

Obstructions for constructing equivariant fibrations

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Let G be a finite group and \mathcal{H} be a family of subgroups of G which is closed under conjugation and taking subgroups. Let B be a G -CW-complex whose isotropy subgroups are in \mathcal{H} and let $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ be a compatible family of H -spaces. A G -fibration over B with the fiber type $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ is a G -equivariant fibration $p: E \rightarrow B$ where $p^{-1}(b)$ is G_b -homotopy equivalent to F_{G_b} for each $b \in B$. In this paper, we develop an obstruction theory for constructing G -fibrations with the fiber type \mathcal{F} over a given G -CW-complex B . Constructing G -fibrations with a prescribed fiber type \mathcal{F} is an important step in the construction of free G -actions on finite CW-complexes which are homotopy equivalent to a product of spheres.

[57S25](#); [55R91](#)

1 Introduction

In 1925, Hopf stated a problem which was later called the topological spherical space form problem: Classify all finite groups which can act freely on a sphere \mathbb{S}^n , $n > 1$. One variant of this problem was solved by Swan [14]. He proved that a finite group acts freely on a finite complex homotopy equivalent to a sphere if and only if it has periodic cohomology. By using Swan's construction and surgery theory, the topological spherical space form problem has been completely solved by Madsen, Thomas and Wall [9]. It turns out that a finite group G acts freely on a sphere if and only if G has periodic cohomology and any element of order 2 in G is central (see [9, Theorem 0.5]).

One of the generalizations of this problem is to classify all finite groups which can act freely on a finite CW-complex homotopy equivalent to a product of k -spheres $\mathbb{S}^{n_1} \times \cdots \times \mathbb{S}^{n_k}$ for some n_1, \dots, n_k . Recently, Adem and Smith [1] gave a homotopy theoretic characterization of cohomological periodicity and as a corollary they obtained a tool to construct free group actions on CW-complexes homotopy equivalent to a product of spheres. More precisely, they have shown that a connected CW-complex X has periodic cohomology if and only if there is a spherical fibration over X with a total space E that has a homotopy type of a finite dimensional CW-complex. As a consequence they proved that if G is a finite group and X is a finite dimensional G -CW-complex whose isotropy subgroups all have periodic cohomology, then there

is a finite dimensional CW-complex Y with a free G -action such that $Y \simeq \mathbb{S}^n \times X$. As remarked in [1], the second result can also be obtained using the techniques given by Connolly and Prassidis in [3]. More recently, Klaus [6] proved that every p -group of rank 3 acts freely on a finite CW-complex homotopy equivalent to a product of three spheres using similar techniques.

The method used by Connolly and Prassidis [3] is to construct a spherical fibration inductively over the skeleta by dealing with cells in each dimension separately. This is a standard strategy in obstruction theory. Note that if there is an orientable G -spherical fibration over the n -skeleton of a CW-complex, then its restriction to the boundary of each $(n+1)$ -cell σ will be an orientable G_σ -fibration with the fiber F_{G_σ} , where G_σ is the isotropy subgroup of σ . Associated to this G_σ -fibration over $\partial\sigma$, there is a classifying map from $\partial\sigma$ to the space $B \text{Aut}_{G_\sigma} F_{G_\sigma}$ where $\text{Aut}_{G_\sigma} F_{G_\sigma}$ is the topological monoid of self G_σ -homotopy equivalences of F_{G_σ} . Combining the attaching map of σ with the classifying map gives us an element in the n -th homotopy group of $B \text{Aut}_{G_\sigma} F_{G_\sigma}$. Therefore we obtain a cellular cochain which assigns a homotopy class in $\pi_n(B \text{Aut}_{G_\sigma} F_{G_\sigma})$ to each $(n+1)$ -cell. This cochain vanishes if and only if the G -fibration over the n -skeleton extends to a G -fibration over the $(n+1)$ -skeleton. In the situation Connolly and Prassidis consider, this cochain can be killed by taking fiber joins. Using this idea, Ünlü [17] gives a concrete cell-by-cell construction of G -spherical fibrations in his thesis.

In obstruction theory, one often has obstructions as cohomology classes which tells when a construction can be performed on the next skeleton after some modifications. In other words, the cohomological obstruction class vanishes if and only if the restriction of the construction to the $(n-1)$ -skeleton extends over the $(n+1)$ -skeleton. Having a cohomological obstruction is better than having a cochain class as an obstruction since a cohomology class is more likely to be zero. Note that if $p: E \rightarrow B$ is a G -fibration and $b \in B^H$ then the fiber $p^{-1}(b)$ is an H -space. When B^H is connected for $H \leq G$, there is an H -space F_H such that for every $b \in B^H$, the fiber $p^{-1}(b)$ is H -homotopy equivalent to F_H . Moreover, if B^H is connected for every $H \leq G$, the family of H -spaces F_H forms a compatible family (see Definition 2.5). In this case, the G -fibration $p: E \rightarrow B$ is said to have the fiber type $\{F_H\}$. In this paper, we notice that the cohomological obstructions for constructing G -fibrations with a given fiber type live in Bredon cohomology of B with coefficients in $\underline{\pi}_{n,\mathcal{F}}$ (see Section 4 for the definition) and we prove the following theorem.

Theorem 1.1 *Let G be a finite group and \mathcal{H} be a family of subgroups of G which is closed under conjugation and taking subgroups. Let B be a G -CW-complex whose isotropy subgroups are in \mathcal{H} such that B^H is simply connected for every*

$H \in \text{Iso}(B)$. Let $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ be a compatible family of finite H -CW-complexes and $p: E_n \rightarrow B^n$ be a G -fibration over the n -skeleton of B with the fiber type $\{F_H\}_{H \in \mathcal{H}}$ where $n \geq 2$.

- (1) There is a cocycle $\alpha_p \in C_{\mathcal{H}}^{n+1}(B; \underline{\pi}_{n, \mathcal{F}})$ which vanishes if and only if p extends to a G -fibration over B^{n+1} with a total space G -homotopy equivalent to a G -CW-complex.
- (2) The cohomology class $[\alpha_p] \in H_{G, \mathcal{H}}^{n+1}(B; \underline{\pi}_{n, \mathcal{F}})$ vanishes if and only if the G -fibration $p|_{B^{n-1}}: p^{-1}(B^{n-1}) \rightarrow B^{n-1}$ extends to a G -fibration over B^{n+1} with a total space G -homotopy equivalent to a G -CW-complex.

Moreover if B is a finite G -CW-complex then the total space of the obtained fibration has the G -homotopy type of a finite G -CW-complex whenever E_n has the G -homotopy type of a finite G -CW-complex.

To prove this theorem we first define an obstruction cochain in the chain complex of Bredon cohomology and show that it is a cocycle. We call this cocycle an obstruction cocycle. Then we show that the difference of obstruction cocycles of any two extensions of the G -fibration $p|_{B^{n-1}}$ is the coboundary of a cochain called the difference cochain. If there is an extension of $p|_{B^{n-1}}$ to a G -fibration over B^{n+1} , then the obstruction cocycle of the restriction of this extension to B^n vanishes. This means that the obstruction cocycle of p is a coboundary and represents a cohomology class which vanishes. This proves the “if” direction of the above theorem.

