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A combinatorial model for the known Bousfield classes

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We give a combinatorial construction of an ordered semiring $\mathcal{A}$, and show that it can be identified with a certain subquotient of the semiring of p–local Bousfield classes, containing almost all of the classes that have previously been named and studied. This is a convenient way to encapsulate most of the known results about Bousfield classes.

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1 Introduction

Fix a prime p, and let $\mathcal{L}$ denote the set of Bousfield classes in the p–local stable category (which can be regarded as an ordered semiring in a natural way). This note is an attempt to organise many of the known results about the structure of $\mathcal{L}$ in a more coherent way.

One of the main open questions about $\mathcal{L}$ is Ravenel's telescope conjecture (TC). The statement will be recalled in Remark 4.2. Many people suspect that TC is false, but this has still not been proven. We will define an ordered semiring $\overline{\mathcal{L}}$ which is, in a certain sense, the largest quotient of $\mathcal{L}$ in which TC becomes true. We will then define (in an explicit, combinatorial way) another ordered semiring $\mathcal{A}$ and a function $\phi\colon \mathcal{A} \to \mathcal{L}$ such that the composite

$$\mathcal{A} \xrightarrow{\phi} \mathcal{L} \xrightarrow{\pi} \overline{\mathcal{L}}$$

is an injective homomorphism of ordered semirings. (However, ϕ itself is probably not a semiring homomorphism, unless TC holds.) For almost all elements $x \in \mathcal{L}$ that have been named and studied, we have $\pi(x) \in \pi\phi(\mathcal{A})$. Thus, $\mathcal{A}$ is a good model for the known part of $\mathcal{L}$.

Remark 1.1 We will mention two exceptions to the idea that $\mathcal{A}$ captures all known phenomena in $\overline{\mathcal{L}}$. First consider the spectra BP/J from Ravenel [15, Definition 2.7], where J is generated by an invariant regular sequence of infinite length. Ravenel shows that for different J these have many different Bousfield classes, but only one of them is in the image of $\phi\colon \mathcal{A}\to\mathcal{L}$. It is unlikely that the situation is any better in $\overline{\mathcal{L}}$. Next, Mahowald, Ravenel and Shick [12] introduce a number of new Bousfield classes in the course of studying the telescope conjecture. It is reasonable to conjecture that their images in $\overline{\mathcal{L}}$ lie in $\pi\phi(\mathcal{A})$, but we have not considered this question carefully.

2 Ordered semirings

Definition 2.1 By an *ordered semiring* we will mean a set $\mathcal{R}$ equipped with elements $0, 1 \in \mathcal{R}$ and binary operations $\vee$ and $\wedge$ such that:

(a) $\vee$ is commutative and associative, with 0 as an identity element.

(b) $\wedge$ is commutative and associative, with 1 as an identity element.

(c) $\wedge$ distributes over $\vee$.

(d) For all $u \in \mathcal{R}$ we have $0\wedge u = 0$ and $1\vee u = 1$ and $u\vee u = u$.

It is easy to check that there is a natural partial order on such an object, where $u\leq v$ if and only if $u\vee v = v$. The binary operations preserve this order, and 0 and 1 are the smallest and largest elements. Moreover, $u\vee v$ is the smallest element satisfying $w\geq u$ and $w\geq v$.

Definition 2.2 Let $\mathcal{R}$ be an ordered semiring. We say that $\mathcal{R}$ is *complete* if every subset $S\subseteq\mathcal{R}$ has a least upper bound $\bigvee S\in\mathcal{R}$. We say that $\mathcal{R}$ is *completely distributive* if, in addition, for all $S\subseteq\mathcal{R}$ and $x\in\mathcal{R}$ we have

$$\bigvee\{x\wedge s \mid s\in S\} = x\wedge\bigvee S.$$

We next recall the definition of the Bousfield semiring $\mathcal{L}$.

Definition 2.3 We write $\mathcal{B}$ for the category of p–local spectra in the sense of stable homotopy theory. This has a coproduct, which is written $X\vee Y$ and is also called the wedge product. There is also a smash product, written $X\wedge Y$. Up to natural isomorphism, both operations are commutative and associative, and the smash product distributes over the wedge product. The p–local sphere spectrum S is a unit for the smash product, and the zero spectrum is a unit for the wedge product.

For any object $E\in\mathcal{B}$ we put

$$\langle E\rangle = \{X\in\mathcal{B} \mid E\wedge X = 0\},$$

and call this the *Bousfield class* of E. We then put

$$\mathcal{L} = \{\langle E\rangle \mid E \in \mathcal{B}\}.$$

(This is a set rather than a proper class, by a theorem of Ohkawa [14; 4].) It is straightforward to check that this has well-defined operations satisfying

$$\begin{aligned}
\langle E\rangle \vee \langle F\rangle &= \langle E \vee F\rangle = \langle E\rangle \cap \langle F\rangle, \\
\langle E\rangle \wedge \langle F\rangle &= \langle E \wedge F\rangle = \{X \mid E \wedge X \in \langle F\rangle\} = \{X \mid F \wedge X \in \langle E\rangle\}.
\end{aligned}$$

It is then easy to check that this gives an ordered semiring, with top and bottom elements

$$1 = \langle S\rangle = \{0\}, \quad 0 = \langle 0\rangle = \mathcal{B}.$$

The resulting ordering of $\mathcal{L}$ is given by $\langle E\rangle \leq \langle F\rangle$ if and only if $\langle E\rangle \supseteq \langle F\rangle$.

Next, recall that $\mathcal{B}$ has a coproduct, written $\bigvee_i X_i$, for any family of objects $(X_i)_{i\in I}$, and these satisfy $W \wedge \bigvee_i X_i \simeq \bigvee_i (W \wedge X_i)$. It follows that $\mathcal{L}$ is completely distributive, with $\bigvee_i \langle E_i\rangle = \langle \bigvee_i E_i\rangle$.

Definition 2.4 Let $\mathcal{R}$ be an ordered semiring, and let $\epsilon \in \mathcal{R}$ be an idempotent element (so $\epsilon \wedge \epsilon = \epsilon$). We put

$$\mathcal{R}/\epsilon = \{a \in \mathcal{R} \mid a \geq \epsilon\}.$$

We define a surjective function $\pi\colon \mathcal{R} \to \mathcal{R}/\epsilon$ by $\pi(a) = a \vee \epsilon$.

Proposition 2.5 *There is a unique ordered semiring structure on $\mathcal{R}/\epsilon$ such that π is a homomorphism. Moreover, if $\phi\colon \mathcal{R} \to \mathcal{S}$ is any homomorphism of ordered semirings with $\phi(\epsilon) = 0$, then there is a unique homomorphism $\overline{\phi}\colon \mathcal{R}/\epsilon \to \mathcal{S}$ with $\overline{\phi} \circ \pi = \phi$.*

Proof The set $\mathcal{R}/\epsilon$ clearly contains 1 and is closed under $\wedge$ and $\vee$. We claim that these operations make $\mathcal{R}/\epsilon$ into an ordered semiring, with ϵ as a zero element. All axioms not involving zero are the same as the corresponding axioms for $\mathcal{R}$. The axioms involving zero say that we should have $\epsilon \vee u = u$ and $\epsilon \wedge u = \epsilon$ for all $u \in \mathcal{R}/\epsilon$, and this follows directly from the definition of $\mathcal{R}/\epsilon$ and the idempotence of ϵ. It is clear that this is the unique structure on $\mathcal{R}/\epsilon$ for which π is a homomorphism. If $\phi\colon \mathcal{R} \to \mathcal{S}$ has $\phi(\epsilon) = 0$ then we can just take $\overline{\phi}$ to be the restriction of ϕ to $\mathcal{R}/\epsilon$. This is clearly a homomorphism, with

$$\overline{\phi}(\pi(a)) = \phi(a \vee \epsilon) = \phi(a) \vee \phi(\epsilon) = \phi(a) \vee 0 = \phi(a),$$

as required. □

Remark 2.6 If $\mathcal{R}$ is complete, or completely distributive, then we find that $\mathcal{R}/\epsilon$ has the same property.

Remark 2.7 In Definition 4.1 we will introduce certain Bousfield classes $a(n) = \langle C_n K'(n)\rangle$ for $n \in \mathbb{N}$, and put $\epsilon(n) = \bigvee_{i<n} a(i)$ for $n \in \mathbb{N}_\infty$. These will all be zero if and only if TC holds. In Lemma 5.18 we will check that $\epsilon(n)$ is idempotent, which allows us to define $\bar{\mathcal{L}} = \varinjlim_{n<\infty} \mathcal{L}/\epsilon(n)$. This will be our main object of study.

Definition 2.8 Let $\mathcal{R}$ be an ordered semiring. An *ideal* in $\mathcal{R}$ is a subset $\mathcal{I} \subseteq \mathcal{R}$ such that

- $0 \in \mathcal{I}$,
- for all $x, y \in \mathcal{I}$ we have $x \vee y \in \mathcal{I}$,
- for all $x \in \mathcal{R}$ and $y \in \mathcal{I}$ we have $x \wedge y \in \mathcal{I}$.

Remark 2.9 Let S be any subset of $\mathcal{R}$, and let $\mathcal{I}$ be the set of elements $x \in \mathcal{R}$ that can be expressed in the form $x = \bigvee_{i=1}^{n} y_i \wedge z_i$ for some $n \in \mathbb{N}$ and $y \in \mathcal{R}^n$ and $z \in S^n$. (This should be interpreted as $x = 0$ in the case $n = 0$.) Just as in the case of ordinary rings, this is the smallest ideal containing S, or in other words, the ideal generated by S.

Lemma 2.10 *Let $\mathcal{R}$ be an ordered semiring.*

(a) *Suppose that every ideal in $\mathcal{R}$ has a least upper bound; then $\mathcal{R}$ is complete.*

(b) *Suppose that $\mathcal{R}$ is complete, and that for $x \in \mathcal{R}$ and every ideal $\mathcal{I} \subseteq \mathcal{R}$ we have $x \wedge \bigvee \mathcal{I} = \bigvee (x \wedge \mathcal{I})$; then $\mathcal{R}$ is completely distributive.*

Proof (a) Let S be a subset of $\mathcal{R}$, and let $\mathcal{I}$ be the ideal that it generates. It is then easy to see that the upper bounds for $\mathcal{I}$ are the same as the upper bounds for S, and $\mathcal{I}$ has a least upper bound by assumption, so this is also a least upper bound for S.

(b) Now suppose we also have an element $x \in \mathcal{R}$, and that $x \wedge \bigvee \mathcal{I} = \bigvee (x \wedge \mathcal{I})$. We find that $x \wedge \mathcal{I}$ is the same as the ideal generated by $x \wedge S$, so

$$\bigvee (x \wedge S) = \bigvee (x \wedge \mathcal{I}) = x \wedge \bigvee \mathcal{I} = x \wedge \bigvee S,$$

as required. □

We next define two canonical subsemirings for any ordered semiring $\mathcal{R}$. This is an obvious axiomatic generalisation of work that Bousfield did for $\mathcal{L}$ in [2].

Definition 2.11 Let $\mathcal{R}$ be an ordered semiring, and let x and y be elements of $\mathcal{R}$. We say that y is a *complement* for x (and vice versa) if $x \vee y = 1$ and $x \wedge y = 0$. If such a y exists, we say that x is *complemented.*

Lemma 2.12 *If x has a complement then it is unique, and we have $x \wedge x = x$.*

Proof Let y be a complement for x. Multiplying the equation $x \vee y = 1$ by x and using $x \wedge y = 0$ gives $x \wedge x = x$.

Now let z be another complement for x. Multiplying the relation $x \vee y = 1$ by z gives $y \wedge z = z$. Multiplying the relation $x \vee z = 1$ by y gives $y \wedge z = y$. Comparing these gives $y = z$. □

This validates the following:

Definition 2.13 For any complemented element x, we write $\neg x$ for the complement.

Definition 2.14 For any ordered semiring $\mathcal{R}$, we put

$$\mathcal{R}_{\mathrm{latt}} = \{x \in \mathcal{R} \mid x \wedge x = x\}, \quad \mathcal{R}_{\mathrm{bool}} = \{x \in \mathcal{R} \mid x \text{ is complemented}\}.$$

Remark 2.15 Let $\phi\colon \mathcal{R} \to \mathcal{S}$ be a homomorphism of ordered semirings. Then it is clear that $\phi(\mathcal{R}_{\mathrm{latt}}) \subseteq \mathcal{S}_{\mathrm{latt}}$. Moreover, if x and y are complements of each other in $\mathcal{R}$, we find that $\phi(x)$ and $\phi(y)$ are complements of each other in $\mathcal{S}$. It follows that $\phi(\mathcal{R}_{\mathrm{bool}}) \subseteq \mathcal{S}_{\mathrm{bool}}$. In other words, both of the above constructions are functorial.

