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Representing a point and the diagonal as zero loci in flag manifolds

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The zero locus of a generic section of a vector bundle over a manifold defines a submanifold. A classical problem in geometry asks to realise a specified submanifold in this way. We study two cases: a point in a generalised flag manifold and the diagonal in the direct product of two copies of a generalised flag manifold. These cases are particularly interesting since they are related to ordinary and equivariant Schubert polynomials, respectively.

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1 Introduction

Let N be a manifold of dimension $2n$. Consider a smooth function $f\colon N \to \mathbb{C}^m$ having $0 \in \mathbb{C}^m$ as a regular value. Then, $M = f^{-1}(0) \subset N$ is a submanifold of codimension $2m$. Conversely, we can ask if a submanifold $M \subset N$ of codimension $2m$ can be realised in this way, or more generally, as the zero locus of a generic section of a rank m complex vector bundle $\xi \to N$. Here, by a generic section, we mean it is transversal to the zero section. We say M is *represented by* ξ if such a bundle ξ exists.

The following example shows that even for the simplest case the question is not as trivial as it may appear to be.

Example 1.1 Consider the representability of a point in S^2. Identify $S^2 = \mathbb{C}P^1$ and let $\gamma^* \to \mathbb{C}P^1$ be the dual of the tautological bundle

$$\gamma = \{([z_0, z_1], (cz_0, cz_1)) \mid [z_0, z_1] \in \mathbb{C}P^1,\ z_0 z_1 \neq 0,\ c \in \mathbb{C}\}.$$

One of its generic sections is given by the projection $(cz_0, cz_1) \mapsto cz_0$, whose zero locus is exactly the south pole $[0, 1] \in \mathbb{C}P^1$. Since S^2 is homogeneous with a transitive $\mathrm{SO}(3)$ action, for any pair of points $x, y \in S^2$ there is an element $g \in \mathrm{SO}(3)$ such that $gx = y$. By choosing g appropriately, we can represent any point in S^2 by $g^*(\gamma^*)$.

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On the other hand, consider the representability of a point in S^{2n} for $n > 2$. Bott's integrality theorem tells that the top Chern class c_n of any rank n complex vector bundle on S^{2n} is divisible by $(n-1)!$ (see for example Konstantis and Parton [8, Proposition 6.1]). However, if there is a rank n bundle with a generic section whose zero locus is a point, the top Chen class of the bundle has to be the generator of $H^{2n}(S^{2n};\mathbb{Z})$. Hence, there is no such bundle. In fact, we can show that a point in S^{2n} is representable if and only if $n \le 2$. A point in S^4 is representable by Lemma 2.2.

From now on, all spaces are assumed to be based, and the basepoints are denoted by pt. The following two submanifolds are particularly interesting (see Pragacz, Srinivas and Pati [13]):

(1) The basepoint $\{\mathrm{pt}\} \subset X$.

(2) The diagonal $\Delta(X) = \{(x,x) \mid x \in X\} \subset X \times X$.

In the language of [13], if any point in X (resp. the diagonal in $X \times X$) is representable, X is said to have property (P_c) (resp. (D_c)). Note that the choice of the basepoint does not make any difference when X is connected since for any pair of points $x, y \in X$ there exists a diffeomorphism $f\colon X \to X$ satisfying $f(y) = x$, so that the bundle and the section for the representability of the point x are pulled back to represent the point y. Note also that when $\Delta(X) \subset X \times X$ is represented by ξ, then $\{\mathrm{pt}\} \subset X$ is represented by $\iota^*(\xi)$, where ι is the inclusion $N \hookrightarrow N \times N$ defined by $\iota(x) = (x, \mathrm{pt})$.

In [13] analogous problems in different settings are considered: in an algebraic setting and in topological settings with complex bundles, real bundles and real oriented bundles. In this note, we focus on the following topological variant:

Problem 1.2 Let X be a (generalised) flag manifold G/P, where G is a complex, connected, simple Lie group and P is a parabolic subgroup. Find a rank $\dim_{\mathbb{C}}(X)$ complex bundle $\xi \to X$ (resp. $\xi \to X \times X$) with a smooth generic section which vanishes exactly at the basepoint (resp. along $\Delta(X)$).

The problem is related to Schubert calculus. The Poincaré dual to the fundamental class of the basepoint defines a cohomology class, which corresponds to the top Schubert class. Similarly, the class of the diagonal can be thought of as a certain restriction of the torus-equivariant top Schubert class (see Section 3).

