

# Spectral order for contact manifolds with convex boundary

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We extend the Heegaard Floer homological definition of spectral order for closed contact 3-manifolds due to Kutluhan, Matić, Van Horn-Morris, and Wand to contact 3-manifolds with convex boundary. We show that the order of a codimension-zero contact submanifold bounds the order of the ambient manifold from above. As the neighborhood of an overtwisted disk has order zero, we obtain that overtwisted contact structures have order zero. We also prove that the order of a small perturbation of a Giroux  $2\pi$ -torsion domain has order at most two, hence any contact structure with positive Giroux torsion has order at most two (and, in particular, a vanishing contact invariant).

[57M27](#), [57R17](#); [57R58](#)

## 1 Introduction

Algebraic torsion of closed contact  $(2n-1)$ -manifolds was defined by Latschev and Wendl [12] via symplectic field theory. It is an invariant with values in  $\mathbb{N} \cup \{\infty\}$  whose finiteness gives obstructions to the existence of symplectic fillings and exact symplectic cobordisms. They also showed that the order of algebraic torsion is zero if and only if the contact homology is trivial — in particular, if the contact structure is overtwisted — and it has order at most one in the presence of positive Giroux torsion. Note that the analytical foundations of symplectic field theory are still under development. Hence, in the appendix to [12], Hutchings provided a similar numerical invariant for contact 3-manifolds via embedded contact homology; however, it is currently unknown whether this is independent of the contact form.

Motivated by the isomorphism between embedded contact homology and Heegaard Floer homology, Kutluhan, Matić, Van Horn-Morris, and Wand [10; 11] defined a Heegaard Floer homological analogue of algebraic torsion for closed contact 3-manifolds called *spectral order* (or *order* for short), and denoted it by  $\mathfrak{o}$ . Their definition uses open book decompositions, and gives a refinement of the Ozsváth–Szabó contact invariant  $c(\xi)$ . Using the fact that an overtwisted contact structure is

supported by an open book whose monodromy is not right-veering, they proved that  $\mathfrak{o}(M, \xi) = 0$  if  $\xi$  is overtwisted.

In this paper, we extend  $\mathfrak{o}$  to contact manifolds with convex boundary, following the definition of Kutluhan et al in the closed case. The definition is in terms of a partial open book decomposition of the underlying sutured manifold supporting the contact structure, and a collection of arcs on the page, containing a basis. This data gives rise to a filtration of the sutured Floer boundary map [8], and the spectral order is the index of the first page of the associated spectral sequence where the distinguished generator representing the contact invariant vanishes, or  $\infty$  otherwise. Then we take the minimum over all partial open books together with a collections of arcs containing a basis. (This extension of the definition of  $\mathfrak{o}$  was also independently observed by Kutluhan et al [11].)

Our first main result is that the spectral order of a codimension-zero contact submanifold gives an upper bound on the spectral order of the ambient manifold.

**Theorem 1** *Let  $(M, \xi)$  be a contact 3–manifold with convex boundary. If  $(N, \xi|_N)$  is a codimension-zero submanifold of  $\text{Int}(M)$  with convex boundary, then*

$$\mathfrak{o}(N, \xi|_N) \geq \mathfrak{o}(M, \xi).$$

We will prove this result in Section 3. As a corollary, we show that if a contact manifold with convex boundary is overtwisted, then it has spectral order zero. This follows immediately from a simple computation that a neighborhood of an overtwisted disk has spectral order zero.

In Section 4, we carry out a computation that shows that the spectral order of a slight enlargement of a Giroux  $2\pi$ –torsion  $T^2 \times I$  has spectral order at most two. In particular, every contact manifold with positive Giroux torsion has vanishing Ozsváth–Szabó invariant, which was proved in the closed case by Ghiggini, Honda, and Van Horn-Morris [4] (the sutured case also follows from their work when combined with [6, Theorem 1.1]). Together with Theorem 1, we obtain the following corollary.

**Theorem 2** *If a contact 3–manifold  $(M, \xi)$  with convex boundary has Giroux  $2\pi$ –torsion, then*

$$\mathfrak{o}(M, \xi) \leq 2.$$

The inequality  $\sigma \leq 1$  was shown in the closed case by Latschev and Wendl [12, Theorem 2] via symplectic field theory, and conjectured in the Heegaard Floer setting in the closed case by Kutluhan et al [11, Question 6.3]. More generally, they asked whether the presence of planar  $k$ -torsion (see [12, Section 3.1] for a definition) implies that the spectral order is at most  $k$ .

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## 2 Spectral order for manifolds with boundary

We first recall the definition of spectral order for closed contact 3-manifolds due to Kutluhan, Matić, Van Horn-Morris, and Wand [11]. Let  $(M, \xi)$  be a closed contact 3-manifold. By the Giroux correspondence theorem [5], the contact structure  $\xi$  is supported by some open book decomposition  $(S, \phi)$  of  $M$ , which is well defined up to positive stabilizations. In particular,  $M$  is identified with  $S \times I / \sim$ , where  $(x, 1) \sim (\phi(x), 0)$  for every  $x \in S$ , and  $(x, t) \sim (x, t')$  for every  $x \in \partial S$  and  $t, t' \in I$ .

An arc basis on  $S$  is a set of pairwise disjoint properly embedded arcs that forms a basis of  $H_1(S, \partial S)$ . A collection of pairwise disjoint arcs  $\underline{a} = \{a_1, \dots, a_n\}$  on  $S$  that contains a basis induces an “overcomplete” Heegaard diagram  $(\Sigma, \alpha, \beta)$  of  $M$ , as follows. We obtain  $\underline{b} = \{b_1, \dots, b_n\}$  by isotoping  $\underline{a}$  such that the endpoints of  $\underline{a}$  are moved in the positive direction along  $\partial S$ , and  $|a_i \cap b_j| = \delta_{ij}$  for  $i, j \in \{1, \dots, n\}$ . Then we set  $\Sigma = (S \times \{\frac{1}{2}\}) \cup_{\partial S} (-S \times \{0\})$ . Furthermore, we let  $\alpha = \{\alpha_1, \dots, \alpha_n\}$  and  $\beta = \{\beta_1, \dots, \beta_n\}$ , where

$$\alpha_i := (a_i \times \{\frac{1}{2}\}) \cup (a_i \times \{0\}) \quad \text{and} \quad \beta_i := (b_i \times \{\frac{1}{2}\}) \cup (\phi(b_i) \times \{0\})$$

for  $i \in \{1, \dots, n\}$ . We also choose a basepoint in each connected component of  $S \setminus \underline{a}$  away from the isotopy between  $\underline{a}$  and  $\underline{b}$ , and denote the set of these on  $S \times \{\frac{1}{2}\} \subset \Sigma$  by  $z$ . Then  $(\Sigma, \alpha, \beta, z)$  is a multipointed Heegaard diagram of  $M$ .

We say that a domain  $D$  in the diagram  $(\Sigma, \alpha, \beta, z)$  connects  $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  if  $\partial(\partial D \cap \alpha) = x - y$  and  $\partial(\partial D \cap \beta) = y - x$ , and  $n_z(D) = 0$  for every  $z \in z$ . We denote by  $D(x, y)$  the set of such domains. If  $x = (x_1, \dots, x_n)$ , then there is a unique permutation  $\pi_x \in S_n$  such that  $x_i \in \alpha_i \cap \beta_{\pi_x(i)}$  for every  $i \in \{1, \dots, n\}$ . Using the above Heegaard diagram, Kutluhan et al [11] defined a function  $J_+$  that assigns an integer

$$J_+(D) = n_x(D) + n_y(D) - e(D) + |x| - |y|$$

to every domain  $D \in D(x, y)$ . Here,  $n_x(D)$  is the sum over all  $p \in x$  of the averages of the coefficients of  $D$  at the four regions around  $p$ , the term  $e(D)$  is the Euler measure of  $D$ , and  $|x|$  and  $|y|$  are the number of cycles in the permutations  $\pi_x$  and  $\pi_y$ , respectively. When  $D$  is a domain of Maslov index 1, the equality  $e(D) = 1 - n_x(D) - n_y(D)$  holds by the work of Lipshitz [13], so the formula becomes

$$J_+(D) = 2(n_x(D) + n_y(D)) - 1 + |x| - |y|.$$

For any topological Whitney disk  $\psi \in \pi_2(x, y)$ , we can define  $J_+(\psi)$  as the value  $J_+(D(\psi))$ , where  $D(\psi)$  is the domain of  $C$ . The function  $J_+$  is additive in that

$$J_+(D_1 + D_2) = J_+(D_1) + J_+(D_2)$$

for every  $D_1 \in D(x_1, x_2)$  and  $D_2 \in D(x_2, x_3)$ . Furthermore,  $J_+(u)$  is always a nonnegative even integer for any  $J$ -holomorphic disk  $u$ . Hence, we have a splitting

$$\widehat{\partial}_{\text{HF}} = \partial_0 + \partial_1 + \partial_2 + \dots$$

of the Heegaard Floer differential  $\widehat{\partial}_{\text{HF}}$ , where  $\partial_i$  is the number of  $J$ -holomorphic disks  $u$  satisfying  $\mu(u) = 1$  and  $J_+(u) = 2i$ . As shown in [11], this gives a spectral sequence

$$E^k(S, \phi, \underline{a}) = H_*(E^{k-1}(S, \phi, \underline{a}), d^{k-1}),$$

induced by the filtered complex

$$\left( C = \bigoplus_{i \in \mathbb{N}} \widehat{\text{CF}}(\Sigma, \beta, \alpha, z)_i, \widehat{\partial} \right).$$

The  $j^{\text{th}}$  coordinate of the differential  $\widehat{\partial}_{\underline{c}}$  for  $\underline{c} = (c_i)_{i \in \mathbb{N}} \in C$  and  $j \in \mathbb{N}$  is defined as

$$(\widehat{\partial}_{\underline{c}})_j = \sum_{i=0}^{\infty} \partial_i c_{i+j},$$

and the filtration is given by

$$\mathcal{F}_p C = \bigoplus_{i=0}^p \widehat{\text{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{z})_i.$$

Note that here we deviate slightly from the definition of Kutluhan et al [11] in that we take the direct sum defining  $C$  over  $\mathbb{N}$  instead of  $\mathbb{Z}$ , but as we shall see, the arising notion of spectral order is exactly the same.

