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The number of fiberings of a surface bundle over a surface

LEI CHEN

For a closed manifold M, let $\mathrm{SFib}(M)$ be the number of ways that M can be realized as a surface bundle, up to π_1–fiberwise diffeomorphism. We consider the case when $\dim(M)=4$. We give the first computation of $\mathrm{SFib}(M)$ where $1<\mathrm{SFib}(M)<\infty$ but M is not a product. In particular, we prove $\mathrm{SFib}(M)=2$ for the Atiyah–Kodaira manifold and any finite cover of a trivial surface bundle. We also give an example where $\mathrm{SFib}(M)=4$.

57R22, 57M50; 57M10, 55N25

1 Introduction

Let M be a closed manifold and for any $g>1$ let S_g denote a closed, connected, orientable surface of genus g. We will call the following number the *surface-fibering number* of M:

$$\mathrm{SFib}(M)=\#\{\text{surface bundles } S_g\to M\to B : g>1,\ B \text{ closed manifold}\}/\sim, \tag{1-1}$$

where two fiberings of M are equivalent if and only if they are π_1*–fiberwise diffeomorphic*; ie if the fundamental groups of their fibers are the same subgroups of $\pi_1(M)$. See Section 2 for more details. The equivalence relation of π_1–fiberwise diffeomorphism is more natural than fiberwise diffeomorphism algebraically because it classifies fiberings based on how $\pi_1(M)$ can be represented as an extension by $\pi_1(S_g)$ for some $g>1$. Also π_1–fiberwise diffeomorphism is finer than fiberwise diffeomorphism; so $\mathrm{SFib}(M)$ is an upper bound for the number of fiberings up to fiberwise diffeomorphism.

In the case $\dim(M)=3$, Thurston [11] classified all possible surface bundle structures on a fixed M using the Thurston norm. His theory implies that if M is a surface bundle over S^1, then $\mathrm{SFib}(M)=\infty$ if and only if $\dim(H^1(M;\mathbb{Q}))>1$. Further, the nonzero $\mathbb{Z}$–points in the so-called "fibered cone" of $H^1(M;\mathbb{R})$ up to scalar multiplication are in one-to-one correspondence with distinct fiberings.

In this paper we study the case where $\dim(M)=4$; in other words, the case where M is a surface bundle over a surface. When the Euler characteristic $\chi(M)$ is positive,

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F E A Johnson [5] proved that $\mathrm{SFib}(M) < \infty$. He also obtained an upper bound for $\mathrm{SFib}(M)$ depending only on $\chi(M)$. For any $N > 1$, Salter [9] constructed an example M_N such that $\mathrm{SFib}(M_N) > N$. His work does not give the exact value of $\mathrm{SFib}(M_N)$ for any N. Salter [9; 10] proved that if the monodromy of a nontrivial bundle $S_g \to M \to B$ is in the Johnson kernel, then $\mathrm{SFib}(M) = 1$. He also proved that if $H^1(M;\mathbb{Q}) \cong H^1(B;\mathbb{Q})$ then $\mathrm{SFib}(M) = 1$.

One beautiful example of a multifibered 4–manifold is the *Atiyah–Kodaira manifold* M_{AK}; see Atiyah's paper [1], Kodaira's paper [6] or Section 3 for the construction. It follows from the construction that M_{AK} has at least two different fiberings:

$$S_6 \to M_{\mathrm{AK}} \to S_{129} \quad \text{and} \quad S_{321} \to M_{\mathrm{AK}} \to S_{3.}$$

It is natural to ask if there are any other fiberings. Our first theorem answers this question in the negative.

Theorem 1.1 (surface-fibering number of M_{AK}) *The Atiyah–Kodaira manifold has precisely two fiberings up to π_1–fiberwise diffeomorphism; that is,* $\mathrm{SFib}(M_{\mathrm{AK}}) = 2$*. In particular, M_{AK} has precisely two fiberings up to fiberwise diffeomorphism.*

As mentioned above, fiberwise diffeomorphism is implied by π_1–fiberwise diffeomorphism. In particular, M_{AK} has two fiberings up to fiberwise diffeomorphism because the two fiberings of M_{AK} have different fibers and thus are clearly not fiberwise diffeomorphic. While M_{AK} has been well-studied in the last 50 years by Atiyah, Hirzebruch, Kodaira and many others, we will show that there are choices involved in the construction, which are parametrized by elements in $H^1(S_{129} \times S_3;\mathbb{Z})$. See Section 3 for details. At the end of Section 3.1, we will pose the question of whether the different Atiyah–Kodaira manifolds we construct are diffeomorphic to one another as smooth manifolds.

Denote the genus of a closed oriented surface S by $g(S)$. We can also compute the surface-fibering number of a finite cover over a product $B \times F$ where B and F are two surfaces with $g(B) > 1$ and $g(F) > 1$.

Theorem 1.2 (finite cover of a trivial bundle) *Let E be a regular finite cover of a trivial bundle $B \times F$ where B and F are two surfaces with $g(B) > 1$ and $g(F) > 1$. Then* $\mathrm{SFib}(E) = 2$.

Salter [9] constructed a certain 4–manifold M_S by performing a section sum of two copies of $S_g \times S_g$; see Section 6 for the construction. He provided four distinct fiberings of M_S; so $\mathrm{SFib}(M_S) \geq 4$. Our next theorem classifies the fiberings of M_S.

Theorem 1.3 (Salter's 4–fibering example) *Salter's example M_S has precisely four fiberings up to π_1–fiberwise diffeomorphism; that is,* $\mathrm{SFib}(M_S) = 4$.

Unlike the Atiyah–Kodaira example, the four fiberings of M_S are actually fiberwise diffeomorphic to one another but not π_1–fiberwise diffeomorphic to one another.

All the known examples have $\mathrm{SFib}(M)$ a power of 2. We conjecture that all the examples that Salter built in [9] have $\mathrm{SFib}(M)$ a power of 2. Therefore, we ask the following question.

Question 1.4 (3–fiberings construction) Is there a surface bundle over a surface with total space M such that $\mathrm{SFib}(M)$ is not a power of 2?

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2 Definition of equivalent fiberings and a criterion for two fiberings

In this section we will introduce the definition of π_1–fiberwise diffeomorphism, which is the equivalence relation we use in defining fibering numbers. We will also give a cohomological criterion for a 4–manifold M to have $\mathrm{SFib}(M) = 2$. In this article, we only discuss surface bundles rather than general fiber bundles. Thus we will use "fiberings" to mean "surface-fiberings". When we talk about fundamental groups in this paper, we omit the base point.