Fulton showed a remarkable result [3, Proposition 7.5]: that the diagonal in $X \times X$ for any type A flag manifold $X = \mathrm{SL}(k+1)/P$ with any parabolic subgroup P is representable. Note that Fulton's result works in a holomorphic setting and is stronger than our topological setting. On the other hand, for full (ie complete) flag manifolds (when $P = B$ is the Borel subgroup) of other Lie types, Pragacz and the author showed [6, Theorem 17] that the basepoint in G/B (and hence the diagonal in $G/B \times G/B$) is not representable unless G is of type A or C. Indeed, we show in this note:

Theorem 1.3 (Proposition 3.2 and Theorem 3.3) *The basepoint (resp. the diagonal) of G/B is representable if and only if G is of type A or C. Moreover, the basepoint of G/P is not representable for any proper parabolic subgroup P when G is of exceptional type.*

Thus, the remaining cases are those of flag manifolds of type B, C and D. In [13, Theorem 12], nonrepresentability of the diagonal is shown for the odd complex quadrics, which are partial flag manifolds of type B. Naturally, we may ask if there is any flag manifold where the basepoint is representable but the diagonal is not. The main result of this note is to give such an example. Namely, we show:

Theorem 1.4 (Theorem 4.2) *Let $\mathrm{Lag}_\omega(\mathbb{C}^{2k})$ be the Lagrangian Grassmannian of maximal isotropic subspaces in the complex symplectic vector space $\mathbb{C}^{2k}$ with a symplectic form ω. The basepoint in $\mathrm{Lag}_\omega(\mathbb{C}^{2k})$ is representable for any k, but its diagonal is not when $k \equiv 2 \bmod 4$.*

We also see that the basepoint is not representable for many type B and D partial flag manifolds (Proposition 4.1 and Remark 4.7).

Throughout this note, $H^*(X)$ stands for the singular cohomology X with integer coefficients. Denote by $M \subset N$ a closed oriented submanifold M of codimension $2m$ embedded in a closed oriented $2n$–manifold N. The cohomology class which is Poincaré dual to the fundamental class of M is denoted by $[M] \in H^{2m}(N)$.

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2 Criteria for representability

We begin with trivial but useful criteria for the representability of submanifolds in general.

Proposition 2.1 *Let N be a closed oriented manifold of dimension $2n$. Assume that $\xi \to N$ represents a submanifold $M \subset N$ of codimension $2m$.*

(1) *The top Chern class $c_m(\xi) \in H^{2m}(N)$ is equal to the class $[M]$.*

(2) *The restriction $\xi|_M$ is isomorphic to the normal bundle $\nu(M)$ of $M \subset N$.*

Note that the converse to (1) does not hold; the equality of classes $c_m(\xi) = [M]$ does not necessarily mean that we can find a generic section whose zero locus is exactly M. However, when M is a point, we can pair zeros with opposite orientations of any generic section to cancel out. This means:

Lemma 2.2 *The basepoint in a closed oriented $2n$–manifold N is representable if and only if there is a rank n complex bundle whose top Chern class is the generator of $H^{2n}(N)$.*

The complex K–theory $K^0(N)$ can be identified with the set of stable equivalence classes of vector bundles over N, where

$$\xi_1 \sim \xi_2 \qquad \Longleftrightarrow \qquad \xi_1 \oplus \mathbb{C}^l \simeq \xi_2 \oplus \mathbb{C}^l \quad \text{for some } l \geq 0.$$

The Chern class $c_m(\xi) \in H^{2m}(N)$ of a bundle $\xi \to N$ depends only on the stable equivalence class of ξ in $K^0(N)$. Therefore, if for a complete set of representatives of $K^0(N)$ there is no bundle whose m^{th} Chern class is equal to the class $[M] \in H^{2m}(N)$, we can conclude that M is not representable.

3 Flag manifolds

From now on, we focus on flag manifolds G/P with the basepoint taken to be the identity coset eP. We assume G is a complex, connected, simple Lie group with a fixed Borel subgroup B containing a maximal compact torus $T \subset B$, and its Weyl group is denoted by $W(G)$. A parabolic subgroup P is a closed subgroup of G containing B. Parabolic subgroups are in one-to-one correspondence with the subgraphs of the Dynkin diagram of G. Denote by K the maximal compact subgroup of G containing T and by H its subgroup $P \cap K$. We have a diffeomorphism $K/H \simeq G/P$ by the Iwasawa decomposition, and in particular, $K/T \simeq G/B$. We use the notation G/P and K/H interchangeably. We also have a diffeomorphism $K/H \simeq \widetilde{K}/p^{-1}(H)$, where $p\colon \widetilde{K} \to K$ is the universal covering. So we can assume K is simply connected if necessary.

The universal flag bundle is denoted by $K/T \overset{c}{\hookrightarrow} BT \to BK$, where BK is the classifying space of K. More generally, we have the universal partial flag bundle $K/H \hookrightarrow BH \to BK$. We say a bundle $K/H \to E \to X$ is a flag bundle if it is a pullback of the universal (partial) flag bundle via a map $X \to BK$. The Atiyah–Hirzebruch homomorphism $H^*(BT) \to K^0(K/T)$ is defined by assigning to a character $\lambda \in \mathrm{Hom}(T, \mathbb{C}^*) \simeq H^2(BT)$ the line bundle $L_\lambda := K \times_T \mathbb{C}_\lambda$ over K/T and extending multiplicatively. Here, denoted by $\mathbb{C}_\lambda$, is the space $\mathbb{C}$ acted by T via λ. This map is known to be surjective when K is simply connected (see [9]).