Recall that a filtered complex

$$\cdots \subseteq \mathcal{F}_{p-1} C \subseteq \mathcal{F}_p C \subseteq \mathcal{F}_{p+1} C \subseteq \cdots$$

induces a spectral sequence by setting

$$Z_p^k = \{x \in \mathcal{F}_p C : \partial x \in \mathcal{F}_{p-k} C\} \quad \text{and} \quad B_p^k = \mathcal{F}_p C \cap \partial \mathcal{F}_{p+k} C.$$

For  $k \in \mathbb{N}$ , the  $k$ -page is the complex  $(E^k = \bigoplus_{p \in \mathbb{Z}} E_p^k, d^k)$ , where

$$E_p^k = \frac{Z_p^k}{Z_{p-1}^{k-1} + B_p^{k-1}},$$

and the differential  $d^k: E_p^k \rightarrow E_{p-k}^k$  is induced by the differential  $\partial$  on the complex  $C$ .

For an open book decomposition  $(S, \phi)$  supporting  $\xi$ , and a collection of arcs  $\underline{\mathbf{a}}$  on  $S$  containing a basis, we denote the induced spectral sequence defined above by  $E^n(S, \phi, \underline{\mathbf{a}})$ . Then note that, for every  $k \in \mathbb{Z}_{>0}$ ,

$$Z_0^k(S, \phi, \underline{\mathbf{a}}) = \{(c_i)_{i \in \mathbb{N}} : c_i = 0 \text{ for } i > 0 \text{ and } \partial_0 c_0 = 0\}.$$

Recall that the contact element  $\text{EH}(\xi) \in \widehat{\text{HF}}(-M)$  is defined as the homology class of the intersection point

$$\mathbf{x}_\xi := (\underline{\mathbf{b}} \cap \underline{\mathbf{a}}) \times \left\{ \frac{1}{2} \right\} \in \mathbb{T}_\beta \cap \mathbb{T}_\alpha,$$

where  $\underline{\mathbf{a}} \times \left\{ \frac{1}{2} \right\}$  and  $\underline{\mathbf{b}} \times \left\{ \frac{1}{2} \right\}$  are subsets of  $\boldsymbol{\alpha}$  and  $\boldsymbol{\beta}$ , respectively, by definition. As there are no nontrivial pseudoholomorphic disks emanating from  $\mathbf{x}_\xi$  in  $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$  that contribute to  $\widehat{\partial}_{\text{HF}}$ , it follows that  $\partial_i \mathbf{x}_\xi = 0$  for every  $i \in \mathbb{N}$ . We often view  $\mathbf{x}_\xi$  as an element of  $C$  supported in degree zero, ie as a sequence  $(d_i)_{i \in \mathbb{N}}$  such that  $d_0 = \mathbf{x}_\xi$  and  $d_i = 0$  for  $i > 0$ . As such,  $\mathbf{x}_\xi \in Z_0^k(S, \phi, \underline{\mathbf{a}})$  for every  $k \in \mathbb{N}$ . The following is [11, Definitions 2.1 and 2.2].

**Definition 3** Let  $(M, \xi)$  be a closed contact 3-manifold. We say that  $\mathfrak{o}(S, \phi, \underline{\mathbf{a}}) = k$  if the distinguished generator

$$x_\xi \in \widehat{\text{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{z})_0,$$

viewed in degree zero, is nonzero in  $E^k(S, \phi, \underline{\mathbf{a}})$ , and zero in  $E^{k+1}(S, \phi, \underline{\mathbf{a}})$ . Then we define the *spectral order of  $(M, \xi)$*  as

$$\mathfrak{o}(M, \xi) = \min\{\mathfrak{o}(S, \phi, \underline{\mathbf{a}}) : (S, \phi) \text{ supports } \xi \text{ and } \underline{\mathbf{a}} \subset S \text{ contains an arc basis}\}.$$

Implicit in the above definition is the choice of an almost complex structure  $J$  on  $\text{Sym}^g(\Sigma)$ . Kutluhan et al [11, Proposition 3.1] showed that  $\mathfrak{o}(S, \phi, \underline{\mathbf{a}}, J)$  is independent of  $J$ , hence we suppress it from our notation throughout.

**Remark** The contact element  $x_\xi$ , viewed in degree zero, vanishes in  $E^{k+1}(S, \phi, \underline{\mathbf{a}})$  if and only if it is contained in

$$B_0^k(S, \phi, \underline{\mathbf{a}}) = \mathcal{F}_0 C \cap \widehat{\partial} \mathcal{F}_k C = \widehat{\text{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})_0 \cap \widehat{\partial} \left( \bigoplus_{i=0}^k \widehat{\text{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})_i \right).$$

This holds precisely if there exist elements  $c_i \in \widehat{\text{CF}}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{z})$  for  $i \in \{0, \dots, k\}$  such that

$$(2-1) \quad \sum_{i=0}^k \partial_i c_i = x_\xi \quad \text{and} \quad \sum_{i=0}^{k-j} \partial_i c_{i+j} = 0 \quad \text{for all } j > 0.$$

Indeed, if we set  $c_i = 0$  for  $i > k$ , then the entries of  $\widehat{\partial}(c_i)_{i \in \mathbb{N}}$  correspond to the left-hand sides in (2-1), and so these equations translate to  $\widehat{\partial}(c_i)_{i \in \mathbb{N}} = (d_j)_{j \in \mathbb{N}}$ , where  $d_0 = x_\xi$  and  $d_j = 0$  for  $j > 0$ . As (2-1) coincides with the equation defining  $\mathcal{B}^k(S, \phi, \underline{\mathbf{a}})$  in [11, page 8], it follows that it does not matter whether we take the direct sum over  $\mathbb{N}$  or  $\mathbb{Z}$  when we define  $\mathfrak{o}$ .

Before extending this definition to manifolds with boundary, we first review the definition of partial open book decompositions, introduced by Honda, Kazez, and Matić [6]. We follow the treatment of Etgu and Ozbagci [2]. An abstract partial open book decomposition is a triple  $\mathcal{P} = (S, P, h)$ , where

- $S$  is a compact, oriented, connected surface with nonempty boundary,
- $P = P_1 \cup \dots \cup P_r$  is a proper subsurface of  $S$  such that  $S$  is obtained from  $\overline{S \setminus P}$  by successively attaching 1-handles  $P_1, \dots, P_r$ ,
- $h: P \rightarrow S$  is an embedding such that  $h|_A = \text{Id}_A$ , where  $A = \partial P \cap \partial S$ .

Given a partial open book decomposition  $(S, P, h)$ , we associate to it a sutured 3-manifold  $(M, \Gamma)$ , as follows. Let  $H = S \times [-1, 0]/\sim$ , where  $(x, t) \sim (x, t')$  for every  $x \in \partial S$  and  $t, t' \in [-1, 0]$ . Furthermore, let  $N = P \times I/\sim$ , where  $(x, t) \sim (x, t')$  for every  $x \in A$  and  $t, t' \in I$ . We obtain the manifold  $M$  by gluing  $(x, 0) \in \partial N$  to  $(x, 0) \in \partial H$  and  $(x, 1) \in \partial N$  to  $(h(x), -1) \in \partial H$  for every  $x \in P$ . The sutures are defined as

$$\Gamma = (\overline{\partial S \setminus \partial P}) \times \{0\} \cup -(\overline{\partial P \setminus \partial S}) \times \{\frac{1}{2}\}.$$

Then

$$\Sigma = (P \times \{0\} \cup -S \times \{-1\})/\sim$$

is a Heegaard surface for  $(M, \Gamma)$ .

Let  $\xi$  be a contact structure on  $M$  such that  $\partial M$  is convex with dividing set  $\Gamma$ . Similarly to the original Giroux correspondence, we say that  $\xi$  is compatible with the partial open book decomposition  $(S, P, h)$  if

- $\xi$  is tight on the handlebodies  $H$  and  $N$ ,
- $\partial H$  is a convex surface with dividing set  $\partial S \times \{0\}$ ,
- $\partial N$  is a convex surface with dividing set  $\partial P \times \{\frac{1}{2}\}$ .