Definition 2.1 (π_1–fiberwise diffeomorphism) Given any closed manifold M, two fiberings $F_1 \to M \xrightarrow{p_1} B_1$ and $F_2 \to M \xrightarrow{p_2} B_2$ are *π_1–fiberwise diffeomorphic* if they satisfy the following conditions:

(1) There exists a diagram

$$\begin{array}{ccc} M & \xrightarrow{a} & M \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow{b} & B_2 \end{array}$$

where a, b are both diffeomorphisms.

(2) We have $a_*(\pi_1(F_1)) = \pi_1(F_1)$, where $a_*\colon \pi_1(M) \to \pi_1(M)$ is the induced map on the fundamental groups.

As equivalence relations, π_1–fiberwise diffeomorphism is finer than fiberwise diffeomorphism or bundle diffeomorphism, where we do not assume the second condition in Definition 2.1. In other words, the numbering of fiberings of M up to fiberwise diffeomorphism is at most $\mathrm{SFib}(M)$. To further classify the fiberings up to fiberwise diffeomorphism, we only need to check all the equivalence classes under π_1–fiberwise diffeomorphism. We use π_1–fiberwise diffeomorphism because it is more natural on the group-theoretic level. Two fiberings $F_1 \to M \xrightarrow{p_1} B_1$ and $F_2 \to M \xrightarrow{p_2} B_2$ are π_1–fiberwise diffeomorphic if and only if $\pi_1(F_1)$ and $\pi_1(F_2)$ are the same subgroups in $\pi_1(M)$. From now on, we call two fiberings equivalent if they are π_1–fiberwise diffeomorphic. We have the following lemma of Salter [9, Lemma 3.3].

Lemma 2.2 *Given any closed* 4*–manifold* M*, if there are two fiberings* $M \xrightarrow{p_1} B_1$ *and* $M \xrightarrow{p_2} B_2$ *that are not equivalent, then* $p_1^*(H^1(B_1;\mathbb{Q})) \cap p_2^*(H^1(B_2;\mathbb{Q})) = \{0\}$.

The following lemma is a cohomological criterion for a 4–manifold M to have $\mathrm{SFib}(M) = 2$.

Lemma 2.3 (criterion for $\mathrm{SFib}(M) = 2$) *Let* $S_{h_1} \to M \xrightarrow{p_1} S_{g_1}$ *and* $S_{h_2} \to M \xrightarrow{p_2} S_{g_2}$ *be two surface bundles over a surface where* $h_1, g_1, h_2, g_2 > 1$ *and* p_1 *is not equivalent to* p_2*. Let* $(p_1, p_2)\colon M \to S_{g_1} \times S_{g_2}$ *be the product. If*

$$(p_1, p_2)^*\colon H^1(S_{g_1} \times S_{g_2};\mathbb{Q}) \to H^1(M;\mathbb{Q})$$

is an isomorphism and if

$$(p_1, p_2)^*\colon H^2(S_{g_1} \times S_{g_2};\mathbb{Q}) \to H^2(M;\mathbb{Q})$$

is injective, then $\mathrm{SFib}(M) = 2$.

Proof Suppose there exists a third fibering $F \to M \xrightarrow{p} B$ such that p is not equivalent to p_1 or p_2. By Lemma 2.2, for any nonzero element $x \in H^1(B;\mathbb{Q})$, we have that $p^*(x) \notin p_1^* H^1(S_{g_1};\mathbb{Q})$ and $p^*(x) \notin p_2^* H^1(S_{g_2};\mathbb{Q})$. Therefore, there exist $a \neq 0 \in p_1^* H^1(S_{g_1};\mathbb{Q})$ and $b \neq 0 \in p_2^* H^1(S_{g_2};\mathbb{Q})$ such that

$$p^*(x) = a + b \in p_1^* H^1(S_{g_1};\mathbb{Q}) \oplus p_2^* H^1(S_{g_2};\mathbb{Q}) \cong H^1(M;\mathbb{Q}).$$

Since $\chi(M) > 0$ and $\chi(F) < 0$, we have $\chi(B) < 0$, implying $g(B) > 1$. Therefore, there is an element $y \neq 0 \in H^1(B;\mathbb{Q})$ which is not a multiple of x but satisfies

$$x \smile y = 0 \in H^2(B;\mathbb{Q}).$$

Suppose that

$$p^*(y) = c + d \in p_1^* H^1(S_{g_1};\mathbb{Q}) \oplus p_2^* H^1(S_{g_2};\mathbb{Q}) \cong H^1(M;\mathbb{Q}).$$

Since $x \smile y = 0$,

$$(a+b) \smile (c+d) = 0 \in (p_1, p_2)^* H^2(S_{g_1} \times S_{g_2};\mathbb{Q}) \subset H^2(M;\mathbb{Q}).$$

By the Künneth formula

$$H^2(S_{g_1} \times S_{g_2};\mathbb{Q}) \cong H^2(S_{g_1};\mathbb{Q}) \oplus \left[H^1(S_{g_1};\mathbb{Q}) \otimes H^1(S_{g_2};\mathbb{Q})\right] \oplus H^2(S_{g_2};\mathbb{Q}),$$

we have $a \smile d + b \smile c = 0$. By skew-commutativity of cup product, $a \smile d = c \smile b$. By the property of the tensor product of vector spaces, the only possibility is that $c = ka$ and $d = kb$ for some $k \in \mathbb{Q}$. Hence y is a multiple of x, which is a contradiction. The result follows. □

3 Description of M_{AK} and the uniqueness problem

In this section we will describe the Atiyah–Kodaira manifold M_{AK} and its monodromy representation. While M_{AK} has been studied intensively in the last 50 years, we will show below that there are choices involved in the construction, which are parametrized by elements in a cohomology group. At the end, we will pose the question of whether the different choices involved determine diffeomorphic manifolds.

3.1 The geometric construction of M_{AK}

We now construct the Atiyah–Kodaira manifold M_{AK}, following Morita [8, Chapter 4.3]. Let S_3 be a surface of genus 3 and let τ be a free $\mathbb{Z}/2\mathbb{Z}$–action on S_3, as in Figure 1. The trivial bundle $S_3 \times S_3$ has two sections: Γ_{id}, the graph of the identity, and Γ_τ, the graph of τ.