We first note how the representability of the basepoint and the diagonal is related to Schubert polynomials. One way to look at Schubert polynomials [1] is that they are elements in $H^*(BT)$ which pull back via $c\colon K/T \to BT$ to the classes of Schubert varieties in K/T. In other words, they are polynomials in the first Chern classes of line bundles on K/T representing the Schubert classes. The top Schubert polynomial represents the class of the basepoint and it is known by [1] that it "produces" all the other Schubert polynomials when the divided difference operators are applied to it. So in a sense, the top Schubert polynomial carries the information of the whole $H^*(K/T)$. This is why we are interested in representing the basepoint. A similar story goes for the (Borel) T–equivariant cohomology $H^*_T(K/T)$, the top double Schubert polynomial and the diagonal, as is explained below.

Let EK be the universal K–space; that is, EK is contractible, on which K acts freely. Then, the classifying spaces are taken to be $BK = EK/K$ and $BT = EK/T$. The Borel construction K/T is defined to be $EK \times_T K/T$, where $[x, gT] = [xt^{-1}, tgT] \in EK \times_T K/T$ for $t \in T$. Consider the commutative diagram

$$\begin{tikzcd}
 & K/T \arrow[r, equal] \arrow[d] & K/T \arrow[d, "c"] \\
K/T \arrow[r] \arrow[d, equal] & EK \times_T K/T \arrow[r, "p_2"] \arrow[d, "p_1"] & BT \arrow[d] \\
K/T \arrow[r, "c"] & BT \arrow[r] & BK
\end{tikzcd}$$

where the lower-right square is a pullback, $p_1([x, gT]) = [x]$ and $p_2([x, gT]) = [xg]$. We have the sequence of maps $K/T \times K/T \xrightarrow{i} EK \times_T K/T \xrightarrow{p} BT \times BT$, where $i(g_1T, g_2T) = [\mathrm{pt}\cdot g_2, g_2^{-1} g_1 T]$ and $p = (p_1, p_2)$. The class of the equivariant point $EK \times_T eT/T$ pulls back via i to the class of the diagonal in $K/T \times K/T$. The class of $EK \times_T eT/T$ corresponds to the class of the top Schubert variety in the equivariant cohomology, which in turn corresponds to the top double Schubert polynomial.

Let us look at the concrete example of $K = U(k)$. Note that $U(k)/T \simeq \mathrm{SU}(k)/T'$ with $T' = T \cap \mathrm{SU}(k)$. Although $U(k)$ is not simple, we consider $U(k)$ for convenience. Lascoux and Schützenberger's top double Schubert polynomial [10]

$$\mathfrak{S}_{w_0}(x, y) = \prod_{1 \le i < j \le k} (x_i - y_j)$$

can be considered as an element in

$$H^*(BT \times BT) \simeq H^*(BT) \otimes H^*(BT) \simeq \mathbb{Z}[x_1, \dots, x_k] \otimes \mathbb{Z}[y_1, \dots, y_k],$$

which pulls back via p to the class of the equivariant point $EK \times_T eT/T$ in the equivariant cohomology $H^*(EK \times_T K/T) = H^*_T(K/T)$. This class further pulls back via i to the class of the diagonal in $H^*(K/T \times K/T)$. For a character $\lambda \in H^2(BT)$, let $\hat{L}_\lambda$ be the line bundle $ET \times_T \mathbb{C}_\lambda \to BT$ such that $c_1(\hat{L}_\lambda) = \lambda$. As $\mathfrak{S}_{w_0}(x, y)$ is a product of linear terms, we can define the rank $\dim_{\mathbb{C}}(K/T) = \frac{1}{2}k(k-1)$ bundle

$$\xi = \bigoplus_{1 \le i < j \le k} \hat{L}_{x_i} \hat{\otimes} \hat{L}^*_{y_j} \to BT \times BT$$

such that its top Chern class is equal to $\mathfrak{S}_{w_0}(x, y)$. We have $c_{k(k-1)/2}(i^* p^*(\xi)) = [\Delta(K/T)] \in H^{k(k-1)}(K/T \times K/T)$.

Similarly, for $K = \mathrm{Sp}(k)$, consider the rank $\dim_{\mathbb{C}}(K/T) = k^2$ bundle

$$\xi = \bigoplus_{1 \le i \le j \le k} \hat{L}_{x_i} \hat{\otimes} \hat{L}_{y_j} \oplus \bigoplus_{1 \le i < j \le k} \hat{L}_{x_i} \hat{\otimes} \hat{L}^*_{y_j} \to BT \times BT.$$

Then, the top Chern class of $p^*(\xi)$ is the equivariant top Schubert class [5, Section 8]. We have $c_{k^2}(i^* p^*(\xi)) = [\Delta(K/T)] \in H^{2k^2}(K/T \times K/T)$. This means that there is a generic section s of ξ such that $[Z(s)] = [\Delta(K/T)]$ but this does not imply the existence of s such that $Z(s)$ is exactly $\Delta(K/T)$. We will show that the diagonal of $\mathrm{Sp}(k)/T$ is actually representable.