For the “only if” direction it suffices to show that for every cochain d there is a G -fibration q over B^n with $q|_{B^{n-1}} = p|_{B^{n-1}}$ such that d is the difference cochain of the extensions p and q of $p|_{B^{n-1}}$. Here the most technical part is the construction of a G -fibration q with these properties. That is because it is not clear how to glue G -fibration $p|_{B^{n-1}}$ to G -fibrations over the n -cells corresponding to the cochain d . For quasifibrations it suffices to take the adjunction of the total spaces to glue two quasifibrations over different base. However, in order to obtain a fibration one needs to put some G -tubes between total spaces of these G -fibrations to create enough space to deal with G -homotopies. We use a generalization of a result due to Tulley [16] to produce a G -fibration q with the required properties.

The paper is organized as follows: Section 2 contains definitions and preliminary results on equivariant fibrations. In Section 3, we give a method to glue G -fibrations over different base spaces by generalizing a construction due to Tulley [16]. Finally, we prove Theorem 1.1 in Section 4.

2 Equivariant fibrations

In this section, we give the basic definitions of the equivariant fibration theory. We refer the reader to Lück [8] and Waner [19] for more details.

Definition 2.1 Let G be a finite group. A G -map $p: E \rightarrow B$ is called a G -fibration if it has the G -homotopy lifting property with respect to every G -space X , that is, given a commutative diagram of G -maps

$$\begin{array}{ccc}
 X \times \{0\} & \xrightarrow{h} & E \\
 \downarrow & \nearrow \tilde{H} & \downarrow p \\
 X \times I & \xrightarrow{H} & B
 \end{array}$$

there exists a G -map $\tilde{H}: X \times I \rightarrow E$ such that $p\tilde{H} = H$ and $\tilde{H}|_{X \times \{0\}} = h$.

Equivalently, a G -map $p: E \rightarrow B$ is a G -fibration if there is a G -map

$$\lambda: \Omega_p = \{(e, \omega) \in E \times B^I \mid p(e) = \omega(0)\} \rightarrow E^I$$

such that $\lambda(e, \omega)(0) = e$ and $p\lambda(e, \omega) = \omega$. The G -map λ is called a G -lifting function. In the nonequivariant theory, Dold [4] proved that being a fibration is a local property. The same proof applies to the equivariant case.

Definition 2.2 A covering \mathcal{U} of G -invariant open sets of B is called *numerable G -covering* if \mathcal{U} is locally finite and there is a G -map $f_U: B \rightarrow I$ such that $U = f_U^{-1}(0, 1]$ for every $U \in \mathcal{U}$.

Theorem 2.3 A G -map $p: E \rightarrow B$ is a G -fibration if there is a numerable G -covering \mathcal{U} of B such that $p|_U: p^{-1}(U) \rightarrow U$ is a G -fibration for each $U \in \mathcal{U}$.

The notion of an equivalence between G -fibrations is defined naturally as follows: Let $p_i: E_i \rightarrow B$ be a G -fibration for $i = 1, 2$. A fiber preserving G -map $f: E_1 \rightarrow E_2$ is called a G -fiber homotopy equivalence if there is a fiber preserving G -map $g: E_2 \rightarrow E_1$ such that the compositions fg and gf are G -homotopy equivalent to identity maps through G -homotopies which are fiber preserving at each time $t \in I$. In this case, we write $p_1 \simeq_G p_2$. As in the nonequivariant case, a fiber preserving G -homotopy equivalence between G -fibrations is a G -fiber homotopy equivalence (see May [10, page 50] for the proof of the nonequivariant case).

In [12], Stasheff proved a classification theorem for nonequivariant fibrations up to fiber homotopy equivalences. When a G -fibration is over a path-connected space with trivial G -action, the fiber at each point in the base has a natural G -space structure and all fibers are G -homotopy equivalent with respect to this structure. In this case, the theory of G -fibrations is essentially the same as the nonequivariant one and we have the following classification theorem.

Theorem 2.4 *Let $\text{Aut}_G(F_G)$ be the monoid of G -equivariant self homotopy equivalences of a finite G -CW-complex F_G . If B is a CW-complex with trivial G -action then there is a one-to-one correspondence between the set of G -fiber homotopy equivalence classes of G -fibrations $p: E \rightarrow B$ with fibers having the G -homotopy type of F_G and the set of homotopy classes of maps $B \rightarrow B \text{Aut}_G(F_G)$. The equivalence is obtained by taking the pullback fibration from the universal fibration over $B \text{Aut}_G(F_G)$.*

This classification theorem can be proved by using the same techniques and ideas from Stasheff [12]. Also, Waner constructs a classifying space for a more general set of equivariant fibrations in [18] and the above theorem can be obtained as a special case of his result.

The monoid $\text{Aut}_G(F_G)$ is not connected in general. However, its connected components are homotopy equivalent via the maps $(\text{Aut}_G(F_G), f) \rightarrow (\text{Aut}_G(F_G), g)$ with $\phi \rightarrow g\phi f^{-1}$ where f^{-1} is the homotopy inverse of f . Furthermore, when the map $\pi_1(B) \rightarrow [F_G, F_G]_G$ is trivial, $B \text{Aut}_G^I(F_G)$ classifies G -fibrations $p: E \rightarrow B$ with trivial G -actions on the base where $\text{Aut}_G^I(F_G)$ is the connected component of identity in $\text{Aut}_G(F_G)$ (see Adem and Smith [1] for nonequivariant case).

For G -fibrations whose G -action on the base is not trivial, we need to consider the collection of equivariant spaces. Note that if p is a G -fibration over B and $b \in B^K$, then the space $p^{-1}(b)$ is closed under K -action and hence a K -space. Moreover, when B^K is connected, the spaces $p^{-1}(b)$ and $p^{-1}(b')$ are K -homotopy equivalent for every $b, b' \in B^K$. On the other hand, when $H^a \leq K$, we have $h(ab) = a(a^{-1}ha)b = ab$ for every $h \in H$ and $b \in B^K$, hence $ab \in B^H$. Clearly, the H -space $p^{-1}(b)$, where the H -action on $p^{-1}(b)$ is given by conjugation and the H -space $p^{-1}(ab)$ are H -homeomorphic. Therefore, when B^H is connected for every isotropy subgroup H of B , the spaces $p^{-1}(b)$ and $p^{-1}(ab)$ are H -homotopy equivalent for every isotropy subgroups H, K with $H^a \leq K$.

Definition 2.5 Let \mathcal{H} be a family of subgroups of G which is closed under conjugation and taking subgroups. A family $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ of H -spaces is said to be a *compatible family* if F_H is H -homotopy equivalent to F_K for every $H, K \in \mathcal{H}$ with $H^a \leq K$ for some $a \in G$ where the H -action on F_K is given by $h \cdot y = a^{-1}hay$.

Let $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ be a compatible family where \mathcal{H} contains the isotropy subgroups of B . We say $p: E \rightarrow B$ is a G -fibration with the fiber type $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ if $F_H \simeq_H p^{-1}(b)$ for every $b \in B^H$ and for every isotropy subgroup H of B . When B^H is connected for every $H \in \mathcal{H}$, every G -fibration over B is a G -fibration with the fiber type \mathcal{F} . However, a G -fibration $p: E \rightarrow B$ does not necessarily have a fiber type.

3 Tulley’s theorem for G -fibrations

The aim of this section is to prove an equivariant version of a theorem due to Tulley (see [16, Theorem 11]). The proof uses the same ideas and methods from [16; 15].