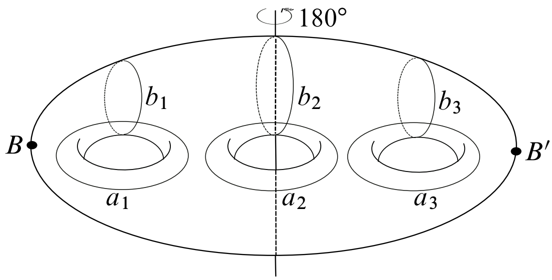


Figure 1: Involution τ

Since the action is free, the two sections are disjoint. The kernel of the surjective homomorphism $\pi_1(S_3) \to H_1(S_3;\mathbb{Z}/2)$ gives a finite cover $i\colon S_{129} \to S_3$. We have the exact sequence

$$1 \to \pi_1(S_{129}) \xrightarrow{i_*} \pi_1(S_3) \to H_1(S_3;\mathbb{Z}/2) \to 1.$$

The pullback surface bundle $i^*(S_3 \times S_3) \cong S_{129} \times S_3$ also has two sections, $S_i = i^*(\Gamma_{\mathrm{id}})$ and $S_\tau = i^*(\Gamma_\tau)$. We have $S_i = \mathrm{graph}(i)$ and $S_\tau = \mathrm{graph}(\tau \circ i)$. The plan now is to characterize $\mathbb{Z}/2$–branched covers over $S_{129} \times S_3$ with branch locus $S_i \cup S_\tau$. We begin by computing the Poincaré dual of the homology class $[S_i] + [S_\tau]$. The Künneth formula gives us

$$H_2(S_{129} \times S_3;\mathbb{Z}/2) \\ \cong H_2(S_{129};\mathbb{Z}/2) \oplus \big[H_1(S_{129};\mathbb{Z}/2) \otimes H_1(S_3;\mathbb{Z}/2)\big] \oplus H_2(S_3;\mathbb{Z}/2).$$

Let $[S_{129}]$ and $[S_3]$ be the fundamental classes of $H_2(S_{129};\mathbb{Z}/2)$ and $H_2(S_3;\mathbb{Z}/2)$, respectively. Pick points $p_0 \in S_{129}$ and $q_0 \in S_3$. Define maps

$$\begin{aligned} e_1\colon S_{129} &\to S_{129} \times S_3 \\ x &\mapsto (x, q_0) \end{aligned} \qquad \text{and} \qquad \begin{aligned} e_2\colon S_3 &\to S_{129} \times S_3 \\ y &\mapsto (p_0, y). \end{aligned}$$

By the computation in [7, Chapter 11] and the fact that i_* induces the zero map on $H_1(-;\mathbb{Z}/2)$, we have $[S_i] = e_{1*}[S_{129}]$ and $[S_\tau] = e_{1*}[S_{129}]$ in $H_2(S_{129} \times S_3;\mathbb{Z}/2)$. Therefore

$$[S_i] + [S_\tau] = e_{1*}[S_{129}] + e_{1*}[S_{129}] = 0 \in H_2(S_{129} \times S_3;\mathbb{Z}/2).$$

Denote the Poincaré dual of $[S_i] + [S_\tau]$ by $\mathrm{PD}([S_i] + [S_\tau])$. By Poincaré duality,

$$\mathrm{PD}([S_i] + [S_\tau]) = 0 \in H^2(S_{129} \times S_3;\mathbb{Z}/2).$$

Let $M := S_{129} \times S_3 - S_i - S_\tau$. We have the long exact sequence in cohomology of the relative pair $(S_{129} \times S_3, M)$:

$$H^1(S_{129} \times S_3, M; \mathbb{Z}/2) \to H^1(S_{129} \times S_3; \mathbb{Z}/2) \to H^1(M; \mathbb{Z}/2) \xrightarrow{\phi} H^2(S_{129} \times S_3, M; \mathbb{Z}/2) \xrightarrow{T} H^2(S_{129} \times S_3; \mathbb{Z}/2). \tag{3-1}$$

By the Thom isomorphism theorem, we have

$$H^1(S_{129} \times S_3, M; \mathbb{Z}/2) = 0$$

and

$$H^2(S_{129} \times S_3, M; \mathbb{Z}/2) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2.$$

Let T and ϕ be the homomorphisms in the exact sequence (3-1). Now $T(1,0) = \mathrm{PD}[S_i]$ and $T(0,1) = \mathrm{PD}[S_\tau]$. Therefore

$$T(1,1) = 0 \in H^2(S_{129} \times S_3; \mathbb{Z}/2).$$

So $\phi^{-1}(1,1)$ is not empty in $H^1(M; \mathbb{Z}/2)$. By the isomorphism

$$\mathrm{Hom}(\pi(M), \mathbb{Z}/2) \cong H^1(M; \mathbb{Z}/2),$$

we have that $H^1(M; \mathbb{Z}/2)$ classifies $\mathbb{Z}/2$–covers of M. Therefore $\phi^{-1}(1,1)$ classifies the $\mathbb{Z}/2$–branched covers of $S_{129} \times S_3$ with branch locus $S_i \cup S_\tau$. Let M_{AK} be one of them. These branched covers are characterized by a subset of $H^1(M; \mathbb{Z}/2)$, which is an affine space over $H^1(S_{129} \times S_3; \mathbb{Z}/2)$ by the exact sequence (3-1). Later we will analyze how an element of $H^1(S_{129} \times S_3; \mathbb{Z}/2)$ affects the monodromy. We also pose a question about the Atiyah–Kodaira construction.

Question 3.1 (uniqueness of Atiyah–Kodaira example) After fixing the trivial bundle $S_{129} \times S_3$ and the two sections S_i and S_τ, there are many choices of branched covers of $S_{129} \times S_3$ with branch locus $S_i \cup S_\tau$. Are the different branched covers diffeomorphic as smooth manifolds?

3.2 The monodromy description of M_{AK}

In this subsection, we provide a second construction of M_{AK} from the point of view of the monodromy representation. For $g > 1$, the monodromy representation determines an S_g–bundle uniquely; see eg Farb and Margalit [4, Chapter 5.6].

Let $S_{g,n}$ be a genus-g surface with n punctures. Let $\mathrm{PMod}_{g,n}$ (resp. $\mathrm{Mod}_{g,n}$) be the pure mapping class group of $S_{g,n}$, ie the group of isotopy classes of orientation-preserving diffeomorphisms of S_g that fix n points individually (resp. as a set). We omit n when $n=0$. Let $\mathrm{PConf}_n(S)$, the *pure configuration space* of a surface S, be the space of ordered n–tuples of distinct points on S. We have a *generalized Birman exact sequence* [4, Theorem 9.1]

$$1 \to \pi_1(\mathrm{PConf}_n(S_g)) \xrightarrow{\text{Push}} \mathrm{PMod}_{g,n} \to \mathrm{Mod}_g \to 1.$$

The two disjoint sections of the bundle $S_3 \times S_3$ give us a map $(\mathrm{id}, \tau)\colon S_3 \to \mathrm{PConf}_2(S_3)$, and hence a monodromy representation

$$\rho_\tau\colon \pi_1(S_3) \to \pi_1(\mathrm{PConf}_2(S_g)) \xrightarrow{\text{Push}} \mathrm{PMod}_{3,2}.$$

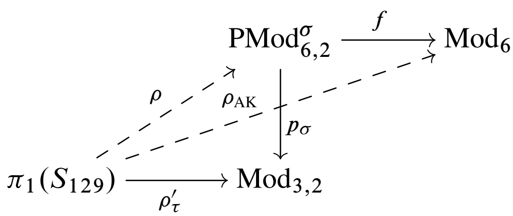
Let $B \in S_3$ and $B' = \tau(B)$. The $\mathbb{Z}/2$–branched covers of S_3 with branch points B and B' are parametrized by a subset of $H^1(S_{3,2};\mathbb{Z}/2)$. Pick any $\mathbb{Z}/2$–branched cover $\pi\colon S_6 \to S_3$ with a deck transformation σ and branch points $\{B, B'\}$.