For this, we recall the following slight generalisation of Proposition 7.5 of Fulton [3] in the current smooth setting:

Proposition 3.1 [6, Theorem 14] *If a point in (the diagonal of) X is representable, then so is any point in (the diagonal of) the total space E of any flag bundle of type A,*

$$\mathrm{SL}(k)/P \hookrightarrow E \xrightarrow{p} X,$$

where P is any parabolic subgroup of $\mathrm{SL}(k)$.

Proof We reproduce a proof for the diagonal case here for the sake of completeness. By universality, $E \xrightarrow{p} X$ is the pullback of the universal flag bundle along some map $f\colon X \to B\mathrm{SL}(k)$. The idea is to consider the pullback diagram

$$\begin{tikzcd} E \times E \arrow[r] \arrow[d, "p\times p"] & BP \times BP \arrow[d] \\ X \times X \arrow[r, "f\times f"] & B\mathrm{SL}(k) \times B\mathrm{SL}(k) \end{tikzcd}$$

and construct a bundle over $E \times E$ as the sum of pullbacks of bundles over $BP \times BP$ and $X \times X$. Let G be $\mathrm{SL}(k)$ and $G/P \to BP \to BG$ be the universal flag bundle, whose fibre we identify with the space of flags $\{0 \subset U_1 \subset U_2 \subset \cdots \subset U_l \subset \mathbb{C}^k\}$. We have the corresponding tautological sequence of bundles on BP,

$$\mathcal{U}_1 \xhookrightarrow{\iota} \mathcal{U}_2 \xhookrightarrow{\iota} \cdots \xhookrightarrow{\iota} \mathcal{U}_l \xhookrightarrow{\iota} \gamma_k \xrightarrow{q} \mathcal{V}_1 \xrightarrow{q} \cdots \xrightarrow{q} \mathcal{V}_l,$$

where γ_k is the pullback of the universal vector bundle over BG and $\mathcal{V}_i = \gamma_k/\mathcal{U}_i$. Denote by $\pi_1, \pi_2\colon BP \times BP \to BP$ the left and the right projections. The following rank $\dim_{\mathbb{C}}(G/P)$ bundle over $BP \times BP$ is defined in [3, Proposition 7.5]:

$$\xi_{BP} = \left\{ \bigoplus_{i=1}^{l} h_i \in \bigoplus_{i=1}^{l} \mathrm{Hom}(\pi_1^*(\mathcal{U}_i), \pi_2^*(\mathcal{V}_i)) \;\middle|\; q \circ h_i = h_{i+1} \circ \iota \text{ for all } i \right\}.$$

Restricted on the fibre product $BP \times_{BG} BP \subset BP \times BP$, this admits a section $s_{BP}\colon BP \times_{BG} BP \to \xi_{BP}$ defined by the tautological map $\pi_1^*(\mathcal{U}_i) \to \pi_1^*(\gamma_k) = \pi_2^*(\gamma_k) \to \pi_2^*(\mathcal{V}_i)$, which vanishes exactly along the diagonal $\Delta(BP) \subset BP \times_{BG} BP$. By a partition of unity argument, we can extend s_{BP} to the whole $BP \times BP$, which we denote by the same symbol s_{BP} (note that this is the place where we have to work in our smooth setting, unlike Fulton's original work in the holomorphic setting).

Let ξ_X be a bundle over $X \times X$ with a generic section s_X which represents the diagonal of X. Note that the pullback bundle $(p\times p)^*(\xi_X)$ admits a section $(p\times p)^*(s_X)$ whose zero locus is $(p\times p)^{-1}(\Delta(X)) = E \times_X E$. The bundle $\xi = (p\times p)^*(\xi_X) \oplus (f\times f)^*(\xi_{BP})$ over E has rank $\frac{1}{2}\dim(E) = \frac{1}{2}(\dim(G/P) + \dim(X))$. The section of ξ defined by $(p\times p)^*(s_X) \oplus (f\times f)^*(s_{BP})$ vanishes exactly along the diagonal as the following is a pullback:

$$\begin{tikzcd} E \times_X E \arrow[r, "f\times f"] & BP \times_{BG} BP \\ \Delta(E) \arrow[r] \arrow[u] & \Delta(BP) \arrow[u] \end{tikzcd}$$

This concludes the proof. $\square$

Proposition 3.2 *When G is of type C, the diagonal of G/P for any parabolic subgroup P of type C (including $P = B$) is representable. Consequently, the basepoint in G/B (resp. the diagonal in $G/B \times G/B$) is representable if and only if G is of type A or C.*