Proposition 2.16 *The set $\mathcal{R}_{\mathrm{latt}}$ is a subsemiring of $\mathcal{R}$. Moreover, for $x, y, z \in \mathcal{R}_{\mathrm{latt}}$ we have $x \leq y \wedge z$ if and only if $x \leq y$ and $x \leq z$, so the $\wedge$ product is just the meet operation for the natural ordering, and this makes $\mathcal{R}_{\mathrm{latt}}$ into a distributive lattice.*

Proof It is clear that $\mathcal{R}_{\mathrm{latt}}$ contains 0 and 1 and is closed under $\wedge$. Now suppose that $x, y \in \mathcal{R}_{\mathrm{latt}}$, and put $z = x \vee y$. Using the commutativity and distributivity of $\wedge$, and the idempotence of x and y, we obtain

$$z \wedge z = x \vee y \vee (x \wedge y).$$

We can rewrite $y \vee (x \wedge y)$ as $(1 \vee x) \wedge y = 1 \wedge y = y$, so $z \wedge z = x \vee y = z$, as required. This proves that $\mathcal{R}_{\mathrm{latt}}$ is a subsemiring.

Now suppose that $x, y, z \in \mathcal{R}_{\mathrm{latt}}$ with $x \leq y$ and $x \leq z$. We then get $x = x \wedge x \leq y \wedge z$, as required. The converse holds in any ordered semiring, so we see that $\wedge$ is just the meet operation, as claimed. □

Proposition 2.17 *The set $\mathcal{R}_{\mathrm{bool}}$ is a subsemiring of $\mathcal{R}_{\mathrm{latt}}$ and is a boolean algebra.*

Proof Lemma 2.12 shows that $\mathcal{R}_{\mathrm{bool}} \subseteq \mathcal{R}_{\mathrm{latt}}$. Note also that if $x \in \mathcal{R}_{\mathrm{bool}}$ then x is a complement for $\neg x$, so $\neg x$ lies in $\mathcal{R}_{\mathrm{bool}}$ as well.

Now suppose that $x_0, x_1 \in \mathcal{R}_{\mathrm{bool}}$, with complements y_0 and y_1. It is then easy to check that $y_0 \wedge y_1$ and $y_0 \vee y_1$ are complements for $x_0 \vee x_1$ and $x_0 \wedge x_1$, showing that $\mathcal{R}_{\mathrm{bool}}$ is closed under $\vee$ and $\wedge$. It also contains 0 and 1, so $\mathcal{R}_{\mathrm{bool}}$ is a subsemiring of $\mathcal{R}_{\mathrm{latt}}$. By one of the standard definitions, a boolean algebra is just a distributive lattice in which every element has a complement, so $\mathcal{R}_{\mathrm{bool}}$ has this structure. □

We can generalise the definition of $\neg x$ as follows. Put

$$A(x) = \{y \mid x \wedge y = 0\}.$$

If y is a complement for x, then it is easy to check that it is the largest element in the set $A(x)$. More generally, if x does not have a complement, but $A(x)$ still has a largest element, then we can define $\neg x$ to be that largest element. If $\mathcal{R}$ is completely distributive then we see that $\bigvee A(x)$ is always an element of $A(x)$ and so qualifies as $\neg x$. In particular, this operation is defined for all elements of the Bousfield semiring, as was already discussed in [2]. However, a homomorphism $\phi\colon \mathcal{R} \to \mathcal{S}$ need not satisfy $\phi(\neg x) = \neg\phi(x)$ in this more general context, even if ϕ preserves infinite joins. In particular, we do not know whether the homomorphisms $\mathcal{A} \to \bar{\mathcal{L}}$ and $\mathcal{L} \to \bar{\mathcal{L}}$ preserve negation. Thus, although we can compute the negation operation in $\mathcal{A}$, this does not provide much information about $\mathcal{L}$, unless we restrict attention to $\mathcal{A}_{\mathrm{bool}}$.

We can generalise still further as follows:

Definition 2.18 Let $\mathcal{R}$ be an ordered semiring, and let x and z be elements of $\mathcal{R}$. Put

$$A(x,z) = \{y \in \mathcal{R} \mid x \wedge y \leq z\}.$$

Then:

- If $A(x,z)$ has a largest element then we denote it by $(x \to z)$, and call it a *Heyting element* for the pair (x,z).
- A *strong Heyting element* for (x,z) is an element $y \in \mathcal{R}$ such that $x \wedge y \leq z \leq y$ and $x \vee y = 1$.

A complement for x is the same as a strong Heyting element for $(x,0)$, and our more general definition of $\neg x$ is just the same as $(x \to 0)$.

Proposition 2.19 (a) *Any strong Heyting element is a Heyting element.*

(b) *If $\mathcal{R}$ is completely distributive, then every pair has a Heyting element.*

(c) *If x is complemented, then $z \vee \neg x$ is a Heyting element for (x, z).*

(d) *Any homomorphism of ordered semirings preserves strong Heyting elements.*

Proof (a) Let y be a strong Heyting element for (x, z). Then $x \wedge y \leq z$, so $y \in A(x, z)$. Let u be any other element of $A(x, z)$, so $x \wedge u \leq z$. Multiplying the relation $x \vee y = 1$ by u gives

$$u = (u \wedge x) \vee (u \wedge y) \leq z \vee y,$$

but we also have $z \leq y$ (as part of the definition of a strong Heyting element) so $u \leq y$, as required.

(b) Suppose that $\mathcal{R}$ is completely distributive, and put $y = \bigvee A(x, z)$. Complete distributivity implies that

$$x \wedge y = \bigvee \{x \wedge u \mid u \in A(x, z)\} \leq z,$$

so $y \in A(x, z)$, and clearly y is the largest element of $A(x, z)$.

(c) Let w be a complement for x, so $w \wedge x = 0$ and $w \vee x = 1$. Put $y = z \vee w \geq z$. Then

$$x \wedge y = (x \wedge z) \vee (x \wedge w) = x \wedge z, \quad x \vee y = z \vee w \vee x = z \vee 1 = 1.$$

It follows that y is a strong Heyting element, as claimed.

(d) This is clear from the definitions. □

If we assume that $x \wedge x = x$ for all x (so that $\mathcal{R} = \mathcal{R}_{\mathrm{latt}}$) then the Heyting elements satisfy a number of additional properties, such as $x \wedge (x \to z) = x \wedge z$. These properties are encapsulated by the definition of a Heyting algebra (see [11, Section 1.1] for example). They do not hold automatically in our more general context, and we have not investigated exactly how much can be rescued.

3 The combinatorial model

Definition 3.1 We put $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$, and give this the obvious order with ∞ as the largest element. We will say that a subset $S \subseteq \mathbb{N}_\infty$ is *small* if $S \subseteq [0, n)$ for some $n \in \mathbb{N}$; otherwise, we will say that S is *big*. We will also say that S is *cosmall* if $\mathbb{N}_\infty \setminus S$ is small, or equivalently S contains $[n, \infty]$ for some n.

Definition 3.2 We put $\mathbb{N}_\omega = \mathbb{N} \cup \{\omega, \infty\}$, with the ordering

$$0 < 1 < 2 < 3 < \cdots < \omega < \infty.$$

Definition 3.3 The set $\mathcal{A}$ has elements as follows:

- For each cosmall subset $T \subseteq \mathbb{N}_\infty$ and each $q \in \mathbb{N}_\infty$ we have an element $t(q,T) \in \mathcal{A}$.
- For each small subset $S \subset \mathbb{N}_\infty$ and each $m \in \mathbb{N}_\omega$ we have an element $j(m,S) \in \mathcal{A}$.
- For each subset $U \subseteq \mathbb{N}_\infty$ we have an element $k(U) \in \mathcal{A}$.

(For the corresponding elements of the Bousfield lattice, see Definition 4.1.)

We write $\mathcal{A}_t$ for the subset of elements of the form $t(q,T)$, and similarly for $\mathcal{A}_j$ and $\mathcal{A}_k$, so that $\mathcal{A} = \mathcal{A}_t \amalg \mathcal{A}_j \amalg \mathcal{A}_k$.

We define commutative binary operations $\vee$ and $\wedge$ on $\mathcal{A}$ as follows:

$$\begin{aligned}
t(q,T) \vee t(q',T') &= t(\min(q,q'), T \cup T'),\\
t(q,T) \vee j(m',S') &= t(q, T \cup S'),\\
t(q,T) \vee k(U') &= t(q, T \cup U'),\\
j(m,S) \vee j(m',S') &= j(\max(m,m'), S \cup S'),\\
j(m,S) \vee k(U') &= \begin{cases} j(m, S \cup U') & \text{if } U' \text{ is small,}\\ k(S \cup U') & \text{if } U' \text{ is big,}\end{cases}\\
k(U) \vee k(U') &= k(U \cup U'),\\
t(q,T) \wedge t(q',T') &= t(\max(q,q'), T \cap T'),\\
t(q,T) \wedge j(m',S') &= \begin{cases} j(m', T \cap S') & \text{if } q \leq m',\\ k(T \cap S') & \text{if } q > m',\end{cases}\\
t(q,T) \wedge k(U') &= k(T \cap U'),\\
j(m,S) \wedge j(m',S') &= k(S \cap S'),\\
j(m,S) \wedge k(U') &= k(S \cap U'),\\
k(U) \wedge k(U') &= k(U \cap U').
\end{aligned}$$

We also put $0 = k(\varnothing)$ and $1 = t(0, \mathbb{N}_\infty)$.

We next give some auxiliary definitions that will help us analyse the structure of $\mathcal{A}$.

Definition 3.4 We write $\mathcal{P}$ for the ordered semiring of subsets of $\mathbb{N}_\infty$, with the operations $\cup$ and $\cap$, and identity elements $0=\varnothing$ and $1=\mathbb{N}_\infty$. We define tail: $\mathcal{A}\to\mathcal{P}$ by

$$\mathrm{tail}(t(q,T))=T,\quad \mathrm{tail}(j(m,S))=S,\quad \mathrm{tail}(k(U))=U.$$

Remark 3.5 Inspection of the definitions shows that $\mathrm{tail}(x\vee y)=\mathrm{tail}(x)\cup\mathrm{tail}(y)$ and $\mathrm{tail}(x\wedge y)=\mathrm{tail}(x)\cap\mathrm{tail}(y)$ for all x and y. Once we have checked that $\mathcal{A}$ is an ordered semiring, this will mean that tail: $\mathcal{A}\to\mathcal{P}$ is a homomorphism of ordered semirings. Inspection of the definitions also shows that

$$\mathrm{tail}(x)=\{i\mid k(i)\wedge x\neq 0\}=\{i\mid k(i)\wedge x=k(i)\}=\{i\mid k(i)\leq x\}.$$

Definition 3.6 We define

$$\mathcal{H}=\{t(q)\mid q\in\mathbb{N}_\infty\}\amalg\{j(m)\mid m\in\mathbb{N}_\omega\}\amalg\{k\},$$

and we define head: $\mathcal{A}\to\mathcal{H}$ in the obvious way.

Remark 3.7 The interaction of the head map with the operations can be summarised as follows:

$$\begin{aligned}
t(q)\vee t(q')&=t(\min(q,q')), & t(q)\wedge t(q')&=t(\max(q,q')),\\
t(q)\vee j(m')&=t(q), & t(q)\wedge j(m')&\in\{j(m'),k\},\\
t(q)\vee k&=t(q), & t(q)\wedge k&=k,\\
j(m)\vee j(m')&=j(\max(m,m')), & j(m)\wedge j(m')&=k,\\
j(m)\vee k&\in\{j(m),k\}, & j(m)\wedge k&=k,\\
k\vee k&=k, & k\wedge k&=k.
\end{aligned}$$

Because of the indeterminate rules for $t(q)\wedge j(m')$ and $j(m)\vee k$, we cannot say that head: $\mathcal{A}\to\mathcal{H}$ is a homomorphism of ordered semirings.

Definition 3.8 We put $\mathbb{N}_*=\{\perp\}\amalg\mathbb{N}_\omega$, and give this the obvious order with $\perp$ as the smallest element. For $m\in\mathbb{N}_*$ and $S\subseteq\mathbb{N}_\infty$ we put

$$\tilde{j}(m,S)=\begin{cases} j(m,S) & \text{if } m>\perp \text{ and } S \text{ is small},\\ k(S) & \text{if } m=\perp \text{ or } S \text{ is big}.\end{cases}$$

Remark 3.9 The elements $\tilde{\jmath}(m,S)$ are distinct, except that $\tilde{\jmath}(m,S)$ is independent of m when S is big. The operations can be rewritten as follows:

$$\begin{aligned}
t(q,T)\vee t(q',T') &= t(\min(q,q'),T\cup T'),\\
t(q,T)\vee \tilde{\jmath}(m',S') &= t(q,T\cup S'),\\
\tilde{\jmath}(m,S)\vee \tilde{\jmath}(m',S') &= \tilde{\jmath}(\max(m,m'),S\cup S'),\\
t(q,T)\wedge t(q',T') &= t(\max(q,q'),T\cap T'),\\
t(q,T)\wedge \tilde{\jmath}(m',S') &= \begin{cases} \tilde{\jmath}(m',T\cap S') & \text{if } q\le m',\\ \tilde{\jmath}(\perp,T\cap S') & \text{if } q>m',\end{cases}\\
\tilde{\jmath}(m,S)\wedge \tilde{\jmath}(m',S') &= \tilde{\jmath}(\perp,S\cap S').
\end{aligned}$$

Proposition 3.10 *$\mathcal{A}$ is an ordered semiring.*

Proof The operations are commutative by construction, and it is immediate from the definitions that $0\vee x=1\wedge x=x\vee x=x$ and $0\wedge x=0$ and $1\vee x=1$. This leaves the associativity and distributivity axioms. Remark 3.5 takes care of the tails, so we just need to worry about the heads. This is just a lengthy but straightforward check of cases, which is most efficiently done using Remark 3.9. (We have also coded a partial formalisation using Maple.) □

The order on $\mathcal{A}$ can be made more explicit as follows:

- We have $t(q,T)\le t(q',T')$ if and only if $T\subseteq T'$ and $q\ge q'$.
- We never have $t(q,T)\le j(m,S)$ or $t(q,T)\le k(U)$.
- We have $j(m,S)\le t(q,T)$ if and only if $S\subseteq T$.
- We have $j(m,S)\le j(m',S')$ if and only if $S\subseteq S'$ and $m\le m'$.
- We have $j(m,S)\le k(U)$ if and only if $S\subseteq U$ and U is big.
- We have $k(U)\le t(q,T)$ if and only if $U\subseteq T$.
- We have $k(U)\le j(m,S)$ if and only if $U\subseteq S$.
- We have $k(U)\le k(U')$ if and only if $U\subseteq U'$.