Then the relative Giroux correspondence theorem says that  $\xi$  is uniquely determined up to contact isotopy, and given such a contact structure  $\xi$ , any two partial open book decompositions compatible with  $\xi$  are related by positive stabilizations.

We now extend the definition of spectral order to manifolds with boundary. Suppose that a contact 3-manifold  $(M, \xi)$  with convex boundary  $\partial M$  and dividing set  $\Gamma$  is given. Then  $(M, \Gamma)$  is a balanced sutured manifold if  $M$  has no closed components. Indeed, every convex surface has a nonempty dividing set. Furthermore,  $\chi(R_+(\Gamma)) = \chi(R_-(\Gamma))$  by [9, Proposition 3.5]. Then we have a compatible partial open book decomposition  $\mathcal{P} = (S, P, h)$ . An arc basis for  $(S, P, h)$  is a set  $\underline{a}$  of properly embedded arcs in  $P$  with endpoints on  $A$  such that  $S \setminus \underline{a}$  deformation retracts onto  $\overline{S \setminus \overline{P}}$ . Similarly to the closed case, a partial open book decomposition of  $M$ , together with a collection of pairwise disjoint arcs  $\underline{a}$  containing a basis and an appropriate choice of basepoints, gives a multipointed sutured Heegaard diagram  $(\Sigma, \alpha, \beta, z)$  of  $(M, \Gamma)$ . Here,  $z$  consists of a basepoint in each component of  $P \setminus \underline{a}$  disjoint from  $\partial P \setminus \partial S$ .

The differential  $\widehat{\partial}_{\text{SFH}}$  of the sutured Floer chain complex counts the number of  $J$ -holomorphic curves  $u$  with  $\mu(u) = 1$ , modulo the  $\mathbb{R}$ -action, that do not intersect the suture  $\Gamma = \partial\Sigma$  and the basepoints  $\mathbf{z}$ . For any topological Whitney disk  $\psi$  from  $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  to  $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$  that does not intersect  $\partial\Sigma$  and  $\mathbf{z}$ , we define the number  $J_+(\psi)$  as in the closed case by

$$J_+(\psi) = n_{\mathbf{x}}(D) + n_{\mathbf{y}}(D) - e(D) + |\mathbf{x}| - |\mathbf{y}|,$$

where  $D = D(\psi)$  is the domain of  $\psi$ . Since the equality  $e(D) = 1 - n_{\mathbf{x}}(D) - n_{\mathbf{y}}(D)$  for  $\mu(D) = 1$  still holds in the sutured case, we get that

$$(2-2) \quad J_+(\psi) = 2(n_{\mathbf{x}}(D) + n_{\mathbf{y}}(D)) - 1 + |\mathbf{x}| - |\mathbf{y}|$$

when  $\mu(\psi) = 1$ . As in the closed case, the function  $J_+$  is clearly additive, and the same argument as in [11, Section 2.2] shows that  $J_+(u)$  is a nonnegative even integer for any  $J$ -holomorphic disk  $u$ .

Hence, we can split the sutured Floer differential  $\widehat{\partial}_{\text{SFH}}$  as

$$\widehat{\partial}_{\text{SFH}} = \partial_0 + \partial_1 + \dots,$$

where  $\partial_r$  counts  $J$ -holomorphic disks  $u$  with  $\mu(u) = 1$  and  $J_+(u) = 2r$ .

Just like in the closed case, the pair  $(\bigoplus_{i \in \mathbb{N}} \text{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{z})_i, \widehat{\partial})$ , where the map  $\widehat{\partial}$  is defined as

$$\widehat{\partial}(c_i)_{i \in \mathbb{N}} = \left( \sum_{i=0}^{\infty} \partial_i c_{i+j} \right)_{j \in \mathbb{N}},$$

is a filtered chain complex. Using its induced spectral sequence, we can define the spectral order of  $(M, \xi)$  in the following way.

**Definition 4** For a contact 3-manifold  $(M, \xi)$  with convex boundary, a compatible partial open book decomposition  $\mathcal{P}$ , and a collection of pairwise disjoint arcs  $\underline{\mathbf{a}}$  containing an arc basis, denote the induced spectral sequence by  $E^k(\mathcal{P}, \underline{\mathbf{a}})$ . We say that  $\mathfrak{o}(\mathcal{P}, \underline{\mathbf{a}}) = k$  if the distinguished generator  $x_\xi \in \text{CF}(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha}, \mathbf{z})_0$  in degree zero remains nonzero in  $E^k(\mathcal{P}, \underline{\mathbf{a}})$ , but vanishes in  $E^{k+1}(\mathcal{P}, \underline{\mathbf{a}})$ . Then we define the *spectral order*

$$\mathfrak{o}(M, \xi) = \min\{\mathfrak{o}(\mathcal{P}, \underline{\mathbf{a}}) : \mathcal{P} \text{ supports } \xi \text{ and } \underline{\mathbf{a}} \text{ contains a basis}\}.$$

This is always a nonnegative integer.

We will need the following lemma for the proof of [Theorem 1](#).

**Lemma 5** Let  $(M, \xi)$  be a contact 3-manifold with (possibly empty) convex boundary and  $B \subset \text{Int}(M)$  a tight contact ball. If  $M_0 = M \setminus \text{Int}(B)$  and  $\xi_0 = \xi|_{M_0}$ , then

$$o(M_0, \xi_0) \geq o(M, \xi).$$

Furthermore, we have equality if  $M$  is closed.

**Proof** We denote by  $\Gamma_B$  the dividing set of  $\xi$  on  $\partial B$ . Let  $\mathcal{P}_0 = (S_0, P_0, h_0)$  be a partial open book decomposition of  $(M_0, \gamma_0)$  supporting  $\xi_0$ , together with a collection of arcs  $\underline{a}_0$  containing a basis, and write  $(\Sigma_0, \beta_0, \alpha_0, z_0)$  for the corresponding based sutured diagram of  $(-M_0, -\Gamma_0)$ .

There is a disk component  $D_B$  of  $\overline{S_0 \setminus P_0}$  corresponding to  $R_+(\Gamma_B)$ , and a disk component  $D'_B$  of  $\overline{S \setminus h(P)}$  corresponding to  $R_-(\Gamma_B)$ . Then  $h$  uniquely extends to a diffeomorphism

$$h: P \cup D_B \rightarrow h(P) \cup D'_B,$$

up to isotopy. If we set  $S = S_0$  and  $P = P \cup D_B$ , then  $\mathcal{P} := (S, P, h)$  is a partial open book of  $(M, \xi)$ . Furthermore,  $\underline{a} = \underline{a}_0$  contains an arc basis for  $\mathcal{P}$ . As  $D_B$  lies in a component of  $P \setminus \underline{a}$  disjoint from  $\partial P \setminus \partial S$ , we need to add a basepoint  $z_B$  here. The based diagram  $(\Sigma, \beta, \alpha, z)$  corresponding to  $(\mathcal{P}, \underline{a})$  is obtained by filling in a boundary component of  $\Sigma_0$  with the disk  $D_B \times \{0\}$ , and taking  $z = z_0 \cup \{z_B\}$ . Hence,  $o(\mathcal{P}, \underline{a}) = o(\mathcal{P}_0, \underline{a}_0)$ , as their defining filtered chain complexes agree. It follows that  $o(M_0, \xi_0) \geq o(M, \xi)$ .

Now suppose that  $M$  is closed. Let  $(S, h)$  be an arbitrary open book decomposition of  $(M, \xi)$ , and  $\underline{a}$  an arbitrary collection of arcs containing a basis. Then each component of  $S \setminus \underline{a}$  is homeomorphic to a disk; let  $D$  be one of them. Consider the partial open book  $\mathcal{P}_0 = (S_0, P_0, h_0)$ , where  $S_0 = S$ ,  $P_0 = S \setminus D$ , and  $h_0 = h|_{P_0}$ . Then  $\mathcal{P}_0$  supports  $(M_0, \xi_0)$ . If  $(\Sigma, \beta, \alpha, z)$  is the diagram arising from  $(S, h, \underline{a})$ , and  $(\Sigma_0, \beta_0, \alpha_0, z_0)$  is the diagram arising from  $(\mathcal{P}_0, \underline{a})$ , then  $\Sigma = \Sigma_0 \cup (D \times \{0\})$ ,  $\beta = \beta_0$ ,  $\alpha = \alpha_0$ , and we obtain  $z_0$  by removing the unique point  $z \cap D$ . Hence,  $o(S, h, \underline{a}) = o(\mathcal{P}_0, \underline{a})$ . Since  $(S, h, \underline{a})$  was arbitrary, we obtain that  $o(M, \xi) \geq o(M_0, \xi_0)$ .  $\square$

Using [Theorem 1](#), we will show that actually equality holds in the first part of [Lemma 5](#).