Proof For $1 \le k' \le k$, let $F^k_{k'} = \mathrm{Sp}(k)/(T^{k'} \cdot \mathrm{Sp}(k-k'))$ be the isotropic flag manifold with respect to a symplectic form ω in $\mathbb{C}^{2k}$:

$$F^k_{k'} = \{0 \subset U_1 \subset U_2 \subset \cdots \subset U_{k'} \subset U_{k'}^{\perp} \subset \cdots \subset U_1^{\perp} \subset \mathbb{C}^{2k} \mid \dim_{\mathbb{C}}(U_i) = i\},$$

where $U_i^{\perp} = \{v \in \mathbb{C}^{2k} \mid \omega(u,v) = 0 \text{ for all } u \in U_i\}$. Denote the tautological bundle on $F^k_{k'}$ corresponding to U_i by $\mathcal{U}_i$. By dropping $U_{k'}$, we obtain a projection $p\colon F^k_{k'} \to F^k_{k'-1}$, which makes $F^k_{k'}$ the projectivisation of $\mathcal{U}^{\perp}_{k'-1}/\mathcal{U}_{k'-1}$ over $F^k_{k'-1}$. By Proposition 3.1, if the diagonal of $F^k_{k'-1}$ is representable, so is that of $F^k_{k'}$. This procedure can be iterated to $F^k_1 = \mathbb{C}P^{2k-1}$, of which the diagonal is representable since it is a type A partial flag manifold.

For a full flag manifold of an arbitrary type, as is reviewed in the introduction, the basepoint in G/B is not representable unless G is of type A or C [6, Theorem 17], and the diagonal is representable when G is of type A [3, Proposition 7.5]. Thus, the second statement follows from the first. □

For exceptional Lie groups, the arguments in [6, Section 6] extend to show:

Theorem 3.3 *When G is of exceptional type, the basepoint in G/P is not representable for any (proper) parabolic subgroup P (including $P = B$).*

Proof By taking the universal covering, we can assume K is simply connected. Let $H = P \cap K$. We shall see that there is no bundle ξ with $c_n(\xi) = u_{2n} \in H^{2n}(K/H)$, where u_{2n} is the generator of the top-degree cohomology. The flag bundle $H/T \hookrightarrow K/T \to K/H$ induces isomorphisms

$$H^*(K/H) \simeq H^*(K/T)^{W(H)}, \quad K^0(K/H) \simeq K^0(K/T)^{W(H)},$$

where $W(H)$ is the Weyl group of H. The universal flag bundle $K/T \xhookrightarrow{c} BT \to BK$ induces a map $c^*\colon H^*(BT) \to H^*(K/T)$, which is compatible with the action of $W(K)$. The Atiyah–Hirzebruch homomorphism $H^*(BT) \to K^0(K/T)$ is also compatible with the action of $W(K)$ and it restricts to a surjection $H^*(BT)^{W(H)} \to K^0(K/T)^{W(H)} \simeq K^0(K/H)$. This asserts that any bundle over K/H stably splits into line bundles when pulled back via $K/T \to K/H$, and hence its Chern classes are polynomials in the elements of $H^2(K/T) \simeq c^*(H^2(BT))$. Let $\tau_{K/H}$ be the smallest positive integer such that $\tau_{K/H} \cdot u_{2n}$ is in the image of

$$c^*\colon H^*(BT)^{W(H)} \to H^*(K/T)^{W(H)} \simeq H^*(K/H),$$

induced by $c^*\colon H^*(BT) \to H^*(K/T)$. Consider the flag bundle

$$H/T \hookrightarrow K/T \to K/H.$$

There is a class $v \in H^*(K/T)$ which restricts to the class of the basepoint in $H^*(H/T)$. Since the class of the basepoint in $H^{2n}(K/T)$ is the product of the pullback of u_{2n} with v, we have

$$\tau_{K/T} \le \tau_{H/T} \cdot \tau_{K/H}.$$

On the other hand, it is known (see [14]) that

$$\tau_{\mathrm{SU}(k)/T} = \tau_{\mathrm{Sp}(k)/T} = 1, \qquad \tau_{\mathrm{Spin}(k)/T} = \begin{cases} 2 & \text{if } 7 \le k \le 12, \\ 4 & \text{if } k = 13, 14, \\ 8 & \text{if } k = 15, 16, \end{cases}$$

$$\tau_{G_2/T} = 2, \quad \tau_{F_4/T} = 6, \quad \tau_{E_6/T} = 6, \quad \tau_{E_7/T} = 12, \quad \tau_{E_8/T} = 2880.$$

Parabolic subgroups are in one-to-one correspondence with subgraphs of the Dynkin diagram. So for any (proper) parabolic subgroup of an exceptional Lie group, we can see $\tau_{K/T} > \tau_{H/T}$ from the list above. Therefore, $t_{K/H} > 1$ and u_{2n} cannot be the Chern class of a bundle. □

4 Grassmannian manifolds

An argument similar to the one in the previous section also works for some G/P with G of classical types. Due to the low-rank equivalences $A_1 = B_1 = C_1 = D_1$, $B_2 = C_2$, $D_2 = A_1 \times A_1$ and $D_3 = A_3$, we assume $k > 2$ for B_k and $k > 3$ for D_k.