We next want to show that $\mathcal{A}$ is completely distributive. Because of Lemma 2.10, we can concentrate on ideals in $\mathcal{A}$.

Definition 3.11 Let $\mathcal{I} \subseteq \mathcal{A}$ be an ideal. We put

$$A = \bigcup \{\operatorname{tail}(u) \mid u \in \mathcal{I}\} \subseteq \mathbb{N}_\infty,$$
$$Q = \{q \in \mathbb{N}_\infty \mid t(q) \in \operatorname{head}(\mathcal{I})\} \subseteq \mathbb{N}_\infty,$$
$$M = \{m \in \mathbb{N}_\omega \mid j(m) \in \operatorname{head}(\mathcal{I})\} \subseteq \mathbb{N}_\omega.$$

We define $\theta(\mathcal{I}) \in \mathcal{A}$ as follows:

(a) If $Q = \varnothing$ and either A is big or M is empty, then $\theta(\mathcal{I}) = k(A)$.

(b) If $Q = \varnothing$ and A is small and M is nonempty but has no largest element, then $\theta(\mathcal{I}) = j(\omega, A)$.

(c) If $Q = \varnothing$ and A is small and M has a largest element, then $\theta(\mathcal{I}) = j(\max(M), A)$.

(d) If $Q \neq \varnothing$, then $\theta(\mathcal{I}) = t(\min(Q), A)$.

(It would be possible to combine cases (b) and (c) in the above definition, but it is more convenient to keep them separate, because they behave differently in various arguments that will be given later.)

Lemma 3.12 *We also have $A = \{i \in \mathbb{N}_\infty \mid k(i) \in \mathcal{I}\}$, and $k(A)$ is the least upper bound for $\mathcal{I} \cap \mathcal{A}_k$.*

Proof If $i \in A$ then there exists $u \in \mathcal{I}$ with $i \in \operatorname{tail}(u)$, which means that $k(i) \wedge u = k(i)$. As $\mathcal{I}$ is an ideal, this means that $k(i) \in \mathcal{I}$. Conversely, if $k(i) \in \mathcal{I}$ then $\{i\} = \operatorname{tail}(k(i)) \subseteq A$. This proves the alternative description of A, and the second claim follows easily from that. □

Lemma 3.13 *In cases* (c) *and* (d) *of Definition 3.11 we have $\theta(\mathcal{I}) \in \mathcal{I}$, and $\theta(\mathcal{I}) \geq u$ for all $u \in \mathcal{I}$, so $\theta(\mathcal{I})$ is the largest element of $\mathcal{I}$.*

Proof We first consider case (c), and put $m_0 = \max(M)$. By the definition of M, there is a small set S_0 such that $j(m_0, S_0) \in \mathcal{I}$. By the definition of A we have $S_0 \subseteq A$. By assumption, the set A is small, and therefore finite. For each $i \in A$ we have $k(i) \in \mathcal{I}$ by Lemma 3.12, and so the element

$$\theta(\mathcal{I}) = j(m_0, A) = j(m_0, S_0) \vee \textstyle\bigvee_{i \in A} k(i)$$

also lies in $\mathcal{I}$. Now consider an arbitrary element $u \in \mathcal{I}$. By assumption we have $Q = \varnothing$, so u is either $j(m, S)$ (for some $m \in M$ and $S \subseteq A$) or $k(S)$ (for some $S \subseteq A$). In all cases it is clear that $u \leq \theta(\mathcal{I})$, as required.

Now consider case (d), and put $q_0 = \min(Q)$. By the definition of Q, there is a cosmall set T_0 such that $t(q_0, T_0) \in \mathcal{I}$. By the definition of A we have $T_0 \subseteq A$, and T_0 is cosmall, so $A = T_0 \amalg A_0$ for some finite set $A_0 \subset \mathbb{N}$. For $i \in A_0$ we have $k(i) \in \mathcal{I}$ by Lemma 3.12, so the element

$$\theta(\mathcal{I}) = t(q_0, A) = t(q_0, T_0) \vee \bigvee_{i \in A_0} k(i)$$

also lies in $\mathcal{I}$. Now consider an arbitrary element $u \in \mathcal{I}$. If $u \in \mathcal{A}_j \amalg \mathcal{A}_k$ then $u = j(m, S)$ or $u = k(S)$ for some $S \subseteq A$, and this gives $u \leq \theta(\mathcal{I})$ (independent of the value of m). If $u \in \mathcal{A}_t$ then $u = t(q, T)$ for some $q \in Q$ and $T \subseteq A$, and we must have $q \geq \min(Q) = q_0$, which again gives $u \leq \theta(\mathcal{I})$, as required. □

Lemma 3.14 *In case* (a) *of Definition 3.11, the element $\theta(\mathcal{I}) = k(A)$ is the least upper bound for $\mathcal{I}$.*

Proof We see from Lemma 3.12 that the element $\theta(\mathcal{I}) = k(A)$ is the least upper bound for $\mathcal{I} \cap \mathcal{A}_k$, so we just need to check that it is an upper bound for all of $\mathcal{I}$. Consider an arbitrary element $u \in \mathcal{I}$. As $Q = \varnothing$ we must have $u = j(m, S)$ or $u = k(S)$ for some $S \subseteq A$. As A is big, it follows that $u \leq k(A)$, as required. □

Lemma 3.15 *In case* (b) *of Definition 3.11, the set M is infinite and contained in $\mathbb{N}$. Moreover, we have $j(m, A) \in \mathcal{I}$ for all $m \in M$, and the element $\theta(\mathcal{I}) = j(\omega, A)$ is the least upper bound for $\mathcal{I}$.*

Proof By assumption, M is a nonempty subset of $\mathbb{N}_\omega$ with no largest element. By inspection, this is only possible if M is an infinite subset of $\mathbb{N}$. Moreover, the set A is small and therefore finite. It follows using Lemma 3.12 that $k(A) \in \mathcal{I}$. If $m \in M$ then $j(m, S_m) \in \mathcal{I}$ for some S_m, which must be a subset of A. It follows that the element $j(m, A) = j(m, S_m) \vee k(A)$ also lies in $\mathcal{I}$.

Now let u be an arbitrary element of $\mathcal{I}$. As $Q = \varnothing$, we must have $u = j(m, S)$ for some $m \in M$ and $S \subseteq A$, or $u = k(S)$ for some $S \subseteq A$. From this it is easy to check that $j(\omega, A)$ is the least upper bound. □

Proposition 3.16 *$\mathcal{A}$ is completely distributive.*

Proof We will use the criteria in Lemma 2.10. Let $\mathcal{I} \subseteq \mathcal{A}$ be an ideal. Lemmas 3.13, 3.14 and 3.15 show that the element $a = \theta(\mathcal{I})$ is always a least upper bound for $\mathcal{I}$. It follows that $\mathcal{A}$ is complete.

Now consider an element $x \in \mathcal{A}$, and put $\mathcal{I}' = x \wedge \mathcal{I}$ and $a' = \bigvee \mathcal{I}'$. It is clear that $x \wedge a$ is an upper bound for $\mathcal{I}'$, so $a' \leq x \wedge a$, and we must show that this is an equality. This is clear from Lemma 3.13 in cases (c) and (d) of Definition 3.11, so we need only consider cases (a) and (b).

In these cases we have $\mathcal{I}' \subseteq \mathcal{I} \subseteq \mathcal{A}_j \cup \mathcal{A}_k$, and also $a \in \mathcal{A}_j \cup \mathcal{A}_k$. Note that

$$\mathrm{tail}(a') = \bigcup\{\mathrm{tail}(u') \mid u' \in \mathcal{I}'\} = \mathrm{tail}(x) \cap \mathrm{tail}(a) = \mathrm{tail}(x \wedge a),$$

so we just need to worry about the head.

Now suppose that x also lies in $\mathcal{A}_j \cup \mathcal{A}_k$. From the definitions we have

$$(\mathcal{A}_j \cup \mathcal{A}_k) \wedge (\mathcal{A}_j \cup \mathcal{A}_k) = \mathcal{A}_k,$$

and it follows that $\mathrm{head}(a') = k = \mathrm{head}(x \wedge a)$, as required.

Now suppose instead that $x = t(q, T)$.

In case (a) we then have $x \wedge a = k(T \cap A)$, and $T \cap A$ is big (because A is big and T is cosmall). Using Lemma 3.12 we see that $k(i) \in x \wedge \mathcal{I}$ for all $i \in T \cap A$, and it follows that $a' \geq k(T \cap A) = x \wedge a$, as required.

Finally, consider case (b) (still with $x = t(q, T)$). Put $M' = \{m' \in M \mid m' \geq q\}$. Using Lemma 3.15 we see that M' is an infinite subset of $\mathbb{N}$, and that $j(m', A) \in \mathcal{I}$ for all $m' \in M'$. In this context we have $x \wedge j(m', A) = j(m', A \cap T)$. It follows that $a' \geq j(m', A \cap T)$ for all $m' \in M'$, and thus that $a' \geq j(\omega, A \cap T) = x \wedge a$, as required. □

Proposition 3.17 $$\mathcal{A}_{\mathrm{latt}} = \mathcal{A}_t \amalg \mathcal{A}_k.$$

Proof Just inspect the definitions to see which elements satisfy $x \wedge x = x$. □

Proposition 3.18 *We have*

$$\mathcal{A}_{\mathrm{bool}} = \{t(0, T) \mid T \textit{ is cosmall}\} \amalg \{k(U) \mid U \textit{ is small}\},$$

with $\neg t(0, T) = k(\mathbb{N}_\infty \setminus T)$ *and* $\neg k(U) = t(0, \mathbb{N}_\infty \setminus U)$.

Proof Inspection of the definitions shows that when $U \subseteq \mathbb{N}_\infty$ is small, we have $t(0, \mathbb{N}_\infty \setminus U) \vee k(U) = t(0, \mathbb{N}_\infty) = 1$ and $t(0, \mathbb{N}_\infty \setminus U) \wedge k(U) = k(\varnothing) = 0$. Thus, the claimed elements all lie in $\mathcal{A}_{\mathrm{bool}}$. Conversely, suppose that x and y are complementary elements of $\mathcal{A}_{\mathrm{bool}}$. We must then have $x \vee y = 1 = t(0, \mathbb{N}_\infty)$. Inspection of the definitions shows that this is only possible if one of x and y has the form $t(0,T)$ for some cosmall T; we may assume without loss that $x = t(0,T)$. We must also have $t(0,T) \wedge y = 0$, and this is only possible if $y = k(U)$ with $U \cap T = \varnothing$. The condition $x \vee y = 1$ now reduces to $T \cup U = \mathbb{N}_\infty$, so we must have $U = \mathbb{N}_\infty \setminus U$. □

Remark 3.19 It is also possible to tabulate all the Heyting elements $(x \to y)$ for $x, y \in \mathcal{A}$, and to determine which of them are strong. Strong Heyting elements in $\mathcal{A}$ will give strong Heyting elements in $\bar{\mathcal{L}}$, but the same cannot be guaranteed for weak Heyting elements. The complete tabulation involves a rather long list of cases, so we will not give all the details here.

4 Basic Bousfield classes

We now introduce notation for various spectra, and the corresponding Bousfield classes. The names that we will use for some of these classes are the same as the names of elements of $\mathcal{A}$. Later we will consider the map $\phi\colon \mathcal{A} \to \mathcal{L}$ that sends each element of $\mathcal{A}$ to the element of $\mathcal{L}$ with the same name.

Definition 4.1 • For $n \in \mathbb{N}$ we let $K(n)$ denote the n^{th} Morava K–theory [10]. In particular, $K(0)$ is the rational Eilenberg–Mac Lane spectrum $H\mathbb{Q}$. We also write $K(\infty)$ for the mod p Eilenberg–Mac Lane spectrum, and $k(n) = \langle K(n) \rangle$.

- For any subset $U \subseteq \mathbb{N}_\infty$ we put $K(U) = \bigvee_{i \in U} K(i)$ and $k(U) = \langle K(U) \rangle$.

- It is a theorem of Mitchell [13] that for each $n \in \mathbb{N}$ we can choose a (p–locally) finite spectrum $U(n)$ of type n, meaning that $K(i)_* U(n) = 0$ if and only if $i < n$. We choose $U(0)$ to be S and $U(1)$ to be the Moore spectrum S/p. We put $F(n) = F(U(n), U(n))$, which is a self-dual finite ring spectrum of type n. Note that $F(0) = S^0$. In all cases we put $f(n) = \langle F(n) \rangle$. As a well-known consequence of the thick subcategory theorem [6, Theorem 7], these Bousfield classes do not depend on the choice of $U(n)$.