Theorem 3.1 *Let $p_1: E_1 \rightarrow B$ and $p_2: E_2 \rightarrow B$ be G -fiber homotopy equivalent G -fibrations. Then there is a G -fibration q over $B \times I$ such that $q|_{B \times \{0\}} = p_1$ and $q|_{B \times \{1\}} = p_2$.*

We call the G -fibration $q: Z \rightarrow B \times I$ in Theorem 3.1 a G -tube between p_1 and p_2 . Let $f: E_1 \rightarrow E_2$ be a fiber preserving G -map between G -fibrations p_1 and p_2 over B . Recall that the mapping cylinder M_f of f is the adjunction space $E_1 \times I \cup_f E_2$ where $f(e) = (e, 0)$ for each $e \in E_1$. We define the G -map $p_f: M_f \rightarrow B$ over B by $p_f(x, s) = p_1(x)$ and $p_f(y) = p_2(y)$ for any $x \in E_1, y \in E_2$, and $s \in I$.

Lemma 3.2 *Let $f: E_1 \rightarrow E_2$ be a fiber preserving G -map between G -fibrations p_1 and p_2 over B . Then the induced G -map $p_f: M_f \rightarrow B$ is a G -fibration.*

Proof The proof is similar to the proof of [15, Theorem 1]. Let $\lambda_i: \Omega_{p_i} \rightarrow E_i^I$ be a G -lifting function for $p_i, i = 1, 2$. Since $\Omega_{p_f} = \Omega_{p_1} \times I \cup_{\tilde{f}} \Omega_{p_2}$ where $\tilde{f}(e, \omega) = (f(e), \omega)$, to show that p_f is a G -fibration it suffices to construct a G -map $\lambda: \Omega_{p_1} \times I \rightarrow M_f^I$ such that $\lambda|_{\Omega_{p_1} \times \{0\}} = \lambda_2 \circ \tilde{f}, p_f \lambda((e, \omega), s) = \omega$, and $\lambda((e, \omega), s)(0) = (e, s)$.

Define $\lambda: \Omega_{p_1} \times I \rightarrow M_f^I$ by

$$\lambda(e, \omega, s)(t) = \begin{cases} (\lambda_1(e, \omega)(t), s - t) & t \leq s, \\ \lambda_2(z, \omega^s)(t - s) & s \leq t, \end{cases}$$

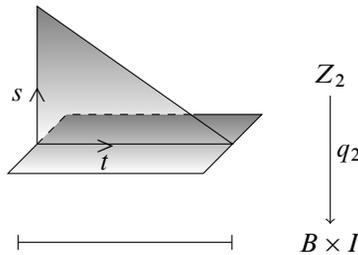
where $z = f \circ \lambda_1(e, \omega)(s)$ and ω^s is given by $\omega^s(t) = \omega(s + t)$ when $s + t \leq 1$ and $\omega^s(t) = 1$, otherwise. Clearly, λ is a continuous G -map which satisfies the relations $\lambda|_{\Omega_{p_1} \times \{0\}} = \lambda_2 \circ \tilde{f}, p_f \lambda((e, \omega), s) = \omega$, and $\lambda((e, \omega), s)(0) = (e, s)$. □

In order to prove [Theorem 3.1](#), it suffices to construct G -fibrations $q_1: Z_1 \rightarrow B \times I$ and $q_2: Z_2 \rightarrow B \times I$ with $q_1|_{B \times \{0\}} = p_1$, $q_1|_{B \times \{1\}} = q_2|_{B \times \{0\}} = p_f$ and $q_2|_{B \times \{1\}} = p_2$ where f is the fiber preserving G -map between p_1 and p_2 . That is because once we have such G -fibrations, we can obtain a G -tube between p_1 and p_2 by gluing q_1 and q_2 as follows. Let $Z = Z_1 \cup_{i_1} M_f \times I \cup_{i_2} Z_2$ where $i_1(x) = (x, 0)$ and $i_2(x) = (x, 1)$ for every $x \in M_f$. Define the G -map $q: Z \rightarrow B \times I$ by

$$q(z) = \begin{cases} (\pi_1(q_1(z)), \frac{1}{3}\pi_2(q_1(z))) & z \in Z_1, \\ (\pi_1(q_2(z)), \frac{2}{3} + \frac{1}{3}\pi_2(q_2(z))) & z \in Z_2, \\ (p_f(x), \frac{1}{3}(1+t)) & z = (x, t) \in M_f \times I, \end{cases}$$

where π_i is the projection map to the i -th coordinate. Since we can extend the lifting functions for q_1 and q_2 to lifting functions for $q|_{B \times [0, 6/9]}$ and $q|_{B \times (4/9, 1]}$, respectively, G -maps $q|_{B \times [0, 6/9]}$ and $q|_{B \times (4/9, 1]}$ are G -fibrations. Therefore $q: Z \rightarrow B \times I$ is a G -fibration by [Theorem 2.3](#).

Proposition 3.3 Let $Z_2 = \{(e, s, t) \in E_1 \times I \times I \mid s + t \leq 1\} \cup \tilde{f} E_2 \times I$ in $M_f \times I$, where $\tilde{f}: E_1 \times \{0\} \times I$ is defined by $\tilde{f}(e, 0, t) = (f(e), t)$. Then $q_2 = (p_f \times \text{id})|_{Z_2}: Z_2 \rightarrow B \times I$ is a G -fibration with $q_2|_{B \times \{0\}} = p_f$ and $q_2|_{B \times \{1\}} = p_2$.

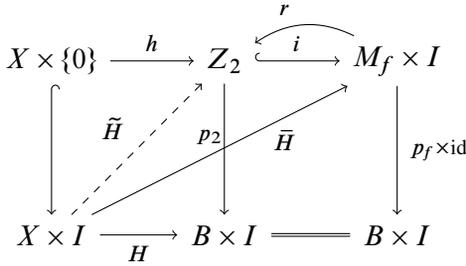


Proof Let $r: M_f \times I \rightarrow Z_2$ be defined by $r|_{E_2 \times I} = \text{id}_{E_2 \times I}$ and

$$r|_{E_1 \times I \times I}(x, s, t) = \begin{cases} (x, s, t) & s + t \leq 1, \\ (x, t) & \text{otherwise.} \end{cases}$$

Then r is a fiber preserving G -retraction. Let $H: X \times I \rightarrow B$ and $h: X \rightarrow Z_2$ be given G -maps with $H|_{X \times \{0\}} = p_2 \circ h$. Since $p_f \times \text{id}$ is a G -fibration, there is a

G -map $\bar{H}: X \times I \rightarrow M_f \times I$ which makes the following diagram commute:

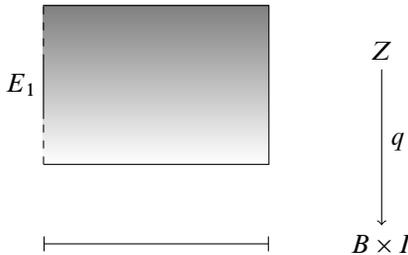


Then the G -map $\tilde{H}: X \times I \rightarrow Z_2$ defined by $\tilde{H} = r\bar{H}$ satisfies $p_2\tilde{H} = H$ and $\tilde{H}|_{X \times \{0\}} = h$. □

Definition 3.4 Let $p_i: E_i \rightarrow B$ be a G -fibration for $i = 1, 2$ with $E_1 \subseteq E_2$ and $p_2|_{E_1} = p_1$. Then p_1 is said to be a G -deformation retract of p_2 if E_1 is a G -deformation retract of E_2 via fiber preserving G -retraction, that is, if there is a G -map $H: E_2 \times I \rightarrow E_2$ such that $H_0 = \text{id}_{E_2}$, $H(e, 1) \in E_1$ and $p_2H(e, t) = p_2(e)$ for every $e \in E_2$. If H also satisfies the relation $H(e, t) = e$ for every $e \in E_1$, we say p_1 is a strong G -deformation retract of p_2 .