Proposition 4.1 *When G is of type B_k $(k > 2)$ or D_k $(k > 3)$ and P is a parabolic subgroup of type A, then the basepoint in G/P is not representable. In particular, the basepoint in the maximal orthogonal Grassmannian $\mathrm{OG}_k(\mathbb{C}^{2k})$ of maximal isotropic subspaces in the complex quadratic vector space $\mathbb{C}^{2k}$ is representable if and only if $k \le 3$.*

Proof If the basepoint in G/P is representable, so would be G/B by Proposition 3.1 applied to the flag bundle $P/B \hookrightarrow G/B \to G/P$. The first statement follows from Proposition 3.2.

For the second statement, recall from [11, Section 1.7] that the connected component of $\mathrm{OG}_k(\mathbb{C}^{2k})$ containing the identity is diffeomorphic to the flag manifold $\mathrm{SO}(2k)/U(k) \simeq \mathrm{SO}(2k-1)/U(k-1)$. Thus, the basepoint is representable if and only if $k \leq 3$. □

The basepoint is representable in G/P if the diagonal is representable in $G/P \times G/P$. The following example shows that the converse is not always true:

Theorem 4.2 *Let $\mathrm{Lag}_\omega(\mathbb{C}^{2k}) \simeq \mathrm{Sp}(k)/U(k)$ be the complex Lagrangian Grassmannian of maximal isotropic subspaces in the complex symplectic vector space $\mathbb{C}^{2k}$ with a symplectic form ω* (see [11, Section 1.7]).

(1) *The basepoint in $\mathrm{Lag}_\omega(\mathbb{C}^{2k})$ is representable.*

(2) *When $k \equiv 2 \bmod 4$, the diagonal in $\mathrm{Lag}_\omega(\mathbb{C}^{2k}) \times \mathrm{Lag}_\omega(\mathbb{C}^{2k})$ is not representable.*

Note that there is a p–local homotopy equivalence $\mathrm{Sp}(k)/U(k) \simeq_p \mathrm{SO}(2k+1)/U(k)$ for odd primes p [4], so 2–torsion plays an important role in our problem. Our proof of the theorem is based on the Steenrod operations, which is similar to that of [13, Theorem 11]. We need a few lemmas.

Lemma 4.3 *The tangent bundle of a flag manifold K/H is*

$$T(K/H) = \bigoplus_{\beta \in \Pi^+ \setminus \Pi_H^+} L_\beta,$$

where Π^+ (resp. Π_H^+) is the set of positive roots of K (resp. H). In particular, for $\mathrm{Sp}(k)/U(k)$, we can take $\Pi^+ = \{2x_i \mid 1 \leq i \leq k\} \cup \{x_i \pm x_j \mid 1 \leq i < j \leq k\}$ and $\Pi_H^+ = \{x_i - x_j \mid 1 \leq i < j \leq k\}$; hence, we have

$$T(\mathrm{Sp}(k)/U(k)) \simeq \left(\bigoplus_i L_{2x_i}\right) \oplus \left(\bigoplus_{i<j} L_{x_i+x_j}\right).$$

Proof The assertion follows from the standard isomorphism

$$T(K/H) \simeq K \times_H (L(K)/L(H)),$$

where $L(K)$ and $L(H)$ are Lie algebras of K and H, respectively. □

Let $2n = \dim(\mathrm{Sp}(k)/U(k)) = k(k+1)$.

Lemma 4.4 (see [12]) *Let c_i (resp. q_i) be elementary symmetric functions in x_j (resp. x_j^2), where $H^*(BT) = \mathbb{Z}[x_1, \dots, x_k]$. Then,*

$$H^*(\mathrm{Sp}(k)/U(k)) \simeq \frac{\mathbb{Z}[c_1, c_2, \dots, c_k]}{(\mathbb{Z}[q_1, q_2, \dots, q_k])^+},$$

where $(\mathbb{Z}[q_1, q_2, \dots, q_k])^+$ is the ideal of positive-degree polynomials in q_j. In particular,

$$u_{2n} = \prod_i x_i \prod_{i<j} (x_i + x_j) = \prod_{i=1}^k c_i,$$

$$u_{2n-2} = \prod_{i=2}^k c_i,$$

$$u_1 = c_1$$

are generators of $H^{2n}(\mathrm{Sp}(k)/U(k))$, $H^{2n-2}(\mathrm{Sp}(k)/U(k))$ and $H^2(\mathrm{Sp}(k)/U(k))$, respectively.