- For $q \in \mathbb{N}$ we recall that the Bott periodicity isomorphism $\Omega SU = BU$ gives a natural virtual vector bundle over $\Omega SU(p^q)$, and the associated Thom spectrum $X(p^q)$ has a natural ring structure. The p–localisation of this has a p–typical summand called $T(q)$ (see [16, Section 6.5]). We will also take $T(\infty) = BP$. Note that $T(0)$ is just the (p–local) sphere spectrum S. In all cases we put $t(q) = \langle T(q) \rangle$ and $t(q;n) = t(q) \wedge f(n)$.

- Now suppose we have $q \in \mathbb{N}$ and a cosmall set $T \subseteq \mathbb{N}_\infty$. For any n such that $[n, \infty] \subseteq T$, we define $t(q, T; n) = t(q; n) \vee k(T)$. We also define $t(q, T) = t(q, T; n_0)$, where n_0 is the smallest integer such that $[n_0, \infty] \subseteq T$.

- For $m \in \mathbb{N}_\infty$ we let $J(m)$ denote the Brown–Comenetz dual of $T(m)$, so there is a natural isomorphism

$$[X, J(m)] \simeq \mathrm{Hom}(\pi_0(T(m) \wedge X), \mathbb{Q}/\mathbb{Z}_{(p)})$$

for all spectra X. We also put $J(\omega) = \bigvee_{m \in \mathbb{N}} J(m)$, and $j(m) = \langle J(m) \rangle$ for all $m \in \mathbb{N}_\omega$. Given a small set S, we put $j(m, S) = j(m) \vee k(S)$.

- For $n \in \mathbb{N}$ we choose a good v_n self-map w_n of $U(n)$. (Here we use Definition 4.5 from [9], which is a slight modification of definitions used in [6; 3]. This means that $w_n \wedge 1 = 1 \wedge w_n$ as endomorphisms of $U(n) \wedge U(n)$, and that $1_{BP} \wedge w_n = v_n^{p^{d_n}} \wedge 1_{U(n)}$ as endomorphisms of $BP \wedge U(n)$ for some $d_n \geq 0$.) We also write w_n for the corresponding element of $\pi_*(F(n))$, and we put $K'(n) = F(n)[w_n^{-1}]$ and $k'(n) = \langle K'(n) \rangle$.

- Now fix $n \in \mathbb{N}$. Let L_n denote the Bousfield localisation functor with respect to the Johnson–Wilson spectrum $E(n)$, and let $C_n X$ denote the fibre of the natural map $X \to L_n X$. We also put $A(n) = C_n K'(n)$ and $a(n) = \langle A(n) \rangle$. Note here that the smash product theorem [17, Theorem 7.5.6] gives $A(n) = K'(n) \wedge C_n S$. We also put $\epsilon(n) = \bigvee_{i<n} a(i)$ for all $n \in \mathbb{N}_\infty$.

Remark 4.2 The original formulation of Ravenel's telescope conjecture [15, Conjecture 10.5] says that $k'(n) = k(n)$ for all $n \in \mathbb{N}$. It is shown in [12, Section 1.3] that this is equivalent to the claim that $K'(n) = L_n K'(n)$, which is in turn equivalent to $a(n) = 0$. These equivalences can also be obtained from Lemma 5.20 below. The formulation $a(n) = 0$ is also used in [7; 8]. We can reformulate it again as $\epsilon(n) = 0$ for all $n \in \mathbb{N}_\infty$, or as $\epsilon(\infty) = 0$.

Remark 4.3 We offer some translations between our notation and that used by some other authors:

(a) In [15, Section 3], Ravenel uses the notation X_n for what we have called $X(p^n)$. He only mentions $T(n)$ in passing, but he calls it T_n. In [16; 17], however, Ravenel uses the same notation as we do here.

(b) We have used the symbol $k(n)$ for the Bousfield class of the spectrum $K(n)$, with homotopy ring $\mathbb{Z}/p[v_n, v_n^{-1}]$. However, many other sources use the symbol $k(n)$ for a certain spectrum with homotopy ring $\mathbb{Z}/p[v_n]$, whose Bousfield class is different from that of $K(n)$. We will instead use the notation $BP\langle n\rangle/I_n$ for this spectrum.

(c) Our finite spectra $U(n)$ and $F(n)$ have type n, and they have the same Bousfield class as any other finite spectrum of type n. In particular, this applies to the Toda–Smith spectra when they exist. The Toda–Smith spectrum of type n is traditionally denoted by $V(n-1)$, but we will call it S/I_n.

(d) Our class $k'(n)$ is often denoted by $\mathrm{Tel}(n)$ or $T(n)$. Our notation is chosen to reflect the fact that $k'(n) = k(n)$ modulo the telescope conjecture.

Remark 4.4 The paper [12] is an incomplete attempt to disprove TC. It involves spectra called $y(n)$ and $Y(n)$, which we will not define here. In Section 3 of that paper, the authors say (in our notation) that $y(n)$ might be the same as $T(n) \wedge S/I_n$ in cases where S/I_n exists, and some of their calculations provide evidence for that possibility. As a closely related possibility, it might be that $\langle y(n)\rangle = t(n) \wedge f(n)$ as Bousfield classes for all n. This would give $\langle Y(n)\rangle = t(n) \wedge k'(n)$. If the strategy in [12] could be completed, it would show that $A(n) \wedge y(n) \neq 0$ for all $n > 1$. If we also knew that $\langle y(n)\rangle = t(n) \wedge f(n)$, we could conclude that $t(n) \wedge a(n) \neq 0$ for $n > 1$. On the other hand, it is known that $t(i) < t(j)$ whenever $i > j$, and that $t(\infty) \wedge a(n) = 0$. One would thus want to ask whether $t(n+1) \wedge a(n)$ is zero or not.

Definition 4.5 We define $\phi\colon \mathcal{A} \to \mathcal{L}$ to be the map that sends each element of $\mathcal{A}$ to the element of $\mathcal{L}$ with the same name.

Definition 4.6 Later we will prove that $\epsilon(n)$ is idempotent for all n. Assuming this for the moment, we can define

$$\bar{\mathcal{L}} = \varinjlim_{n<\infty} \mathcal{L}/\epsilon(n).$$

We write π for the canonical quotient map $\mathcal{L} \to \bar{\mathcal{L}}$, and we put $\bar{\phi} = \pi\phi\colon \mathcal{A} \to \bar{\mathcal{L}}$.

We will need some properties of the spectra $T(q)$.

Lemma 4.7 *The spectrum $T(q)$ is (-1)–connected, and each homotopy group is finitely generated over $\mathbb{Z}_{(p)}$.*

Proof As $X(p^q)$ is the Thom spectrum of a virtual bundle of virtual dimension zero, it is certainly (-1)–connected. It is a standard calculation that

$$H_*(X(p^q)) = \mathbb{Z}[b_i \mid 0 < i \leq p^q],$$

with $|b_i| = 2i$. Using this and the Atiyah–Hirzebruch spectral sequence

$$H_i(X(p^q); \pi_j(S)) \Rightarrow \pi_{i+j}(X(p^q)),$$

we see that the homotopy groups of $X(p^q)$ are finitely generated over $\mathbb{Z}$. As $T(q)$ is a summand in $X(p^q)_{(p)}$, we deduce that it is (-1)–connected, with homotopy groups that are finitely generated over $\mathbb{Z}_{(p)}$. □

Lemma 4.8 *For $q \geq r$ we have*

$$T(q)_*T(r) = T(q)_*[t_1, \dots, t_r]$$

(with $|t_i| = 2(p^i - 1)$).

The literature contains various similar and closely related results, but we have not been able to find this precise version.

Proof By construction [16, Section 6.5], there is a map $i_q\colon T(q) \to BP$ which induces an isomorphism from $BP_*T(q)$ to the subring $BP_*[t_1, \dots, t_q]$ of the ring $BP_*BP = BP_*[t_i \mid i > 0]$. This implies that the connectivity of the map i_q is $|t_{q+1}|-1$, which is strictly greater than $|t_r|$. The connectivity of the map

$$i_q \wedge 1\colon T(q) \wedge T(r) \to BP \wedge T(r)$$

is at least as large as that of i_q, so the elements $t_i \in BP_*T(r)$ have unique preimages in $T(q)_*T(r)$, which we also denote by t_i. These give us a map

$$\alpha\colon T(q)_*[t_1, \dots, t_r] \to T(q)_*T(r).$$

From the description of $BP_*T(r)$ it follows easily that $H_*(T(r)) = \mathbb{Z}_{(p)}[t_1, \dots, t_r]$, so we have an Atiyah–Hirzebruch spectral sequence

$$H_*(T(r); T(q)_*) = T(q)_*[t_1, \dots, t_r] \Rightarrow T(q)_*T(r).$$

The map α provides enough permanent cycles to show that the spectral sequence collapses, and it follows that α is an isomorphism. □

Lemma 4.9 *If $m \leq m' \leq \infty$, then $T(m)$ can be expressed as the homotopy inverse limit of a tower of spectra $Q(r)$, where the fibre of the map $Q(r+1) \to Q(r)$ is a product of suspended copies of $T(m')$, and $Q(r) = 0$ for $r < 0$.*

Proof This is essentially a standard construction with generalised Adams resolutions.

Let $j\colon M \to S$ denote the fibre of the unit map $S \to T(m')$ (where S denotes the p–local sphere). Recall that $H_*T(m')$ is a polynomial ring over $\mathbb{Z}_{(p)}$ with generators t_i in degree $2(p^i-1)$ for $1 \leq i \leq m'$. It follows that M is $(d-1)$–connected, where $d = 2(p-1) > 0$. Now put $N(r) = M^{(r)} \wedge T(m)$. We can use j to make these into a tower. We let $P(r)$ denote the cofibre of the map $N(r+1) \to N(r)$, which is $T(m') \wedge M^{(r)} \wedge T(m)$. We also let $Q(r)$ denote the cofibre of the map $N(r) \to N(0) = T(m)$. Connectivity arguments show that $T(m)$ is the homotopy inverse limit of the spectra $Q(r)$. We know, from Lemma 4.8,

$$T(m')_*T(m') = T(m')_*[t_i \mid 1 \leq i \leq m'],$$
$$T(m')_*T(m) = T(m')_*[t_i \mid 1 \leq i \leq m].$$

It follows that the spectra $T(m') \wedge T(m')$ and $T(m') \wedge T(m)$ are free modules over $T(m')$, and thus that the same is true of $P(r)$. This means that $P(r) = \bigvee_i \Sigma^{d_i} T(m')$ for some sequence (d_i). It is also easy to see that $P(r)$ is of finite type, so $d_i \to \infty$, so $P(r)$ can also be described as $\prod_i \Sigma^{d_i} T(m')$. Note also that the fibre of the map $Q(r+1) \to Q(r)$ is the same as $P(r)$, by the octahedral axiom. □

We will also need the following fact about $K'(n)$:

Lemma 4.10 *$K'(n)$ admits a ring structure such that the natural map $F(n) \to K'(n)$ is a ring map.*

The standard way to prove this is to show that $K'(n)$ is a Bousfield localisation of $F(n)$. We will give essentially the same argument, formulated in a more direct way.

Proof If Y is a finite spectrum of type $n+1$ then $1_{DY} \wedge w_n$ induces a nilpotent endomorphism of $MU \wedge (DY \wedge F(n))$, so the nilpotence theorem tells us that $1_{DY} \wedge w_n$ is itself nilpotent, which implies that the spectrum $F(Y, K'(n)) = DY \wedge F(n)[w_n^{-1}]$ is zero.

Now let Q be the cofibre of the natural map $F(n) \to K'(n)$. It is not hard to see that this is a homotopy colimit of spectra isomorphic to $F(n)/w_n^k$, which are finite and of type $n+1$. Using this, we see that $F(Q, K'(n)) = 0$. It follows inductively that the restriction maps

$$F(K'(n)^{(r)}, K'(n)) \to F(F(n)^{(r)}, K'(n))$$

are isomorphisms for all $r \geq 0$. Using the case $r = 2$, we see that the map

$$F(n) \wedge F(n) \xrightarrow{\text{mult}} F(n) \to K'(n)$$

extends in a unique way over $K'(n) \wedge K'(n)$. Using the cases $r = 3$ and $r = 1$, we see that this extension gives an associative and unital product. $\square$

5 Relations in $\mathcal{L}$

We first recall some basic general facts about Bousfield classes:

Proposition 5.1 (a) *If R is a ring spectrum then $\langle R\rangle \wedge \langle R\rangle = \langle R\rangle$. Moreover, if M is any R–module spectrum then $\langle M\rangle = \langle R\rangle \wedge \langle M\rangle \leq \langle R\rangle$.*

(b) *Let K be a ring spectrum such that all nonzero homogeneous elements of K_* are invertible. Then for any X we have either $K_*X = 0$ and $\langle K\rangle \wedge \langle X\rangle = 0$, or $K_*X \neq 0$ and $\langle K\rangle \wedge \langle X\rangle = \langle K\rangle$ and $\langle X\rangle \geq \langle K\rangle$.*

(c) *Let X be a spectrum, and let $v\colon \Sigma^d X \to X$ be a self-map with cofibre X/v and telescope $X[v^{-1}]$. Then $\langle X\rangle = \langle X/v\rangle \vee \langle X[v^{-1}]\rangle$.*

(d) *Let T and X be spectra such that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$. Then $T \wedge IX = 0$ if and only if $T \wedge I(X/p) = 0$ if and only if $F(T, X/p) = 0$.*

(e) *Suppose again that the homotopy groups of X are finitely generated over $\mathbb{Z}_{(p)}$, and that they are not all torsion groups. Then $\langle X\rangle = \langle X_p^\wedge\rangle = \langle H\mathbb{Q}\rangle \vee \langle X/p\rangle$.*

Proof None of this is new, but we will give brief proofs for the convenience of the reader.

