarse component of X is bounded by T . Then for every invariant entourage S of X with $T \subseteq S$ we have an isomorphism

$$P_S(X) \cong \bigsqcup_{i \in \pi_0(X)} \Delta^{X_i}$$

of quasimetric spaces, where X_i is the component labelled by i and Δ^{X_i} is the set of finitely supported probability measures on X_i .

Let π be a choice of a subset of $\pi_0(X)$ of representatives of the orbit set $\pi_0(X)/G$. For every i in π we let G_i be the stabilizer of i . This group acts simplicially on X_i . Then we obtain a G -equivariant isomorphism of G -quasimetric spaces

$$P_S(X) \cong \coprod_{i \in \pi} G \times_{G_i} \Delta^{X_i}.$$

Let i be in π and x_i be a point in X_i . The set $Gx_i \cap X_i$ is finite since the G -action on X is discontinuous and X_i belongs to the minimal bornology compatible with the coarse structure of X . Hence, $Gx_i \cap X_i$ is the set of vertices of a simplex Δ_i in Δ^{X_i} . The barycenter b_i of Δ_i is fixed under the action of G_i .

There is a unique G -equivariant map $b: \pi_0(X) \rightarrow P_S(X)$ such that $b(x_i) = b_i$ for all i in π . We consider $\pi_0(X)$ as a discrete quasimetric space. The map b is a homotopy equivalence. Indeed, an inverse is given by the map $p: P_S(X) \rightarrow \pi_0(X)$ which sends every point μ in $P_S(X)$ to the coarse component represented by any choice of a vertex of a simplex containing μ . Then $p \circ b = \text{id}_{\pi_0(X)}$ and $b \circ p$ is homotopic to $\text{id}_{P_S(X)}$ by a unit-speed homotopy. The cone functor sends this homotopy to a coarse homotopy.

We equip $\pi_0(X)$ with the bornology induced from the bornology of X via the map b . It follows from the compatibility of the coarse and the bornological structure on X that the projection p is proper.

We conclude that

$$E(\mathcal{O}(P_S(X))) \simeq E(\mathcal{O}(\pi_0(X))).$$

Since $\pi_0(X)$ is a discrete G -bornological coarse space the cone $\mathcal{O}(\pi_0(X))$ is a flasque G -bornological coarse space. Consequently, $0 \simeq E(\mathcal{O}(\pi_0(X)))$. This implies

$$0 \simeq E(\mathcal{O}(P_S(X))). \quad \square$$

Proof of Theorem 3.20 The theorem is an immediate consequence of Corollary 3.55 and Proposition 3.56: The class of G -coarse spaces with G -FDC is by definition the smallest class of G -coarse spaces which is closed under decomposition and contains all semibounded spaces. Moreover, any decomposition of G -bornological space with discontinuous G -action yields families of G -bornological coarse spaces with discontinuous G -action. \square

References

- [1] U Bunke, A Engel, *Homotopy theory with bornological coarse spaces*, preprint (2017) [arXiv](#)
- [2] U Bunke, A Engel, D Kasprowski, C Winges, *Equivariant coarse homotopy theory and coarse algebraic K -homology*, preprint (2017) [arXiv](#)
- [3] U Bunke, A Engel, D Kasprowski, C Winges, *Injectivity results for coarse homology theories*, preprint (2018) [arXiv](#)
- [4] U Bunke, A Engel, D Kasprowski, C Winges, *Transfers in coarse homology*, preprint (2018) [arXiv](#)
- [5] U Bunke, D Kasprowski, C Winges, *Split injectivity of A -theoretic assembly maps*, preprint (2018) [arXiv](#) To appear in Int. Math. Res. Not.
- [6] E Guentner, R Tessera, G Yu, *A notion of geometric complexity and its application to topological rigidity*, Invent. Math. 189 (2012) 315–357 [MR](#)
- [7] E Guentner, R Tessera, G Yu, *Discrete groups with finite decomposition complexity*, Groups Geom. Dyn. 7 (2013) 377–402 [MR](#)
- [8] D Kasprowski, *On the K -theory of groups with finite decomposition complexity*, PhD thesis, Universität Münster (2014) Available at <http://www.math.uni-bonn.de/people/daniel/ThesisKasprowski.pdf>
- [9] D Kasprowski, *On the K -theory of linear groups*, Ann. K-Theory 1 (2016) 441–456 [MR](#)
- [10] D A Ramras, R Tessera, G Yu, *Finite decomposition complexity and the integral Novikov conjecture for higher algebraic K -theory*, J. Reine Angew. Math. 694 (2014) 129–178 [MR](#)

UN, AE: Fakultät für Mathematik, Universität Regensburg
Regensburg, Germany

DK, CW: Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn
Bonn, Germany

ulrich.bunke@mathematik.uni-regensburg.de,
alexander.engel@mathematik.uni-regensburg.de,
kasprowski@uni-bonn.de, winges@math.uni-bonn.de

Received: 7 June 2018 Revised: 22 January 2019