Let $\mathrm{PMod}_{6,2}^\sigma$ be the centralizer of σ in $\mathrm{PMod}_{6,2}$. We have a map $p_\sigma\colon \mathrm{PMod}_{6,2}^\sigma \to \mathrm{Mod}_{3,2}$. By the construction, $\pi_1(S_{129})$ acts trivially on $H^1(S_{3,2};\mathbb{Z}/2)$; this can also be seen by computing the $\rho_\tau(\pi_1(S_3))$–action on $H^1(S_{3,2};\mathbb{Z}/2)$. The monodromy

$$\rho'_\tau := \rho_\tau|_{\pi_1(S_{129})}\colon \pi_1(S_{129}) \to \pi_1(S_3) \to \mathrm{Mod}_{3,2}$$

admits a lift to $\mathrm{PMod}_{6,2}^\sigma$ as in the diagram

(3-2)



ie there exists ρ such that $p_\sigma \circ \rho = \rho'_\tau$. Let $f\colon \mathrm{PMod}_{6,2} \to \mathrm{Mod}_6$ be the forgetful map and let $\rho_{\mathrm{AK}} = \rho \circ f$ be the monodromy representation of a lift. The geometric construction depends on some noncanonical parameters; similarly, this phenomenon reappears when we consider the monodromy representation.

Remark 3.2 The lift ρ in diagram (3-2) is not unique! Let $\{g_i, h_i\}$ be the generators of $\pi_1(S_{129})$ such that $\pi_1(S_{129}) = \langle g_i, h_i \mid \prod_i [g_i, h_i] = 1\rangle$. Because σ commutes with any element in the set $\{G_i = \rho_{\mathrm{AK}}(g_i), H_i = \rho_{\mathrm{AK}}(h_i)\}$, we could multiply σ with a subset of $\{G_i, H_i\}$ to obtain a new monodromy representation. For example, $\{G_i\sigma, H_i\}$ is a new monodromy representation. Among all the different monodromy representations, are the total spaces of the surface bundles diffeomorphic to each other?

4 The proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by computing $H^1(M_{\mathrm{AK}};\mathbb{Q})$.

4.1 The lift of a square of a point push

In this subsection, we will determine the lifts of some elements of $\pi_1(S_{129})$ to Mod_6 under the branched cover. For any simple closed curve c, we denote the Dehn twist about c by T_c. For any loop L at the base point B, denote the point-pushing map on L by $\mathrm{Push}(L)$. Let a be the loop in Figure 2. We have $\mathrm{Push}(a) = T_x T_y^{-1}$; see eg [4, Fact 4.7].

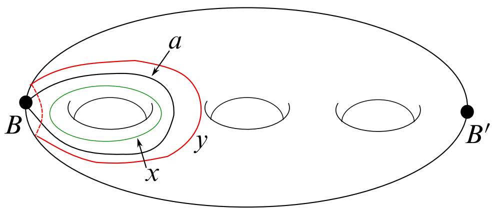


Figure 2: Point-pushing. The lighter colors (green and red) represent x and y and the darker color (black) represents a.

Since B is one of the branched points of the $\mathbb{Z}/2$–cover $\pi\colon S_6 \to S_3$, one of the curves x or y will lift to two copies and the other will lift to a single copy. The curve a will have two lifts, which we call a^- and a^+. Since $\mathrm{Push}(a)^2$ acts trivially on $H_1(S_{3,2};\mathbb{Z}/2)$, the action $\mathrm{Push}(a)^2$ lifts to an action on S_6. Let $\mathrm{Lift}(\mathrm{Push}(a)^2)$ be the lift of the point-pushing action on S_6. For any two curves c_1, c_2, let $i(c_1, c_2)$

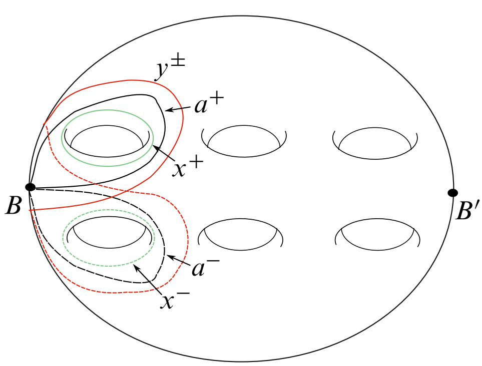


Figure 3: Lifts of a, x, y with the same colorings as Figure 2

be the algebraic intersection number of c_1 and c_2. In the following computation, we will use the letters a, x, y to represent either the curves or the homology classes that the curves represent.

Lemma 4.1 *Pick a direction on a and assign directions on a^+, a^- accordingly. For any $c \in H_1(S_6;\mathbb{Q})$, there are two possibilities for the action of $\mathrm{Lift}(\mathrm{Push}(a)^2)$ on c:*

$$\mathrm{Lift}(\mathrm{Push}(a)^2)(c) = c \pm i(c, a^+ - a^-)(a^+ - a^-)$$

or

$$\mathrm{Lift}(\mathrm{Push}(a)^2)(c) = \sigma_*(c \pm i(c, a^+ - a^-)(a^+ - a^-)).$$

Proof Suppose without loss of generality that $\mathrm{Lift}(x) = x^+ \cup x^-$ and $\mathrm{Lift}(y) = y^\pm$. By looking at the action locally, we have that $\mathrm{Lift}(T_x^2) = T_{x^-}^2 T_{x^+}^2$ and $\mathrm{Lift}(T_y^2) = T_{y^\pm}$. Therefore

$$\mathrm{Lift}(\mathrm{Push}(a)^2) = \mathrm{Lift}(T_x^2 T_y^{-2}) = T_{x^-}^2 T_{x^+}^2 T_{y^\pm}^{-1}.$$

Since x^+ and x^- are homotopic to a^+ and a^- on S_6, respectively, we have $x^+ = a^+$ and $x^- = a^-$ as homology classes. Since $x^+, x^-, y^\pm$ bound a pair of pants, there exists an orientation of $y^\pm$ such that as a homology class, $y^\pm = a^+ + a^-$. Thus we have the following computation on the action of the homology:

$$\begin{aligned} T_{x^+}^2 T_{x^-}^2 T_{y^\pm}^{-1}(c) &= c - i(c, y^\pm) y^\pm + i(c, x^+) 2x^+ + i(c, x^-) 2x^- \\ &= c - i(c, a^+ + a^-)(a^+ + a^-) + i(c, a^+) 2a^+ + i(c, a^-) 2a^- \\ &= c + i(c, a^- - a^+)(a^- - a^+). \end{aligned} \tag{4-1}$$