### 3 Inequality of spectral orders

The goal of this section is to prove [Theorem 1](#) from the introduction. We first briefly recall the construction of the contact gluing map  $\Phi$  on sutured Floer homology, defined

by Honda, Kazez, and Matic' [6]. Let  $(M, \Gamma_M)$  be a sutured manifold, and let  $(N, \Gamma_N)$  be a sutured submanifold of  $\text{Int}(M)$ . Furthermore, let  $\xi$  be a contact structure on  $M \setminus \text{Int}(N)$  with convex boundary and dividing set  $\Gamma_M$  on  $\partial M$  and dividing set  $\Gamma_N$  on  $\partial N$ . We can suppose that  $M \setminus N$  has no isolated components; ie every component of  $M \setminus N$  intersects  $\partial M$ . Indeed, by Lemma 5, if we remove a tight contact ball from each isolated component,  $\mathfrak{o}(M, \xi)$  does not decrease.

Choose a collar neighborhood  $Z \simeq \partial N \times I$  of  $\partial N$  in  $M \setminus \text{Int}(N)$  such that  $Z \cap N = \partial N \times \{0\}$ , on which the contact structure  $\xi$  is  $I$ -invariant, and write  $N'$  to denote  $M \setminus \text{Int}(N \cup Z)$ . Let  $\Sigma_{N'}$  be a Heegaard surface compatible with  $\xi|_{N'}$ , and let  $\Sigma_Z$  be a Heegaard surface compatible with  $\xi|_Z$ . Then, for any sutured Heegaard diagram  $\mathcal{H} = (\Sigma, \beta, \alpha)$  of  $(N, \Gamma_N)$  that is contact-compatible near  $\partial N$  in the sense of Honda, Kazez, and Matic' [6], the union  $\Sigma \cup \Sigma_Z \cup \Sigma_{N'}$  is a Heegaard surface for  $(M, \Gamma_M)$ , and we can complete  $\alpha$  and  $\beta$  to attaching sets of  $(M, \Gamma_M)$  by adding  $\alpha'$  and  $\beta'$  compatible with  $\xi|_{N' \cup Z}$ . We write

$$\mathcal{H}' = (\Sigma \cup \Sigma_Z \cup \Sigma_{N'}, \beta \cup \beta', \alpha \cup \alpha').$$

Then the map

$$\begin{aligned} \Phi_\xi: \text{CF}(\mathcal{H}) &\rightarrow \text{CF}(\mathcal{H}'), \\ \mathbf{y} &\mapsto (\mathbf{y}, \mathbf{x}'), \end{aligned}$$

is a chain map, where  $\mathbf{x}' \in \mathbb{T}_{\beta'} \cap \mathbb{T}_{\alpha'}$  is the canonical representative of the contact class  $\text{EH}(\xi|_{N' \cup Z})$ . Note that this construction makes sense even if we replace Heegaard diagrams with multipointed Heegaard diagrams.

**Proof of Theorem 1** As in the statement of Theorem 1, let  $(M, \xi)$  be a contact 3-manifold with convex boundary and dividing set  $\Gamma_M$ , and let  $N$  be a codimension-zero submanifold of  $\text{Int}(M)$ , also with convex boundary and dividing set  $\Gamma_N$ . Then let  $\mathcal{P}_N = (S_N, P_N, h_N)$  be a partial open book decomposition of  $(N, \xi|_N, \Gamma_N)$ , together with a choice of an arc basis  $\underline{a}_N$ , and let  $\mathcal{H} = (\Sigma, \beta, \alpha)$  be the corresponding diagram of  $(-N, -\Gamma_N)$ .

Let  $\zeta$  be an  $I$ -invariant contact structure on  $\partial N \times I$  such that  $\partial N \times \{t\}$  is convex with dividing set  $\Gamma_N$  for every  $t \in I$ . According to the remark after [6, Lemma 4.1], we can first extend  $\mathcal{H}$  to a diagram of  $(-N \cup (\partial N \times [0, 2]), -\Gamma_N \times \{2\})$  that is contact-compatible near  $\partial N \times \{2\}$ , by gluing two Heegaard surfaces arising from certain special partial open book decompositions of  $(\partial N \times I, \zeta)$ . We denote the resulting contact-compatible diagram  $(\Sigma \cup \Sigma_\zeta, \beta \cup \beta_\zeta, \alpha \cup \alpha_\zeta)$ . Then, using Step 2 of [6, Section 4],

and as explained above, we can further extend this to a diagram

$$\mathcal{H}' = (\Sigma \cup \Sigma_\xi \cup \Sigma_Z \cup \Sigma_{N'}, \beta \cup \beta_\xi \cup \beta', \alpha \cup \alpha_\xi \cup \alpha')$$

of  $(-M, -\Gamma_M)$ . Analogously to the gluing map, we obtain a chain map

$$\begin{aligned} \Phi: \text{CF}(\mathcal{H}) &\rightarrow \text{CF}(\mathcal{H}'), \\ \mathbf{y} &\mapsto (\mathbf{y}, \mathbf{x}'), \end{aligned}$$

where  $\mathbf{x}' \in \mathbb{T}_{\beta_\xi \cup \beta'} \cap \mathbb{T}_{\alpha_\xi \cup \alpha'}$  is the canonical representative of the contact class  $\text{EH}(\xi|_{(\partial N \times [0,2]) \cup Z \cup N'})$ . As  $\mathcal{H}$  is not necessarily contact-compatible, we do not claim that  $\Phi$  is the contact gluing map under naturality, but this is not necessary for our purposes. By construction,  $\Phi$  maps the contact class  $\mathbf{x}_{\xi|_N}$  to the contact class  $\mathbf{x}_\xi$ . Note that this construction of Honda, Kazez, and Matic [6] actually gives a partial open book  $\mathcal{P} = (S, P, h)$  supporting  $(M, \xi)$  and an arc basis  $\underline{\mathbf{a}}$  that extend  $\mathcal{P}_N$  and  $\underline{\mathbf{a}}_N$ , respectively.

Now consider the case when  $\underline{\mathbf{a}}_N$  is not an arc basis, but a collection of pairwise disjoint arcs that contains an arc basis. Then we need to choose basepoints  $\mathbf{z}$  such that every connected component of  $P_N \setminus \bigcup \underline{\mathbf{a}}_N$  that does not intersect  $\partial P_N \setminus \partial S_N$  has exactly one basepoint. The gluing process can be applied to this case without modification, to get a collection of pairwise disjoint arcs  $\underline{\mathbf{a}}$  in  $P$ . After gluing, every connected component of  $P \setminus \bigcup \underline{\mathbf{a}}$  disjoint from  $\partial P \setminus \partial S$  contains exactly one basepoint, since such a component must come from  $P_N \setminus \bigcup \underline{\mathbf{a}}_N$ , and other components do not contain a basepoint. Hence, the data  $(\mathcal{P}, \underline{\mathbf{a}}, \mathbf{z})$  satisfies the conditions needed to define its order. The proof of the fact that the gluing map is a chain map between Floer chain complexes [6] also applies to this case without further modification, for the same reason.

**Lemma 6** *Let  $\Phi$  be as above. Then the map*

$$\bar{\Phi}: \bigoplus_{i \in \mathbb{N}} \text{CF}(\mathcal{H})_i \rightarrow \bigoplus_{i \in \mathbb{N}} \text{CF}(\mathcal{H}')_i$$

defined by  $\bar{\Phi}((c_i)_{i \in \mathbb{N}}) = (\Phi(c_i))_{i \in \mathbb{N}}$  is a filtered chain map, hence induces a morphism  $(\Phi^k)_{k \in \mathbb{N}}$  of spectral sequences; ie  $\Phi^0 = \bar{\Phi}$ , and

$$\Phi^k: E^k(\mathcal{P}_N, \underline{\mathbf{a}}_N) \rightarrow E^k(\mathcal{P}, \underline{\mathbf{a}})$$

is a chain map for every  $k \in \mathbb{N}$  such that the map induced on homology is  $\Phi^{k+1}$ .

**Proof** Let  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\alpha$ . Any holomorphic disk  $u$  from  $(\mathbf{x}, \mathbf{x}')$  to  $(\mathbf{y}, \mathbf{x}')$  in  $\text{CF}(\mathcal{H}')$  is actually a holomorphic disk from  $\mathbf{x}$  to  $\mathbf{y}$  in  $\text{CF}(\mathcal{H})$ ; ie its domain  $D := D(u)$  is zero outside  $\Sigma$ ; see [6, page 12]. Since the Euler measure and the point measure of  $D$

depend only on the nonzero coefficients, the Maslov index of  $u$  in  $\mathcal{H}$  and in  $\mathcal{H}'$  are the same. Suppose that  $\mu(u) = 1$ . Then, in  $\mathcal{H}'$ , we have

$$\begin{aligned} J_+(u) &= 2(n_{(x,x')} (D) + n_{(y,x')} (D)) - 1 + |(x, x')| - |(y, x')| \\ &= 2(n_x(D) + n_y(D)) - 1 + |x| + |x'| - |y| - |x'| \\ &= 2(n_x(D) + n_y(D)) - 1 + |x| - |y|. \end{aligned}$$

This is the same as the value of  $J_+(u)$  in  $\mathcal{H}$ . Hence  $\Phi$  preserves the  $J_+$  filtration.