(a) It is immediate from the definitions that $\langle X \wedge Y\rangle \leq \langle X\rangle$ and $\langle X \wedge Y\rangle \leq \langle Y\rangle$. Similarly, it is clear that $\langle X\rangle \leq \langle Y\rangle$ whenever X is a retract of Y. If R is a ring and M is an R–module then M is a retract of $R \wedge M$ (via the unit map $\eta \wedge 1\colon M \to R \wedge M$ and the multiplication $R \wedge M \to M$), so $\langle M\rangle \leq \langle R \wedge M\rangle$. On the other hand, we have $\langle R \wedge M\rangle \leq \langle R\rangle$ and $\langle R \wedge M\rangle \leq \langle M\rangle$. Putting this together gives

$\langle M\rangle=\langle R\rangle\wedge\langle M\rangle\leq\langle R\rangle$, as claimed. Taking $M=R$ gives $\langle R\rangle\wedge\langle R\rangle=\langle R\rangle$. This is all covered by [2, Section 2.6; 15, Proposition 1.24].

(b) A slight adaptation of standard linear algebra shows that all graded modules over K_* are free. If M is a K–module then we can choose a basis $\{e_i\}_{i\in I}$ for M_* over K_*, and this will give a map $f\colon \bigvee_{i\in I}\Sigma^{|e_i|}K\to M$ of K–modules such that $\pi_*(f)$ is an isomorphism, which means that f is an equivalence. Thus, if $M_*\neq 0$ then $\langle M\rangle=\langle K\rangle$. Taking $M=K\wedge X$ gives claim (b). This is all covered in [6, Section 1.3].

(c) First note that if $X=0$ then it is clear that $X/v=0$ and $X[v^{-1}]=0$. Conversely, if $X/v=0$ then v is an equivalence, so $X[v^{-1}]=X$; so if $X[v^{-1}]$ is also 0, then $X=0$. Thus, we have $X=0$ if and only if $X/v=X[v^{-1}]=0$. Now let T be an arbitrary spectrum, and put

$$w=1_T\wedge v\colon T\wedge X\to T\wedge X,$$

so $T\wedge(X/v)=(T\wedge X)/w$ and $T\wedge X[v^{-1}]=(T\wedge X)[w^{-1}]$. By applying our first claim to w, we see that $T\wedge X=0$ if and only if $T\wedge(X/v)=T\wedge X[v^{-1}]=0$. In other words, we have $\langle X\rangle=\langle X/v\rangle\vee\langle X[v^{-1}]\rangle$, as claimed. This is [15, Lemma 1.34].

(d) First, we have $\pi_k(IX)=\mathrm{Hom}(\pi_{-k}(X),\mathbb{Q}/\mathbb{Z}_{(p)})$. Using the fact that $\pi_{-k}(X)$ is finitely generated, we see that this is a torsion group. It follows that $(IX)[p^{-1}]=0$, so (c) gives $\langle IX\rangle=\langle (IX)/p\rangle$. On the other hand, I converts cofibrations to fibrations (with arrows reversed), giving $(IX)/p=\Sigma I(X/p)$, so $\langle IX\rangle=\langle I(X/p)\rangle$, so $T\wedge IX=0$ if and only if $T\wedge I(X/p)=0$. Next, we note that each homotopy group $\pi_k(X/p)$ is finite, which implies that the natural map

$$\pi_k(X/p)\to\mathrm{Hom}\big(\mathrm{Hom}(\pi_k(X/p),\mathbb{Q}/\mathbb{Z}_{(p)}),\mathbb{Q}/\mathbb{Z}_{(p)}\big)$$

is an isomorphism, so the natural map $X/p\to I^2(X/p)$ is an equivalence. This gives

$$\pi_kF(T,X/p)=\pi_kF(T,I^2(X/p))=\mathrm{Hom}\big(\pi_{-k}(T\wedge I(X/p)),\mathbb{Q}/\mathbb{Z}_{(p)}\big).$$

It is well known that for an abelian group A we have

$$A=0\iff\mathrm{Hom}(A,\mathbb{Q}/\mathbb{Z}_{(p)})=0,$$

so $F(T,X/p)=0$ if and only if $T\wedge I(X/p)=0$, as claimed. (This is essentially covered by [15, Section 2].)

(e) As a special case of (c) we have $\langle X\rangle=\langle X[p^{-1}]\rangle\vee\langle X/p\rangle$. As everything is implicitly p–local we see that $X[p^{-1}]$ is a module over $S[p^{-1}]=S\mathbb{Q}=H\mathbb{Q}$, with homotopy groups $\pi_*(X)\otimes\mathbb{Q}\neq 0$, so $\langle X[p^{-1}]\rangle=\langle H\mathbb{Q}\rangle$, so $\langle X\rangle=\langle H\mathbb{Q}\rangle\vee\langle X/p\rangle$. Now let Y denote the p–completion of X, which can be constructed as the cofibre of the natural map $F(S\mathbb{Q},X)\to X$. As X is assumed to have finite type, we just have $\pi_*(Y)=\mathbb{Z}_p\otimes\pi_*(X)$, and this is again not a torsion group, so $\langle Y\rangle=\langle H\mathbb{Q}\rangle\vee\langle Y/p\rangle$. Moreover, as $F(S\mathbb{Q},X)$ is a module over $S\mathbb{Q}$ we see that $F(S\mathbb{Q},X)/p=0$ and so $X/p=Y/p$, which gives $\langle Y\rangle=\langle X\rangle$. $\square$

We next recall some relations between the elements named in Definition 4.1. Again, many of these results are in the literature, but it seems useful to collect proofs in one place.

Lemma 5.2 *For any $n\in\mathbb{N}_\infty$ and any spectrum X, we have either $K(n)_*X=0$ and $k(n)\wedge\langle X\rangle=0$, or $K(n)_*X\neq 0$ and $k(n)\wedge\langle X\rangle=k(n)$ and $\langle X\rangle\geq k(n)$.*

Proof This is a standard instance of Proposition 5.1(b). $\square$

Lemma 5.3 *For all i we have $k(i)\wedge k(i)=k(i)$, and $k(i)\wedge k(i')=0$ for $i\neq i'$. Thus, $k(U)\wedge k(U')=k(U\cap U')$ and $k(U)\vee k(U')=k(U\cup U')$ (for all $U,U'\subseteq\mathbb{N}_\infty$).*

Proof The first claim holds because $K(i)$ is a ring spectrum, and the second can be deduced from the fact that over $K(i)_*K(i')$ we have two isomorphic formal group laws of different heights. It is also proved as part of [15, Theorem 2.1]. The remaining claims are clear from the first two. $\square$

Lemma 5.4 *The elements $t(q)$, $f(n)$, $t(q;n)$, $k(i)$, $k(U)$ and $k'(i)$ all satisfy $u\wedge u=u$.*

Proof We have already seen the cases $k(i)$ and $k(U)$. The spectra $T(q)$ and $F(n)$ have ring structures by construction, and $K'(n)$ is also a ring by Lemma 4.10, so all remaining claims follow from Proposition 5.1(a). $\square$

Lemma 5.5 *For all $i\in\mathbb{N}_\infty$ and $n\in\mathbb{N}$ we have $k(i)\wedge f(n)=0$ if $i<n$, and $k(i)\wedge f(n)=k(i)$ if $i\geq n$.*

Proof The spectrum $F(n)$ was defined to have type n, which means by definition that $K(i)_*F(n)=0$ if and only if $i<n$. The claim follows from this together with Lemma 5.2. $\square$

Lemma 5.6 *For all $q \leq q' \leq \infty$ we have $t(q) \geq t(q')$ and $t(q) \wedge t(q') = t(q')$.*

Proof There is a morphism $T(q) \to T(q')$ of ring spectra, which makes $T(q')$ into a $T(q)$–module spectrum. □

Lemma 5.7 *For all $q, i \in \mathbb{N}_\infty$ we have $t(q) \geq k(i)$ and $t(q) \wedge k(i) = k(i)$.*

Proof There is a morphism $T(q) \to T(\infty) = BP \to K(i)$ of ring spectra. □

Corollary 5.8 *For all $q \in \mathbb{N}_\infty$ and $n \in \mathbb{N}$ and $U \subseteq \mathbb{N}_\infty$ we have $t(q;n) \wedge k(U) = k(U \cap [n,\infty])$.*

Proof This is clear from Lemmas 5.5 and 5.7. □

Lemma 5.9 *Let $X \in \mathcal{B}$ be such that $\pi_i(X)$ is torsion for all i, and $\pi_i(X) = 0$ for $i > 0$. Then $\langle X \rangle \leq k(\infty)$.*

Proof Put

$$\mathcal{C} = \{X \mid \langle X \rangle \leq k(\infty)\} = \{X \mid X \wedge Z = 0 \text{ whenever } K(\infty) \wedge Z = 0\}.$$

This is closed under cofibres, coproducts and retracts, and it follows that it is closed under homotopy colimits of sequences. It contains $K(\infty) = H\mathbb{Z}/p$ by definition, so it contains HA whenever $pA = 0$ (by coproducts), so it contains HA whenever $p^d A = 0$ (by cofibres), so it contains HA whenever A is torsion (by sequential colimits). Thus, if X is a torsion spectrum, we see that all the Postnikov sections $X[-d] = \Sigma^{-d} H(\pi_{-d} X)$ lie in $\mathcal{C}$, so $X[-d,0] \in \mathcal{C}$ for all $d \geq 0$ (by induction and cofibres), so $X = X[-\infty, d] \in \mathcal{C}$ (by sequential colimits). □

Lemma 5.10 *For all $m \in \mathbb{N}_\omega$ we have $j(m) \leq k(\infty)$.*

Proof For $m \neq \omega$ we have $J(m) = IT(m)$, and $T(m)$ is (-1)–connected with finitely generated homotopy groups, so Lemma 5.9 applies to $J(m)$. As $J(\omega) = \bigvee_{i \in \mathbb{N}} J(i)$, the claim holds for $m = \omega$ as well. □

Lemma 5.11 *If $m \leq m' \in \mathbb{N}_\omega$, then $j(m) \leq j(m')$.*

Proof The case $m' = \omega$ is immediate from the definition of $J(\omega)$, and the case $m = \omega$ will follow from the cases $m \in \mathbb{N}$, so we may assume that $\omega \notin \{m, m'\}$.

We must show that if $X \wedge J(m') = 0$ then $X \wedge J(m) = 0$. In view of Lemma 4.7, we can translate these statements using part (d) of Proposition 5.1. We must now show that if $F(X, T(m')/p) = 0$ then $F(X, T(m)/p) = 0$. This translated statement follows easily from Lemma 4.9. □

Lemma 5.12 *For all $m \in \mathbb{N}_\omega$ and $q \in \mathbb{N}_\infty$ and $n \in \mathbb{N}$ we have $j(m) \leq k(\infty) \leq t(q;n)$. Moreover, if $m < q$ then we have $t(q) \wedge j(m) = 0$, but if $m \geq q$ then $t(q) \wedge j(m) = j(m)$.*

Most of the statements with $m = 0$ are contained in [8, Lemma 7.1].

Proof We know from Lemma 5.10 that $j(m) \leq k(\infty)$, and from Lemma 5.7 that $k(\infty) \leq t(q)$, and from Lemma 5.5 that $k(\infty) \leq f(n)$. It follows that

$$k(\infty) = k(\infty) \wedge k(\infty) \leq t(q) \wedge f(n) = t(q;n),$$

as claimed.

For the remaining statements, the case $m = \omega$ follows easily from the cases $m \in \mathbb{N}$. We will therefore assume that $m \in \mathbb{N}_\infty$.

Suppose that $m \geq q$. Then $T(m)$ is naturally a $T(q)$–module, so $J(m) = I(T(m))$ is naturally a $T(q)$–module, which implies (by Proposition 5.1(b)) that $t(q) \wedge j(m) = j(m)$.

We now just need to show that when $m < q$ we have $T(q) \wedge J(m) = 0$. By Proposition 5.1(d), this is equivalent to $F(T(q), T(m)/p) = 0$. If $q = \infty$ then this is [15, Lemma 3.2(b)]. If $q < \infty$ then we can use Lemma 5.11 to reduce to the case $m = q - 1$, which is [15, Lemma 3.2(a)]. □

Lemma 5.13 *For all $n \in \mathbb{N}_\infty$ and $m \in \mathbb{N}_\omega$ we have $k(n) \wedge j(m) = 0$.*

Proof First suppose that $n < \infty$, so $k(n) \wedge k(\infty) = 0$. We have $j(m) \leq k(\infty)$ by Lemma 5.10, so $k(n) \wedge j(m) = 0$.