To show that there is a G -tube $q_1: Z_1 \rightarrow B \times I$ between p_1 and p_f , we need the following proposition. The nonequivariant version of this proposition is proved by Tulley [16] and used in a recent paper by Steimle [13].

Proposition 3.5 If p_1 is a strong G -deformation retract of p_2 then the G -map $q = (p_2 \times \text{id})|_Z: Z \rightarrow B \times I$ where $Z = \{(e, t) \in E_2 \times I \mid e \in E_1 \text{ when } t > 0\}$ is a G -fibration.



Proof We are using the same approach that is used in the proof of Theorem 3.1 in Langston [7]. Let $H: E_2 \times I \rightarrow E_2$ be a strong G -deformation retraction of the

G -fibration p_2 onto p_1 . Let $\lambda: \Omega_{p_2} \rightarrow E_2^I$ be a G -lifting function for p_2 . Define a G -map $\lambda': \Omega_q \rightarrow Z^I$ by $\pi_2(\lambda'((e, \omega_2(0)), (\omega_1, \omega_2)))(t) = \omega_2(t)$ and

$$\pi_1(\lambda'((e, \omega_2(0)), (\omega_1, \omega_2)))(t) = \begin{cases} H(\lambda(e, \omega_1)(t), t/\omega_2(t)) & \omega_2(t) > 0, \omega_2(t) \geq t, \\ e & t = \omega_2(t) = 0, \\ H(\lambda(e, \omega_1)(t), 1) & t > 0, t \geq \omega_2(t). \end{cases}$$

Clearly, $q\lambda'((e, \omega_2(0)), (\omega_1, \omega_2)) = (\omega_1, \omega_2)$ and $\lambda'((e, \omega_2(0)), (\omega_1, \omega_2))(0) = e$. Therefore we only need to check the continuity of $\pi_1\lambda'$ at $t = 0$. For this it suffices to show that the adjoint map $\widetilde{\pi_1\lambda'}: \Omega_q \times I \rightarrow E_2$ is continuous at $t = 0$.

Let $(e_\alpha, \omega_{1,\alpha}, \omega_{2,\alpha}, t_\alpha)$ be a net converging to $(e, \omega_1, \omega_2, 0)$. Let U be an open neighborhood of $e \in E_1$. Since $H: E_2 \times I \rightarrow E_2$ is continuous, $V = H^{-1}(U)$ is open. Since $(e, t) \in V$ for every $t \in I$, there are open neighborhoods $A_t \ni e$ and $V_t \ni t$ such that $A_t \times V_t \subseteq V$, for all $t \in I$. Since I is compact, there exist t_1, \dots, t_n such that $I = \bigcup_{i=1}^n V_{t_i}$. Then $A = \bigcap_{i=0}^n A_{t_i}$ is an open neighborhood of e with the property that $H(A \times I) \subseteq U$. Since $\widetilde{\lambda}$ is continuous, there is β such that $\widetilde{\lambda}(e_\alpha, \omega_{1,\alpha}, t_\alpha) \in A$ for every $\beta > \alpha$. Therefore $\widetilde{\pi_1\lambda'}(e_\alpha, \omega_{1,\alpha}, \omega_{2,\alpha}, t_\alpha) \in U$ for every $\alpha > \beta$ as desired. \square

It is proved in [11, Corollary 2.4.2] that when $f: E_1 \rightarrow E_2$ is a homotopy equivalence then E_1 is a strong deformation retract of M_f . The same proof applies to G -fibrations (see [5, Lemma 2.5.2]). Therefore, by Proposition 3.5, there is a G -tube q_1 between p_1 and p_f . Note that p_2 is also a strong G -deformation retract of p_f and one can also use Proposition 3.5 to construct a G -tube between p_2 and p_f . This completes the proof of Theorem 3.1. As an immediate corollary, we have:

Corollary 3.6 *Let B_1, B_2 and B be topological spaces such that $B \subseteq B_1 \cap B_2$. If $p_1: E_1 \rightarrow B_1$ and $p_2: E_2 \rightarrow B_2$ are G -fibrations with $p_1|_B \simeq_G p_2|_B$ then there is a G -fibration over $B_1 \cup_{i_1} (B \times I) \cup_{i_2} B_2$ extending p_1 and p_2 where $i_j: B \times \{j\} \rightarrow B_j$ are inclusions.*

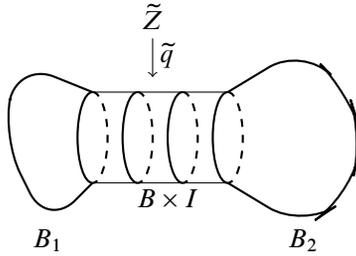
Proof By Theorem 3.1, there is a G -tube $q: Z \rightarrow B \times I$ between $p_1|_B$ and $p_2|_B$. Without loss of generality, we can consider q over $B \times [\frac{1}{3}, \frac{2}{3}]$ with $q|_{B \times \{1/3\}} = p_1$ and $q|_{B \times \{2/3\}} = p_2$. Let

$$\widetilde{Z} = E_1 \cup_{k_1} (p_1^{-1}(B) \times [0, \frac{1}{3}]) \cup_{m_1} Z \cup_{m_2} (p_2^{-1}(B) \times [\frac{2}{3}, 1]) \cup_{k_2} E_2,$$

where $k_1: p_1^{-1}(B) \times \{0\} \rightarrow E_1$, $k_2: p_1^{-1}(B) \times \{1\} \rightarrow E_2$ and $m_1: p_1^{-1}(B) \times \{\frac{1}{3}\} \rightarrow Z$, $m_2: p_2^{-1}(B) \times \{\frac{2}{3}\} \rightarrow Z$ are the inclusions. Define a G -map

$$\widetilde{q}: \widetilde{Z} \rightarrow B_1 \cup_{i_1} (B \times I) \cup_{i_2} B_2$$

by $\tilde{q}|_{E_i} = p_i$, $\tilde{q}|_{Z \times [1/3, 2/3]}(z, t) = q(z)$ and for $j = 1, 2$ by $\tilde{q}(e, t) = p_j(e)$ when $t \in [2(j-1)/3, (2j-1)/3]$.



Clearly the restriction of \tilde{q} to the following subsets are G -fibrations

$$\{B_1 \cup_{i_1} B \times [0, \frac{2}{9}], B \times [\frac{1}{9}, \frac{5}{9}], B \times [\frac{4}{9}, \frac{8}{9}], B \times [\frac{7}{9}, 1] \cup_{i_2} E_2\}.$$

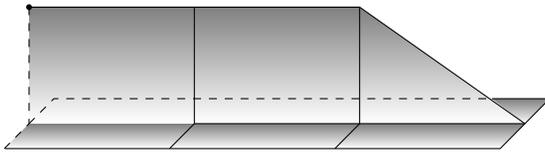
Therefore, by [Theorem 2.3](#), \tilde{q} is a G -fibration. □

Theorem 3.7 Let $p_i: E_i \rightarrow B$, $i = 1, 2$, be G -fiber homotopy equivalent G -fibrations such that E_1 and E_2 have the G -homotopy type of a G -CW-complex. Then there is a G -tube q between p_1 and p_2 whose total space is G -homotopy equivalent to a G -CW-complex.