Now, by the definition of the differential  $\partial_i$ , the map  $\Phi$  commutes with  $\partial_i$  for all  $i \in \mathbb{N}$ . Hence, it commutes with the total differential  $\widehat{\partial}$ , and so  $\bar{\Phi}$  is a filtered chain map. Therefore  $\bar{\Phi}$  induces a morphism  $(\Phi^k)_{k \in \mathbb{N}}$  between the corresponding spectral sequences. □

Since  $\Phi(x_{\xi|_N}) = x_\xi$ , Lemma 6 implies that if  $x_{\xi|_N}$  vanishes in  $E^k(\mathcal{P}_N, \underline{a}_N)$ , then it also vanishes in  $E^k(\mathcal{P}, \underline{a})$ . Hence, by the definition of the spectral order,

$$o(\mathcal{P}_N, \underline{a}_N) \geq o(\mathcal{P}, \underline{a}) \geq o(M, \xi).$$

Taking the minimum of over all possible choices of  $(\mathcal{P}_N, \underline{a}_N)$ , we get that

$$o(N, \xi|_N) \geq o(M, \xi),$$

as required. This concludes the proof of Theorem 1. □

We are now in a position to strengthen Lemma 5.

**Corollary 7** *Let  $(M, \xi)$  be a connected contact 3–manifold with (possibly empty) convex boundary and  $B \subset \text{Int}(M)$  a tight contact ball. If  $M_0 = M \setminus \text{Int}(B)$  and  $\xi_0 = \xi|_{M_0}$ , then*

$$o(M_0, \xi_0) = o(M, \xi).$$

**Proof** We have already shown the closed case in Lemma 5, so we can suppose that  $\partial M \neq \emptyset$ . Let  $M' \subset \text{Int}(M)$  be a codimension-zero submanifold of  $M$  with convex boundary, such that  $(M', \xi')$  is contactomorphic to  $(M, \xi)$ , where  $\xi' = \xi|_{M'}$ . Since  $M$  is connected, we can assume that  $B \subset \text{Int}(M) \setminus M'$ . If we apply Theorem 1 to the sequence  $M' \subset M_0 \subset M$ , we obtain that

$$o(M', \xi') \geq o(M_0, \xi_0) \geq o(M, \xi).$$

As  $(M, \xi)$  and  $(M', \xi')$  are contactomorphic,  $o(M, \xi) = o(M', \xi')$ , and the result follows. □

## 4 Calculation of upper bounds on some spectral orders

Let  $(M, \xi)$  be a contact 3-manifold with convex boundary. Suppose that  $(M, \xi)$  is overtwisted. Then, by definition, it contains an embedded overtwisted disk  $\Delta$ . This has a standard neighborhood; ie there exists a neighborhood  $U \supset \Delta$  such that  $(U, \xi|_U)$  is contactomorphic to a neighborhood of the disk  $\Delta_{\text{std}} = \{z = 0, \rho \leq \pi\}$  inside the standard overtwisted contact structure on  $\mathbb{R}^3$ , which is defined as follows [1]:

$$\xi_{\text{OT}} = \ker(\cos \rho dz + \rho \sin \rho d\phi).$$

Inside  $U$ , we can perturb  $\Delta$  to a convex surface  $D$ . Take a neighborhood  $V = D \times [-1, 1]$  such that  $\xi|_{\text{Int}(M)}$  is  $\mathbb{R}$ -invariant. After rounding its edges, we obtain an open subset  $V_0 \simeq D^3$  such that the dividing set  $\Gamma_{V_0}$  on  $\partial V_0$  is given by three disjoint curves. Honda, Kazez, and Matić [7, Example 1] gave a partial open book decomposition of  $N = \bar{V}_0$ , and the corresponding Heegaard diagram is shown in Figure 1. This diagram can be used to show that  $\mathfrak{o}(M, \xi) = 0$ , which was proven by Kutluhan et al [10] in the closed case using the fact that an overtwisted contact structure admits an open book whose monodromy is not right-veering.

**Remark 8** It is convenient and customary to present the sutured diagram  $(\Sigma, \beta, \alpha)$  arising from a partial open book decomposition  $(S, P, h)$  and arcs basis  $\underline{a}$  on the surface  $-S \times \{0\} \subset \Sigma$ . Instead of gluing in  $P \times \{0\}$ , for each  $a \in \underline{a}$ , we identify the opposite edges of  $N(a) \cap \partial S$  for a regular neighborhood  $N(a)$  of  $a$  in  $S$ . This is possible since  $P = N(\underline{a})$ .

**Proposition 9** *If  $N$  is the standard neighborhood of an overtwisted disk in the contact manifold  $(M, \xi)$  as above, then*

$$\mathfrak{o}(N, \xi|_N) = 0.$$

**Proof** Honda, Kazez, and Matić [7, Example 1] computed that  $c(N, \xi|_N) = 0$ ; we extend their proof. Consider the partial open book decomposition of  $(N, \xi)$  shown in Figure 1. The contact element  $\text{EH}(N, \xi|_N)$  is represented by the point  $x$ , which is zero in homology because  $\partial y = x$ . The only  $J$ -holomorphic curve from  $y$  to  $x$  is the bigon, which satisfies  $J_+ = 0$ . Hence  $\mathfrak{o}(N, \xi|_N) \leq 0$ .  $\square$

**Theorem 10** *If the contact manifold  $(M, \xi)$  with convex boundary is overtwisted, then  $\mathfrak{o}(M, \xi) = 0$ .*

**Proof** We have  $\mathfrak{o}(M, \xi) \leq \mathfrak{o}(N, \xi|_N) = 0$  by Theorem 1 and Proposition 9.  $\square$

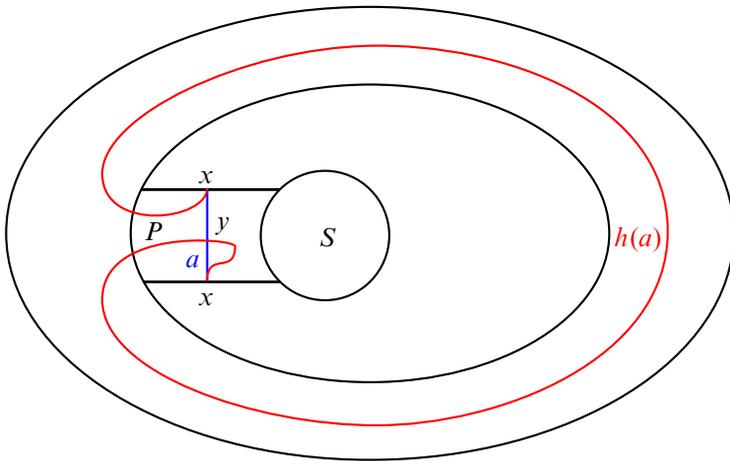


Figure 1: A sutured Heegaard diagram arising from a partial open book decomposition of a neighborhood of an overtwisted disk. We obtain the Heegaard surface by identifying the two horizontal arcs.

We now consider the case when  $(M, \xi)$  has Giroux  $2\pi$ -torsion. Recall that a contact manifold  $(M, \xi)$  has  $2\pi$ -torsion if it admits an embedding

$$(M_{2\pi}, \eta_{2\pi}) = (T^2 \times [0, 1], \ker(\cos(2\pi t) dx - \sin(2\pi t) dy)) \hookrightarrow (M, \xi).$$

The boundary of  $(M_{2\pi}, \eta_{2\pi})$  is not convex. However, as in [4, Lemma 5], if it embeds in  $(M, \xi)$ , then there exist small  $\epsilon_0, \epsilon_1 > 0$  such that the slightly extended domain

$$(M', \eta') = (T^2 \times [-\epsilon_0, 1 + \epsilon_1], \ker(\cos(2\pi t) dx - \sin(2\pi t) dy))$$

also embeds inside  $(M, \xi)$  so that  $T^2 \times \{-\epsilon_0\}$  and  $T^2 \times \{1 + \epsilon_1\}$  are pre-Lagrangian tori with integer slopes  $s_0$  and  $s_1$  that form a basis of  $H_1(T^2)$ . By the work of Ghiggini [3], we can perturb  $\partial M'$  to get a new contact submanifold  $\tilde{M}$  such that  $\partial \tilde{M}$  is convex, and the slopes of the dividing sets are  $s_0$  and  $s_1$ . After a change of coordinates in  $\tilde{M}$ , we can assume these slopes are 0 and  $\infty$ .

The contact manifold  $\tilde{M}$  is not minimally twisting and consists of five basic slices, which means that we can construct a partial open book decomposition of it by attaching four bypasses to a partial open book diagram of a basic slice, which can be found in Examples 4, 5, and 6 of [7]. The diagram we get is shown in Figure 2.