Proof Let $X = \mathrm{Sp}(k)/U(k)$. Since $H_*(\mathrm{Sp}(k))$ has no torsion, by [2] we have

$$H^*(X) \simeq \frac{H^*(BT)^{W(U(k))}}{(H^+(BT)^{W(\mathrm{Sp}(k))})},$$

where $(H^+(BT)^{W(\mathrm{Sp}(k))})$ is the ideal generated by the positive-degree Weyl group invariants. Since $W(U(k)) \curvearrowright H^*(BT)$ is permutation and $W(\mathrm{Sp}(k)) \curvearrowright H^*(BT)$ is signed permutation, we have

$$H^*(X) \simeq \frac{\mathbb{Z}[c_1, c_2, \dots, c_k]}{(\mathbb{Z}[q_1, q_2, \dots, q_k])^+}.$$

By the degree reason, it is easy to see that $\prod_{i=1}^k c_i \in H^{2n}(X)$, $\prod_{i=2}^k c_i \in H^{2n-2}(X)$ and $c_1 \in H^2(X)$ are generators. The Euler characteristic $\chi(X)$ is equal to

$$\frac{|W(\mathrm{Sp}(k))|}{|W(U(k))|}$$

as the cells in the Bruhat decomposition of X are indexed by the cosets

$$W(\mathrm{Sp}(k))/W(U(k)).$$

Since

$$\prod_i (2x_i) \prod_{i<j} (x_i + x_j) = c_n(TX) = \chi(X) u_{2n} = \frac{|W(\mathrm{Sp}(k))|}{|W(U(k))|} u_{2n} = 2^k u_{2n},$$

we have $u_{2n} = \prod_i x_i \prod_{i<j} (x_i + x_j)$. □

Lemma 4.5 *When $k \equiv 2 \bmod 4$, any bundle $\xi \to \mathrm{Sp}(k)/U(k)$ representing the basepoint in $\mathrm{Sp}(k)/U(k)$ is spin.*

Proof We show $c_n(\xi) = \pm u_{2n}$ implies $c_1(\xi) \equiv w_2(\xi) = 0 \bmod 2$, where $w_2(\xi)$ is the second Stiefel–Whitney class. Since $H^*(\mathrm{Sp}(k)/U(k))$ has no torsion,

$$H^*(\mathrm{Sp}(k)/U(k); \mathbb{Z}/2\mathbb{Z}) \simeq H^*(\mathrm{Sp}(k)/U(k); \mathbb{Z}) \otimes \mathbb{Z}/2\mathbb{Z}.$$

We use the same symbol for an integral class and its mod 2 reduction, and the equations below are meant to hold in $H^*(\mathrm{Sp}(k)/U(k); \mathbb{Z}/2\mathbb{Z})$. By Wu's formula, we have $\mathrm{Sq}^2(c_i) = c_1 c_i + (2i-1)(i-1)c_{i+1}$. Since $c_i^2 \in \mathbb{Z}/2\mathbb{Z}[q_1, q_2, \dots, q_k]$, by Lemma 4.4 we have

$$\mathrm{Sq}^2 u_{2n-2} = \mathrm{Sq}^2\left(\prod_{i=2}^{k} c_i\right) = (k-1)\prod_{i=1}^{k} c_i = u_{2n}.$$

Set $c_1(\xi) = au_1$ and $c_{n-1}(\xi) = bu_{n-1}$ for some $a, b \in \mathbb{Z}$. Since $k \equiv 2 \bmod 4$, we have $n \equiv 1 \bmod 2$. Again by Wu's formula, we have

$$bu_n = \mathrm{Sq}^2(c_{n-1}(\xi)) = c_1(\xi)c_{n-1}(\xi) + c_n(\xi) = (ab+1)u_n.$$

So $b(a+1) \equiv 1$, and hence $a \equiv 0 \bmod 2$. □

Proof of Theorem 4.2 Denote $\mathrm{Sp}(k)/U(k)$ by X.

(1) Consider the bundle

$$\hat{\xi} = \left(\bigoplus_i L_{x_i}\right) \oplus \left(\bigoplus_{i<j} L_{x_i+x_j}\right)$$

over $\mathrm{Sp}(k)/T$. Since $\hat{\xi}$ is invariant under the action of $W(U(k))$, there is a bundle ξ over $\mathrm{Sp}(k)/U(k)$ which pulls back to $\hat{\xi}$ via the projection $\mathrm{Sp}(k)/T \to \mathrm{Sp}(k)/U(k)$. Then, $c_n(\xi) = \prod_i x_i \prod_{i<j}(x_i + x_j) = u_{2n}$ is a generator of the top-degree cohomology by Lemma 4.4. By Lemma 2.2, the basepoint is represented by ξ.