In the case where $\mathrm{Lift}(y) = y^+ \cup y^-$ and $\mathrm{Lift}(x) = x^\pm$, we have

$$\mathrm{Lift}(\mathrm{Push}(a)^2)(c) = c - i(c, a^+ - a^-)(a^+ - a^-).$$

Since every element has two lifts that differ by the deck transformation σ, we have the second possibility. □

4.2 The eigendecomposition of the action of σ_* on $H^1(S_6;\mathbb{Q})$

In this subsection, we will discuss the eigendecomposition of the action of σ_* on $H^1(S_6;\mathbb{Q})$ and determine the image of $\pi^*\colon H^1(S_3;\mathbb{Q}) \to H^1(S_6;\mathbb{Q})$ induced from the $\mathbb{Z}/2$–branched cover $\pi\colon S_6 \to S_3$. The action of the deck transformation σ on S_6 induces a decomposition of $H_1(S_6;\mathbb{Q})$ by the eigenvalues of the σ_*–action on $H_1(S_6;\mathbb{Q})$. Since σ is an involution, the eigenvalues of σ_* are $\{\pm 1\}$. Let H^+

be the eigenspace of σ associated with eigenvalue $+1$ and H^- the eigenspace of σ associated with eigenvalue -1. Then there is a direct sum

$$H_1(S_6;\mathbb{Q}) = H^- \oplus H^+.$$

Via the universal coefficient theorem, $f \in H^1(S_6;\mathbb{Q})$ corresponds to a functional $f\colon H_1(S_6;\mathbb{Q}) \to \mathbb{Q}$.

Claim 4.2 *A functional $f\colon H_1(S_6;\mathbb{Q}) \to \mathbb{Q}$ belongs to $\pi^* H^1(S_3;\mathbb{Q})$ if and only if $H^- \subset \ker(f)$.*

Proof It is classical that $\pi^*\colon H^1(S_3;\mathbb{Q}) \to H^1(S_6;\mathbb{Q})^{\mathbb{Z}/2}$ is an isomorphism; see eg [2, Theorem III.2.4]. Then

$$\begin{aligned} f \in H^1(S_6;\mathbb{Q})^{\mathbb{Z}/2} &\iff \sigma^*(f) = f \\ &\iff \sigma^*(f)(x) = f(x) \quad \text{for any } x \in H_1(S_6;\mathbb{Q}) \\ &\iff f(\sigma_*(x) - x) = 0 \quad \text{for any } x \in H_1(S_6;\mathbb{Q}). \end{aligned}$$

Since σ_* is an involution, we know that the subspace of $H_1(S_6;\mathbb{Q})$ that is spanned by $\{\sigma_*(x) - x : x \in H_1(S_6;\mathbb{Q})\}$ is H^-. Therefore $f \in \pi^* H^1(S_3;\mathbb{Q})$ if and only if $H^- \subset \ker(f)$. □

In Figure 4, we have a geometric description of a basis $\{a_1^+, a_1^-, \dots\}$ of $H_1(S_6;\mathbb{Q})$ where a_i^+ and a_i^- or b_i^+ and b_i^- are each other's images under the σ_* action.

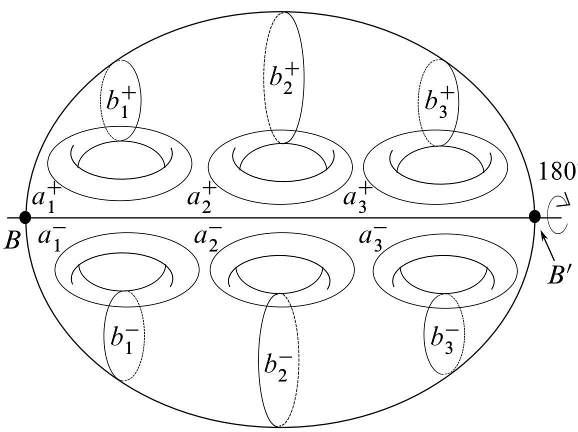


Figure 4: Deck transformation σ

4.3 The $\pi_1(S_{129})$–invariant cohomology

Let a_1 be a simple loop based at B as in Figure 1. Since a_1 does not intersect $\tau(a_1)$, the monodromy action of a_1 on $S_{3,2}$ is the product of point-pushings at B and B' of a_1 and $\tau(a_1)$. By the monodromy description of M_{AK}, we have that $\rho_{\mathrm{AK}}(a_1^2) = \mathrm{Lift}(\mathrm{Push}(a_1)^2\,\mathrm{Push}(\tau(a_1))^2)$.

Lemma 4.3 *Let $f \in H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})}$ be an invariant cohomology class. Then f satisfies $f(a_1^+ - a_1^-) = 0$ and $f(a_3^+ - a_3^-) = 0$.*

Proof By Lemma 4.1, $\mathrm{Push}(a_1)^2\,\mathrm{Push}(\tau(a_1))^2$ has two possible lifts that differ by the deck transformation σ.

Case 1 For any $c \in H_1(S_6;\mathbb{Q})$,

$$\rho_{\mathrm{AK}}(a_1^2)(c) = c \pm i(c, a_1^+ - a_1^-)(a_1^+ - a_1^-) \pm i(c, a_3^+ - a_3^-)(a_3^+ - a_3^-).$$

Since f is invariant under the action of $\rho_{\mathrm{AK}}(a_1^2)$, we have $f(\rho_{\mathrm{AK}}(a_1^2)(c)) = f(c)$ for any $c \in H_1(S_6;\mathbb{Q})$. After evaluating f on both sides, we obtain

$$f(c) = f(c) \pm i(c, a_1^+ - a_1^-) f(a_1^+ - a_1^-) \pm i(c, a_3^+ - a_3^-) f(\tilde{a}_3 - a_3^+).$$

Equivalently,

$$i(c, a_1^+ - a_1^-) f(a_1^+ - a_1^-) \pm i(c, a_3^+ - a_3^-) f(a_3^+ - a_3^-) = 0.$$

However, $a_1^+ - a_1^-$ and $a_3^+ - a_3^-$ are independent elements in $H_1(S_6;\mathbb{Q})$, so we can find c such that $i(c, a_1^+ - a_1^-) = 0$ and $i(c, a_3^+ - a_3^-) = 1$. Therefore we must have $f(a_3^+ - a_3^-) = 0$. By the same argument, $f(a_1^- - a_1^+) = 0$.