Applying the Sarkar–Wang algorithm [14] to this diagram along the dashed arcs in Figure 3, we obtain the one in Figure 4. It is easy to check that every region that does not intersect the boundary is either a bigon or a quadrilateral. In Figure 4, the  $\beta$ -curves

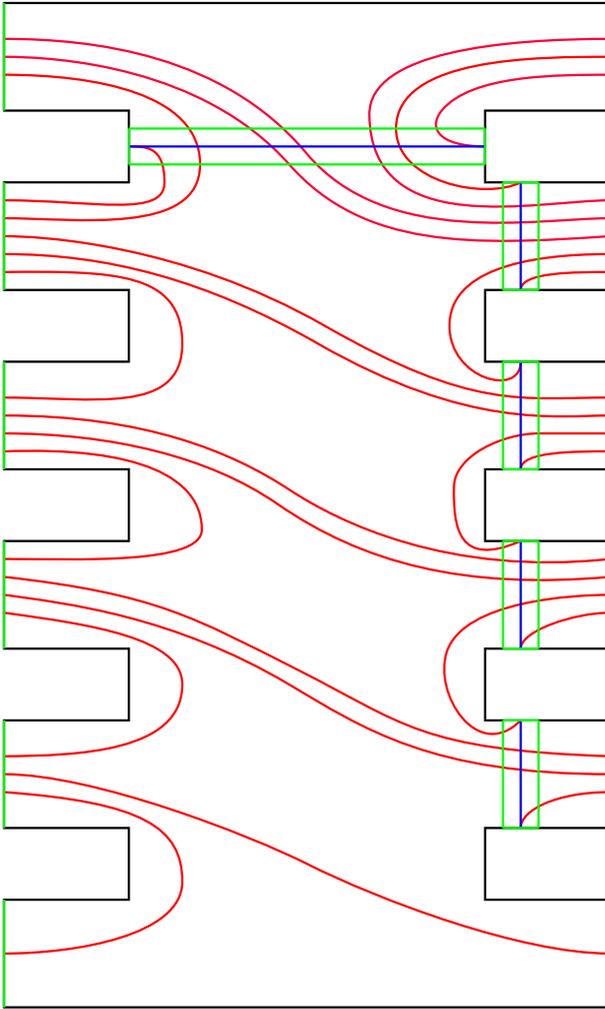


Figure 2: A sutred diagram arising from a partial open book decomposition of a neighborhood of a Giroux torsion domain. The opposite green arcs in the boundary are identified.

are shown in red and the  $\alpha$ -curves in blue, and the opposite green arcs in the boundary of the surface are identified. The intersection points between  $\alpha$  and  $\beta$  are labeled  $x_1, \dots, x_{18}$  from right to left along the horizontal blue arc, and along the four vertical blue arcs they are labeled from top to bottom  $y_1, \dots, y_{15}$ ,  $z_1, \dots, z_{10}$ ,  $w_1, \dots, w_6$ , and  $v_1, \dots, v_3$ , respectively.

The contact element  $\text{EH}(\xi|_{\widetilde{M}})$  is represented by the unordered tuple  $(x_1, y_1, z_1, w_1, v_1)$ .

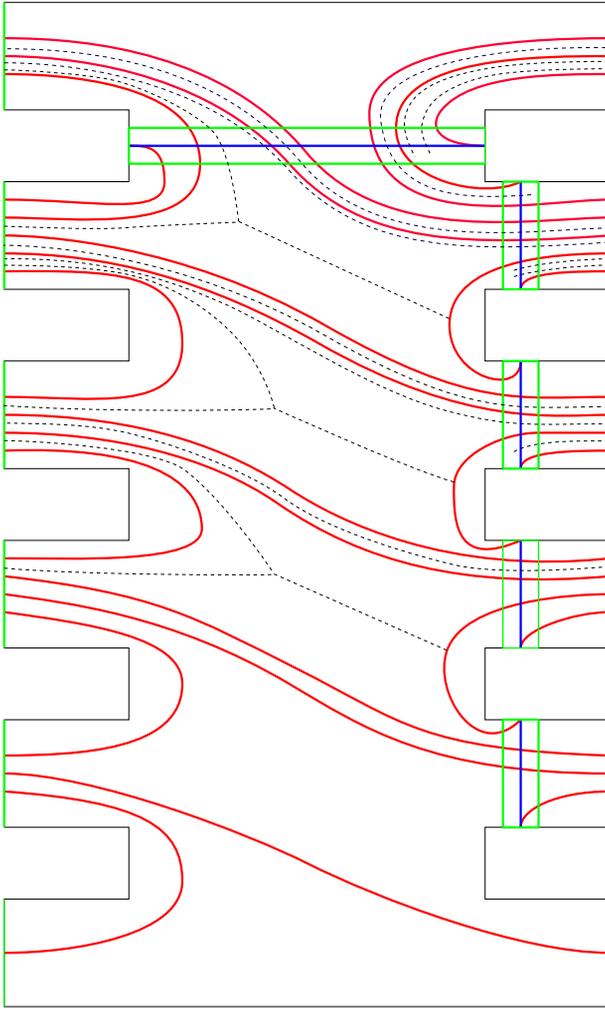


Figure 3: We apply the Sarkar–Wang algorithm by isotoping the red curves along the dashed arcs.

We now directly prove that the contact invariant of  $\tilde{M}$  is zero and calculate its spectral order with respect to the given diagram, thus giving an upper bound on  $\mathfrak{o}(\tilde{M})$ .

If  $Q$  is a quadrilateral component of  $\Sigma \setminus (\alpha \cup \beta)$  disjoint from  $\partial\Sigma$  with corners  $c_1, c_2, c_3, c_4 \in \alpha \cap \beta$ , then we say that  $c_1, c_3$  are its from-corners and  $c_2, c_4$  are its to-corners if

$$\partial(\partial Q \cap \alpha) = c_1 + c_3 - c_2 - c_4.$$

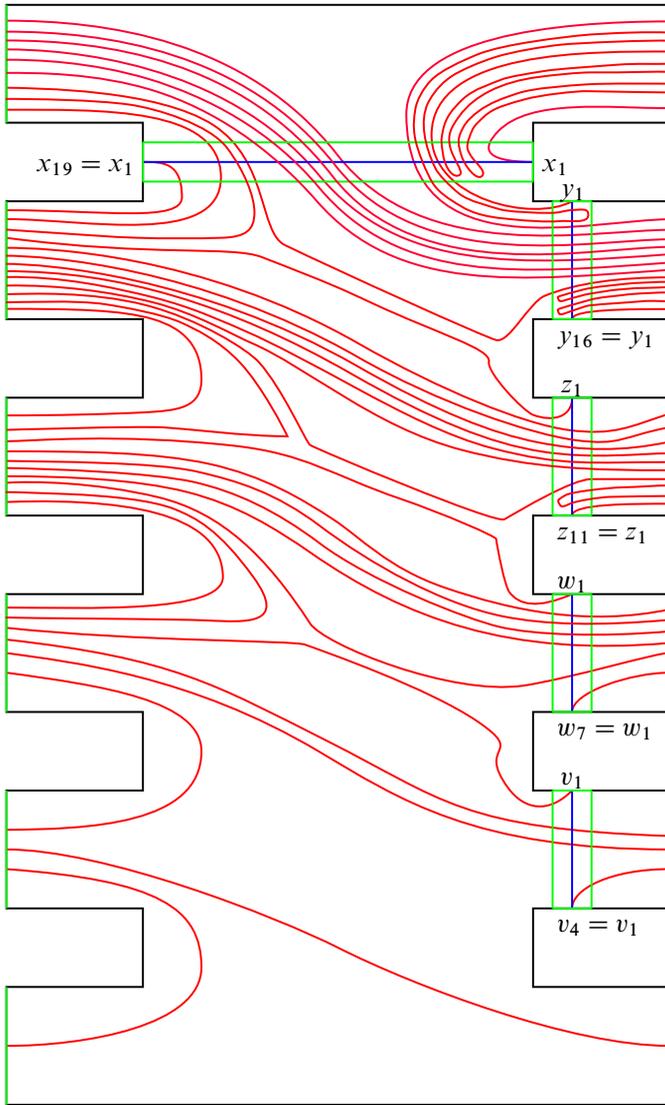


Figure 4: The diagram after applying the Sarkar–Wang algorithm

For any generator  $(c_1, c_3, \dots) \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ , the coefficient of  $(c_2, c_4, \dots)$  in the boundary  $\partial(c_1, c_3, \dots)$  is the number of such quadrilaterals.