Now consider the case where $n = \infty$ and $m \in \mathbb{N}$. We have $t(m+1) \wedge j(m) = 0$ by Lemma 5.12, but $k(\infty) \leq t(m+1)$ by Lemma 5.7, so $k(\infty) \wedge j(m) = 0$. The case $m = \omega$ follows from this.

Finally, consider the case where $n = m = \infty$. Here the claim is that $H/p \wedge IBP = 0$, or equivalently that $F(H/p, BP/p) = 0$. This is the first step in the proof of Theorem 2.2 of [15]. □

Lemma 5.14 *For all $i, j \in \mathbb{N}$ we have $k'(i) \wedge k'(i) = k'(i)$, but $k'(i) \wedge k'(j) = 0$ for $i \neq j$. We also have $k'(i) \wedge f(j) = 0$ if $i < j$, and $k'(i) \wedge f(j) = k'(i)$ if $i \geq j$. Finally, we have $f(n) = k'(n) \vee f(n+1)$.*

Proof If $i < j$ then v_i is nilpotent in $MU_*(F(i) \wedge F(j))$, so the nilpotence theorem tells us that $w_i \wedge 1_{F(j)}$ is nilpotent as a self-map of $F(i) \wedge F(j)$, so $K'(i) \wedge F(j) = 0$, so $k'(i) \wedge f(j) = 0$. It is clear that $k'(j) \leq f(j)$, so we also have $k'(i) \wedge k'(j) = 0$ when $i < j$. By symmetry, this actually holds whenever $i \neq j$.

Next, Proposition 5.1(c) gives

$$f(n) = \langle F(n)/v \rangle \vee \langle F(n)[v^{-1}] \rangle = \langle F(n)/v \rangle \vee k'(n).$$

The thick subcategory theorem shows that $\langle F(n)/v \rangle = f(n+1)$, so we obtain $f(n) = f(n+1) \vee k'(n)$ (and we saw above that $f(n+1) \wedge k'(n) = 0$). An induction based on this shows that $1 = f(0) = f(j) \vee \bigvee_{m<j} k'(m)$. We can multiply this by $k'(i)$ and use the relations that we have already established to get $k'(i) \wedge k'(i) = k'(i)$ if $i < j$, and $k'(i) \wedge f(j) = k'(i)$ for $i \geq j$. □

The next result is closely related to [7, Section 1].

Lemma 5.15 *For all $n \in \mathbb{N}$ we have $k(n) = t(\infty) \wedge k'(n) = t(\infty) \wedge k(n)$ and $t(\infty) \wedge a(n) = 0$.*

Proof By construction, the spectrum $T(\infty) \wedge K'(n)$ is obtained by inverting the self-map $u = 1_{BP} \wedge w_n$ of $BP \wedge F(n)$. However, we chose w_n to be good, which means that u is the same as $v_n \wedge 1_{F(n)}$, so $T(\infty) \wedge K'(n) = v_n^{-1} BP \wedge F(n)$. We also know from [15, Theorem 2.1] that $\langle v_n^{-1} BP \rangle = \langle E(n) \rangle = \bigvee_{i \leq n} k(i)$, and it follows that $t(\infty) \wedge k'(n) = k(n)$, as claimed. This is the same as $t(\infty) \wedge k(n)$ by Lemma 5.7.

Now recall that $A(n) = K'(n) \wedge C_n S$, and by definition we have $E(n) \wedge C_n S = 0$. As $\langle T(\infty) \wedge K'(n) \rangle = \langle E(n) \rangle$, this gives

$$T(\infty) \wedge A(n) = T(\infty) \wedge K'(n) \wedge C_n S = 0,$$

so $t(\infty) \wedge a(n) = 0$. □

Lemma 5.16 *For all $n \in \mathbb{N}$ we have $k(n) \leq k'(n)$ and $k'(n) \wedge k(n) = k(n)$, whereas $k'(n) \wedge k(m) = 0$ for $m \neq n$.*

Proof Multiply the equations in Lemma 5.14 by $t(\infty)$ and then use Lemma 5.15. □

Lemma 5.17 *For all $n, n' \in \mathbb{N}$ we have*

$$f(n) \vee f(n') = f(\min(n, n')) \quad \textit{and} \quad f(n) \wedge f(n') = f(\max(n, n')).$$

Proof Recall that $F(n)$ has type n, which means that $K(i)_* F(n)$ is zero for $i < n$, and nonzero for $i \geq n$. It follows that $F(n) \vee F(n')$ has type $\min(n, n')$, and $F(n) \wedge F(n')$ has type $\max(n, n')$. The thick subcategory theorem tells us that the Bousfield class of a finite p–local spectrum depends only on its type, so $f(n) \vee f(n') = f(\min(n, n'))$ and $f(n) \wedge f(n') = f(\max(n, n'))$. □

Lemma 5.18 *The elements $a(n)$ satisfy*

$$a(n) \wedge a(n) = a(n) \leq k'(n) \quad \text{and} \quad a(n) \wedge a(m) = 0$$

for $m \neq n$. Thus, the element $\epsilon(n) = \bigvee_{i<n} a(i)$ is idempotent for all $n \in \mathbb{N}_\infty$.

This is also proved in [7, Section 1].

Proof First, the smash product theorem [17, Theorem 7.5.6] tells us that $A(n) = K'(n) \wedge C_n S$, so $a(n) \leq k'(n)$, so $a(n) \wedge a(m) = 0$ for $m \neq n$ by Lemma 5.14. We also have $C_n S \wedge C_n S = C_n(C_n S) = C_n S$ by the basic theory of Bousfield localisation, and in combination with Lemma 5.14 this gives $a(n) \wedge a(n) = a(n)$. □

Corollary 5.19 *For all n in $\mathbb{N}$ we have $\epsilon(n) \wedge f(n) = 0$.*

Proof As $\epsilon(n) = \bigvee_{i<n} a(i)$, it will be enough to show that $a(i) \wedge f(n) = 0$ for $i < n$. The lemma shows that $a(i) \leq k'(i)$, and $k'(i) \wedge f(n) = 0$ by Lemma 5.14, so $a(i) \wedge f(n) = 0$, as required. □

Lemma 5.20 *The elements $a(n)$ satisfy $a(n) \wedge k(m) = 0$ for all n and m, and $k'(n) = k(n) \vee a(n)$.*

This is also proved in [7, Section 1].

Proof We saw in Lemma 5.15 that $t(\infty) \wedge a(n) = 0$, and $k(m) \leq t(\infty)$ (even for $m = \infty$) by Lemma 5.7, so $k(m) \wedge a(n) = 0$. Next, it follows from the smash product theorem that $L_n S \wedge X = 0$ if and only if $L_n X = 0$ if and only if $E(n) \wedge X = 0$, which means that $\langle L_n S \rangle = \langle E(n) \rangle$. This is also the same as $\bigvee_{i=0}^{n} k(i)$, by [15, Theorem 2.1]. We can multiply by $k'(n)$ and use Lemma 5.16 to get $\langle L_n K'(n) \rangle = k(n)$.

We also have a fibration

$$A(n) = C_n S \wedge K'(n) \to K'(n) \to L_n S \wedge K'(n),$$

which easily gives

$$k'(n) = \langle K'(n)\rangle = \langle A(n)\rangle \vee \langle L_n S \wedge K'(n)\rangle = a(n) \vee k(n),$$

as claimed. □

Lemma 5.21 *For all $m, m' \in \mathbb{N}_\omega$ we have $j(m) \wedge j(m') = 0$.*

Proof Lemma 5.10 gives $j(m) \leq k(\infty)$, and Lemma 5.13 gives $k(\infty) \wedge j(m') = 0$, so $j(m) \wedge j(m') = 0$. □

Lemma 5.22 *If $m \in \mathbb{N}_\omega$ and $U \subseteq \mathbb{N}$ is infinite then $j(m) \leq k(U)$.*

Proof In view of Lemma 5.11 we may assume that $m = \infty$, so $T(m) = BP$. Suppose that $K(U) \wedge X = 0$. Hovey proved as [7, Corollary 3.5] that $BP_p^\wedge$ is $K(U)$–local, so the spectrum $BP/p = (BP_p^\wedge)/p$ is also $K(U)$–local, so $F(X, BP/p) = 0$. It follows by Proposition 5.1(d) that $J(\infty) \wedge X = 0$. We conclude that $j(\infty) \leq k(U)$, as claimed. □

Corollary 5.23 *If $m \in \mathbb{N}_\omega$, and $U \subseteq \mathbb{N}_\infty$ is big then $j(m) \leq k(U)$.*

Proof This is just the conjunction of Lemmas 5.10 and 5.22. □

Lemma 5.24 *For all $n \in \mathbb{N}$ and $m \in \mathbb{N}_\omega$ we have $f(n) \wedge j(m) = j(m)$, and $a(n) \wedge j(m) = k'(n) \wedge j(m) = 0$.*

The statements with $m = 0$ are contained in [8, Lemma 7.1].

Proof The claims for $m = \omega$ follow easily from the claims for $m \in \mathbb{N}$, so we will assume that $m \in \mathbb{N}_\infty$.

We first prove that $f(n) \wedge j(m) = j(m)$. This is immediate when $n = 0$, and follows from Proposition 5.1(d) when $n = 1$, so we can suppose that $n > 1$. It is clear that $f(n) \wedge j(m) \leq j(m)$, so we just need the reverse inequality. Suppose that $X \wedge F(n) \wedge J(m) = 0$, or equivalently $F(X \wedge F(n), T(m)/p) = 0$. We chose $F(n)$ to be self-dual, so $F(X, F(n) \wedge T(m)/p) = 0$. By the thick subcategory theorem, we can replace $F(n)$ here by any other finite spectrum of type n, so in particular $F(X, F(n-1)/w_{n-1}^k \wedge T(m)/p) = 0$ for all k. As $n > 1$, a connectivity argument shows that $F(n-1) \wedge T(m)/p$ is the homotopy inverse limit of the spectra $F(n-1)/w_{n-1}^k \wedge T(m)/p$, so we see that $F(X, F(n-1) \wedge T(m)/p) = 0$. By reversing the previous steps, we get $X \wedge F(n-1) \wedge J(m) = 0$. This gives $f(n) \wedge j(m) = f(n-1) \wedge j(m)$, which is the same as $j(m)$ by induction.

We can multiply the relation $f(n+1) \wedge j(m) = j(m)$ by $k'(n)$ and use

$$k'(n) \wedge f(n+1) = 0$$

(from Lemma 5.14) to get $k'(n) \wedge j(m) = 0$. We also have $a(n) \leq k'(n)$ by Lemma 5.20, so $a(n) \wedge j(m) = 0$. □

6 The main theorem

By considering the phenomena in Lemma 6.3 below, we see that ϕ is unlikely to preserve either $\vee$ or $\wedge$ unless TC holds. However, if we pass to $\bar{\mathcal{L}}$ then we have the following:

Theorem 6.1 *The map $\bar{\phi} = \pi\phi\colon \mathcal{A} \to \bar{\mathcal{L}}$ is an injective homomorphism of ordered semirings.*

Proof See Corollaries 6.5 and 6.9 below. □

We must show that the rules in Definition 3.3 are valid as equations in $\bar{\mathcal{L}}$. In fact, most of them are already valid in $\mathcal{L}$:

Lemma 6.2 *The rules for $t \wedge j$, $t \wedge k$, $j \wedge j$, $j \wedge k$, $k \wedge k$, $j \vee j$, $j \vee k$ and $k \vee k$ are all valid in $\mathcal{L}$.*

(More concisely, these are all the rules where the right-hand side does not involve t.)

Proof • Consider the element $x = t(q, T) \wedge j(m', S')$. Let n be minimal such that $[n, \infty] \subseteq T$. Then x is the wedge of terms $u_1 = t(q; n) \wedge j(m')$ and $u_2 = t(q; n) \wedge k(S')$ and $u_3 = k(T) \wedge j(m')$ and $u_4 = k(T \cap S')$. Corollary 5.8 tells us that $u_2 = k(S' \cap [n, \infty]) \leq u_4$. We have $u_3 = 0$ by Lemma 5.13, and $u_1 = t(q) \wedge j(m')$ by Lemma 5.24. If $q \leq m'$ then Lemma 5.12 gives $u_1 = j(m')$ and so $x = u_1 \vee u_4 = j(m', T \cap S')$. If $q > m'$ then the same lemma gives $u_1 = 0$ and so $x = u_4 = k(T \cap S')$.

• Consider the element $x = t(q, T) \wedge k(U')$. This is the wedge of the terms $u_1 = t(q; n) \wedge k(U') = k(U' \cap [n, \infty])$ and $u_2 = k(T) \wedge k(U') = k(T \cap U') \geq u_1$, so $x = u_2 = k(T \cap U')$, as required.

- Consider the element $x = j(m,S) \wedge j(m',S')$. This is the wedge of terms $u_1 = j(m) \wedge j(m')$ and $u_2 = j(m) \wedge k(S')$ and $u_3 = k(S) \wedge j(m')$ and $u_4 = k(S \cap S')$. The first three terms are zero by Lemmas 5.21 and 5.13, so $x = u_4 = k(S \cap S')$.

- Consider the element $x = j(m,S) \wedge k(U')$. This is the wedge of terms $u_1 = j(m) \wedge k(U')$ and $u_2 = k(S \cap U')$. We have $u_1 = 0$ by Lemma 5.13, so $x = u_2 = k(S \cap U')$.