Case 2 For any $c \in H_1(S_6;\mathbb{Q})$,

$$\rho_{\mathrm{AK}}(a_1^2)(c) = \sigma_*(c \pm i(c, a_1^+ - a_1^-)(a_1^+ - a_1^-) \pm i(c, a_3^+ - a_3^-)(a_3^+ - a_3^-)).$$

Since f is invariant under the action of a_1^2, we have $f(\rho_{\mathrm{AK}}(a_1^2)(c)) = f(c)$. After evaluating f on both sides, we obtain

$$f(c) = f(\sigma_*(c)) \pm i(c, a_1^- - a_1^+) f(\sigma_*(a_1^- - a_1^+)) \pm i(c, a_3^+ - a_3^-) f(\sigma_*(a_3^+ - a_3^-)).$$

If we set $c = a_1^-$ and $c = a_3^-$, respectively, we obtain

$$f(a_1^-) = f(a_1^+) \quad \text{and} \quad f(a_3^-) = f(a_3^+). \qquad \square$$

This allows us to determine the full invariant cohomology $H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})}$.

Lemma 4.4 *We have the isomorphism*

$$H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})} \cong H^1(S_3;\mathbb{Q}).$$

Proof By using the same argument as in Lemma 4.3 on $(b_1)^2$ and $(b_2a_1)^2$, we obtain that $f(b_3^+ - b_3^-) = 0$, $f(b_1^+ - b_1^-) = 0$ and

$$f((b_2+a_1)^+ - (b_2+a_1)^-) = 0.$$

Since we already have $f(a_1^+ - a_1^-) = 0$, we obtain that $f(b_2^+ - b_2^-) = 0$.

From the above discussion, any $f \in H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})}$ is zero on the 5–dimensional space spanned by

$$a_1^+ - a_1^-, \quad b_1^+ - b_1^-, \quad b_3^+ - b_3^-, \quad b_3^+ - b_3^-, \quad b_2^+ - b_2^-.$$

Therefore $\dim(H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})}) \le 7$. Since $\pi^* H^1(S_3;\mathbb{Q})) \subset H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})}$, we have that $\dim(H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})}) \ge 6$. Via the Serre spectral sequence of the fiber bundle $S_6 \to M_{\mathrm{AK}} \to S_{129}$, we have

$$1 \to H^1(S_{129};\mathbb{Q}) \to H^1(M_{\mathrm{AK}};\mathbb{Q}) \to H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})} \to 1.$$

This implies

$$\dim(H^1(M_{\mathrm{AK}};\mathbb{Q})) = \dim(H^1(S_{129};\mathbb{Q})) + \dim(H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})}).$$

As a branched cover of an algebraic surface along an algebraic curve, the manifold M_{AK} is itself an algebraic surface; thus M_{AK} is a Kähler manifold. Therefore, $\dim(H^1(M_{\mathrm{AK}};\mathbb{Q}))$ must be an even number. So we have $\dim(H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})}) = 6$, which shows

$$H^1(S_6;\mathbb{Q})^{\pi_1(S_{129})} \cong H^1(S_3;\mathbb{Q}).$$ □

4.4 Finishing the proof of Theorem 1.1

Proof of Theorem 1.1 Since $M_{\mathrm{AK}} \xrightarrow{P} S_{129} \times S_3$ is a finite branched cover, the induced map $P^*\colon H^4(S_3 \times S_{129};\mathbb{Q}) \to H^4(M_{\mathrm{AK}};\mathbb{Q})$ is an isomorphism. By Poincaré duality, $P^*\colon H^k(S_3 \times S_{129};\mathbb{Q}) \to H^k(M_{\mathrm{AK}};\mathbb{Q})$ is injective for any k. Via Lemma 4.4, we also have

$$H^1(M_{\mathrm{AK}};\mathbb{Q}) \cong H^1(S_3;\mathbb{Q}) \oplus H^1(S_{129};\mathbb{Q}).$$

Therefore M_{AK} satisfies the assumption of Lemma 2.3, and hence $\mathrm{SFib}(M_{\mathrm{AK}}) = 2$. □

5 Fibering number of a finite cover of a trivial bundle

In this section, we will prove Theorem 1.2. Let E be a regular finite cover of $B \times F$ where B and F are surfaces with $g(B) > 1$ and $g(F) > 1$. Let $p_1\colon E \to B$ and $p_2\colon E \to F$ be the projections. Denote the image of $p_{1*}\colon \pi_1(E) \to \pi_1(B)$ by $\mathrm{Im}(p_1)$ and the image of $p_{2*}\colon \pi_1(E) \to \pi_1(F)$ by $\mathrm{Im}(p_2)$.

Lemma 5.1 *With the above definitions and assumptions, we have*

$$H^1(E;\mathbb{Q}) \cong H^1(\mathrm{Im}(p_1);\mathbb{Q}) \oplus H^1(\mathrm{Im}(p_2);\mathbb{Q}).$$

Proof All spaces involved here are $K(\pi,1)$ spaces; we will sometimes switch between cohomology of groups and that of spaces. Let $\pi_1(\widetilde{F})$ be the kernel of $p_{1*}\colon \pi_1(E) \to \mathrm{Im}(p_1)$. Since $\pi_1(E)$ is a finite-index normal subgroup of $\mathrm{Im}(p_1) \times \mathrm{Im}(p_2)$, we have that $\pi_1(\widetilde{F})$ is a finite-index normal subgroup of $\mathrm{Im}(p_2)$. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\widetilde{F}) & \longrightarrow & \pi_1(E) & \longrightarrow & \mathrm{Im}(p_1) & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow{=} & & \\ 1 & \longrightarrow & \mathrm{Im}(p_2) & \longrightarrow & \mathrm{Im}(p_1)\times\mathrm{Im}(p_2) & \longrightarrow & \mathrm{Im}(p_1) & \longrightarrow & 1 \end{array} \tag{5-1}$$

The group $\pi(E)$ acts on $H^1(\widetilde{F};\mathbb{Q})$ by conjugation. Since $\mathrm{Im}(p_1)$ commutes with $\mathrm{Im}(p_2)$ and $p_{1*}\colon \pi_1(E) \to \mathrm{Im}(p_2)$ is surjective, we have that

$$H^1(\widetilde{F};\mathbb{Q})^{\mathrm{Im}(p_2)} \cong H^1(\widetilde{F};\mathbb{Q})^{\pi_1(E)}.$$

Since $\pi_1(\widetilde{F})$ is a finite-index subgroup of $\mathrm{Im}(p_2)$, we have that

$$H^1(\widetilde{F};\mathbb{Q})^{\mathrm{Im}(p_2)} \cong H^1(\mathrm{Im}(p_2);\mathbb{Q}).$$

By the top exact sequence of (5-1), we obtain

$$0 \to H^1(\mathrm{Im}(p_1);\mathbb{Q}) \to H^1(E;\mathbb{Q}) \to H^1(\widetilde{F};\mathbb{Q})^{\pi(E)} \cong H^1(\mathrm{Im}(p_2);\mathbb{Q}) \to 0.$$