Since the only quadrilateral whose to-corners are in  $\{x_1, y_1, z_1, w_1, v_1\}$  is  $y_1 y_2 z_1 z_2$ , we get that

$$\partial(x_1, y_2, z_2, w_1, v_1) = (x_1, y_1, z_1, w_1, v_1) + (x_1, y_3, z_2, w_1, v_1),$$

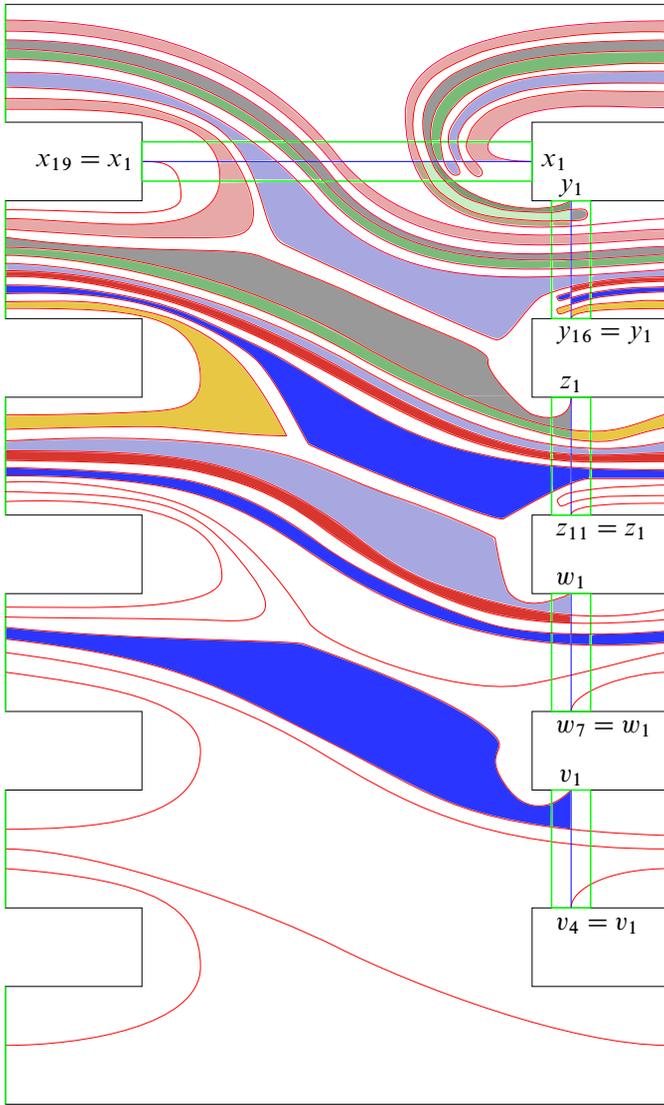


Figure 5: The quadrilaterals and bigons relevant to the computation (shaded)

where the last term comes from the bigon  $y_2y_3$ . This quadrilateral and bigon are shaded gray in Figure 5.

The only quadrilateral whose to-corners are in  $\{x_1, y_2, z_2, w_1, v_1\}$  is  $x_1x_2y_3y_4$ , and we have that

$$\partial(x_2, y_4, z_2, w_1, v_1) = (x_1, y_3, z_2, w_1, v_1) + (x_3, y_4, z_2, w_1, v_1),$$

where the last term comes from the bigon  $x_2x_3$ . This quadrilateral and bigon are shaded pink in [Figure 5](#).

The only quadrilateral whose to-corners are in  $\{x_3, y_4, z_2, w_1, v_1\}$  is  $x_3x_4w_1w_2$ , and we have that

$$\partial(x_4, y_4, z_2, w_2, v_1) = (x_3, y_4, z_2, w_1, v_1) + (x_5, y_4, z_2, w_2, v_1),$$

where the last term comes from the bigon  $x_4x_5$ . This quadrilateral and bigon are shaded light blue in [Figure 5](#).

The only quadrilateral whose to-corners are in  $\{x_5, y_4, z_2, w_2, v_1\}$  is  $x_5x_6z_2z_3$ , shaded green in [Figure 5](#), and we have that

$$\partial(x_6, y_4, z_3, w_2, v_1) = (x_5, y_4, z_2, w_2, v_1) + (x_9, y_1, z_3, w_2, v_1),$$

where the last term comes from the quadrilateral  $x_6x_9y_4y_1$ .

The only quadrilateral whose to-corners are in  $\{x_9, y_1, z_3, w_2, v_1\}$  is  $y_1y_{15}z_3z_2$ , and we have that

$$\partial(x_9, y_{15}, z_2, w_2, v_1) = (x_9, y_1, z_3, w_2, v_1) + (x_9, y_{14}, z_2, w_2, v_1),$$

where the last term comes from the bigon  $y_{15}y_{14}$ . This quadrilateral and bigon are shaded yellow in [Figure 5](#).

The only quadrilateral whose to-corners are in  $\{x_9, y_{14}, z_2, w_2, v_1\}$  is  $y_{14}y_{13}v_1v_2$ , and we have that

$$\partial(x_9, y_{13}, z_2, w_2, v_2) = (x_9, y_{14}, z_2, w_2, v_1) + (x_9, y_{12}, z_2, w_2, v_2),$$

where the last term comes from the bigon  $y_{13}y_{12}$ . This quadrilateral and bigon are shaded blue in [Figure 5](#).

The only quadrilateral whose to-corners are in  $\{x_9, y_{12}, z_2, w_2, v_2\}$  is  $y_{12}y_{11}w_2w_3$ , shown in red, and we have that

$$\partial(x_9, y_{11}, z_2, w_3, v_2) = (x_9, y_{12}, z_2, w_2, v_2).$$

Hence, over  $\mathbb{F}_2$ ,

$$\begin{aligned} &\partial((x_1, y_2, z_2, w_1, v_1) + (x_2, y_4, z_2, w_1, v_1) + (x_4, y_4, z_2, w_2, v_1) + (x_6, y_4, z_3, w_2, v_1) \\ &\quad + (x_9, y_{15}, z_2, w_2, v_1) + (x_9, y_{13}, z_2, w_2, v_2) + (x_9, y_{11}, z_2, w_3, v_2)) \\ &= (x_1, y_1, z_1, w_1), \end{aligned}$$

which is exactly  $x_{\xi|_{\widetilde{M}}}$ . Thus  $\text{EH}(\xi|_{\widetilde{M}}) = 0$ , so the spectral order of  $\widetilde{M}$  is finite.

type	name	$2(n_x + n_y)$	$ \mathbf{x}  -  \mathbf{y} $	$J_+$
quadrilateral	$y_1 y_2 z_1 z_2$	2	-1	0
	$x_1 x_2 y_3 y_4$	2	-1	0
	$x_3 x_4 w_1 w_2$	2	-1	0
	$x_5 x_6 z_2 z_3$	2	1	2
	$x_6 x_9 y_4 y_1$	2	-1	0
	$y_1 y_{15} z_3 z_2$	2	-1	0
	$y_{14} y_{13} v_1 v_2$	2	-1	0
	$y_{12} y_{11} w_2 w_3$	2	1	2
bigon	$y_2 y_3$	1	0	0
	$x_2 x_3$	1	0	0
	$x_4 x_5$	1	0	0
	$y_{15} y_{14}$	1	0	0
	$y_{13} y_{12}$	1	0	0

Table 1: The  $J$ -holomorphic disks used in the calculations and the values used to compute their  $J_+$

**Remark** This result, together with the gluing map of [6], gives an explicit computational proof of the fact that the contact invariant of any contact 3-manifold with Giroux  $2\pi$ -torsion vanishes, which was proven in the closed case by Ghiggini, Honda, and Van Horn-Morris [4].

**Remark** In [7, Example 6(c)], Honda, Kazez, and Matić showed that if we only attach four bypasses to a basic slice, ie if the contact structure is minimally twisting, then the contact invariant is nonzero because it embeds in the unique Stein fillable contact structure on  $T^3$ , which already has nonzero contact invariant. This can also be shown explicitly using a computation analogous to, but simpler than the one above. Hence, it is necessary to enlarge the Giroux  $2\pi$ -torsion domain a bit to obtain vanishing of the contact element.

**Proposition 11** For the perturbed Giroux  $2\pi$ -torsion domain  $\tilde{M}$ , we have

$$o(\tilde{M}, \xi|_{\tilde{M}}) \leq 2.$$

**Proof** The complete list of the  $J$ -holomorphic disks used in the calculations above and the values used to compute their  $J_+$  are given in Table 1. If we label the  $\alpha$ - and

$\beta$ -curves so that  $x_1 \in \alpha_1 \cap \beta_1$ ,  $y_1 \in \alpha_2 \cap \beta_2$ ,  $z_1 \in \alpha_3 \cap \beta_3$ ,  $w_1 \in \alpha_4 \cap \beta_4$ , and  $v_1 \in \alpha_5 \cap \beta_5$ , then

$$x_1, x_9, y_4, y_{16} \in \beta_1, \quad x_6, y_1, z_2 \in \beta_2, \quad z_1, x_2, x_3, y_2, y_3, y_{11}, w_2 \in \beta_3, \\ x_4, x_5, y_{14}, y_{15}, z_3, w_1, v_2 \in \beta_4, \quad \text{and} \quad y_{12}, y_{13}, w_3, v_1 \in \beta_5.$$

Furthermore,  $x_i \in \alpha_1$ ,  $y_i \in \alpha_2$ ,  $z_i \in \alpha_3$ ,  $w_i \in \alpha_4$ , and  $v_i \in \alpha_5$  for any  $i$ . Note that if there is a bigon connecting  $\mathbf{x}, \mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ , then  $|\mathbf{x}| = |\mathbf{y}|$ . Using this,

$$\begin{aligned} |(x_1, y_1, z_1, w_1, v_1)| &= |(1)(2)(3)(4)(5)| = 5, \\ |(x_1, y_2, z_2, w_1, v_1)| &= |(1)(23)(4)(5)| = 4 = |(x_1, y_3, z_2, w_1, v_1)|, \\ |(x_2, y_4, z_2, w_1, v_1)| &= |(132)(4)(5)| = 3 = |(x_3, y_4, z_2, w_1, v_1)|, \\ |(x_4, y_4, z_2, w_2, v_1)| &= |(1432)(5)| = 2 = |(x_5, y_4, z_2, w_2, v_1)|, \\ |(x_6, y_4, z_3, w_2, v_1)| &= |(12)(34)(5)| = 3, \\ |(x_9, y_1, z_3, w_2, v_1)| &= |(1)(2)(34)(5)| = 4, \\ |(x_9, y_{15}, z_2, w_2, v_1)| &= |(1)(243)(5)| = 3 = |(x_9, y_{14}, z_2, w_2, v_1)|, \\ |(x_9, y_{14}, z_2, w_2, v_2)| &= |(1)(2543)| = 2 = |(x_9, y_{12}, z_2, w_2, v_2)|, \\ |(x_9, y_{11}, z_2, w_3, v_2)| &= |(1)(23)(45)| = 3. \end{aligned}$$

From Table 1, we see that every  $J$ -holomorphic disk  $u$  used to compute the differential satisfies  $J_+(u) \leq 2$ ; see (2-2).