(2) Assume that $\xi' \to X \times X$ represents the diagonal $\Delta(X)$. By Proposition 2.1(2), the pullback of ξ' along $\Delta\colon X \to X \times X$ is isomorphic to the normal bundle $\nu(\Delta)$, which is isomorphic to TX. On the other hand, the pullback of ξ' along the inclusion to each factor $i_1, i_2\colon X \to X \times X$ represents the class of the basepoint, where $i_1(x) = (x, \mathrm{pt})$ and $i_2(x) = (\mathrm{pt}, x)$. Since $i_1^* \otimes i_2^*\colon H^2(X \times X) \simeq H^2(X) \otimes H^2(X)$, we see

$$c_1(TX) = c_1(\Delta^*(\xi)) = \Delta^*(c_1(\xi)) = c_1(i_1^*(\xi)) + c_1(i_2^*(\xi)) \equiv 0 \bmod 2$$

by Lemma 4.5. However, $c_1(TX) = (k+1)u_1$ by Lemma 4.3 and this contradicts that k is even. □

Corollary 4.6 *Let G be of type C_k and P be of type A. The basepoint in G/P is representable.*

Proof Note that any type A parabolic subgroup P is contained in the maximal parabolic subgroup P_k of type A_{k-1}. Apply Proposition 3.1 to the flag bundle $P_k/P \hookrightarrow G/P \to G/P_k$, where $G/P_k \simeq \mathrm{Sp}(k)/U(k)$. □

Remark 4.7 A result of Totaro [15] shows $\tau_{\mathrm{Spin}(2k+1)/T} = \tau_{\mathrm{Spin}(2k+2)/T} = 2^{u(k)}$, where $u(k)$ is either $k - \lfloor \log_2(\binom{k+1}{2} + 1) \rfloor$ or that expression plus 1. Let G be of type B_k (resp. D_{k+1}), so that its compact form is $\mathrm{Spin}(2k+1)$ (resp. $\mathrm{Spin}(2k+2)$). Since any parabolic subgroup of G is a product of type B (resp. of type D) and type A subgroups, the basepoint is not representable in G/P for any P when $u(k-1) < u(k)$ by the same argument as in the proof of Theorem 3.3. Note that $u(k-1) = u(k)$ rarely occurs when k gets bigger. A list of $u(k)$ for small k is given in [15].

For example, let $Q_l = \{x \in \mathbb{C}P^{l+1} \mid x_1^2 + \cdots + x_{l+2}^2 = 0\}$ be the complex quadric. In [13, Theorem 12], it is shown that the diagonal in Q_l is not representable for any odd l. Since Q_l is isomorphic to the real oriented Grassmannian (see [7, page 280])

$$\widetilde{\mathrm{Gr}}_2(\mathbb{R}^{l+2}) := \mathrm{SO}(l+2)/(\mathrm{SO}(2) \times \mathrm{SO}(l)),$$

the basepoint in Q_l is not representable for many l. For example, $0 = u(2) < u(3) = 1$ shows that the basepoint in Q_5 is not representable as $\tau_{\mathrm{Spin}(7)/T} \le \tau_{H/T} \cdot \tau_{\mathrm{Spin}(7)/H}$, and hence $2 \le \tau_{\mathrm{Spin}(7)/H} = \tau_{Q_5}$, where H is the inverse image of $\mathrm{SO}(2) \times \mathrm{SO}(5)$ under the covering $\mathrm{Spin}(7) \to \mathrm{SO}(7)$. Note the low-rank equivalences $Q_1 = \mathbb{C}P^1$, $Q_2 = \mathbb{C}P^1 \times \mathbb{C}P^1$, $Q_3 = \mathrm{Lag}_\omega(\mathbb{C}^4)$, $Q_4 = \mathrm{Gr}_2(\mathbb{C}^4)$ and $Q_6 = \mathrm{OG}_4(\mathbb{C}^8)$. So, up to $l \le 6$, the basepoints are representable for Q_1, Q_2, Q_3 and Q_4 but not for Q_5 or Q_6.

type of G	A	B	C	D	exceptional
point for G/B	○	×	○	×	×
point for G/P (P of type A)	○	×	○	×	×
point for G/P (otherwise)	○	?	?	?	×
diagonal for G/B	○	×	○	×	×
diagonal for G/P (P of type A)	○	×	?	×	×
diagonal for G/P (otherwise)	○	?	?	?	×

Table 1: Summary of representability

Theorem 4.2 shows that the converse to Proposition 3.1 does not hold in general; even when the diagonal of the total space of the type A flag bundle $U(2)/T \to \mathrm{Sp}(2)/T \to \mathrm{Sp}(2)/U(2)$ is representable, that of the base space is not representable. This makes it difficult to complete the study of representability for partial flag manifolds of type B, C and D. The current status of the problem is summarised in Table 1. The partial information obtained in this note on the entries with the symbol "?" suggests that a case-by-case analysis may be necessary to settle the remaining cases.

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