The lemma follows. □

Proof of Theorem 1.2 Since $\pi_1(E)$ is a finite-index subgroup of $\mathrm{Im}(p_1)\times\mathrm{Im}(p_2)$, we obtain that $H^4(\mathrm{Im}(p_1)\times\mathrm{Im}(p_2);\mathbb{Q}) \to H^4(E;\mathbb{Q})$ is an isomorphism. By Poincaré duality, $H^k(\mathrm{Im}(p_1)\times\mathrm{Im}(p_2);\mathbb{Q}) \to H^k(E;\mathbb{Q})$ is injective for every k. More specifically, $H^2(\mathrm{Im}(p_1)\times\mathrm{Im}(p_2);\mathbb{Q}) \subset H^2(E;\mathbb{Q})$. Therefore E satisfies the assumptions of Lemma 2.3, which shows that $\mathrm{SFib}(E) = 2$. □

6 An example with exactly four fiberings

Now we deal with an example of Salter [10] and we prove that it has exactly four fiberings. As we mentioned before, the equivalence relation we choose is π_1–fiberwise diffeomorphism, not fiberwise diffeomorphism. Under fiberwise diffeomorphism, M_S only has one fibering.

Let $\triangle$ be the diagonal in $S_g \times S_g$. Let $M_S = (S_g \times S_g - \triangle) \cup_\theta (S_g \times S_g - \triangle)$, where θ is the identification of the boundaries of the two copies of $S_g \times S_g - \triangle$. Each copy of $S_g \times S_g - \triangle$ has two fiberings, p_1 and p_2, where p_i is the projection onto the i^{th} coordinate. Therefore M_S has four obvious fiberings: $\{(p_i, p_j) : i, j = 1, 2\}$.

There is a subtlety in defining (p_1, p_2) and (p_2, p_1) in the smooth category, but the details will be immaterial here. See [10, Section 2].

Lemma 6.1 *With the notation as in the previous paragraph and for $g \geq 2$,*

$$H^1(S_g \times S_g - \triangle; \mathbb{Q}) \cong p_1^*(H^1(S_g; \mathbb{Q})) \oplus p_2^*(H^1(S_g; \mathbb{Q})).$$

Proof By the Thom isomorphism theorem, $H^1(S_g \times S_g, S_g \times S_g - \triangle; \mathbb{Q}) = 0$. The long exact sequence of relative cohomology

$$0 \to H^1(S_g \times S_g; \mathbb{Q}) \to H^1(S_g \times S_g - \triangle; \mathbb{Q}) \to H^1(S_g \times S_g, S_g \times S_g - \triangle; \mathbb{Q}) = 0$$

tells us that $H^1(S_g \times S_g; \mathbb{Q}) \cong H^1(S_g \times S_g - \triangle; \mathbb{Q})$. By the Künneth formula,

$$H^1(S_g \times S_g - \triangle; \mathbb{Q}) \cong p_1^*(H^1(S_g; \mathbb{Q})) \oplus p_2^*(H^1(S_g; \mathbb{Q})).$$

This completes the proof. □

Let

$$\text{add: } H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \to H^1(S_g; \mathbb{Q})$$

be the addition of elements in the abelian group $H^1(S_g; \mathbb{Q})$.

Lemma 6.2 *For $g > 1$, we have the following exact sequences:*

$$0 \to H^1(M_S; \mathbb{Q}) \xrightarrow{E^1} H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \oplus H^1(S_g; \mathbb{Q}) \xrightarrow{\text{add}} H^1(S_g; \mathbb{Q}) \to 0 \tag{6-1}$$

and

$$0 \to H^2(M_S; \mathbb{Q}) \xrightarrow{E^2} H^2(S_g \times S_g - \triangle; \mathbb{Q}) \oplus H^2(S_g \times S_g - \triangle; \mathbb{Q}).$$

Proof Let M_1 and M_2 be the two copies of $S_g \times S_g - \triangle$ in the construction of M_S. Define $N := M_1 \cap M_2$; this is a circle bundle over S_g. The circle bundle N has Euler number $2-2g \neq 0$. By the Serre spectral sequence of the circle bundle $S^1 \to N \to S_g$, we have

$$H^1(N;\mathbb{Q}) = H^1(S_g;\mathbb{Q}).$$

The map

$$H_1(N;\mathbb{Q}) = H_1(S_g;\mathbb{Q}) \to H_1(S_g \times S_g;\mathbb{Q}) = H_1(S_g;\mathbb{Q}) \oplus H_1(S_g;\mathbb{Q})$$

is induced by the diagonal embedding. Therefore

$$H^1(S_g \times S_g;\mathbb{Q}) = H^1(S_g;\mathbb{Q}) \oplus H^1(S_g;\mathbb{Q}) \to H^1(N;\mathbb{Q}) = H^1(S_g;\mathbb{Q})$$

is the addition of the two elements (dual to the diagonal map). Consequently, we have a long exact sequence coming from the Mayer–Vietoris pair (M_1, M_2):

$$\text{(6-2)} \quad 0 \to H^1(M_S;\mathbb{Q}) \xrightarrow{E^1} H^1(S_g \times S_g - \triangle;\mathbb{Q}) \oplus H^1(S_g \times S_g - \triangle;\mathbb{Q}) \\ \xrightarrow{s^*} H^1(N;\mathbb{Q}) \to H^2(M_S;\mathbb{Q}) \xrightarrow{E^2} H^2(S_g \times S_g - \triangle;\mathbb{Q}) \oplus H^2(S_g \times S_g - \triangle;\mathbb{Q}).$$

We know s^* is surjective, therefore E^1 and E^2 are injective. □

By the short exact sequence (6-1), we identify $H^1(M_S;\mathbb{Q})$ as a subspace of

$$H^1(S_g;\mathbb{Q}) \oplus H^1(S_g;\mathbb{Q}) \oplus H^1(S_g;\mathbb{Q}) \oplus H^1(S_g;\mathbb{Q}).$$

We use x, y, z, w to represent the coordinates. Therefore any element $a \in H^1(M_S;\mathbb{Q})$ can be written as

$$a = (x, y, z, w) \in H^1(S_g;\mathbb{Q}) \oplus H^1(S_g;\mathbb{Q}) \oplus H^1(S_g;\mathbb{Q}) \oplus H^1(S_g;\mathbb{Q})$$

such that $x + y + z + w = 0 \in H^1(S_g;\mathbb{Q})$. We also identify $H^1(S_g \times S_g - \triangle;\mathbb{Q})$ with $H^1(S_g;\mathbb{Q}) \oplus H^1(S_g;\mathbb{Q})$ by Lemma 6.1. Any element $a' \in H^1(S_g \times S_g - \triangle;\mathbb{Q})$ can be written as $a' = (x, y) \in H^1(S_g;\mathbb{Q}) \oplus H^1(S_g;\mathbb{Q})$.