Proof Let $f: E_1 \rightarrow E_2$ be a G -fiber homotopy equivalence. Recall that the total space of the G -tube q we constructed is $Z = Z_1 \cup_{i_1} M_f \times [\frac{1}{3}, \frac{2}{3}] \cup Z_2$ where

$$Z_1 = \{(e, t) \in M_f \times [0, \frac{1}{3}] \mid e \in M_f \text{ if } 0 < t \leq \frac{1}{3}, e \in E_1 \text{ if } t = 0\},$$

$$Z_2 = \{(x, s, t) \in E_1 \times I \times [\frac{2}{3}, 1] \mid s + 3t \leq 3\} \cup_f E_2 \times [\frac{2}{3}, 1].$$



Clearly, Z is a strong G -deformation retract of M_f . On the other hand, M_f is G -homotopic to E_2 and hence it has a G -homotopy type of a G -CW-complex. □

Corollary 3.8 A G -fibration $p: E \rightarrow \mathbb{S}^{n-1}$ over $(n-1)$ -sphere with trivial G -action on the base extends to a G -fibration over a disk if and only if it is G -fiber homotopy equivalent to a trivial G -fibration.

Proof Since \mathbb{D}^n is contractible, the “only if” part holds. For the “if” part, let p be G -fiber homotopy equivalent to a trivial fibration. Consider \mathbb{D}^n as the cone of \mathbb{S}^{n-1} ,

that is, $\mathbb{D}^n = \mathbb{S}^{n-1} \times [0, 2]/\sim$ where $(y, 2) \sim *$. Let $B_1 = \mathbb{S}^{n-1} \times [1, 2]/\sim$, and $B_2 = B = \mathbb{S}^{n-1} \times \{1\}$. Then $\varepsilon|_B \simeq p$ where $\varepsilon: B_1 \times F_G \rightarrow B_1$ where $F_G = p^{-1}(x)$ for some $x \in \mathbb{S}^{n-1}$. By [Corollary 3.6](#), there is a G -fibration extending p . \square

Remark In [\[3\]](#), the statement of [Corollary 3.8](#) appears on page 137 but in there the total spaces were attached directly which results in a G -quasifibration from which one gets a G -fibration by taking the corresponding Hurewicz fibration.

4 Constructing G -fibrations

In this section, we introduce an obstruction theory for constructing G -fibrations over G -CW-complexes and we prove [Theorem 1.1](#). An adequate cohomology theory to develop such an obstruction theory is Bredon cohomology. Let us first fix some notation for Bredon cohomology. We refer the reader to Bredon [\[2\]](#) and Lück [\[8\]](#) for more detailed information about Bredon cohomology.

Let G be a finite group and \mathcal{H} be a family of subgroups of G which is closed under conjugation and taking subgroups. The orbit category $\text{Or}_{\mathcal{H}}(G)$ relative to the family \mathcal{H} is defined as the category whose objects are left cosets G/H where $H \in \mathcal{H}$ and whose morphisms are G -maps from G/H to G/K . Recall that any $a \in G$ with $H^a \leq K$ induces a G -map $\hat{a}: G/H \rightarrow G/K$ defined by $\hat{a}(H) = aK$ and conversely, any G -map from G/H to G/K is of this form.

Let B be a G -CW-complex whose isotropy subgroups are all in \mathcal{H} . A coefficient system for the Bredon cohomology is a contravariant functor $M: \text{Or}_{\mathcal{H}}(G) \rightarrow \mathcal{A}b$, where $\mathcal{A}b$ is the category of abelian groups. A morphism $T: M \rightarrow N$ between two coefficient systems is a natural transformation of functors. Note that a coefficient system is a $\mathbb{Z} \text{Or}_{\mathcal{H}}(G)$ -module with the usual definition of modules over a small category. Since the $\mathbb{Z} \text{Or}_{\mathcal{H}}(G)$ -module category is an abelian category, the usual notions for doing homological algebra exist.

Given a local coefficient system $M: \text{Or}_{\mathcal{H}}(G) \rightarrow \mathcal{A}b$, one defines the cochain complex $C_{\mathcal{H}}^*(B; M)$ of B with coefficients in M as follows: Let $C_{\mathcal{H}}^n(B; M)$ be the submodule of $\bigoplus_{H \in \mathcal{H}} \text{Hom}_{\mathbb{Z}}(C_n(B^H; \mathbb{Z}), M(G/H))$ formed by elements $(f(H))_{H \in \mathcal{H}}$ which makes the following diagram commute:

$$\begin{array}{ccc} C_n(B^K; \mathbb{Z}) & \xrightarrow{f(K)} & M(G/K) \\ \bar{a}_n \downarrow & & M(\hat{a}) \downarrow \\ C_n(B^H; \mathbb{Z}) & \xrightarrow{f(H)} & M(G/H) \end{array}$$

for any $H, K \in \mathcal{H}$ with $H^a \leq K$. Here $\bar{a}: B^K \rightarrow B^H$ is given by $\bar{a}(x) = ax$ for any $x \in B^K$ and \bar{a}_* denotes the induced map between the chain complexes. The coboundary map $\delta: C_{\mathcal{H}}^n(B; M) \rightarrow C_{\mathcal{H}}^{n+1}(B; M)$ is defined by $(\delta f)(H)(\tau) = f(H)(\partial\tau)$ for any $H \in \mathcal{H}$ and for any $(n+1)$ -cell τ of B^H .

Definition 4.1 The Bredon cohomology $H_{G, \mathcal{H}}^*(B; M)$ of G -CW-complex B with coefficients in M is defined as the cohomology of the cochain complex $C_{\mathcal{H}}^*(B; M)$.

Let $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ be a compatible family. Since \mathcal{F} is compatible, we can consider the universal K -fibration $u_K: E_K \rightarrow B \text{Aut}_K^I(F_K)$ with trivial K -action on the base as an H -fibration with the fiber F_H via conjugation action whenever $H^a \leq K$. Let $\tilde{a}: B \text{Aut}_K^I(F_K) \rightarrow B \text{Aut}_H^I(F_H)$ be the classifying map of this fibration. Define a contravariant functor $\underline{\pi}_{n, \mathcal{F}}: \text{Or}_{\mathcal{H}}(G) \rightarrow \mathcal{A}b$ by letting

$$\begin{aligned} \underline{\pi}_{n, \mathcal{F}}(G/H) &= \pi_n(B \text{Aut}_H^I(F_H)), \\ \underline{\pi}_{n, \mathcal{F}}(\hat{a}: G/H \rightarrow G/K) &= \pi_*(\tilde{a}): \pi_n(B \text{Aut}_K^I(F_K)) \rightarrow \pi_n(B \text{Aut}_H^I(F_H)). \end{aligned}$$