- We know from Lemma 5.3 that $k(U) \wedge k(U') = k(U \cap U')$.

- Put $x = j(m,S) \vee j(m',S')$. Then $x = j(m) \vee j(m') \vee k(S \cup S')$, but $j(m) \vee j(m') = j(\max(m,m'))$ by Lemma 5.21, which gives $x = j(\max(m,m'), S \cup S')$.

- Put $x = j(m,S) \vee k(U') = j(m) \vee k(S \cup U')$. If U' is big then so is $S \cup U'$, so $j(m) \leq k(S \cup U')$ by Corollary 5.23, so $x = k(S \cup U')$. On the other hand, we are assuming that S is small, so if U' is small then $S \cup U'$ will also be small, so $j(m, S \cup U')$ is defined and is equal to x.

- We know from Lemma 5.3 that $k(U) \vee k(U') = k(U \cup U')$. □

For the remaining rules, we have the following modified statement:

Lemma 6.3 *The following rules are valid in $\mathcal{L}$ (provided that n is large enough for the terms on the left to be defined):*

$$\begin{aligned} t(q,T;n) \wedge t(q',T';n) &= t(\max(q,q'), T \cap T'; n), \\ t(q,T;n) \vee t(q',T';n) &= t(\min(q,q'), T \cup T'; n), \\ t(q,T;n) \vee j(m',S') &= t(q, T \cup S'; n), \\ t(q,T;n) \vee k(U') &= t(q, T \cup U'; n). \end{aligned}$$

Proof • Consider the element $x = t(q,T;n) \wedge t(q',T';n)$. This is the wedge of the terms $u_1 = t(q;n) \wedge t(q';n')$ and $u_2 = t(q;n) \wedge k(T') = k(T' \cap [n,\infty])$ and $u_3 = k(T) \wedge t(q';n) = k(T \cap [n,\infty])$ and $u_4 = k(T \cap T')$. We are assuming that $[n,\infty) \subseteq T$ and $[n,\infty) \subseteq T'$, so $u_2, u_3 \leq u_4$. We also have $u_1 = t(\max(q,q');n)$ by Lemmas 5.6 and 5.17. This leaves $x = t(\max(q,q'), T \cap T'; n)$.

- We have $t(q,T;n) \vee t(q',T';n) = t(q;n) \vee t(q';n) \vee k(T \cup T')$, and, by Lemma 5.6, $t(q;n) \vee t(q';n) = t(\min(q,q');n)$, which leaves $t(\min(q,q'), T \cup T'; n)$.

- Put $x = t(q,T;n) \vee j(m',S')$. Then $x = t(q;n) \vee k(T \cup S') \vee j(m')$, but $j(m') \leq k(\infty) \leq t(q;n)$ by Lemma 5.10, so we can drop that term, giving $x = t(q, T \cup S'; n)$.

- We have $t(q,T;n) \vee k(U') = t(q;n) \vee k(T) \vee k(U') = t(q;n) \vee k(T \cup U') = t(q,T \cup U';n)$. □

Lemma 6.4 *In $\bar{\mathcal{L}}$ the element $t(q,T;n)$ is independent of the choice of n.*

Proof It is clear that $t(q,T;n) \geq t(q,T;n+1)$ in $\mathcal{L}$, and it will suffice to show that this becomes an equality in $\bar{\mathcal{L}}$. We have

$$f(n) = f(n+1) \vee k'(n) = f(n+1) \vee k(n) \vee a(n)$$

by Lemmas 5.14 and 5.20. In conjunction with Lemma 5.7 this gives

$$t(q;n) = t(q;n+1) \vee k(n) \vee (t(q) \wedge a(n)).$$

However, we are assuming that $[n,\infty] \subseteq T$, so $n \in T$, so $k(n) \vee k(T) = k(T)$, so

$$t(q,T;n) = t(q,T;n+1) \vee (t(q) \wedge a(n)).$$

The extra term is less than or equal to $\epsilon(n+1)$, so it is killed by the homomorphism $\mathcal{L} \to \mathcal{L}/\epsilon(n) \to \bar{\mathcal{L}}$. □

Corollary 6.5 *All the relations in Definition 3.3 are valid as equations in $\bar{\mathcal{L}}$, so the map $\bar{\phi} = \pi\phi\colon \mathcal{A} \to \bar{\mathcal{L}}$ is a homomorphism of semirings.*

Proof This is clear from Lemmas 6.2, 6.3 and 6.4. □

Remark 6.6 As well as $\bar{\mathcal{L}}$, we can also consider the object $\hat{\mathcal{L}} = \mathcal{L}/\epsilon(\infty)$. The canonical map $\mathcal{L} \to \hat{\mathcal{L}}$ then factors through $\bar{\mathcal{L}}$, so we see that the composite $\mathcal{A} \to \mathcal{L} \to \hat{\mathcal{L}}$ is also a homomorphism of ordered semirings. This has the advantage that $\hat{\mathcal{L}}$ is completely distributive, which we cannot prove for $\bar{\mathcal{L}}$. However, we do not know whether the map $\mathcal{A} \to \hat{\mathcal{L}}$ is injective.

Definition 6.7 We recall that $\mathcal{P}$ denotes the set of subsets of $\mathbb{N}_\infty$, and we define maps $\sigma_i\colon \mathcal{L} \to \mathcal{P}$ by

$$\sigma_1(x) = \{i \in \mathbb{N}_\infty \mid k(i) \wedge x \neq 0\} = \{i \in \mathbb{N}_\infty \mid k(i) \wedge x = k(i)\} = \{i \in \mathbb{N}_\infty \mid x \geq k(i)\},$$

$$\sigma_2(x) = \{i \in \mathbb{N}_\infty \mid j(i) \wedge x \neq 0\},$$

$$\sigma_3(x) = \{i \in \mathbb{N}_\infty \mid x \geq j(i)\}.$$

(The three versions of σ_1 agree by Lemma 5.2.)

Lemma 6.8 *There are maps $\overline{\sigma}_r\colon \overline{\mathcal{L}} \to \mathcal{P}$ (for $r = 0, 1, 2$) with $\overline{\sigma}_r \circ \pi = \sigma_r\colon \mathcal{L} \to \mathcal{P}$.*

Proof Lemmas 5.20 and 5.24 show that $k(i) \wedge \epsilon(\infty) = j(i) \wedge \epsilon(\infty) = 0$. It follows that when $r \leq 2$ we have $\sigma_r(\epsilon(n) \vee x) = \sigma_r(x)$ for all $n \in \mathbb{N}_\infty$ and all $x \in \mathcal{L}$. This means that σ_1 and σ_2 factor through $\overline{\mathcal{L}}$ (or even $\widehat{\mathcal{L}}$), as claimed.

Now consider σ_3. If $x \geq j(i)$, then of course $\epsilon(n) \vee x \geq j(i)$ for all $n \in \mathbb{N}$. Conversely, suppose that $n \in \mathbb{N}$ and $\epsilon(n) \vee x \geq j(i)$. It follows that

$$f(n) \wedge (\epsilon(n) \vee x) \geq f(n) \wedge j(i).$$

The right side is $j(i)$ by Lemma 5.24. On the left side, we have $f(n) \wedge \epsilon(n) = 0$ by Corollary 5.19. We therefore have $x \geq f(n) \wedge x \geq j(i)$. Putting this together, we see that $\sigma_3(\epsilon(n) \vee x) = \sigma_3(x)$ for all $n \in \mathbb{N}$, so σ_3 factors through $\overline{\mathcal{L}}$. (It is not clear, however, whether σ_3 factors through $\widehat{\mathcal{L}}$.) $\square$

Corollary 6.9 *The map $\overline{\phi}\colon \mathcal{A} \to \overline{\mathcal{L}}$ is injective.*

Proof It is easy to check the following table of values of the maps σ_r:

	σ_1	σ_2	σ_3
$t(q, T)$	T	$[q, \infty]$	$[0, \infty]$
$j(m, S)$	S	$\varnothing$	$[0, m] \cap \mathbb{N}_\infty$
$k(U)$ (small)	U	$\varnothing$	$\varnothing$
$k(U)$ (big)	U	$\varnothing$	$[0, \infty]$

(In particular, we have $\sigma_3(j(\omega, S)) = \mathbb{N}$ but $\sigma_3(j(\infty, S)) = \mathbb{N}_\infty$.) Now consider an element $x \in \mathcal{A}$ with $\sigma\phi(x) = u \in \mathcal{P}^3$. We see that:

- If $u_2 \neq \varnothing$ then $x = t(\min(u_2), u_1)$.
- If $u_2 = \varnothing$ and u_1 is small and $u_3 \neq \varnothing$ then $x = j(\sup(u_3), u_1)$ (where the supremum is taken in $\mathbb{N}_\omega$).
- If $u_2 = \varnothing$ and u_1 is small and $u_3 = \varnothing$ then $x = k(u_1)$.
- If $u_2 = \varnothing$ and u_1 is big then $x = k(u_1)$.

This means that the composite $\sigma\phi$ is injective, but this is the same as $\overline{\sigma}\overline{\phi}$, so $\overline{\phi}$ is injective. $\square$

We do not know whether $\overline{\mathcal{L}}$ is complete. However, we do have the following partial result:

Proposition 6.10 *Let S be any subset of $\mathcal{A}$, and put $a = \bigvee S \in \mathcal{A}$. Then $\overline{\phi}(a)$ is the least upper bound for $\overline{\phi}(S)$ in $\overline{\mathcal{L}}$.*

Proof Let V denote the set of upper bounds for $\overline{\phi}(S)$. As $\overline{\phi}$ is a homomorphism of ordered semirings, it is clear that $\overline{\phi}(a) \in V$. We must show that $\overline{\phi}(a)$ is the smallest element of V.

Now let $\mathcal{I}$ denote the ideal in $\mathcal{A}$ generated by S, so a is also the least upper bound for $\mathcal{I}$. This means that $a = \theta(\mathcal{I})$, where θ is as in Definition 3.11.

Note that the set

$$S' = \{x \in \mathcal{A} \mid \overline{\phi}(x) \leq v \text{ for all } v \in V\}$$

is an ideal containing S, so it contains $\mathcal{I}$. This means that V is also the set of upper bounds for $\overline{\phi}(\mathcal{I})$.

In cases (c) and (d) of Definition 3.11, Lemma 3.13 tells us that $a \in \mathcal{I}$, and the claim follows immediately from this. We therefore need only consider cases (a) and (b), in which $\mathcal{I} \subseteq \mathcal{A}_j \cup \mathcal{A}_k$.

Now let v be an element of $\mathcal{L}$ such that the image $\pi(v) \in \overline{\mathcal{L}}$ lies in V. This means that for all $x \in \mathcal{I}$ there exists $n \in \mathbb{N}$ such that $\epsilon(n) \vee v \geq \phi(x)$ in $\mathcal{L}$. We must show that there exists m such that $\epsilon(m) \vee v \geq \phi(a)$.

Recall that

$$A = \bigcup_{x \in \mathcal{I}} \operatorname{tail}(x) \subseteq \mathbb{N}_\infty,$$

and Lemma 3.12 tells us that $k(i) \in \mathcal{I}$ for all $i \in A$. This means that $\epsilon(n) \vee v \geq k(i)$ for some $n \in \mathbb{N}$. Lemma 5.20 tells us that $\epsilon(n) \wedge k(i) = 0$, so we can multiply the above relation by $k(i)$ to get

$$v \geq k(i) \wedge v \geq k(i) \wedge k(i) = k(i)$$

in $\mathcal{L}$. This holds for all $i \in A$, and the element $k(A) \in \mathcal{L}$ is by definition the least upper bound in $\mathcal{L}$ of the elements $k(i)$ with $i \in A$, so we get $v \geq k(A)$ in $\mathcal{L}$. In case (a), this is already the desired conclusion.