We will need the following algebraic lemma [3, Lemma 3.7] on the cup product structure of $H^*(S_g \times S_g - \triangle;\mathbb{Q})$.

Lemma 6.3 *Let $r, s \in H^1(S_g \times S_g - \triangle;\mathbb{Q})$ be two independent elements. If $r \smile s = 0$, then $r, s \in p_i^*(H^1(S_g;\mathbb{Q}))$ for some $i \in \{1, 2\}$.*

Proof of Theorem 1.3 From the naturality of cup product, we have the following commutative diagram:

$$\begin{array}{ccc} \Lambda^2 H^1(M_S;\mathbb{Q}) & \xrightarrow{\Lambda^2 E^1} & \Lambda^2\big(H^1(S_g\times S_g-\triangle;\mathbb{Q})\oplus H^1(S_g\times S_g-\triangle;\mathbb{Q})\big) \\ \Big\downarrow \text{cup} & & \Big\downarrow \text{cup} \\ H^2(M_S;\mathbb{Q}) & \xrightarrow{E^2} & H^2(S_g\times S_g-\triangle;\mathbb{Q})\oplus H^2(S_g\times S_g-\triangle;\mathbb{Q}) \end{array} \tag{6-3}$$

Let $S_h\to E\xrightarrow{p} B$ be some fibering. Since $\chi(M_S)>0$ and $\chi(S_h)<0$, we compute that $\chi(B)<0$, and hence that $g(B)>1$. Define $H:=p^*(H^1(B;\mathbb{Q}))$. Since $g(B)>1$, there exist linearly independent $b,b'\in H^1(B;\mathbb{Q})$ such that $b\smile b'=0$. Let

$$p^*(b)=(x,y,z,w),\qquad p^*(b')=(x',y',z',w')\in H.$$

By Lemma 6.2 and diagram (6-3), $p^*(b)\smile p^*(b')=0$ if and only if both

$$(x,y)\smile(x',y')=0\in H^2(S_g\times S_g-\triangle;\mathbb{Q}),$$
$$(z,w)\smile(z',w')=0\in H^2(S_g\times S_g-\triangle;\mathbb{Q}).$$

By Lemma 6.3, we have the following possibilities: one of (1) and (1′) must be true and one of (2) and (2′) must be true.

(1) (x,y) and (x',y') are dependent in $H^1(S_g\times S_g-\triangle;\mathbb{Q})$.

(1′) $x=x'=0$ or $y=y'=0$.

(2) (z,w) and (z',w') are dependent in $H^1(S_g\times S_g-\triangle;\mathbb{Q})$.

(2′) $z=z'=0$ or $w=w'=0$.

In the original four fiberings, the subspaces $\{(p_i,p_j)^*(H^1(S_g;\mathbb{Q})): i,j=1,2\}$ satisfy the following:

- $(p_1,p_1)^*(H^1(S_g;\mathbb{Q}))$ contains all elements with $y=0$ and $w=0$.
- $(p_1,p_2)^*(H^1(S_g;\mathbb{Q}))$ contains all elements with $y=0$ and $z=0$.
- $(p_2,p_1)^*(H^1(S_g;\mathbb{Q}))$ contains all elements with $x=0$ and $w=0$.
- $(p_2,p_2)^*(H^1(S_g;\mathbb{Q}))$ contains all elements with $x=0$ and $z=0$.

If (1′) and (2′) are true for $p^*(b)$, then $p^*(b)$ belongs to one of the four spaces above. By Lemma 2.2, the fibering $S_h\to E\xrightarrow{p} B$ must be one of the four original fiberings. Thus to conclude the proof of Theorem 1.3, it suffices to prove the following claim:

Claim 6.4 *There exists an element in the subspace H satisfying* $(1')$ *and* $(2')$.

We now prove the claim. We assume that no element in H satisfies $(1')$ and $(2')$. Since $g(B) > 1$, for any element $a \in H$, the dimension of the subspace $\{h \in H : a \smile h = 0\}$ is at least 3. We break our discussion into three cases.

Case 1 **There is an element $a = (x, y, z, w) \in H$ such that x, y, z and w are all nonzero** Find $b \in H$ such that $b \smile a = 0$. Via Lemma 6.3, we have that $a = (kx, ky, lz, lw)$ for $k, l \in \mathbb{Q}$. However, the subspace $\{(kx, ky, lz, lw) : k, l \in \mathbb{Q}\}$ is only 2–dimensional; this contradicts the fact that the dimension of the subspace $\{h \in H : a \smile h = 0\}$ is at least 3.

Case 2 **There exists an element $a = (x, y, z, w) \in H$ such that $x = y = 0$ and $z, w \neq 0$** We know that $z + w = 0$ by Lemma 6.2. Let $b = (x', y', z', w') \in H$. If $b \smile a = 0$, then $(z', w') = k(z, w)$ for $k \in \mathbb{Q}$ by Lemma 6.3. The dimension of the space $\{(0, 0, mz, mw) : m \in \mathbb{Q}\}$ is 1, so there exists $b = (x', y', kz, kw) \in \{h \in H : a \smile h = 0\}$ such that x' or y' is nonzero. Since $z + w = 0$, we have that $x' + y' = 0$. Thus x' and y' are both nonzero. Let $l \in \mathbb{Q}$ such that $l + k \neq 0$. The linear combination $la + b = (x', y', (l + k)z, (l + k)w) \in H$ has all coordinates nonzero; this reduces to Case 1.

Case 3 **Every nonzero element $(x, y, z, w) \in H$ has exactly one coordinate equal to zero** If two elements $a, b \in H$ have different coordinates zero, we could find a linear combination $ka + lb \in H$ for $k, l \in \mathbb{Q}$ such that all coordinates of $ka + lb$ are nonzero; this reduces to Case 1. Therefore all elements in H have the same coordinate equal to zero.

Assume without loss of generality that every element $(x, y, z, w) \in H$ only has $w = 0$. There are independent elements $a = (x, y, z, 0)$, $b = (x', y', z', 0) \in H$ such that $a \smile b = 0$. By Lemma 6.2 and Lemma 6.3, we have $(x', y') = k(x, y)$ for $k \in \mathbb{Q}$. Since a, b are independent, we know that $z' \neq kz$. Then the nonzero element $k(x, y, z, 0) - (x', y', z', 0) = (0, 0, kz - z', 0) \in H$ only has one coordinate nonzero. This is a contradiction to the assumption of Case 3.

This completes the proof of the claim, hence the theorem. □

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Department of Mathematics, University of Chicago
Chicago, IL, United States

chenlei@math.uchicago.edu

http://math.uchicago.edu/~chenlei

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