From now on we assume that B^H is simply connected for every $H \in \text{Iso}(B)$, where $\text{Iso}(B)$ is the set of isotropy subgroups of B . Let $p: E \rightarrow B_n$ be a G -fibration over the n -skeleton of B with the fiber type $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ where $n \geq 2$. For every $H \in \mathcal{H}$, the map $p_H: p^{-1}(B_n^H) \rightarrow B_n^H$ is an H -fibration. In particular, the H -fibration p_H is classified by a map $\phi_{p, H}: B_n^H \rightarrow B \text{Aut}_H^I F_H$ when H is an isotropy subgroup. For an arbitrary $H \in \mathcal{H}$ and $(n+1)$ -cell σ of B_n^H with an attaching map $f_\sigma: \mathbb{S}^n \rightarrow B_n^{G_\sigma} \subseteq B_n^H$, the G_σ -fibration $f_\sigma^*(p_H)$ is classified by the map $\text{Res}_H^{G_\sigma} \circ \phi_{p, G_\sigma} \circ f_\sigma$. Here, $\text{Res}_H^{G_\sigma}: B \text{Aut}_{G_\sigma}^I F_{G_\sigma} \rightarrow B \text{Aut}_H^I F_H$ is induced by the relation $H \leq G_\sigma$. When $H \in \text{Iso}(B)$, the maps $\text{Res}_H^{G_\sigma} \circ \phi_{p, G_\sigma} \circ f_\sigma$ and $\phi_{p, H} \circ f_\sigma$ are homotopic since they both classify the H -fibration $f_\sigma^*(p_H)$. Moreover, when $H, K \in \mathcal{H}$ with $H^a \leq K$, we have $\tilde{a} \circ \text{Res}_K^{G_\sigma} \simeq \text{Res}_H^{G_\sigma} \circ \tilde{a}$ for every $(n+1)$ -cell σ of B^K .

We define

$$\alpha_p \in \prod_{H \in \mathcal{H}} \text{Hom}_{\mathbb{Z}}(C_{n+1}(B^H), \pi_n(B \text{Aut}_H^I F_H))$$

by $\alpha_p(H)(\sigma) = (\text{Res}_H^{G_\sigma})_*[\phi_{p, G_\sigma} \circ f_\sigma]$ for every $H \in \mathcal{H}$ and for every $(n+1)$ -cell σ of B^H with an attaching map $f_\sigma: \mathbb{S}^n \rightarrow B_n^{G_\sigma}$. For α_p to be a cochain, the following

diagram must commute up to homotopy

$$\begin{array}{ccccc}
 \mathbb{S}^n & \xrightarrow{f_\sigma} & B_n^{G_\sigma} & \xrightarrow{\phi_{p,G_\sigma}} & B \operatorname{Aut}_{G_\sigma}^I F_{G_\sigma} \\
 \parallel & & \bar{a} \downarrow & & \tilde{a} \downarrow \\
 \mathbb{S}^n & \xrightarrow{f_{a\sigma}} & B_n^{G_{a\sigma}} & \xrightarrow{\phi_{p,G_{a\sigma}}} & B \operatorname{Aut}_{G_{a\sigma}}^I F_{G_{a\sigma}}
 \end{array}$$

for every $a \in G$. The first square commutes because B has a G -CW-complex structure. Since \tilde{a} is the classifying map of the $G_{a\sigma}$ -fibration u_{G_σ} and ϕ_{p,G_σ} is the classifying map of p_{G_σ} , the pullback of the universal $G_{a\sigma}$ -fibration $u_{G_{a\sigma}}$ by the composition $\tilde{a} \circ \phi_{p,G_\sigma}$ is $G_{a\sigma}$ -fiber homotopy equivalent to the fibration p_{G_σ} considered as an $G_{a\sigma}$ -fibration. On the other hand, the pullback of the $G_{a\sigma}$ -fibration $p_{G_{a\sigma}}$ by \bar{a} is $G_{a\sigma}$ -fiber homotopy equivalent to the fibration p_{G_σ} , so $(\phi_{p,G_{a\sigma}} \circ \bar{a})^* u_{G_{a\sigma}} \simeq_{G_{a\sigma}} p_{G_\sigma}$. Therefore the $G_{a\sigma}$ -fibrations $(\tilde{a} \circ \phi_{p,G_\sigma})^* u_{G_{a\sigma}}$ and $(\phi_{p,G_{a\sigma}} \circ \bar{a})^* u_{G_{a\sigma}}$ are $G_{a\sigma}$ -fiber homotopy equivalent. By Theorem 2.4, the maps $\tilde{a} \circ \phi_{p,G_\sigma}$ and $\phi_{p,G_{a\sigma}} \circ \bar{a}$ are homotopic and hence α_p is a cochain in $C_{\mathcal{H}}^{n+1}(B, \underline{\pi}_n, \mathcal{F})$.

Proposition 4.2 α_p is a cocycle.

Proof For an $H \in \operatorname{Iso}(B)$, $\alpha_p(H) \in C^{n+1}(B^H, \pi_n(B \operatorname{Aut}_H^I F_H))$ is the obstruction cochain for extending the map $\phi_{p,H}: B_n^H \rightarrow B \operatorname{Aut}_H^I F_H$ to the $(n+1)$ -skeleton of B^H . Therefore, by classical obstruction theory, we have

$$(\delta\alpha_p)(H)(\tau) = \delta(\alpha_p(H))(\tau) = 0$$

for any $(n+2)$ -cell τ of B^H . On the other hand, for arbitrary $H \in \mathcal{H}$, we have $(\delta\alpha_p)(H)(\tau) = (\operatorname{Res}_H^{G_\tau})_*(\delta\alpha_p(G_\tau)(\tau)) = 0$. So, $\delta\alpha_p = 0$. □

We call α_p the *obstruction cocycle*. From now on we assume that the total spaces of G -fibrations that we consider have the homotopy type of a G -CW-complex unless otherwise stated.

Proposition 4.3 A G -fibration $p: E_n \rightarrow B_n$ with the fiber type $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$, where F_H is a finite H -CW-complex, extends to a G -fibration over the $(n+1)$ -skeleton B_{n+1} if and only if $\alpha_p = 0$. Moreover if E_n has the G -homotopy type of a finite G -CW-complex then the total space of the fibration that we obtain has the G -homotopy type of a finite G -CW-complex.

Proof If the obstruction cocycle is zero, then $[\phi_{G_\sigma} \circ f_\sigma] = 0$ for any $(n+1)$ -cell σ of B . By Theorem 2.4, the restriction $p|_{\partial\sigma}$ is G_σ -fiber homotopy equivalent to

the trivial G_σ -fibration $\varepsilon: \partial\sigma \times F_{G_\sigma} \rightarrow \partial\sigma$. Let $\beta_\sigma: \partial\sigma \times F_{G_\sigma} \rightarrow p^{-1}(\partial\sigma)$ be the G_σ -fiber homotopy equivalence between them. By [Corollary 3.8](#), $p|_{\partial\sigma}$ extends to a G_σ -fibration $\tilde{p}_\sigma: Z \rightarrow \sigma$ over σ . Let us define

$$E' = \left(\coprod_{\sigma \in I_{n+1}} G \times_{G_\sigma} (Z \cup_{i_1} (p^{-1}(\partial\sigma) \times I)) \right) \cup_{i_2} E,$$

where I_{n+1} is the set of representatives of G -orbits of $(n+1)$ -cells and i_j 's are the corresponding inclusions for $j = 1, 2$. Let $q: E' \rightarrow B_{n+1}$ be defined by relations

$$q|_Z = \tilde{p}_\sigma, \quad q|_{p^{-1}(\partial\sigma) \times I} = p|_{\partial\sigma} \times \text{id} \quad \text{and} \quad q|_E = p.$$

By [Theorem 2.3](#), the G -map q is a G -fibration.