Finally, we consider case (b), in which A is small but the set

$$M = \{m \in \mathbb{N}_\omega \mid j(m) \in \operatorname{head}(\mathcal{I})\}$$

is an infinite subset of $\mathbb{N}$. Lemma 3.15 tells us that $j(m, A) \in \mathcal{I}$ for all $m \in M$. Thus, for each $m \in M$ there exists $n \in \mathbb{N}$ such that $\epsilon(n) \vee v \geq j(m, A) \geq j(m)$. We now multiply this relation by $f(n)$. Corollary 5.19 tells us that $f(n) \wedge \epsilon(n) = 0$, and Lemma 5.24 gives $f(n) \wedge j(m) = j(m)$, so we have

$$v \geq f(n) \wedge v \geq j(m).$$

Now recall that $j(i) \leq j(i+1)$ for $i \in \mathbb{N}$, and that the element $j(\omega) \in \mathcal{L}$ is by definition the join in $\mathcal{L}$ of these elements $j(i)$. We therefore have $v \geq j(\omega)$ in $\mathcal{L}$, and we have already seen that $v \geq k(A)$, so $v \geq j(\omega) \vee k(A) = j(\omega, A) = a$, as required. □

7 Index of popular Bousfield classes

We next give a list of spectra X together with corresponding elements $x \in \mathcal{A}$. We write $X = x$ to indicate that $\langle X \rangle = \phi(x)$ in $\mathcal{L}$, or $X \simeq x$ to indicate that $\pi\langle X \rangle = \pi\phi(x)$ in $\bar{\mathcal{L}}$. As usual, everything is implicitly p–localised:

(1) $$0 = k(\varnothing),$$

(2) $$S = S_p^\wedge = T(0) = t(0, \mathbb{N}_\infty),$$

(3) $$S/p = S/p^\infty = t(0, [1, \infty]),$$

(4) $$F(n) = t(0, [n, \infty]),$$

(5) $$H\mathbb{Q} = S\mathbb{Q} = I(H\mathbb{Q}) = k(\{0\}),$$

(6) $$H/p = H/p^\infty = I(H) = I(H/p) = I(BP\langle n\rangle) = k(\{\infty\}),$$

(7) $$H = k(\{0, \infty\}),$$

(8) $$v_n^{-1} F(n) = K'(n) \simeq k(\{n\}),$$

(9) $$T(q) = t(q, \mathbb{N}_\infty),$$

(10) $$BP = BP_p^\wedge = T(\infty) = t(\infty, \mathbb{N}_\infty),$$

(11) $$P(n) = BP/I_n = t(\infty, [n, \infty]),$$

(12) $$B(n) = v_n^{-1} P(n) = K(n) = M_n S = k(\{n\}),$$

(13) $$IB(n) = IK(n) = k(\{n\}),$$

(14) $$E(n) = v_n^{-1} BP\langle n\rangle = v_n^{-1} BP = L_n S = k([0, n]),$$

(15) $$\widehat{E(n)} = L_{K(n)} S = k([0, n]),$$

(16) $$C_n S \simeq t(0, [n+1, \infty]),$$

(17) $$BP\langle n\rangle = k([0, n] \cup \{\infty\}),$$

(18) $$BP\langle n\rangle / I_n = k(\{n, \infty\}),$$

(19) $$KU = KO = k(\{0,1\}),$$

(20) $$kU = kO = k(\{0,1,\infty\}),$$

(21) $$\mathrm{Ell} = \mathrm{TMF} = k(\{0,1,2\}),$$

(22) $$I(S) = I(T(0)) = I(F(n)) = j(0,\varnothing),$$

(23) $$I(S_p^\wedge) = I(S/p^\infty) = j(0,\{0\}),$$

(24) $$I(T(m)) = I(T(m)\wedge F(n)) = j(m,\varnothing).$$

Proof (1) Clear from the definitions.

(2) Clear from the definitions together with Proposition 5.1(e).

(4) Clear from the definitions.

(3) From (4) we see that $t(0,[1,\infty])$ is the same as

$$F(1) = F(U(1),U(1)) = F(S/p,S/p) = D(S/p)\wedge S/p,$$

and it is easy to check that this has the same Bousfield class as S/p. Now S/p^∞ can be described as the homotopy colimit of the spectra S/p^n, or as the cofibre of the map $S\to S[p^{-1}]$. From the first description (together with the cofibrations $S/p^n\to S/p^{n+1}\to S/p$) we see that $\langle S/p^\infty\rangle\le\langle S/p\rangle$. The second description shows that $S/p^\infty\wedge\Sigma^{-1}S/p\simeq S/p$, which gives $\langle S/p\rangle\le\langle S/p^\infty\rangle$, so we have $\langle S/p^\infty\rangle=\langle S/p\rangle$.

(5) By definition we have $H\mathbb{Q}=k(\{0\})$, and it is standard that this is the same as $S\mathbb{Q}$. Moreover, $I(H\mathbb{Q})$ is a module over $H\mathbb{Q}$, so Proposition 5.1(b) tells us that the Bousfield class is the same as $H\mathbb{Q}$ provided that $I(H\mathbb{Q})\ne 0$. By definition we have $\pi_0(I(H\mathbb{Q}))=\mathrm{Hom}(\mathbb{Q},\mathbb{Q}/\mathbb{Z}_{(p)})$, which is nontrivial, as required.

(6) By definition we have $H/p=k(\{\infty\})$, and this is the same as H/p^∞ as a consequence of (3). Note that $\pi_*(I(H))=\mathrm{Hom}(\pi_{-*}(H),\mathbb{Z}/p^\infty)$, which is a copy of $\mathbb{Z}/p^\infty$ concentrated in degree zero, so $I(H)=H/p^\infty$. A similar argument gives $I(H/p)=H/p$. We next consider the classes $u(n)=\langle I(BP\langle n\rangle)\rangle$. These start with $u(0)=\langle IH\rangle=k(\infty)$, so it will suffice to prove that $u(n)=u(n-1)$ when $n>0$. Proposition 5.1(c) gives

$$u(n)=\langle I(BP\langle n\rangle)/v_n\rangle\vee\langle I(BP\langle n\rangle)[v_n^{-1}]\rangle,$$

and the first term is the same up to suspension as $u(n-1)$. The second term is the colimit of the spectra $\Sigma^{-k|v_n|}IBP\langle n\rangle$, which is trivial because the homotopy of $IBP\langle n\rangle$ is concentrated in nonpositive degrees. The claim follows.

(7) Proposition 5.1(e) gives $\langle H\rangle = \langle H/p\rangle \vee \langle H\mathbb{Q}\rangle$, and this is $k(\{0,\infty\})$ by definition.

(8) We have $K'(n) = v_n^{-1}F(n)$ by definition. If TC fails, then this may be different from $k(\{n\})$ in $\mathcal{L}$. However, Lemma 5.20 tells us that $k'(n) = a(n) \vee k(n)$, so $K'(n)$ and $k(n)$ have the same image in $\bar{\mathcal{L}}$, as we indicate by writing $K'(n) \simeq k(n)$.

(9) True by definition.

(10) Clear from the definitions together with Proposition 5.1(e).

(11) We have $P(n) = BP/I_n$ by definition. Now consider a more general spectrum of the form BP/J, where J is generated by an invariant regular sequence of length n. Ravenel proved as [15, Theorem 2.1(g)] that $\langle BP/J\rangle = \langle P(n)\rangle$. Using the theory of generalised Moore spectra [17, Chapter 6; 9, Chapter 4] we see that for suitable J there is a finite spectrum S/J of type n such that $BP/J = BP \wedge S/J$. By the thick subcategory theorem we have $\langle S/J\rangle = f(n)$ and so $\langle P(n)\rangle = \langle BP\rangle \wedge f(n) = t(\infty, [n,\infty])$.

(12) We have $K(n) = k(\{n\})$ by definition, and the spectrum $B(n) = v_n^{-1}P(n)$ has the same Bousfield class by [15, Theorem 2.1(a)]. We will discuss M_nS under (14).

(13) We first note that $IK(n)$ is a $K(n)$–module, and all $K(n)$–modules are free, and the homotopy groups of $IK(n)$ have the same order as those of $K(n)$, so $IK(n) \simeq K(n)$ as spectra, so certainly $\langle IK(n)\rangle = k(\{n\})$. Next, note that $IB(n)$ is a $B(n)$–module, so

$$\langle IB(n)\rangle = \langle B(n)\rangle \wedge \langle IB(n)\rangle = \langle K(n)\rangle \wedge \langle IB(n)\rangle,$$

and this is either zero or $k(\{n\})$ by Lemma 5.2. It cannot be zero because

$$\pi_0(IB(n)) = \mathrm{Hom}(\pi_0 B(n), \mathbb{Q}/\mathbb{Z}_{(p)}) \neq 0,$$

so it must be $k(\{n\})$, as claimed.

(14) We have $E(n) = v_n^{-1}BP\langle n\rangle$ by definition. This has the same Bousfield class as $v_n^{-1}BP$ and as $K(\{0,\dots,n\})$ by parts (b) and (d) of [15, Theorem 2.1]. Note that $E(n) \wedge X = 0$ if and only if $L_nX = 0$, which is equivalent to $L_nS \wedge X = 0$ by [17, Theorem 7.5.6]. This means that L_nS also has the same Bousfield class. Finally, recall that $M_nS = C_{n-1}L_nS = C_{n-1}S \wedge L_nS$. This gives

$$\langle M_nS\rangle = \big\langle \textstyle\bigvee_{i\le n} K(i) \wedge C_{n-1}S\big\rangle = \big\langle \textstyle\bigvee_{i\le n} C_{n-1}(K(i))\big\rangle.$$

Here $K(i)$ is $E(n-1)$–local for $i<n$, and $E(n-1)$–acyclic for $i=n$, which gives $\langle M_nS\rangle = k(n)$, as claimed in (12).

(15) This is part of [9, Proposition 5.3] (where $\widehat{E(n)}$ is denoted by E, and $L_{K(n)}S$ is denoted by $\widehat{L}S$).

(16) We know from (4) that $t(0,[n+1,\infty]) = f(n+1)$, and we must show that this becomes the same as C_nS in $\bar{\mathcal{L}}$. Put

$$e=\langle E(n)\rangle = \langle L_nS\rangle = \bigvee_{i\le n} k(i) e' = \bigvee_{i\le n} k'(i), \quad f=f(n+1), \quad f'=\langle C_nS\rangle.$$

An induction based on Lemma 5.14 gives $1=f(0)=e'\vee f$, with $e'\wedge f=0$. Next, C_nS is by definition the fibre of the localisation map $S\to L_nS$, and this fibre sequence gives $1=e\vee f'$. Moreover, C_nS and $F(n+1)$ are both $E(n)$–acyclic by construction, so $e\wedge f=e\wedge f'=0$ and $f'\wedge f=f$. We now have

$$\begin{aligned} f &= f\wedge 1 = f\wedge(e\vee f') = f\wedge f',\\ f' &= f'\wedge 1 = f'\wedge(e'\vee f) = (f'\wedge e')\vee(f'\wedge f) = (f'\wedge e')\vee f. \end{aligned}$$

All of this is valid in $\mathcal{L}$. If we pass to $\bar{\mathcal{L}}$ then e and e' become the same by (8), so $f'\wedge e' = f'\wedge e = 0$, so $f=f'$, as required.

(17) This is [15, Theorem 2.1(e)].

(18) By the same argument as for (11), we have $\langle BP\langle n\rangle/I_n\rangle = \langle BP\langle n\rangle\rangle \wedge f(n)$. Using (17) and Lemma 5.5, this reduces to $k(\{n,\infty\})$.

(19) The spectrum KU is Landweber exact with strict height 1, so it is Bousfield equivalent to $E(1)$ by [7, Corollary 1.12]. It is a theorem of Wood [18] that $KU = KO/\eta$, where $\eta\in\pi_1(KO)$ is the Hopf map. This is essentially equivalent to [1, Proposition 3.2], and the same paper proves the standard fact that $\eta^3=0$ in $\pi_*(KO)$. As η is nilpotent we find that KU generates the same thick subcategory as KO, and thus has the same Bousfield class.

(20) We can take connective covers in the theorem of Wood to see that $kU = kO/\eta$, so $\langle kU\rangle = \langle kO\rangle$. If v denotes the Bott element in $\pi_2(kU)$ then we have $kU/v = H$ and $kU[v^{-1}] = KU$, so $\langle kU\rangle = \langle KU\rangle \vee \langle H\rangle$, which is $k(\{0,1,\infty\})$ by (7) and (19).

(21) Here Ell is intended to denote any of the standard Landweber exact versions of elliptic cohomology, which all have strict height 2, so $\mathrm{Ell} = k(\{0,1,2\})$ by [7, Corollary 1.12]. At primes $p>2$ the spectrum TMF is itself a version of Ell and so

$\langle \mathrm{TMF} \rangle = k(\{0,1,2\})$. For $p \in \{2,3\}$ it is known [5] that there is a finite spectrum X of type 0 such that $\mathrm{TMF} \wedge X = E(2)$, so we again have the same Bousfield class.

(22) We have $I(S) = I(T(0)) = j(0,\varnothing)$ by definition. For any finite spectrum X, it is easy to see that $I(X) = DX \wedge I(S)$. As $F(n)$ is self-dual we have $I(F(n)) = F(n) \wedge I(S)$, and this has Bousfield class $j(0,\varnothing)$ by Lemma 5.24.

(23) First, we have

$$I(S_p^\wedge)/p = \Sigma I((S_p^\wedge)/p) = \Sigma I(S/p),$$

which gives

$$\langle I(S_p^\wedge) \rangle \geq \langle I(S/p) \rangle = \langle I(S) \rangle = j(0,\varnothing).$$

Next, there is a natural map $i\colon S \to S_p^\wedge$, with cofibre X say. We find that $\pi_k(X) = 0$ for $k \neq 0$, but that $\pi_0(X) = \mathbb{Z}_p/\mathbb{Z}_{(p)}$, which is a nontrivial rational vector space. This gives a fibration $IX \to I(S_p^\wedge) \to IS$, giving

$$\langle I(S_p^\wedge) \rangle \leq \langle IX \rangle \vee \langle IS \rangle.$$

Here IS is torsion and IX is rational and nontrivial, so it follows that $I(S_p^\wedge)$ is not torsion, and so $\langle I(S_p^\wedge) \rangle \geq \langle HQ \rangle = k(\{0\})$. Putting this together, we get $\langle I(S_p^\wedge) \rangle = \langle IS \rangle \vee \langle H\mathbb{Q} \rangle = j(0,\{0\})$, as claimed. A similar proof works for S/p^∞, using the defining cofibration $S \to S\mathbb{Q} \to S/p^\infty$.

(24) We have $I(T(m)) = j(m,\varnothing)$ by definition, and this is the same as $I(T(m) \wedge F(n))$ by Lemma 5.24 and the self-duality of $F(n)$. □

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