For simplicity, we write  $(i, j, k, l, m)$  for the generator  $(x_i, y_j, z_k, w_l, v_m)$ . Then let

$$\begin{aligned} b_0 &= (1, 2, 2, 1, 1) + (2, 4, 2, 1, 1) + (4, 4, 2, 2, 1) + (6, 4, 3, 2, 1) \\ &\quad + (9, 15, 2, 2, 1) + (9, 13, 2, 2, 2), \\ b_1 &= (6, 4, 3, 2, 1) + (9, 15, 2, 2, 1) + (9, 13, 2, 2, 2) + (9, 11, 2, 3, 2), \\ b_2 &= (9, 11, 2, 3, 2), \end{aligned}$$

considered as chains with  $\mathbb{Z}_2$  coefficients. Using Table 1,

$$\begin{aligned} \partial_0 b_0 &= (1, 1, 1, 1, 1) + (5, 4, 2, 2, 1) + (9, 12, 2, 2, 2), \\ \partial_0 b_1 &= (9, 12, 2, 2, 2) \quad \text{and} \quad \partial_1 b_1 = (5, 4, 2, 2, 1) + (9, 12, 2, 2, 2), \\ \partial_0 b_2 &= 0, \quad \partial_1 b_2 = (9, 12, 2, 2, 2), \quad \text{and} \quad \partial_2 b_2 = 0. \end{aligned}$$

Hence  $\partial_0 b_0 + \partial_1 b_1 = (1, 1, 1, 1, 1)$ ,  $\partial_0 b_1 + \partial_1 b_2 = 0$ , and  $\partial_0 b_2 = 0$ . So, if we set  $b_i = 0$  for every  $i > 2$ , then  $\widehat{\partial}(b_i)_{i \in \mathbb{N}} = (c_i)_{i \in \mathbb{N}}$ , where  $c_0 = (1, 1, 1, 1, 1)$  represents

the EH class, and  $c_i = 0$  for every  $i > 0$ . By (2-1), the element  $x_{\xi|_{\tilde{M}}}$  lies in  $B_0^2$ , and hence vanishes in  $E^3$ ; ie  $\mathfrak{o}(\tilde{M}, \xi|_{\tilde{M}}) \leq 2$ , as claimed.  $\square$

As an immediate corollary, we obtain Theorem 2 from the introduction.

**Theorem 2** *If a contact 3–manifold  $(M, \xi)$  with convex boundary has Giroux  $2\pi$ –torsion, then*

$$\mathfrak{o}(M, \xi) \leq 2.$$

**Proof** If the Giroux  $2\pi$ –torsion domain  $M_{2\pi}$  embeds in  $M$ , then the perturbed domain  $\tilde{M}$  also embeds in  $M$ , by the argument outlined at the beginning of this section. Then Theorem 1 and Proposition 11 imply that

$$\mathfrak{o}(M, \xi) \leq \mathfrak{o}(\tilde{M}, \xi|_{\tilde{M}}) \leq 2. \quad \square$$

## 5 Open questions

We raise some questions that naturally arise from the discussions above. First, as in the case of closed contact 3–manifolds, we would like to know how the spectral order  $\mathfrak{o}(\mathcal{P}, \underline{a})$  depends on the choice of partial open book decomposition  $\mathcal{P}$  and arc system  $\underline{a}$ .

**Remark** Given two possible choices of partial open book decompositions  $(\mathcal{P}, \underline{a})$  and  $(\mathcal{P}', \underline{a}')$  for a given contact 3–manifold  $(M, \xi)$  with convex boundary, it is natural to ask whether  $\mathfrak{o}(\mathcal{P}, \underline{a}) = \mathfrak{o}(\mathcal{P}', \underline{a}')$ . In the closed case, according to Kutluhan et al [11], the number  $\mathfrak{o}(S, \phi, \underline{a})$  does not depend on the isotopy class of the arc basis  $\underline{a}$ , but if two arc bases differ by an arc-slide, the corresponding values of  $\mathfrak{o}$  might not be the same. Since our definition of  $\mathfrak{o}$  is a direct generalization of the original one, the same holds in our case.

Now, given the inequality  $\mathfrak{o}(N, \xi|_N) \geq \mathfrak{o}(M, \xi)$ , whenever  $(N, \xi|_N)$  is a compact codimension-zero submanifold of  $(M, \xi)$  with convex boundary, we are led to the following question.

**Question 1** *If a contact 3–manifold  $(N, \xi)$  with convex boundary satisfies the inequality  $\mathfrak{o}(M, \xi_M) \leq k$  for every closed contact 3–manifold  $(M, \xi_M)$  in which  $(N, \xi)$  embeds, do we have  $\mathfrak{o}(N, \xi) \leq k$ ?*

An affirmative answer to [Question 1](#) would imply that the inequality  $\mathfrak{o}(N, \xi|_N) \leq \mathfrak{o}(M, \xi)$  is sharp and cannot be improved without giving extra conditions even when  $M$  is assumed to be closed. We can ask the following question regarding the spectral order of planar torsion domains.

**Question 2** Is there a way to prove that the order of a Giroux torsion domain is at most 1, instead of 2?

The upper bound to the spectral order of a Giroux torsion domain is predicted to be 1 by [[11](#), Question 6.3], since a Giroux torsion domain is a planar torsion domain of order 1. However, our computation only allows us to prove that it is at most 2. If the above question has an affirmative answer, then we must be able to prove it via explicit computation by starting from a complete system of arcs, and then duplicating the arcs, one by one. The problem is that the resulting diagram is too large for practical computation by hand.

Finally, probably the most interesting question in this area is whether the converse of [Theorem 10](#) holds, analogously to [[11](#), Question 6.1].

**Question 3** Does  $\mathfrak{o}(M, \xi) = 0$  imply that  $\xi$  is overtwisted?

## References

- [1] **Y Eliashberg**, *Classification of overtwisted contact structures on 3-manifolds*, Invent. Math. 98 (1989) 623–637 [MR](#)
- [2] **T Etgü, B Özbağcı**, *Partial open book decompositions and the contact class in sutured Floer homology*, Turkish J. Math. 33 (2009) 295–312 [MR](#)
- [3] **P Ghiggini**, *Tight contact structures on Seifert manifolds over  $T^2$  with one singular fibre*, Algebr. Geom. Topol. 5 (2005) 785–833 [MR](#)
- [4] **P Ghiggini, K Honda, J Van Horn-Morris**, *The vanishing of the contact invariant in the presence of torsion*, preprint (2007) [arXiv](#)
- [5] **E Giroux**, *Géométrie de contact: de la dimension trois vers les dimensions supérieures*, from “Proceedings of the International Congress of Mathematicians, II” (L Tatsien, editor), Higher Ed., Beijing (2002) 405–414 [MR](#)
- [6] **K Honda, WH Kazez, G Matic**, *Contact structures, sutured Floer homology and TQFT*, preprint (2008) [arXiv](#)
- [7] **K Honda, WH Kazez, G Matic**, *The contact invariant in sutured Floer homology*, Invent. Math. 176 (2009) 637–676 [MR](#)

- [8] **A Juhász**, *Holomorphic discs and sutured manifolds*, *Algebr. Geom. Topol.* 6 (2006) 1429–1457 [MR](#)
- [9] **A Juhász**, *The sutured Floer homology polytope*, *Geom. Topol.* 14 (2010) 1303–1354 [MR](#)
- [10] **C Kutluhan, G Maticić, J Van Horn-Morris, A Wand**, *Algebraic torsion via Heegaard Floer homology*, preprint (2015) [arXiv](#)
- [11] **C Kutluhan, G Maticić, J Van Horn-Morris, A Wand**, *Filtering the Heegaard Floer contact invariant*, preprint (2016) [arXiv](#)
- [12] **J Latschev, C Wendl**, *Algebraic torsion in contact manifolds*, *Geom. Funct. Anal.* 21 (2011) 1144–1195 [MR](#) With an appendix by Michael Hutchings
- [13] **R Lipshitz**, *A cylindrical reformulation of Heegaard Floer homology*, *Geom. Topol.* 10 (2006) 955–1096 [MR](#)
- [14] **S Sarkar, J Wang**, *An algorithm for computing some Heegaard Floer homologies*, *Ann. of Math.* 171 (2010) 1213–1236 [MR](#)

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