Since all the fibers and the base space have the G -homotopy type of a G -CW-complex, E' is G -homotopy equivalent to a G -CW-complex. More precisely, for each orbit representative $\sigma \in I_{n+1}$, we can deform $G \times_{G_\sigma} (Z \cup_{i_1} (p^{-1}(\partial\sigma) \times I)) \cup_{i_2} E$ to $G \times_{G_\sigma} (\sigma \times F_{G_\sigma}) \cup_{\beta_\sigma} E$ relative to E via a G -map as shown in [Figure 1](#). \square

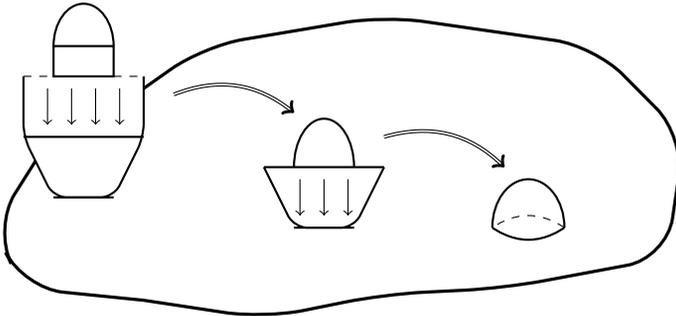


Figure 1

[Proposition 4.3](#) proves the first part of [Theorem 1.1](#). The second part of the theorem says that if α_p is cohomologous to zero, that is, $\alpha_p = \delta d$ for some cochain $d \in C_{\mathcal{H}}^n(B, \underline{\pi}_n, \mathcal{F})$ then the G -fibration $p|_{B_{n-1}}: p^{-1}(B_{n-1}) \rightarrow B_{n-1}$ extends to a G -fibration over B_{n+1} . In order to prove this, we redefine p over the n -skeleton relative to the $(n-1)$ -skeleton in such a way that the obstruction cocycle of this new G -fibration vanishes.

Let p_1 and p_2 be G -fibrations over B_n whose restrictions to B_{n-1} are G -fiber homotopy equivalent. Then by [Corollary 3.6](#), there is a G -fibration $q: Z \rightarrow (B \times I)_n$ with $q|_{B_n \times \{0\}} = p_1$ and $q|_{B_n \times \{1\}} = p_2$. Let $\Psi_{q,H}: (B^H \times I)_n \rightarrow B \text{Aut}_H^I F_H$ be the classifying map of the fibration q_H for $H \in \text{Iso}(B)$. Note that the composition $\Psi_{q,H} \circ i_j$ gives a classifying map for the H -fibration p_j^H where i_j 's are the corresponding inclusions. As before, the map $\text{Res}_H^{G_{ae}} \circ \Psi_{q,G_{ae}} \circ f_{ae}$ is homotopic to $\tilde{\alpha} \circ \text{Res}_K^{G_e} \circ \Psi_{q,G_e} \circ f_e$ for

every $(n+1)$ -cell e of $B^K \times I$ with the attaching map $f_e: \mathbb{S}^{n-1} \rightarrow B_n^{G\sigma}$ and for every $H^a \leq K$. Therefore, the map $d_{p_1,q,p_2} \in \prod_{H \in \mathcal{H}} \text{Hom}_{\mathbb{Z}}(C_n(B^H), \pi_n(B \text{Aut}_H^I F_H))$ defined for an n -cell τ of B^H by

$$d_{p_1,q,p_2}(H)(\tau) = (-1)^{n+1} (\text{Res}_H^{G\tau})_* [\Psi_{q,G\tau} \circ f_{\tau \times I}],$$

is a cochain in $C_{\mathcal{H}}^n(B; \underline{\pi}_n, \mathcal{F})$. We call d_{p_1,q,p_2} the *difference cochain*. As in the standard theory, we have the following.

Proposition 4.4 $\delta d_{p_1,q,p_2} = \alpha_{p_1} - \alpha_{p_2}$.

Proof It suffices to show that $d_{p_1,q,p_2}(H) = \alpha_{p_1}(H) - \alpha_{p_2}(H)$ for every $H \in \text{Iso}(B)$. Let $o_{\Psi} \in C^{n+1}(B^H \times I, \pi_n(B \text{Aut}_H^I F_H))$ be the obstruction cocycle to the extension of $\Psi_{q,H}$ to $(B^H \times I)_{n+1}$. Then $\delta o_{\Psi} = 0$. On the other hand, $o_{\Psi}(H)(\theta) = [\Psi_{q,H} \circ f_{\theta}]$ for every $(n+1)$ -cell θ of $B^H \times I$. Therefore, as in the proof of the corresponding result in the standard theory, we have

$$\begin{aligned} 0 &= (\delta o_{\Psi})(H)(\sigma \times I) \\ &= o_{\Psi}(H)(\partial \sigma \times I) + (-1)^{n+1} (o_{\Psi}(H)(\sigma \times \{1\}) - o_{\Psi}(H)(\sigma \times \{0\})) \\ &= [\Psi_{q,H} \circ f_{\partial \sigma \times I}] + (-1)^{n+1} ([\Psi_{q,H} \circ f_{\sigma \times \{1\}}] - [\Psi_{q,H} \circ f_{\sigma \times \{0\}}]) \\ &= (-1)^{n+1} (d_{p_1,q,p_2}(H)(\partial \sigma) + \alpha_{p_2}(H)(\sigma) - \alpha_{p_1}(H)(\sigma)) \\ &= (-1)^{n+1} (\delta d_{p_1,q,p_2}(H)(\sigma) + \alpha_{p_2}(H)(\sigma) - \alpha_{p_1}(H)(\sigma)) \end{aligned}$$

for any $(n+1)$ -cell σ of B^H . This implies that for every $H \in G$ and for every $(n+1)$ -cell σ of B^H , we have

$$\delta d_{p_1,q,p_2}(H)(\sigma) = (\alpha_{p_1} - \alpha_{p_2})(H)(\sigma)$$

and hence $\delta d_{p_1,q,p_2} = \alpha_{p_1} - \alpha_{p_2}$. □

Proposition 4.4 immediately implies the “if” direction of **Theorem 1.1(2)**. To see this, note that if the G -fibration $p|_{B_{n-1}}$ extends to a G -fibration \tilde{p} over B_{n+1} then

$$\delta d_{p,q,\tilde{p}}|_{B_n} = \alpha_p - \overbrace{\alpha_{\tilde{p}}|_{B_n}}^0 = \alpha_p,$$

hence the cohomology class $[\alpha_p]$ vanishes. For the “only if” direction, we need the following observation.

Proposition 4.5 Let $p: E \rightarrow B_n$ be a G -fibration over B_n with the fiber type $\mathcal{F} = \{F_H\}_{H \in \mathcal{H}}$ where F_H is a finite H -CW-complex for every $H \in \mathcal{H}$. Then for every $d \in C_{\mathcal{H}}^n(B; \underline{\pi}_n, \mathcal{F})$, there is a G -fibration $q: Z \rightarrow (B \times I)_n$ such that $d_{p,q,\tilde{p}} = d$ where $\tilde{p} = q|_{B_n \times \{1\}}$. Moreover if E has the G -homotopy type of a finite G -CW-complex then the space $q^{-1}(B_n \times \{1\})$ has the G -homotopy type of a finite G -CW-complex.

Proof For an n -cell τ of B , the G_τ -map $\bar{p}_\tau: p^{-1}(\bar{\tau}) \cup p^{-1}(\partial\tau) \times I \rightarrow \bar{\tau} \cup \partial\tau \times I$, where $\bar{p}_\tau|_{p^{-1}(\bar{\tau})} = p|_{p^{-1}(\bar{\tau})}$ and $\bar{p}_\tau|_{p^{-1}(\partial\tau) \times I} = p|_{p^{-1}(\partial\tau)} \times \text{id}$, is a G_τ -fibration and it is classified by the map $\phi_{p, G_\tau} \circ \pi_1$ where $\pi_1: \bar{\tau} \cup \partial\tau \times I \rightarrow \bar{\tau}$ is the projection to the first coordinate. Let E_τ be the pullback of \bar{p}_τ by the map

$$\bar{f}_\tau = (f_\tau \circ \pi_1, f_\tau \times \text{id}): \mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I \rightarrow \bar{\tau} \cup \partial\tau \times I$$

where $f_\tau: (\mathbb{D}^n, \mathbb{S}^{n-1}) \rightarrow (\bar{\tau}, \tau)$ is the characteristic map of τ .

Let $X_1 = \{(x, 1) \in \mathbb{D}^n \times \{1\} \mid \frac{1}{2} \leq |x| \leq 1\}$, $X_2 = \{(x, 1) \in \mathbb{D}^n \times \{1\} \mid \frac{1}{4} \leq |x| \leq \frac{1}{2}\}$, and $X_3 = \{(x, 1) \in \mathbb{D}^n \times \{1\} \mid 0 \leq |x| \leq \frac{1}{4}\}$. Let $p_1: E_\tau^1 \rightarrow X_1$ be the induced G_τ -fibration $(\bar{f}_\tau f)^*(\bar{p}_\tau)$ where $f: X_1 \rightarrow \mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I$ is given by

$$f(x, 1) = \begin{cases} (2x - x/|x|, 4|x| - 3) & \frac{3}{4} \leq |x| \leq 1, \\ (2x - x/|x|, 0) & \frac{1}{2} \leq |x| \leq \frac{3}{4}. \end{cases}$$

Note that $p_1|_{\mathbb{S}_1^{n-1} \times \{1\}} = f_\tau^*(\bar{p}|_{\partial\tau})$ and $p_1|_{\mathbb{S}_{1/2}^{n-1} \times \{1\}}$ is the trivial G_τ -fibration with the fiber $F = p^{-1}(f_\tau(0))$, where \mathbb{S}_r^{n-1} is the $(n-1)$ -sphere of radius r .

Since $d(G_\tau)(\tau) \in \pi_n(B \text{Aut}_{G_\tau}^I F_{G_\tau})$, it is represented by a map

$$\Psi_\tau: (\mathbb{D}_{1/4}^n, \mathbb{S}_{1/4}^{n-1}) \rightarrow (B \text{Aut}_{G_\tau}^I F_{G_\tau}, *).$$

Let $u'_{G_\tau} = E \text{Aut}_{G_\tau}^I F_{G_\tau} \times_{\text{Aut}_{G_\tau}^I F_{G_\tau}} F \rightarrow B \text{Aut}_{G_\tau}^I F_{G_\tau}$. Then the restriction of the G_τ -fibration $\Psi_\tau^*(u'_{G_\tau}): E_\tau^2 \rightarrow X_3$ to $\mathbb{S}_{1/4}^{n-1} \times \{1\}$ is the same as the trivial G_τ -fibration with the fiber F . By gluing these fibration with the trivial one over X_2 , we obtain a G_τ -fibration $E_\tau^1 \cup F \times X_2 \cup E_\tau^2 \rightarrow \mathbb{D}^n \times \{1\}$ over $\mathbb{D}^n \times \{1\}$. Let $\tilde{p}_\tau: \tilde{E}_\tau \rightarrow \tau$ be the corresponding G_τ -fibration over τ . As in the proof of [Proposition 4.3](#), the G -map

$$\begin{aligned} \tilde{E} &= \left(\coprod_{\tau \in I_n} G \times_{G_\tau} (\tilde{E}_\tau \cup_{i_1} p^{-1}(\partial\tau) \times I) \right) \cup_{i_2} p^{-1}(B_{n-1}) \\ &\quad \tilde{p} \downarrow \\ &\quad B_n \end{aligned}$$

is a G -fibration over B_n . Moreover, when E has the G -homotopy type of a G -CW-complex, so does \tilde{E} .

Let $q: E \cup (p^{-1}(B_{n-1}) \times I) \cup \tilde{E} \rightarrow (B \times I)_n$ be the G -fibration defined by $q|_E = p$, $q|_{\tilde{E}} = \tilde{p}$, and $q|_{p^{-1}(B_{n-1}) \times I} = p|_{B_{n-1}} \times \text{id}$. Then $d_{p,q,\tilde{p}}(G_\tau)(\tau)$ is represented by the classifying map

$$\tilde{\Psi}: \mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I \cup X_1 \cup X_2 \cup X_3 \rightarrow B \text{Aut}_{G_\tau} F_{G_\tau},$$

where

$$\begin{aligned} \widetilde{\Psi}|_{\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I} &= \phi_{p,G_\tau} \pi_1 \bar{f}_\tau, & \widetilde{\Psi}|_{X_1} &= \phi_{p,G_\tau} \pi_1 \bar{f}_\tau f \\ \widetilde{\Psi}|_{X_2} &= c_{\phi_{p,G_\tau}(f_\tau(0))}, & \widetilde{\Psi}|_{X_3} &= \Psi_\tau. \end{aligned}$$

Here, $c_{\phi_{p,G_\tau}(f_\tau(0))}$ is the constant map at $\phi_{p,G_\tau}(f_\tau(0))$. Since $\widetilde{\Psi}|_{\mathbb{D}^n \times \{0\} \cup \mathbb{S}^{n-1} \times I \cup X_1}$ is homotopic to the constant map $c_{\phi_{p,G_\tau}(f_\tau(0))}$ relative to $\mathbb{S}^{n-1}_{1/2} \times \{1\}$, the classifying map $\widetilde{\Psi}$ also represents $d(G_\tau)(\tau)$. Therefore, we have $d = d_{p,q,\tilde{p}}$. \square

Now we can prove the “only if” of the main theorem as follows.

Proof of Theorem 1.1 It only remains to show that if α_p is cohomologous to zero then there is a G -fibration over B_{n+1} which extends $p|_{B_{n-1}}$. Let $\alpha_p = \delta d$ for some $d \in \text{Hom}(C_n(B), \underline{\pi}_{n,F})$. By Proposition 4.5, there is a G -fibration q over $B \times I$ such that $d = d_{p,q,\tilde{p}}$ where $\tilde{p} = q|_{B_n \times \{1\}}$. Since $\alpha_p = \delta d = \alpha_p - \alpha_{\tilde{p}}$, we have $\alpha_{\tilde{p}} = 0$ and hence \tilde{p} extends to a G -fibration over B_{n+1} . \square

Remark In Theorem 1.1, one can replace the assumption that B^H is simply connected for every $H \in \mathcal{H}$ with the assumption that the map $\pi_1(B^H) \rightarrow [F_H, F_H]_H$ is trivial for every $H \in \mathcal{H}$. In applications, one often has fibers which are homotopy equivalent to spheres and one can take fiber joins to make this map trivial.

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